The Granular Origins of Tail Dispersion Risk*

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August 20, 2024

Abstract

We study tail risk in the cross-section of asset prices at high frequencies. The tail behavior of the cross-section depends on whether a systematic jump event occurred. If so, the cross-sectional return tail is governed by assets' exposures to the systematic event while, otherwise, it is determined by idiosyncratic jumps. An estimator for the tail shape of the cross-sectional distribution displays distinct properties with and without systematic jumps. We show empirically that shocks to the cross-sectional tail shape are a source of priced risk: fat idiosyncratic tails are favored by investors, while fat-tailed exposures to systematic jumps are disliked.

Keywords: cross-sectional asset pricing, dispersion, high-frequency data, power law, tail risk.

*We thank Rob Engle, Xavier Gabaix, Rene Garcia, Markus Pelger, Harald Uhlig, Mark Watson and numerous participants at various presentations for helpful comments and suggestions. Andersen and Todorov gratefully acknowledge financial support from the Kenney Fund at the Kellogg School of Management.

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1 Introduction

Tail risk in asset returns plays an important role in financial economics. In particular, there is extensive evidence showing that investors demand compensation for bearing tail risk. While most of the focus in the existing asset pricing literature has been on tail risk in the time series, our objective in this study is cross-sectional tail risk in stock returns at high frequencies. This effort aims to exploit the rich information contained in the cross-section of equity returns for asset pricing models.[1]

The focus on high-frequency returns allows us to establish, in a model-free non-parametric manner, a connection between the cross-sectional tail risk and the underlying features of the assets' return dynamics. Specifically, we show that the tail behavior of the cross-sectional return distribution differs depending on whether the time interval contains a systematic jump event or not. The latter is defined as a jump with a pervasive effect on the cross-section of asset prices, i.e., it affects a nontrivial fraction of the stocks. Examples of systematic jumps are times when the market portfolio jumps or, more generally, when systematic return factors exhibit a jump.[2] When a systematic jump is present within a short time interval, the tails of the cross-sectional return distribution are governed by the assets' exposure to that jump risk. When this is not the case, the tails of the cross-sectional return distribution are determined by the tail properties of the idiosyncratic jump risk in asset prices. On the contrary, if we consider returns over coarser time intervals, the cross-sectional tail behavior will be governed by a mixture of these two sources of tail risk as well as the properties of the time-varying asset volatility. The use of high-frequency data, therefore, is critical in disentangling the different sources of tail risk in asset prices.

We develop inference tools for assessing the cross-sectional tail risk in asset prices at high frequencies. Specifically, assuming the cross-sectional return tails obey an approximate power law, we characterize the tail behavior via two parameters only – the tail scale and tail shape. We propose non-parametric estimators of the tail

[1]Investors pay close attention to the cross-sectional return distribution, as noted, e.g., in the *Financial Times* article "Narrow markets should humble macro forecasters" from August 7, 2023, on how the recent equity market gains were distributed across the cross-section of stocks.

[2]While many systematic jumps can be linked to observable factors, Jacod et al. (2024) show that the cross-section of asset returns often is exposed to systematic events not readily associated with jumps in observable factors.

shape parameters using a large cross-section of returns. The estimators have different asymptotic behavior depending on whether the time interval contains a systematic jump or not due to the very different manifestation of idiosyncratic and systematic jumps in high-frequency data. On the one hand, estimating the tail behavior from exposure to systematic jumps is infeasible unless they materialize within our high-frequency intervals. On the other hand, whenever such a jump occurs, a nontrivial fraction of the assets jump and can be used for tail estimation. In contrast, the likelihood of an idiosyncratic jump is proportional to the interval length, which shrinks asymptotically in our setting. Nonetheless, in this scenario, we can exploit a set of consecutive time intervals for estimation. This is not feasible for systematic jumps because they, by definition, are rare events that we must pool over longer time periods to enable formal analysis. We establish the asymptotic properties of our tail estimators under the condition that both the number of stocks and the sampling frequency diverge. We also propose a goodness-of-fit test for the power law in the tails based on a Kolmogorov-Smirnov (KS) statistic.

Implementing our inference procedures, the goodness-of-fit test suggests that the power law provides a good approximation to the tail features of the cross-sectional return distributions for the S&P 500 index constituents at the 10-minute frequency between 2003 and 2022. We document nontrivial variation in the time series of the tail shape indices over the sample period. In addition, the time-series variation in the cross-sectional tail shape indices with and without systematic events differ, and their dynamics are distinct from that of the market volatility as well as the common idiosyncratic volatility, i.e., the cross-sectional average of idiosyncratic volatility.

These differences in time-series behavior imply that shocks to the tail shape parameter of the cross-sectional high-frequency return distribution may constitute a distinct source of systematic risk that is of concern to investors. We investigate this hypothesis through formal asset pricing tests. For this analysis, we use daily returns for the universe of all traded stocks between 2004 and 2022, except for the exclusion of micro-cap and penny stocks, following common practice in the literature. Hence, we enlarge the cross-section in the asset pricing test significantly relative to the one used for constructing the tail shape indices, with the latter requiring returns that are robust to market microstructure frictions. Our interest is whether innovations (shocks) to the cross-sectional tail shape indices are priced sources of risk. Towards this end, we estimate the assets' exposure towards these shocks using daily returns and our daily time series of tail shape indices. We then implement a classic sorting exercise on the tail shape betas and check if the generated spread of the high-minus-low portfolios can be rationalized with exposures to existing systematic factors.

We find that stocks with high exposure to positive shocks to the systematic jump tail shape, i.e., stocks that perform relatively well when the tails fatten, have low future returns. The return spread between the Low and High quintile portfolios is economically large and remains statistically significant after controlling for a number of systematic risk factors including the Market, Fama-French three/five/six factors (FF3/FF5/FF6, Fama and French (1993, 2015, 2018)), the tail risk factor by Kelly and Jiang (2014), the idiosyncratic risk factor of Ang et al. (2006)), and the common idiosyncratic volatility factor of Herskovic et al. (2016). The portfolio performance is robust to different weighting schemes (equal/value weight) and portfolio holding windows (one/three months). The results suggest that investors dislike fat tails in the distribution of assets' exposure to systematic jumps and, vice versa, favor thin tails. Economically, it implies that periods of elevated cross-sectional dispersion at systematic jump events are viewed as bad scenarios and hedging them requires a risk premium in equilibrium. Of course, this is also consistent with the standard view that investors are averse to increased return dispersion induced by systematic risk.

In contrast, our analysis reveals a striking and significant reversal in the sign of the risk premium when we consider shocks to the tail shape index of the idiosyncratic jumps. These tail shocks are also priced, but the price of risk is now negative. That is, investors react favorably to a fattening of the idiosyncratic jump tails in asset returns. The return spread between the High and Low quintile portfolios sorted on the idiosyncratic tail shape betas is positive and statistically significant after controlling for our set of common risk factors. These portfolio sorting results are robust to both the equal/value weighting schemes and portfolio holding window (one/three months). This finding is harder to rationalize within standard economic models in which higher volatility and/or jump risk typically is disliked by investors. However, this type of result has been obtained in a number of prior studies, with both behavioral and rational explanations having been put forth. The behavioral perspective notes that many investors have an element of lottery-like preferences, as documented in equity and option market settings by, e.g., Boyer and Vorkink (2014); Blau et al. (2016) and Filippou et al. (2018). Prior theoretical research has also explored the modeling of lottery-like preferences and their pricing implications; for example, the optimal belief model by Brunnermeier et al. (2007) and the cumulative prospect theory by Barberis and Huang (2008). Given heterogeneity for the preference for skewness, this readily leads to equilibrium cross-sectional asset pricing effects, where some agents sacrifice a degree of mean-variance optimality in exchange for exposure to positive skewness, including idiosyncratic skewness, see, for example, Mitton and Vorkink (2007). A supplementary perspective, rooted in heterogeneous firm characteristics, emphasizes the real growth options and operational flexibility of firms as a feature leading to positive idiosyncratic skewness. Such firms have longer-term convex payoffs and generate asymmetric, lottery-like upside potential for investors, as documented by, e.g., Cao et al. (2008), Del Viva et al. (2017), and Ho et al. (2023).

The finding of a significant premium for tail shape risk that changes sign - depending on whether the cross-sectional return dispersion stems from systematic or idiosyncratic jumps - demonstrates the necessity of treating price increments with and without systematic jumps differently. A natural question is whether these tail shape risks are related? We find that the correlation between the tail shape shocks and the high-versus-low portfolios sorted on the different tail-shape betas is very weak. Consequently, the economic mechanisms explaining these pricing effects differ, as alluded to above. From a practical point of view, it implies that one can obtain even stronger performance by exploiting these pricing effects jointly. Towards this end, we construct a simple equally-weighted portfolio with the two sorted portfolios, namely, the HmL portfolio sorted on idiosyncratic jump tail shape shock betas and the LmH portfolio sorted on systematic jump tail shape shock betas. The combined portfolio has zero-net cost and achieves additional diversification through the weighting scheme. We confirm that this portfolio, as hypothesized, delivers a higher Sharpe ratio than the two individual tail-risk sorted portfolios.

