Differentiated Public Goods*

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Abstract

A model of differentiated public goods helps resolve all three empirical puzzles regarding charity donations documented in Andreoni's (1988) "The Limits of Altruism". Government contributions to charitable activities can even crowd in, instead of crowd out, private giving. A diverse society is conceptually different from a polarized one, with more (less) charity donations in the former (latter) than in a homogeneous society. Like any profit-seeking entrepreneurs, activist entrepreneurs also face a product-design problem of how to package differentiated public goods into bundles to maximize donations.

KEYWORDS: public goods, altruism, crowding in, diversity, polarization, product design, bundling

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1 Introduction

It has long been a puzzle in economics why people donate so much to charity. The fact that people donate, in itself, is not a puzzle—a simple model of altruism can explain that. Donating to charity is like contributing to a public good. When people are altruistic and care about animal rights, they will willingly contribute to the public good of animal rights protection. Yet altruism has it limits. When there are many citizens who are all altruistic and care about animal rights, they will start free-riding each other and scale down their individual contributions. Economic theory predicts that this free-riding effect is very pervasive, so much so that it is puzzling why people, albeit being altruistic, donate so much.

In a seminal paper aptly titled "The Limits of Altruism", Andreoni (1988) enumerates three empirical facts that classical models of altruism cannot explain:

1. "First, there is vast participation. According to two national surveys, over 85% of all households make donations to charities." Classical models, instead, predict very narrow participation—in a large economy, only the very rich will make donations.

2. "Second, both aggregate and individual gifts are large. [. . .] In total, the charitable sector of the American economy accounts for about 20% of GNP. Average giving was over $200 per household in 1971, ranging from $70 for the lowest income quartile to $350 for the highest quartile." Classical models, instead, predicts that most people will donate $0, except for the very rich.

3. Third, "[e]conometric studies indicate that a one dollar increase in government contributions to 'charitable activities' is associated with a decrease in private giving of only 5 to 28 cents." Classical models, instead, predicts that a one dollar increase in government contributions will decrease private giving by (almost, if not exactly) one dollar (Warr, 1982; Roberts, 1984). Intuitively, altruistic people care about outcomes: Are animal rights adequately protected? Are the poor and sick adequately tended? If the government has already delivered the outcomes they want, they have no further reason to contribute.

Since Andreoni (1988), economists have suggested different theories to resolve the puzzle. For example, people may donate simply because of warm glow (Andreoni, 1989, 1990); and people sometimes do good deeds to assure themselves that they are indeed good people (Bénabou and Tirole, 2002, 2006).

While we do not deny the validity and importance of these complementary theories, we also believe the importance of altruism has been under-appreciated, due to a simplifying assumption in classical models. Specifically, classical models typically lump all charitable activities into a single public good. When citizens make charity donations, they all donate to this single public good, and hence everyone is free-riding each other. In reality, there is a garden variety of charitable activities, and different citizens may care more about different ones—some citizens care more about quality journalism, some care more about animal rights, some care more about cultural preservation, and yet some care more about environmental conservation. If different citizens specialize in donating to different charities, the free-rider problem will be much weaker than previously predicted. Yes, my donation to the protection of animal rights benefits you too, because you care about that too, albeit not as much as I do. But since you do not donate to the protection of animal rights (because you care more about and hence specialize in donating to environmental conservation), I cannot free-ride you when it comes to protecting animal rights, which I care more about.

This paper will demonstrate how a simple model of differentiated public goods can help resolve all three empirical puzzles regarding charity donations as documented by Andreoni (1988). In Section 2, we start with an illustrative example with two citizens and two differentiated public goods. The illustrative example features an equilibrium that achieves the first best notwithstanding the presence of public goods.

In Section 3, we extend the example into a model of large economy, and show how (1) the society as a whole contributes a non-vanishing share of its total wealth to public goods, and (2) a non-vanishing share of citizens make strictly positive contributions to public goods, even as we go to the large-economy limit. Both predictions differ from what classical models predict, but are in line with what we observe in the real world.

In Section 4, we show how government contributions to public goods do not necessarily crowd out, and may even crowd *in*, private giving. Econometricians who observe a mixture of these crowding-in and crowding-out instances will document an average crowding-out effect far weaker than what classical models predict.

Section 5 explores how the government should allocate its budget among different public goods. It can be better to eschew allocating the budget to a public good that is more plagued by the free-rider problem, but instead to a public good that is less plagued but is considered by many as complementary to another public good that is plagued by the free-rider problem.

Section 6 compares a homogeneous, a diverse, and a polarized societies. A diverse society has the most private giving, and can have a higher welfare than a homogeneous society. A polarized society has less private giving than a diverse society, and has the lowest welfare.

Section 7 studies an entrepreneurial activist trying to solicit donations for a public good. Like any profit-seeking entrepreneurs, she faces a product-design problem of how to package different components into bundles to maximize donations.

Section 8 concludes.

2 An Illustrative Example

Consider an economy with two citizens, 1 and 2, and two differentiated public goods, X and Y. Each citizen i has a total budget of 1 that can be divided into his private consumption c_i, his contribution x_i to public good X, and his contribution y_i to public good Y; i.e., his budget constraint is $c_i + x_i + y_i \leq 1$. Let x be the total contribution to public good X (i.e., $x = x_1 + x_2$), and likewise for y. By the non-rivalrous nature of public goods, x is also the amount of public good X that every citizen manages to enjoy, and likewise for y.

Citizens 1 and 2, however, have different tastes regarding these two different public goods, with citizen 1 caring more about X and citizen 2 caring more about Y. For example, suppose citizen 1 aggregates these two public goods using the Leontief aggregation function of $G_1 = \min\{x, 2y\}$, meaning that every unit of public good Y is best consumed with 2 units of public good X. Meanwhile, citizen 2 uses a different Leontief aggregation function: $G_2 = \min\{2x, y\}$. Note that implicit in the Leontief aggregation function is the assumption that citizens regard different public goods as complements. We shall return to this assumption later.

Each citizen i then aggregate c_i and G_i with the same utility function, say, $u_i = c_i + f(G_i)$ with some concave function $f(\cdot)$ that satisfies $f'(0) > 1.5$ and $f'(1) < 1$.

Suppose the citizens make their public-good contributions simultaneously. This is then a perfect-information simultaneous-move game. It is easy to guess the (unique) equilibrium of this game. Since the two citizens are symmetric (albeit different), it is easy to guess that the equilibrium is also symmetric in the sense that $x = y$. But citizen 1 cares more about public good x, and hence when $x = y$, he would rather scale back his contribution y_1 to public good Y and move the money to public good X, unless y_1 has already hit the lower bound of 0. Therefore, in equilibrium, it must be the case that citizen 1 contributes only to public good X, and citizen 2 only to Y; i.e., $x_1 = x = y = y_2$ and $y_1 = 0 = x_2$.

With $x = y$, citizen 1's utility is $u_1 = c_1 + f(G_1) = c_1 + f(\min\{x, 2y\}) = c_1 + f(x)$. He willingly contributes x_1 to public good X up to the level $x_1 = x$ only if $f'(x) = 1$. A symmetric argument, with citizen 2 and public good Y replacing citizen 1 and public good X, respectively, suggests that $f'(y) = 1$ as well. We hence obtain the equilibrium level of public good provision: $x = (f')^{-1}(1) = y$.[1]

In contrast to almost all other games of public good provision, there is *no* suboptimal provision in the above equilibrium. Indeed, the equilibrium levels of public goods implement the first best. A social planner who can dictate $\{(x_i, y_i)\}_{i=1,2}$ and maximizes $u_1 + u_2$ will choose x and y to maximize $f(\min\{x, 2y\}) + f(\min\{2x, y\}) - x - y$. This would lead her to choose $x = (f')^{-1} = y$ as well.

The result that the equilibrium level of public good provision is as high as the first best level is of course very extreme. We have pulled many strings in this illustrative example to cook up this extreme result in order to make our point as sharp as possible. There are many ways to kill this extreme result. For example, if there are multiple clones of citizen 1 (and likewise multiple clones of citizen 2), then, while all clones of citizen 1 (2) continue to contribute only to public good X (Y), and hence there is sill no free-riding across groups, there will be free-riding within groups—each clone of citizen 1 will free-ride other clones of citizen 1, and likewise for clones of citizen 2. Such within-group free-riding will drive a wedge between the equilibrium level of public good provision and the first best level. For another example, if we replace the Leontief aggregation function with something smoother, then, at the equilibrium level of public good provision, an extra dollar citizen 1 contributes to public good X will generate positive marginal utility for citizen 2, an externality that citizen 1 does not internalize. Once again, there will be a wedge between

[1] The assumption of $f'(1) < 1$ guarantees an interior solution; i.e., $x = y < 1$. The assumption of $f'(0) > 1.5$ guarantees that it is not an equilibrium for both citizens not to make any contribution.

the equilibrium level of public good provision and the first best level.