Related Literature

Our study is situated within a broader literature that explores the power-law tail behavior of various economic and financial variables. Power-law tail patterns have been observed in domains ranging from city sizes (Kingsley Zipf (1932); Gabaix (1999); Eeckhout (2004)), income distributions of companies (Okuyama et al. (1999)), firm sizes (Axtell (2001)), macroeconomic disasters (Barro and Jin (2011)), and stock trading volume (Gopikrishnan et al. (2000)). These studies find that power-law tail behavior is crucial for comprehending key mechanisms in economics and finance like the source of aggregate economic fluctuations. For example, Gabaix et al. (2003, 2006) argue that heavy-tailed financial returns and the stock market crashes can be explained by concentrated trades by large market participants. The heavy-tailedness in firm sizes is used to explain the aggregate economic movements in the US market (Gabaix (2011)) and the international market (Di Giovanni and Levchenko (2012)). Acemoglu et al. (2012, 2017) show that when there exist fat-tailed inter-sectoral input-output linkages, micro-economic idiosyncratic shocks may lead to sizable aggregate fluctuation and systematic macroeconomic tail risks.

Our paper is closely related to studies exploring the power-law behavior of stock returns. Financial returns are known to conform to a heavy-tailed distribution that can be accommodated by a Pareto distribution; see, e.g., Mandelbrot (1963); Fama (1965); Gopikrishnan et al. (1999); Gabaix (2012). Bollerslev and Todorov (2011) propose a non-parametric estimator of tail-shape risk, which is the tail-shape parameter of the power law based on high-frequency data of univariate processes. Notably, substantial time-variation and serial dependence is found in the market tail-risk index by Bollerslev and Todorov (2011) and Bollerslev et al. (2015), with the latter using option data for estimation. Unlike these studies, our focus is on the cross-sectional tail behavior in a large number of assets and its implications for asset prices.

Our focus on the tails of the cross-sectional asset return distribution is reminiscent of and inspired by the approach in Kelly and Jiang (2014), who study the time-varying cross-sectional tail risk estimated from daily data. They find empirically that the daily tail shape index can be used in stock return prediction. Different from Kelly and Jiang (2014), we investigate the cross-sectional tail behavior of returns at high frequencies. We find that this behavior differs both from a statistical and economic point of view depending on whether the time interval contains a systematic jump event or not. In that sense, we explore the granular origin of cross-sectional tail risk in asset prices.[3]

Finally, our work is closely connected to the general literature on tail risk and time-varying volatility risk, as well as their effect on the cross-section of asset prices. The literature on tail risk and asset pricing is extensive. Jump tail risk, and in particular the left jump tail, is shown to be helpful in predicting future returns in the U.S. market (Bollerslev et al. (2015); Andersen et al. (2015)) and internationally (Andersen et al. (2020, 2021). Lin and Todorov (2019) find that aggregate idiosyncratic asymmetric jump variations predict future equity returns. Several studies show that tail risks help explain the cross-section of asset returns. Cremers et al. (2015) find that stocks with high exposure to aggregate jump risk have contemporaneous high expected returns in the cross-section. Bollerslev et al. (2016) show that jump betas entail significant risk premiums. Bollerslev et al. (2020) find stocks with high positive-minus-negative jump volatilities have high returns. Ang et al. (2006) find that stocks with low (idiosyncratic) volatilities have high positive returns. Bali et al. (2011) explain the puzzling effect of idiosyncratic risk using lottery-like preferences and find that stocks with historically high maximum returns are overpriced and have low future returns in the cross-section. Herskovic et al. (2016) evaluate the aggregated idiosyncratic risk and find that shocks to the common idiosyncratic volatility (CiV) are negatively priced. We contribute to this strand of the literature by examining the distinct pricing implications of shocks to the tail shape of the cross-sectional asset return distribution stemming from systematic versus idiosyncratic high-frequency jumps.

Outline

The remainder of the paper is organized as follows. Section 2 introduces our model setup. In Section 3, we compare the cross-sectional and time-series asset return tail behavior. Section 4 presents our theoretical inference results. We assess the cross-sectional asset return tail dispersion risk empirically in Section 5. The asset pricing

[3] In recent work, Almeida et al. (2023) also explore the cross-sectional high-frequency return tails. They focus exclusively on large downside moves and do not distinguish between systematic and idiosyncratic jumps. They find that stocks' exposure to the physical left tail has little predictive power for cross-sectional returns, while the corresponding risk-neutralized tail returns do have significant explanatory power along this dimension.

implications of the cross-sectional asset tail risk variation are explored in Section 6. Section 7 gives the concluding remarks. Appendix A outlines the proofs of the main theoretical results. All additional mathematical details and proofs, a Monte Carlo study as well as additional empirical results are collected in Appendices B–F of the supplementary materials.

2 Setup

We denote the price of an asset i at time t by P_{it} for $i = 1, ..., N$. The log-price $p_{it} = \log(P_{it})$ has the following general dynamics,

$$p_{it} = p_{i0} + \int_0^t \alpha_{is} ds + \int_0^t \boldsymbol{\beta}_{i,s}^\top d\mathbf{W}_s + \int_0^t \widetilde{\sigma}_{is} d\widetilde{W}_{is} + J_{it} + \widetilde{J}_{it}, \tag{2.1}$$

where $\mathbf{W}_t = (W_t^1, ..., W_t^k)^\top$, for some positive integer k and $W_t^1, ..., W_t^k$, $\widetilde{W}_{1t}$,..., $\widetilde{W}_{Nt}$ are independent standard Brownian motions, J_{it} and $\widetilde{J}_{it}$ are the systematic and idiosyncratic jump components of the asset prices, respectively. The formal definition of these processes is given in Appendix C. On an intuitive level, however, the difference between systematic and idiosyncratic jumps is clear: the former are pervasive in the sense that they arrive together and impact a nontrivial fraction of the cross-section of asset prices, while the latter arrive independently. Our setup is very general and does not involve any assumption regarding the source of risk driving the systematic jumps, i.e., we do not link the systematic jumps to jumps of observable systematic risk factors. Similarly, we do not make an assumption about the source of systematic diffusive risk. This is an important generality of our setup given the large number of alternative systematic factors put forth in the asset pricing literature.

As we will see later, the cross-sectional tail behavior of asset prices at high frequencies depends critically on whether the given interval contains a systematic jump or not. The timing of systematic jumps can be consistently estimated using the method of Jacod et al. (2024). We assume that this identification of systematic jump locations has been performed.

Towards this end, suppose we sample the asset prices over the fixed interval $[0, 1]$ at equidistant times $0, 1/n, 2/n, ..., 1$ and denote the length of the sampling interval by $\Delta_n = 1/n$ and the log-price increment by $\Delta_j^n p_i = p_{i,j\Delta_n} - p_{i,(j-1)\Delta_n}$, for $j = 1, ..., n$. Let us denote with T_n the set of indices of the high-frequency increments containing the systematic jumps, with $\widehat{T}_n$ denoting a consistent estimator of this set. Formal definitions of T_n and $\widehat{T}_n$ are given in Appendix C. Our tail inference will be performed separately on $\widehat{T}_n$ and its compliment set $\widehat{T}_n^c$, i.e., on the set of increments with and without detected systematic jumps.

We focus on the cross-sectional tail behavior of high-frequency returns which, in turn, is linked to the tail behavior of systematic and idiosyncratic jumps in asset prices. This tail risk can vary over time in a stochastic way and we will assume that this variation is adapted to the σ-algebra $\mathcal{C}$ of "common shocks", which contains various aggregate level shocks. Intuitively, $\mathcal{C}$ captures everything that is related to systematic risk in the economy.

We will assume that the tails of the large jumps have regular variation. Similar to Bollerslev and Todorov (2011), this assumption is formulated in terms of jumps in the asset price level (recall that p_{it} denotes the log-price). More specifically, for a generic function $g : \mathbb{R} \to \mathbb{R}$, we denote $g^{\pm}(x) = \frac{g(\pm \log(1+x))}{1+x}$, and the tail of the measure $g^{\pm}(x)$, henceforth indicated by the corresponding capital letter, is then,

$$G^{\pm}(x) := \int_x^{\infty} g^{\pm}(u)du, \quad \text{for some } x > 0.$$

If g is a probability density of a random variable X, then $G^{\pm}(x)$ are the positive and negative tail probabilities of $e^X - 1$ (for $X > 0$) and of $e^{-X} - 1$ (for $X < 0$).