But the main insight of our illustrative example remains: the facts that public goods are differentiated, and that different citizens care more about different ones, *go a long way in explaining why the free-rider problem is much weaker than classical models predict*.

3 Large Economy

Our illustrative example above has only two citizens. Since empirical puzzles regarding charity donations are all expressed in the context of a large economy (Andreoni, 1988), it is hence important to extent our illustrative example to a large economy.

If we increase the number of citizens while keeping the number of differentiated public goods at two, apparently within-group (that is, within the group of citizens who contribute only to, say, public good X) free-riding effect will kick in very fast, rendering both the proportion of contributing citizens and average contributions converging to zero, replicating the empirical failures of classical models.

However, in a large economy it can only be natural to have a large variety of public goods as well—some citizens care more about quality journalism, some care more about animal rights, some care more about cultural preservation, and yet some care more about environmental conservation. In a large economy where both the number of citizens and the number of differentiated public goods increase to infinity, the insight of our illustrative example will be reserved. In particular, free-riding is not going to be so pervasive as to drive the proportion of donating citizens to zero.

To illustrate this, let's consider a sequence of societies, $k = 1, 2, \ldots$, each characterized by a vector (n_k, m_k), where n_k is the number of citizens in the society, and m_k is the number of different *types* (to be explained below) these citizens are evenly divided into. Specifically, $t_k := n_k/m_k$ is an integer, and there are exactly t_k citizens belonging to each type $\theta \in \Theta_k$, where Θ_k is the set of possible types, with $|\Theta_k| = m_k$.

We assume that both n_k and m_k increase in k without bound, and that $t_\infty := \lim_{k\to\infty} t_k$ exists but is possibly infinite. We are interested in two cases. The first is that $m_k = o(n_k)$, which will behave similarly to classical models.[2] The second is that $m_k = O(n_k)$ but not

[2]Our model, however, does not nest the classical models. Classical models feature $m_k \equiv 1$, whereas our model features m_k increasing in k without bound.

$m_k = o(n_k)$, which will behave similarly to the illustrative example in Section 2. These two cases correspond to two possible ways in which New York City grows. The first is that it grows by admitting immigrants from more or less the same origins of its existing residents. The second is that it grows by admitting immigrants from origins vastly different from those of its existing residents. The former results in a homogeneous megacity. The latter results in a pluralistic society.

There is a countably infinite set, Ω, of differentiated public goods, with ω being a generic element. For any k, the type set Θ_k is a subset of Ω. Each citizen $i \in \{1, 2, \ldots, n_k\}$ has a total budget of 1 that can be divided into his private consumption, c_i, and his contribution $x_i(\omega)$ to each public good ω; i.e., his budget constraint is $c_i + \sum_\omega x_i(\omega) \leq 1$. The total contribution to (same as the provision of) public good ω is then $x(\omega) = \sum_i x_i(\omega)$.

A citizen with type $\theta \in \Theta_k \subset \Omega$ aggregates the differentiated public goods using the following aggregation function:

$$G_\theta = A \cdot g(x(\theta)) + \sum_{\omega \neq \theta} a \cdot g(x(\omega)),$$

where $x(\omega)$ is the provision of (same as the total contribution to) public good ω, A and a are two constants satisfying $A > a > 0$, and $g(\cdot)$ is strictly increasing and strictly concave function satisfying $g'(0) < \infty$ and $\lim_{x\to\infty} g'(x) = 0$. In other words, a type-θ citizen cares more about public good θ, and cares equally little about every other public good.

A type-θ citizen i then aggregates c_i and G_θ using a common utility function:

$$U = u(c_i) + H(G_\theta),$$

where $u(\cdot)$ is strictly increasing and concave, and $H(\cdot)$ is strictly increasing. We also assume that $u(\cdot)$ is strictly concave in the current section, but find it more convenient to work with a linear $u(\cdot)$ in many examples in later sections. In later sections, we will also study the implications of the extra assumption that $H(\cdot)$ is strictly convex. In the current section, however, we do not make this extra assumption, and allow for $H(\cdot)$ to have any shape except for being asymptotically linear. Specifically, we assume that $h := \lim_{G\to\infty} H'(G)$ exists and is strictly positive but finite, and $\lim_{G\to\infty} H''(G) = 0$. Finally, we assume the following:

$$h \cdot A \cdot g'(0) > u'(1) > h \cdot a \cdot g'(0). \tag{1}$$

Variants of condition (1) will appear repeatedly in later sections. Their exact meanings differ slightly, but all are meant to guarantee that a citizen only contributes to his favorite public good but not to any others.