Conditional on $\mathcal{C}$, the systematic jumps in the asset prices are assumed to be identically and independently distributed in the cross-section at each jump time p, with conditional jump distribution given by $f_p(x)$. We assume regular variation for $f_p(x)$. That is,

$$\frac{F_p^{\pm}(x+u)}{F_p^{\pm}(x)} \approx (1 + u/x)^{-1/\xi_S^{\pm}}, \quad \text{as } x, u \to +\infty, \tag{2.2}$$

for some tail shape parameters $\xi_S^{\pm}$, which are $\mathcal{C}$-adapted random variables. We make this approximation formal in Appendix C. Our assumption of an i.i.d. jump size distributions in the cross-section, conditional on $\mathcal{C}$, is natural if we think of the available stocks as being drawn randomly from a population with an infinite set of assets, see, e.g., Gagliardini et al. (2016) for a detailed discussion of this perspective.

We make a similar regular variation assumption for the idiosyncratic jumps. Specifically, letting $\nu_{t,i}(x)$ denote the time-varying jump intensity of $\widetilde{J}_{it}$, we assume,

$$\frac{\mathbb{E}(\mathcal{V}^{\pm}_{t,i}(x+u)|\mathcal{C})}{\mathbb{E}(\mathcal{V}^{\pm}_{t,i}(x)|\mathcal{C})} \approx (1+u/x)^{-1/\xi^{\pm}_{I}}, \quad \text{as } x, u \to +\infty, \tag{2.3}$$

for some tail shape parameters $\xi^{\pm}_{I}$, which are $\mathcal{C}$-adapted random variables. Note that in the case of idiosyncratic jumps, our assumption is for the $\mathcal{C}$-conditional expected jump intensity. In this manner, we can accommodate settings in which the tail properties of individual stocks differ in the cross-section.

3 Time-Series versus Cross-Sectional Tails

It is useful to contrast the tails in the time-series and cross-section of asset prices. We start with the systematic jumps. It is common in many asset pricing models to assume that systematic jumps obey a linear factor structure. For example, if,

$$\Delta J_{ip} = \gamma_i \times f_p,$$

for γ_i being factor loadings and f_p the value of a factor at jump time p, then f_p is adapted to the common shock σ-algebra $\mathcal{C}$, and we have only a single realization of it at time p. Hence, for this jump factor model, the cross-sectional tail behavior of asset returns at time p is governed by the tail behavior of the jump factor loadings, γ_i, and not by the factor tails. On the other hand, for systematic jumps, the time-series tail behavior of a specific stock is determined by the tail behavior of the factor f_t and not the factor loading, assuming, of course, that the latter does not vary over time.

Moving next to the case of idiosyncratic jumps, suppose that these jumps follow time-changed compound Poisson distributions with jump size governed by a double-exponential distribution. Specifically, assume,

$$\nu_{t,i} \;=\; \phi_{t,i}\,(e^{-\lambda_i^{-}|x|}1_{\{x<0\}} + e^{-\lambda_i^{+}|x|}1_{\{x>0\}}),$$

for some constants $\lambda_i^{\pm} > 0$ and a stochastic process $\phi_{t,i}$, which controls the time-varying probability of jump arrivals. Then the time-series tail behavior of the idiosyncratic jumps for stock i are determined by the two parameters $\lambda_i^{\pm}$. Now, if these parameters are drawn from a uniform distribution on $[\lambda_l^{\pm}, \lambda_h^{\pm}]$, the cross-sectional tail behavior of the idiosyncratic jumps will be governed by $\lambda_l^{\pm}$ – the tail parameters representing the fattest tails in the universe of stocks.

4 Inference for Return Tails at High Frequency

This section develops inference tools for exploring the tails of cross-sectional return distributions at high frequency. Section 4.1 deals with the case in which the interval contains a systematic jump event, while Section 4.2 handles the case without systematic jumps in the interval. Finally, Section 4.3 introduces a goodness-of-fit test for the power law of the cross-sectional return distribution tails. Without loss of generality, our exposition focuses on the positive systematic jump tails. For this reason, we remove the superscript "+" in the notation in this section.

4.1 Systematic Jump Tail Decay Index Estimation

We first consider the case where the high-frequency intervals involved in the estimation contain systematic jumps. For such increments, the leading tail component of the asset returns are the systematic jumps. Idiosyncratic jumps may also be present, but they are (asymptotically) rare, as the probability of an idiosyncratic jump in a given stock is approximately proportional to the length of the interval and, hence, they do not distort the inference. As a result, we may develop our inference tools by focusing strictly on the systematic jump and ignoring the presence of other asset price components. Given the simple returns,

$$R_{ij} \;=\; \frac{P_{ij\Delta_n} - P_{i(j-1)\Delta_n}}{P_{i(j-1)\Delta_n}}\,, \tag{4.1}$$

we propose the following estimator of ξ_S^+,

$$\widehat{\xi}_S \;=\; \frac{1}{\widehat{M}_N^S} \sum_{i=1}^{N} \sum_{j \in \widehat{\mathcal{T}}_n} \log(R_{ij}/\rho_N^S) \mathbf{1}_{\{R_{ij} > \rho_N^S\}}, \tag{4.2}$$

where

$$\widehat{M}_N^S \;=\; \sum_{i=1}^{N} \sum_{j \in \widehat{\mathcal{T}}_n} \mathbf{1}_{\{R_{ij} > \rho_N^S\}}, \tag{4.3}$$

and for some sequence $\rho_N^S \to \infty$. This is simply the Peak-Over-Threshold (POT) estimator. Theorem 1 provides the central limit theorem (CLT) for $\widehat{\xi}_S$. In its statement and henceforth, we denote $\mathcal{C}$-conditional convergence in law with $\overset{\mathcal{L}|\mathcal{C}}{\to}$. The latter is the usual convergence in law, but applied to conditional distributions.

Theorem 1 *For the process $\{p_{it}\}_{i \geq 1}$ defined in (2.1), assume Assumptions 1–3, 5 and Condition SJ hold. Then*

$$\sqrt{\widehat{M}_N^S}\,(\widehat{\xi}_S - \xi_S) \;\overset{\mathcal{L}|\mathcal{C}}{\to}\; N\big(0, \xi_S^2\big), \tag{4.4}$$

where $N(0, \sigma^2)$ denotes normal distribution with zero mean and variance σ^2.

A feasible CLT follows readily by replacing ξ_S^+ in the variance term by $\widehat{\xi}_S^+$. We note that the number of large systematic jumps over a given interval is fixed but since systematic jumps are pervasive in the cross-section, we can use most of the asset returns for inference at jump events. The convergence rate is determined by the number of returns exploited in the estimation. This, in turn, depends on how accurately the power law describes the jump tail distribution and the sampling frequency, as this determines the size of the "residual" components of the asset prices. The restriction on the number of observations used in estimation is provided in Condition SJ.

4.2 Idiosyncratic Jump Tail Decay Index Estimation

We now turn to the case where the high-frequency increments used in estimation do not contain systematic jumps. In this scenario, the tail behavior of the cross-sectional return distribution is governed by the idiosyncratic jumps. From (2.3), the idiosyncratic jump tails can be approximated by the power law with tail decay parameter ξ_I. Therefore, we propose the following estimator of ξ_I,

$$\widehat{\xi}_I \;=\; \frac{1}{\widehat{M}_N^I} \sum_{i=1}^{N} \sum_{j \in \widehat{\mathcal{T}}_n^c} \log(R_{ij}/\rho_N^I) \mathbf{1}_{\{R_{ij} > \rho_N^I\}}, \tag{4.5}$$

where

$$\widehat{M}_N^I \;=\; \sum_{i=1}^{N} \sum_{j \in \widehat{\mathcal{T}}_n^c} \mathbf{1}_{\{R_{ij} > \rho_N^I\}}, \tag{4.6}$$

and for some sequence $\rho_N^I \to \infty$. The CLT for $\widehat{\xi}_I$ is given in the next theorem.

Theorem 2 *For the process $\{p_{it}\}_{i \geq 1}$ defined in equation (2.1), invoke Assumptions 1, 2, 4, 5 and suppose Condition IJ holds. Then*

$$\sqrt{\widehat{M}_N^I}\, (\widehat{\xi}_I - \xi_I) \;\stackrel{\mathcal{L}|\mathcal{C}}{\to}\; N\big(0, 2\,\xi_I^2\big). \tag{4.7}$$

A feasible CLT follows by replacing ξ_I in the variance term by $\widehat{\xi}_I$. While this result appears similar to that for $\widehat{\xi}_S$, there are critical differences. The number of stocks exhibiting idiosyncratic jumps over a single time interval is asymptotically small, as the sampling interval shrinks towards zero, because the jump probability is approximately proportional to the length of the short interval. This is unlike the systematic jump scenario, since the latter materialize for a nontrivial number of assets simultaneously. Furthermore, the set of high-frequency increments without a systematic jump grows asymptotically, because the number of systematic jumps is fixed. Thus, we can pool many more increments for estimation of the idiosyncratic tail distribution than for the systematic jump tails. Finally, note that the variance term in the CLT also differs from that of $\widehat{\xi}_S$ by a factor of 2. This is because the idiosyncratic jumps arrive randomly in the cross-section of stocks. By contrast, when we condition on a systematic jump, there is no randomness in the inference due to the jump arrival.

As for $\widehat{\xi}_S$, the rate of convergence of $\widehat{\xi}_I$ is governed by the number of increments employed for inference. This, in turn, hinges on the approximation error by the power law in the tails and the size of the discretization error, with the latter depending on the sampling frequency. This feature is captured by Condition IJ.

4.3 Goodness-of-Fit Test for Cross-Sectional Return Tails

This section develops a goodness-of-fit test for the tail power law based on a version of the Kolmogorov-Smirnov (KS) test statistic (see, e.g., Clauset et al. (2009)).

For the systematic jumps, our KS statistic is given by,

$$D_N^S \;=\; \sup_x \left| F_{N,t}^S(x) - P_{N,t}^S(x) \right|, \tag{4.8}$$

where $F_N^S(x)$ is the empirical tail distribution of the systematic jumps,

$$F_N^S(x) \;=\; \frac{1}{\widehat{M}_N^S} \sum_{i=1}^{N} \sum_{j \in \widehat{T}_n} \mathbf{1}_{\{R_{ij} > x\}}, \quad \text{for } x \geq \rho_N^S, \tag{4.9}$$

and $P_N^S(x)$ is the tail probability implied by the estimated Pareto distribution,

$$P_N^S(x) \;=\; (x/\rho_N^S)^{-1/\widehat{\xi}_S}, \quad \text{for } x \geq \rho_N^S, \tag{4.10}$$

with $\widehat{\xi}_S$ defined in equation (4.2).

The next theorem characterizes the key property of our proposed KS statistic.

Theorem 3 *Suppose the assumptions of Theorem 1 apply with $\tau_S(x) \equiv 0$, for all $x \geq \rho_N^S$. Then,*

$$\sqrt{\widehat{M}_N^S}\, D_{N,t}^S \;\overset{\mathcal{L}|\mathcal{C}}{\to}\; \mathcal{K}^S,$$

where $\mathcal{K}^S$ is defined in Appendix D.

The limit distribution $\mathcal{K}^S$ is that of a KS statistic based on a sample of the same size as that used for our estimator based on the exact power-law tail distribution. We can estimate the quantiles of this distribution via simulation.

By replacing the subindex S with I and the set T_n with T_n^c, we obtain corresponding results for the goodness-of-fit test for the power law characterization of the idiosyncratic jump tails. The theoretical results are provided in Appendix E.

5 Empirical Evidence for Cross-Sectional Tails

5.1 Data and Systematic Jump Detection

For tail estimation, we use the S&P 500 Index constituent stocks over 2003–2022. The high-frequency price data are obtained from the TAQ database. We sample prices each 10 minutes from 09:35 EST to 15:55 EST using the previous-tick approach and obtain 38 intraday 10-minute log returns. Following common data cleaning procedures, "bounce backs" are removed. Finally, we use the SPY ETF for the S&P 500 Index as our market proxy. Holidays and half-trading days are excluded. In total, we have about 450 stocks per day and there are 4,993 full trading days in our sample.

We conduct our analysis on market-neutral asset returns, i.e., the returns of from being long an individual stock and short the market index. It is readily seen that, subject to mild local boundedness conditions on the market index, the theoretical results in Section 4 continue to apply when the estimator is based on market-neutral rather than raw returns. We adopt this approach because it facilitates identification of the non-market systematic jump risk, see Jacod et al. (2024).

We start with detection of the systematic jump times, i.e., determining the set $\widehat{T}_n$. The systematic jumps can be split into those that trigger jumps in the market index and those that do not. For the former, we use the high-frequency observations on the SPY index and a standard truncation procedure. For the non-market systematic jumps, we use the method proposed by Jacod et al. (2024), see equations (15)–(17) in that paper. Further details on the systematic jump detection are provided in Appendix B. Upon applying these jump detection procedures, we find that 7% of the days in our sample contain market jumps and 9% involve non-market systematic jumps. In total, the daily systematic jump rate (market and non-market) is 12%. That is, there is one systematic jump about every 8 trading days.

5.2 The Tails of the Cross-Sectional Return Distributions

5.2.1 In the Presence of Systematic Jumps

We first focus on the high-frequency intervals containing systematic jumps. In order to obtain a reasonable large sample size, we aggregate the systematic jumps for a year. For illustration, we pick two years, 2008 and 2016, representing one case with a high and one with a moderate level of volatility. We display the distribution of the 10-minute S&P 500 stock market-neutral returns, when systematic jumps are present during each of these two years in Figure 1. For ease of comparison, we standardize the returns to have a unit standard deviation within each cross-section. The plot reveals that the cross-sectional distributions of the systematic jumps in both years are heavy-tailed. We also note that the year 2016 features a somewhat higher number of detected systematic jumps than 2008.

We next fit a power law to the right and left return tails using the procedures from Section 4. The returns used for estimation are those above the 95-th cross-sectional quantile. In Figure 1, we also plot the logarithm of the empirical tail probabilities along with the fitted values implied by the estimated Pareto distributions. We observe that the cross-sectional return tails appear well approximated by power laws. Our estimates for ξ_t in 2016 are 0.43 for the left and 0.48 for the right tail. Those for 2008 are 0.65 for the left and 0.64 for the right tail. These estimates imply near symmetric tails in both years, while the tails in 2008 are somewhat fatter than in 2016.

5.2.2 Absent Systematic Jumps

We now turn to returns that do not contain systematic jumps. Since they constitute the vast majority of our observations, we can perform this analysis at the daily level. For illustration, we pick two representative days, 2008-10-10 and 2016-03-04, stemming from the years we used above for analysis of the systematic jump tails. On October 10, 2008, market volatility is elevated, while on March 4, 2016, it is about average. We plot the distribution of the 10-minute market-neutral returns on these days in Figure 2. There are no detected systematic jumps on either of the two days. Once more, we standardize the returns to have a cross-sectional unit standard

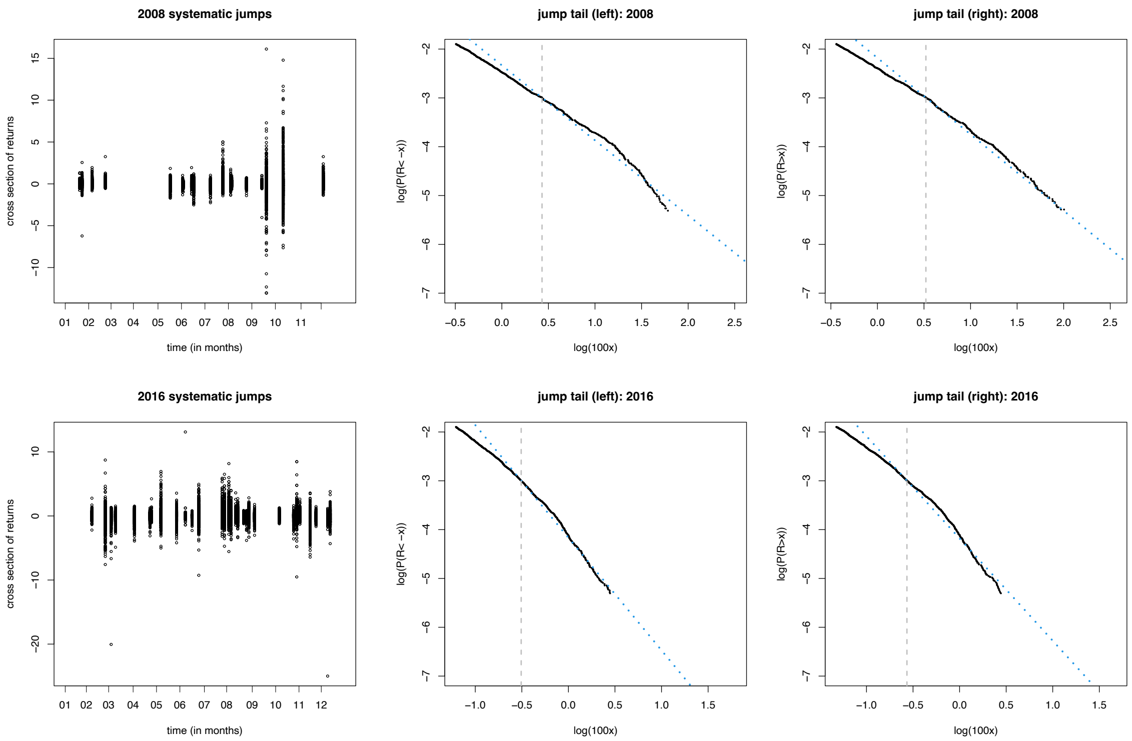

Figure 1: Left: Distribution of S&P 500 Index stocks' 10-minute market-neutral returns when there are systematic jumps for the year 2008 and year 2016. The returns are normalized to have a standard deviation of one. Middle and right: Empirical tail distribution and the fitted Pareto tail of S&P 500 Index stocks' 10-minute returns when there are systematic jumps during the year 2008 and the year 2016.

deviation. The plot corroborates the hypothesis of fat-tailed cross-sectional return distributions. Although the levels of volatility on the two days are very different, the heavy-tailedness of the two cross-sectional return distributions appear similar.