For any k, there is a complete-information simultaneous-move public-good provision game, where citizens choose how to allocate their budgets simultaneously. Our solution concept is pure-strategy Nash equilibrium. We further focus on Nash equilibria that are symmetric, where different citizens with different types play the same strategy up to relabelling. We refer to a pure-strategy symmetric Nash equilibrium simply as *an equilibrium*.

We first establish the existence of a specific kind of equilibrium in a large economy.

Lemma 1 *There exist constants $\underline{x} > 0$ and M such that, for any society (n, m) with $m > M$, there exists an equilibrium where, for every $\theta \in \Theta$, the total contribution $x(\theta)$ to public good θ is at least $\underline{x}$. Moreover, every type-θ citizen contributes only to public good θ and not to any other public goods.*

The proof of Lemma 1 is relegated to Appendix A. Since we do not want to impose the assumption that $H(\cdot)$ is concave (in later sections, we will actually assume that it is strictly convex), we cannot guarantee the convexity (in the space of pure strategies) of a generic citizen's best-response correspondence. We hence cannot apply Kakutani's Fixed Point Theorem to establish the existence of a pure strategy equilibrium. However, the asymptotic linearity of $H(\cdot)$ guarantees that the composite function $H(A \cdot g(\cdot) + z)$ is concave for large enough z, which is what we need to prove the existence of an equilibrium in a large economy.

Without further restrictions on the shape of $H(\cdot)$ as a function of G in the range where G is small, we do not know whether or not there also exist other equilibria where the total contribution to any public good is vanishingly small in a large economy. What Lemma 1 establishes is the existence of a specific kind of equilibrium, namely one where the total contribution to any public good is at least some minimal amount. When every other public good $\theta' \neq \theta$ receives at least a minimal amount of total contribution, type-θ citizens will perceive a large aggregate public good G_θ in a large economy. This allows us to take advantage of the asymptotic property of $H(\cdot)$ and argue that these citizens as a group will also have incentives to contribute a minimal amount to public good θ as well.

Finally, that every type-θ citizen contributes only to public good θ but not to any other public goods is a direct consequence of condition (1).

3.1 Homogeneous Megacity

A homogeneous megacity refers to the large-k limit of a sequence of societies, $k = 1, 2, \ldots$, where $m_k = o(n_k)$. In this case, $t_\infty := \lim_{k\to\infty} t_k = \infty$.

Proposition 1 *Suppose $m_k = o(n_k)$. For any $\epsilon > 0$, for any society with large enough k, in any equilibrium, every citizen contributes to public goods a share of his budget smaller than ϵ.*

Part of the Proof: The full proof is relegated to Appendix A. Here we provide only part of the proof. Suppose there exists an $\epsilon > 0$ (which is independent of k) such that, for every K, there exists a society $k > K$ such that there exists an equilibrium where every citizen contributes to public goods a share of his budget larger than ϵ.[3] Suppose, furthermore, the equilibrium takes the form as the one in Lemma 1; i.e., every type-θ contributes only to public good θ and not to any other public good. As K and hence the corresponding k go to infinity, $G_\theta > (m_k - 1) \cdot a \cdot g(t_k \epsilon)$ (for any type $\theta \in \Theta_k$) will also go to infinity, and hence $H'(G_\theta)$ will go to h. Meanwhile, $x(\theta) > (t_k - 1)\epsilon$ will also goes to infinity, and hence $g'(x(\theta))$ will go to 0. Therefore, in any such society with large enough k, we must have $u'(1) > H'(G_\theta) \cdot A \cdot g'(x(\theta))$ and hence no type-θ citizen will have incentives to contribute to public good θ, a contradiction. ■

Since citizens are symmetric, if every citizen contributes a vanishingly small share of his budget to public goods, the society as a whole contributes a vanishingly small share of its total wealth to all public goods combined. This is the same as one of the three predictions of the classical models that were, as Andreoni (1988) pointed out, at odds with empirical facts.

3.2 Pluralistic Society

A pluralistic society refers to the large-k limit of a sequence of societies, $k = 1, 2, \ldots$, where $m_k = O(n_k)$ but not $m_k = o(n_k)$. In this case, $t_\infty := \lim_{k\to\infty} t_k < \infty$.

[3] Recall that we are focusing on symmetric equilibria. Therefore, if one citizen contributes to public goods a share of his budget larger than ϵ in an equilibrium, every citizen does in that equilibrium.