Next, we go on to fit power laws to the tails of the cross-sectional return distributions using the approach developed in Section 4.2. In Figure 2, we also plot the logarithm of the empirical tail probabilities, together with the fitted values by the estimated Pareto distributions. Similar to the case of systematic jumps, we see that the power law provides a good fit for the tails of the cross-sectional return distributions in the absence of systematic jumps. The slope of the Pareto tail is given by $-\alpha_t = -1/\xi_t$. A flatter fitted tail probability line thus indicates a fatter tail distribution. Our estimates of the tail shape index, ξ_t, on March-4-2016 are 0.524 for

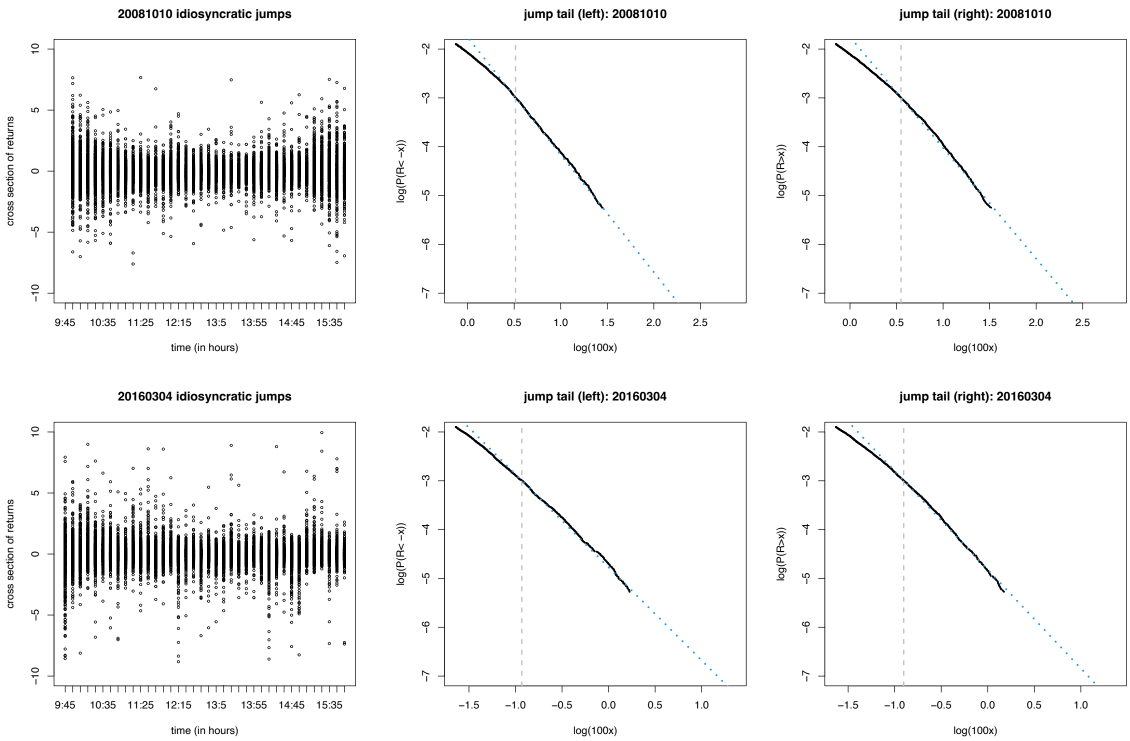

Figure 2: Left: Distribution of S&P 500 stock 10-minute market-neutral returns on 2008-10-10 and 2016-03-04. Systematic jumps are not detected on either of the two days. The returns are standardized to have a standard deviation of one. Middle and right: Empirical tail distribution and the fitted Pareto tail of S&P 500 Index stocks' 10-minute returns of the dates 2008-10-10 and the date 2016-03-04.

left tail and 0.495 for right tail. Those for October-10-2008 are 0.415 and 0.440 for the left and right tails, respectively. This implies that the tails of the cross-sectional return distribution on March-4-2016 are slightly fatter than those on October-10-2008 despite volatility being much higher on October-10-2008.

5.3 Goodness-of-Fit Test for Tails

We now test formally whether the tails of the cross-sectional return distributions are approximated well by the Pareto distribution, performing the test separately for the tails of returns with and without systematic jumps. We use three different cutoffs for the tails corresponding to 7%, 5%, and 3% return quantiles. The test for the system-

Table 1: Goodness-of-fit test results for the power law of the jump tails. We report the proportions of rejections between 2003 and 2022. We test the power-law for the systematic jump tails using data in a moving one-year window and the idiosyncratic jump tail based on daily samples. The significant level is set to 5% and 1%.

Significant level	Power Law Tail Rejection Rate			
	Systematic jump		Idiosyncratic jump	
Goodness-of-Fit Test Rejection Rate				
	left tail	right tail	left tail	right tail
significance level$=$ 5%				
$\bar{\nu}_\phi(\rho_N) = 0.07$	0.039	0.123	0.171	0.164
$\bar{\nu}_\phi(\rho_N) = 0.05$	0.039	0.075	0.114	0.099
$\bar{\nu}_\phi(\rho_N) = 0.03$	0.013	0.052	0.073	0.064
significance level$=$ 1%				
$\bar{\nu}_\phi(\rho_N) = 0.07$	0.013	0.044	0.056	0.053
$\bar{\nu}_\phi(\rho_N) = 0.05$	0.004	0.018	0.027	0.027
$\bar{\nu}_\phi(\rho_N) = 0.03$	0.000	0.000	0.016	0.014

atic jumps is based on a 252-day moving window, while the test for the idiosyncratic jumps is performed on a daily basis. Table 1 summarizes the percentage of days when the power law distribution is rejected over our sample. Overall, the results show that the tails of the cross-sectional return distribution are well approximated by a power law when one looks deep in the tails, including the 5% level used for our empirical work below. Indeed, the rejection rates of the test are low and roughly match the nominal significance levels of the test for the highest levels of truncation.

5.4 Time Series of the Cross-Sectional Tail Shape Indices

This section explores time-series variation in the cross-sectional tail shape. The systematic tail shape index is estimated over the past year from 10-minute returns containing systematic jumps.[4] The idiosyncratic tail shape index is estimated daily.

[4] We have also checked the index estimated using moving window of 22 days. The results are noisier and, thus, less informative due to the limited sample size.

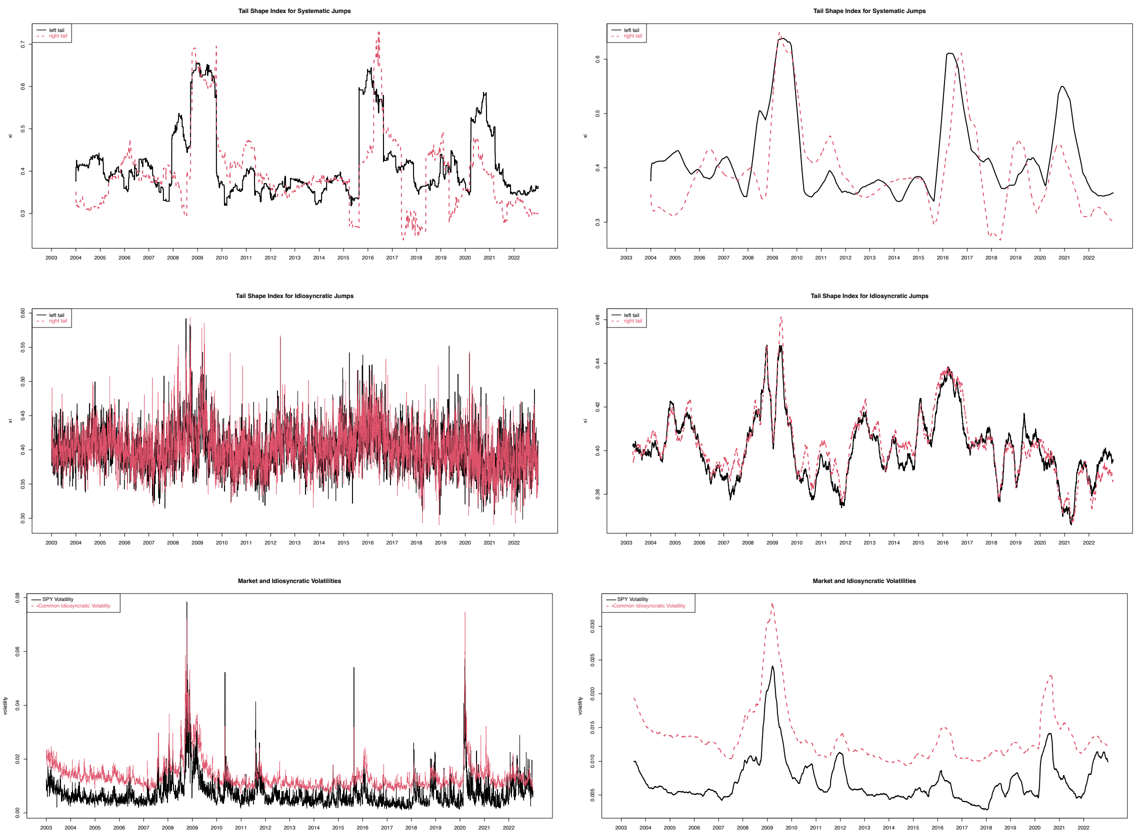

Figure 3: Top: Time series of estimated systematic jump tail shape indices over the past 252 days. Middle: Daily idiosyncratic jump tail shape indices. Bottom: Daily market volatilities and common idiosyncratic volatilities. The left column corresponds to the raw time series and the right to their 132-day moving averages.

Figure 3 (left panel) depicts the time series of the cross-sectional tail shape indices. For reference, we plot the daily market and common idiosyncratic volatilities in the bottom panel. The former estimate is the square-root of the daily realized variance of the SPY index. The common idiosyncratic volatility estimate is the square-root of the cross-sectional average of daily idiosyncratic realized variances, defined as the realized asset return variances minus their market beta estimates times the market return. To better assess the low-frequency pattern in the series, the left panel of Figure 3 provides 6-month (132-day) moving averages of the respective time series.

The figures warrant several comments. First, there is considerable variation in the tail shape of the cross-sectional return distributions over time. The range of the shape indices is between 0.3 and 0.6 for systematic jumps and between 0.35 and 0.5 for idiosyncratic jumps. The associated shape parameter $\alpha = 1/\xi$ is in the range 2–3.5, which is comparable to the shape parameter estimates in typical financial time series such as the ones displayed in Figure 4 below.[5] Second, the left and right tail shapes appear similar. This is easier to see for the returns that do not contain systematic jumps. The discrepancies between the left and right tail estimates are more sizable or returns including systematic jumps, but this is to be expected given the noise stemming from the smaller sample size.[6] Third, there is little correlation between the variation of the tail shapes for returns with and without systematic jumps. For example, the cross-sectional return distribution is fat-tailed during the crisis of 2008, regardless of whether the returns included systematic jumps or not, while the opposite conclusion applies after the onset of the pandemic-related turbulence in 2020. Finally, the dynamics of the tail shapes of the cross-sectional return distribution seems distinct from that of market and common idiosyncratic volatility.

We now compare the time-series variation in the tail shape of the cross-sectional return distribution at times of systematic jumps with the time-series tail shape index estimate of the market factor (proxied by the SPY ETF). The latter time series is depicted in Figure 4. The time-series tail index of SPY is obtained from 10-

[5] Comparing our results with the tail index estimates obtained from the ordinary least squares method in Gabaix et al. (2006), we find the range of the shape parameters to be similar.

[6] We also check the systematic jumps separately based on SPY jumps, and latent systematic jumps. For both types of systematic jumps, the right and left tails appear similar. There is slightly larger variation in SPY jumps, but again the estimates are likely quite noisy.

minute returns with a rolling window of one year, re-estimated monthly. We draw 6-month (132-day) moving averages of the series to assess the low-frequency pattern and compare it with the cross-sectional tail shape indices in the top panel of Figure 3 above. Recall our discussion in Section 3 about the difference in time-series and cross-sectional tail risk associated with systematic jumps drawn from a linear factor model. The current illustration highlights the manifestation of these discrepancies in applied work. Indeed, the dynamics of the time series and cross-sectional tail shapes of systematic jumps do appear to be very different.

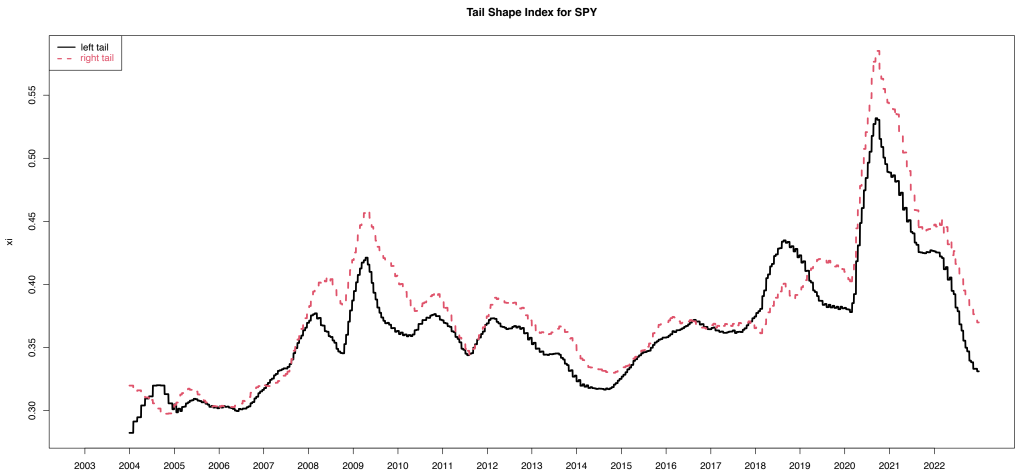

Figure 4: Time series of 132-day moving averages of the tail shape index of the SPY ETF (above).

6 Asset Pricing Implications

Given the significant and distinct time-series variation in the tail shape of the cross-sectional return distribution, we now ask whether it is a source of priced risk. We address this issue using the entire pool of stocks traded on NYSE/AMEX/NASDAQ between 2003 and 2022. The daily stock return data is obtained from CRSP, excluding only micro-cap stocks with size below the 20% quantile of the NYSE breakpoints and stocks with a share price under $5, as is standard in the literature.

6.1 Pricing of Shocks to the Systematic Jump Tails

We start by assessing whether shocks to ξ_S are priced. Because the empirical evidence in the previous section revealed no substantive differences between the left and right tails, we combine the two estimates to improve the efficiency of the inference. That is, we use $\bar{\xi}_{S;t} = 2/(\alpha^L_{S;t} + \alpha^R_{S;t})$, where $\alpha^L_{S;t} = 1/\xi^-_{S;t}$, and $\alpha^R_{S;t} = 1/\xi^+_{S;t}$. We then compute the change between adjacent non-overlapping periods of h days,

$$\Delta\xi_{S;t,h} \;=\; \frac{1}{h}\sum_{j=t-h+1}^{t}\bar{\xi}_{S;j} - \frac{1}{h}\sum_{j=t-2h+1}^{t-h}\bar{\xi}_{S;j}. \tag{6.1}$$

We focus on monthly tail dispersion risk innovations and correspondingly set $h = 22$ (with the unit of time now being one trading day) in the analysis henceforth. Given our definitions, a positive (negative) tail shape shock represents an increase (decrease) in the fat-tailedness of the cross-sectional return distribution during times of systematic jumps. We estimate the exposure of the returns to such systematic jump tail shocks using the following standard time-series regression with $h = 22$,

$$R_{i,t,h} \;=\; a_i + b_i\,\Delta\xi_{S;t,h} + \varepsilon_{i,t}\,, \tag{6.2}$$

where $R_{i,t,h}$ is the h-day cumulative returns from day $t - h + 1$ to t.

For each day t, we run the regression in equation (6.2) using the data from the past 1260 days and get the estimated loadings $(\hat{b}_i)_{i\geq 1}$. We then sort the stocks into 5 groups based on $(\hat{b}_i)_{i\geq 1}$. We form equal- and value-weighted portfolios for each quintile group and track the out-of-sample portfolio returns. Beyond an out-of-sample holding window of one month ($H = 22$), we also explored $H = 66$. We summarize the portfolio performance in Table 2 for the one-month holding window. Results for the three-month holding window are qualitatively similar and deferred to Appendix H.

Panel A of Table 2 reports the out-of-sample annualized average returns for the 5 quintile portfolios and the low-minus-high portfolio (LmH_{SysJ}) that goes long/short the quintile portfolio with the lowest/highest exposure to systematic jump tail dispersion risk shocks. We also report the t-statistics of the returns based on Newey-West standard errors using 22 lags. We observe a monotonic increase in the average returns

Table 2: Returns of systematic jump tail shock beta-sorted portfolios. The table reports out-of-sample annualized average returns (in percentage) and t-statistics for portfolios sorted on the cross-sectional jump tail beta. We sort stocks daily into quintile portfolios using jump tail betas that are estimated using daily returns over the past five years. The holding window of the portfolio is one month. We use all stocks traded on NYSE/AMEX/NASDAQ exchange with price above $5 and stock capitalization beyond the 20% NYSE size breakpoint. We consider both equal- and value-weighted portfolios. The out-of-sample period is between the years 2004 and 2022. Panel A reports the portfolios' average betas to the jump tail shocks and annualized returns, all in percentage, and Panel B reports the alphas under CAPM, FF3/FF5/FF6 models, while controlling for the tail-risk factor (Kelly and Jiang (2014)), the CiV factor (Herskovic et al. (2016)), the idiosyncratic risk factor (Ang et al. (2006)), and the HmL_{IdioJ} portfolio sorted using idiosyncratic jump shock exposures. The t-statistics are computed based on Newey-West (Newey and West (1987)) standard errors with lag length equal to the holding window.

		Equal-weight		**Value-weight**	
		Panel A: Mean Portfolio Returns			
Portfolio	**Beta**	**Ret**	**t-stat**	**Ret**	**t-stat**
H	1.77	10.74	2.59	10.91	3.02
Q4	-0.05	13.00	3.47	10.23	3.34
Q3	-0.93	13.66	3.59	13.50	4.58
Q2	-1.83	14.44	3.71	15.82	5.10
L	-3.81	15.87	3.53	16.14	4.24
LmH_{SysJ}		5.13	2.67	5.23	2.11

	Panel B: Abnormal Return of LmH_{SysJ} Portfolio			
Measure	**Ret**	**t-stat**	**Ret**	**t-stat**
CAPM alpha	3.66	1.92	3.91	1.55
FF3 alpha	3.90	2.08	4.35	1.74
FF5 alpha	4.46	2.31	5.13	2.00
FF6 alpha	5.11	2.66	5.82	2.29
CAPM+tailrisk alpha	4.07	2.02	4.82	1.86
FF6+tailrisk alpha	5.02	2.54	5.92	2.28
CAPM+CiV alpha	4.32	1.83	5.90	1.79
FF6+CiV alpha	5.25	2.19	6.59	1.98
CAPM+idiorisk alpha	4.84	2.19	5.57	2.02
FF6+idiorisk alpha	5.02	2.54	6.23	2.42
CAPM+HmL_{IdioJ} alpha	4.02	2.13	4.98	1.93
FF6+HmL_{IdioJ} alpha	5.43	2.89	6.94	2.64

from the high beta portfolio to the low beta portfolio, based on both equal- and value-weighting schemes. The equal-weighted LmH_{SysJ} portfolio has an annualized return of 5.13% with a t-statistic of 2.67, that is, equivalent to an annualized Sharpe ratio (SR) of 0.62, while the value-weighted LmH_{SysJ} portfolio has an annualized return of 5.23% with a t-statistic of 2.11, or a SR of 0.49.

We further evaluate the significance of the LmH_{SysJ} portfolio's returns after controlling for a variety of systematic risk factors used in previous research. We consider the following factors: the Market, Fama-French three/five/six factors (FF3/FF5/FF6, Fama and French (1993, 2015, 2018)). In addition, we include the tail risk factor by Kelly and Jiang (2014), the idiosyncratic risk factor (idiorisk) of Ang et al. (2006))[7], and the CiV factor of Herskovic et al. (2016).[8] Furthermore, we include the HmL portfolio sorted using idiosyncratic jump tail shape shock betas, introduced in Section 6.2 below, as a control factor. The results are summarized in Panel B of Table 2. We see that the LmH_{SysJ} portfolio has a significant alpha controlling for most of the factors. For example, under FF6, the alpha of the equal-weighted LmH_{SysJ} portfolio is 5.11% ($t = 2.66$), the alpha of the value-weighted LmH_{SysJ} portfolio is 5.82% ($t = 2.29$). After controlling for the tail-risk factor of Kelly and Jiang (2014), the CiV factor of Herskovic et al. (2016), the idiosyncratic risk factor of Ang et al. (2006), and the effect of idiosyncratic jump tail risks (HmL_{IdioJ}), to be introduced below, the alphas of our LmH_{SysJ} portfolio remain significant.

We next compute the pairwise correlation between the tail dispersion risk innovations used for our portfolio sorting and the control factors explored in Panel B of Table 2. We summarize the results in Table 3. It shows that the systematic jump tail-shape shocks have little correlation with the common systematic risk factors, corroborating the hypothesis that systematic jump tail shape shocks represent risks not captured by prior advocated asset pricing factors.

Finally, as a robustness check, in Appendix I, we report portfolio sorting results using different truncation levels in the systematic jump detection procedure. The findings are qualitatively similar to those reported here.

[7]Replicated tail-risk beta LS sorted portfolio of Kelly and Jiang (2014) and the idiosyncratic risk factor portfolio are obtained from Chen and Zimmermann (2021).

[8]The CiV-beta LS portfolio monthly returns are obtained from `https://bernardherskovic.com/data/`

Table 3: Correlations between the innovation in jump tail dispersion risks and the return factors. Reported values are pairwise correlation between the innovations of systematic jump risks, $Inno\xi_S$, idiosyncratic jump risks $Inno\xi_S$, obtained from equation (6.1), and return factors that include the Market (Mkt), FF factors (SMB, HML, RMW, CMA, MOM), the tail-risk factor (Kelly and Jiang (2014)) denoted by tailrisk, the CiV factor (Herskovic et al. (2016)) denoted by CIV.LS, and the idiosyncratic risk factor (Ang et al. (2006)) denoted by idiorisk. In addition, we include the signals used to construct the CiV factor, that is the common idiosyncratic shocks (CiV.shock). The evaluation period is between 2004 and 2022.

	$Inno\xi_I$	Mkt	SMB	HML	RMW	CMA	MOM	tailrisk	CIV.LS	CIV.shock	idiorisk
$Inno\xi_S$	0.008	0.003	-0.003	0.007	-0.005	0.03	0.004	-0.03	0.086	0.042	0.007
$Inno\xi_I$	–	-0.084	-0.006	0.011	-0.011	-0.009	-0.026	0.007	-0.022	0.134	-0.011

Overall, our finding of a negative price response to positive ξ_S shocks implies that investors view periods, in which systematic jumps generate more extreme returns in the cross-section, as unfavorable states of the world. Stocks that do well during such times serve as hedges and require a lower risk premium. This is intuitive from a standard portfolio perspective, as systematic jumps constitute a major source of risk for strategies exposed to systematic risk factors. Enhanced and time-varying return dispersion induced by systematic jumps exacerbates the possibility of poor portfolio performance and renders efficient diversification more difficult. Increasing dispersion risk during times of large systematic return jump events is therefore disliked by investors. We further note that this phenomenon is akin to the beta risk documented in Boloorforoosh et al. (2020) who find that low market-betas tend to increase along with market risk and hence require an additional risk premium. The finding about the relation between beta dynamics and volatility is broadly consistent with the well documented observations that the returns will become more synchronized during volatile period (Solnik et al. (1996); Andersen et al. (2001)).

6.2 Pricing of Shocks to the Idiosyncratic Jump Tails

We next study whether shocks to the tail shape parameter of idiosyncratic jump risk requires compensation. Note that, just like the strength of average idiosyncratic volatility, shocks to the tail shape parameter of the idiosyncratic jump risk is a form of aggregate risk. As for the systematic jump tails, we measure the idiosyncratic jump tail shape shocks using the innovations in the jump tail index $\xi_{I;t}$. Specifically, we use $\bar{\xi}_{I;t} = 2/(\alpha^L_{I;t} + \alpha^R_{I;t})$, where $\alpha^L_{I;t} = 1/\xi^-_{I;t}$, and $\alpha^R_{I;t} = 1/\xi^+_{I;t}$ represent the parameters estimated from the left and right idiosyncratic jump tails, respectively. We have performed the same analysis using separate left and right tail shape indices, generating similar results to those based on the average of the two reported here.

Following the portfolio formation approach of Section 6.1, we sort stocks on the basis of their exposures to $\bar{\xi}_{I;t}$. We summarize the results in Table 4. The out-of-sample evaluation period is between 2004 and 2022, and the holding-window for the portfolio is one month. Table 4 reveals that, unlike the portfolios sorted on exposure to systematic jump tail shape risk, there is a decreasing pattern in the average returns from the high to the low beta portfolio, both for equal-weighting and value-weighting schemes. The high (low) beta portfolio has positive (negative) exposures to the idiosyncratic jump shocks. The equal-weighted HmL_{IdioJ} portfolio, that is, the portfolio that goes long/short the quintile portfolio with the highest/lowest exposure to idiosyncratic tail shape shocks, has annualized returns of 3.19% with a t-statistic of 1.98 (annualized SR of 0.46). The returns for the value-weighted portfolio is 3.98% with a t-statistic of 1.92 (annualized SR of 0.45). Furthermore, the HmL_{IdioJ} portfolio's alphas are significant after controlling for a variety of common systematic risk factors. Sorting results for a three-month holding period are similar to those for one month and are provided in the Appendix H.

Consistent with the significance of the alphas of the HmL_{IdioJ} portfolios, the shocks to the idiosyncratic jump tail shape index appear weakly and insignificantly correlated with the common systematic risk factors, as seen from Table 3.

We check the robustness of the portfolio sorting results for systematic jump detection for alternative thresholds in Appendix I. We find the results for idiosyncratic jump sorted portfolios consistent across the different tuning parameter settings.

Overall, our results suggest that positive shocks to ξ_I (a fattening of the idiosyncratic jump tail) are viewed favorably by investors, i.e., times featuring thicker idiosyncratic jump tails are good states of the world. Therefore, stocks that do well when ξ_I is high trade at a premium and should earn low expected future returns. How do we rationalize this pricing result which is the exact opposite to our finding from

Table 4: Returns of idiosyncratic jump tail shock beta-sorted portfolios. The table reports out-of-sample annualized average returns (in percentage) and t-statistics for portfolios sorted based on cross-sectional jump tail beta. We sort stocks each day into quintile portfolios using jump tail betas. The jump tail betas are estimated using daily returns of the past five years. The holding window of the portfolio is one month. We use all stocks traded on NYSE/AMEX/NASDAQ exchange with price above $5 and stock capitalization beyond the 20% NYSE size breakpoint. Stocks are formed into equal- and value-weighted portfolios. The out-of-sample period is between 2004 and 2022. Panel A reports the portfolios' average betas to the idiosyncratic jump tail shocks and annualized returns, all in percentages, and Panel B reports the alphas under CAPM, FF3/FF5/FF6 models, controlling for the tail-risk factor (Kelly and Jiang (2014)), the CiV factor (Herskovic et al. (2016)), the idiosyncratic risk factor (Ang et al. (2006)), and the low-minus-high portfolio sorted using systematic jump tail exposures (LmH$_{SysJ}$). The t-statistics are based on Newey-West (Newey and West (1987)) standard errors using a lag length equals to the holding window.

		Equal-weight		**Value-weight**	
		Panel A: Average Portfolio Returns			
Portfolio	**Beta**	**Ret**	**t-stat**	**Ret**	**t-stat**
H	6.19	15.31	3.73	15.43	4.61
Q4	2.73	13.98	3.70	13.30	4.57
Q3	0.76	13.47	3.55	10.69	3.72
Q2	-1.19	13.19	3.41	11.11	3.66
L	-5.26	12.12	2.73	11.46	3.14
HmL$_{IdioJ}$		3.19	1.98	3.98	1.92

	Panel B: Abnormal Return of HmL$_{IdioJ}$ Portfolio			
Measure	**Ret**	**t-stat**	**Ret**	**t-stat**
CAPM alpha	4.17	2.26	4.90	1.98
FF3 alpha	4.06	2.17	4.37	1.72
FF5 alpha	4.10	2.20	4.86	1.87
FF6 alpha	5.11	2.66	4.72	1.82
CAPM+tailrisk alpha	4.45	2.37	5.91	2.31
FF6+tailrisk alpha	4.44	2.37	5.92	2.28
CAPM+CiV alpha	3.82	1.86	5.52	2.33
FF6+CiV alpha	4.11	2.02	5.76	2.35
CAPM+idiorisk alpha	3.94	2.20	5.09	2.01
FF6+idiorisk alpha	4.22	2.32	5.65	2.16
CAPM+LmH$_{SysJ}$ alpha	4.46	2.58	5.25	2.23
FF6+LmH$_{SysJ}$ alpha	4.42	2.63	4.99	2.07

portfolio sorts based on ξ_S? We first reiterate that, although ξ_I is linked to idiosyncratic jumps, shocks to ξ_I still represent aggregate risk, as they impact the extent to which the cross-section experiences more or less extreme idiosyncratic jumps. Apparently, many investors prefer exposure to this type of common risk, which is consistent with lottery-type preferences, implying that there is an attraction to stocks that can generate huge positive returns (winning the lottery). For such investors, episodes in which idiosyncratic jump tails grow fatter provide better upside potential and hence are preferred by them. Lottery-like preferences has been evidenced in both equity and option market contexts, as documented in studies such as Boyer and Vorkink (2014); Blau et al. (2016) and Filippou et al. (2018). Moreover, Ho et al. (2023) and the references therein find that firms with a high degree of growth options and operational flexibility generate longer-term convex payoffs with lottery-style features.

6.3 Portfolio Combination

The results in Sections 6.1 and 6.2 demonstrate that the LmH_{SysJ} and HmL_{IdioJ} portfolios sorted based on the exposure to systematic and idiosyncratic jump tail shape shocks earn positive expected returns in the future. In addition, Table 3 shows that shocks to ξ_S and ξ_I appear nearly uncorrelated. Hence, not surprisingly, the HmL_{IdioJ} and LmH_{SysJ} portfolio returns are weakly related, with sample correlation coefficients equalling -0.11 and -0.07 for the equal- and value-weighted portfolios. Consequently, combining them should improve performance, as it does for Asness et al. (2013), who document substantial gains, in terms of Sharpe ratios, from combining value and momentum portfolios. Towards this end, we evaluate a simple equal-weighted combination of the HmL_{IdioJ} and LmH_{SysJ} portfolios. Denoting the return of the HmL_{IdioJ} (LmH_{SysJ}) portfolio of idiosyncratic (systematic) jump tail exposure by r_t^{IdioJ} (r_t^{SysJ}), the return of the combined portfolio (Comb) is given by,

$$r_t^{Comb} = 0.5\, r_t^{IdioJ} + 0.5\, r_t^{SysJ}.$$

We report the performance of this combined portfolio in Table 5.

Table 5 confirms that the performance of the combined portfolio improves relative to the individual portfolios in terms of generating higher return significance and,

Table 5: Return performance of the combined portfolio. The table reports average returns and abnormal returns under alternative linear factor models over our 2004-2022 sample. The portfolio holding period is one month. We also report t-statistics based on Newey-West (Newey and West (1987)) standard errors using a lag length equals to the holding period.

	Equal-weight		**Value-weight**	
Measure	**Ret**	**t-stat**	**Ret**	**t-stat**
MEAN	3.57	3.58	3.91	2.97
CAPM alpha	3.04	2.88	3.47	2.43
FF3 alpha	3.16	3.05	3.67	2.56
FF5 alpha	3.37	3.12	4.15	2.81
FF6 alpha	3.67	3.43	4.50	3.09
CAPM+tailrisk alpha	3.22	2.93	4.15	2.80
FF6+tailrisk alpha	3.64	3.30	4.72	3.12
CAPM+CiV alpha	3.82	1.86	5.52	2.33
FF6+CiV alpha	4.12	2.02	5.76	2.35
CAPM+idiorisk alpha	3.63	3.11	4.50	2.96
FF6+idiorisk alpha	3.51	1.80	4.85	3.24

hence, a higher Sharpe ratio. For example, the t-statistic of the mean return is 3.58 (annualized SR is 0.83) for the combined one-month holding-period portfolio under the equal-weighted scheme and 2.97 (annualized SR is 0.70) for value-weighted scheme. These numbers substantially exceed the counterparts for the individual HmL_{IdioJ} and LmH_{SysJ} portfolios.

7 Conclusion

We develop a framework for estimation of cross-sectional tail risks, which are captured by the power-law shape index for the jump tails. The estimators are constructed using high-frequency data from a large cross-section of assets. We prove asymptotic normality for the tail-shape index estimators and propose a goodness-of-fit test for the adequacy of the power law for capturing the systematic and idiosyncratic jump tails, respectively. Empirically, we find that both of these tail dispersion risk factors evolve differently from the concurrent return volatilities. Furthermore, the systematic and idiosyncratic jump tails exhibit distinct time-series dynamics and both carry significant risk premiums, but with opposite signs. The pricing effect of these jump tail dispersion risks cannot be rationalized through their interaction and they remain significant when we control for a number of popular cross-sectional factor pricing models.

Appendix A Outline of Strategy Behind the Proofs

In this section, we outline the proofs for the main theoretical results. The details behind the proofs of the theorems and Lemmas are provided in Appendix F of the supplementary material.

The proof of Theorem 1 is divided into two steps. In the first step, we verify the CLT for the POT estimator of the tail index based on the infeasible systematic jumps (Lemma 2). In the second step, we show that replacing the infeasible systematic jumps with the estimated ones from high-frequency returns will not affect the CLT (Lemma 3). The proof of Theorem 2 proceeds similarly. That is, we first verify the CLT for the estimator based on the infeasible idiosyncratic jumps (Lemma 4), and then we use Lemma 3 to show that the result still holds after replacing the infeasible idiosyncratic jumps with the ones estimated from the high-frequency returns.

The proof of Theorem 3 (and similarly Theorem 4 for idiosyncratic jump tails' goodness-of-fit test given in Appendix E) is obtained by first establishing the asymptotic distribution of the infeasible KS statistics based on the infeasible jump observations, and then applying Lemma 5, which shows that the difference in the KS statistics based on the estimated and true jumps is asymptotically negligible.

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