Proposition 2 *Suppose $m_k = O(n_k)$ but not $m_k = o(n_k)$. There exists $s > 0$ (which is independent of k) such that, for any society with large enough k, there exists an equilibrium where every citizen contributes at least an s share of his budget to public goods.*

Proof: Proposition 2 follows immediately from Lemma 1. Let $s = \underline{x}/t_\infty > 0$, and consider any k large enough so that $m_k > M$, where $\underline{x}$ and M are as stated in Lemma 1. By Lemma 1, there exists an equilibrium where every public good $\theta \in \Theta_k$ receives a total contribution of at least $\underline{x}$, and from type-θ citizens only. Therefore, every type-θ citizen contributes at least an s share of his budget to public goods. ∎

In a pluralistic society, in contrast to a homogeneous megacity, the society as a whole contributes a non-vanishing share of its total wealth to all public goods combined. This is in line with what we observe in the real world.

3.3 Heterogeneous Wealth

So far we have only considered symmetric citizens. It is easy to extend our model to allow for citizens with heterogeneous wealth.

Suppose, for any society with vector (n_k, m_k), there is a common empirical distribution F_k of wealth in every sub-population of citizens with the same type, with the maximum wealth equal to 1. Adapt the definition of a symmetric equilibrium to this setting by requiring only that citizens with the same wealth play the same strategy up to relabelling. Lemma 1 continues to hold in this setting, with a slightly modified proof (see Appendex A). In any equilibrium satisfying the description in Lemma 1, for any $\theta \in \Theta_k$, some type-θ citizens, at least the richest one among them, must be contributing to public good θ. Therefore, the proportion of citizens making strictly positive contributions to public goods is at least $1/t_k$. In a pluralistic society, that minimal share converges to $1/t_\infty > 0$ as k goes to infinity.

Proposition 3 *Suppose $m_k = O(n_k)$ but not $m_k = o(n_k)$. There exists $s > 0$ (which is independent of k) such that, for any society with large enough k, there exists an equilibrium where at least an s share of citizens make strictly positive contributions to public goods.*

In a pluralistic society, in contrast to what classical models predicted, a non-vanishing share of citizens make strictly positive contributions to public goods. This is in line with what we observe in the real world.

4 Crowding In vs Crowding Out

A model of differentiated public goods can also help address the puzzle of why a one dollar increase in government contributions to charitable activities does not decrease private giving by (almost, if not exactly) one dollar. To understand how, let's first review why the classic theory (with only one public good) predicts a dollar-by-dollar crowding out effect.

Example 1: Consider a one-citizen economy where, in the absence of the government, the citizen optimally splits his budget of \$1 by spending \$1/2 on private consumption and contributing \$1/2 to the single public good.[4] Suppose the government levies a tax of \$1/2 on this citizen and contributes the tax revenue to the public good. The citizen will reason that his optimal split is still achieved, albeit indirectly through the government. So he will spend the remaining \$1/2 on private consumption, and consume the \$1/2 public good provided by the government. In other words, his original private contribution is crowded out dollar-by-dollar. ■

It is true that the classical theory does not predict that the crowding-out effect is always dollar-by-dollar. If the government instead levies a tax of \$2/3 on the citizen and contribute the tax revenue to the public good, this will crowd out the citizen's private contribution by at most \$1/2, and hence the crowding-out effect is not dollar-by-dollar. However, for small government contributions the crowding-out effect remains dollar-by-dollar.

Example 2: To illustrate how introducing differentiated public goods can help destroy the prediction of dollar-by-dollar crowding-out effect, it suffices to continue with the example of a one-citizen economy. Suppose there are two differentiated public goods: θ and ω. Suppose the citizen is of type θ, and has the utility function

$$U = c + A \cdot g(x(\theta)) + a \cdot g(x(\omega));$$

[4] One may complain that there cannot be any meaningful "public" good in a one-citizen economy. But this (silly) example helps bring home the fact that the "public-ness" of a good plays no role in the reasoning.

i.e., both $u(\cdot)$ and $H(\cdot)$ are identity functions. Suppose

$$A \cdot g'(0) > 1 > A \cdot g'(1) \tag{2}$$

and

$$1 \geq a \cdot g'(0). \tag{3}$$

Then, in the absence of the government, the citizen contributes $\chi^* \in (0, 1)$ to public good θ, contributes nothing to public good ω, and spends the remaining $1 - \chi^*$ on private conumption, where χ^* solves the first order condition

$$A \cdot g'(\chi) = 1. \tag{4}$$

Suppose the government levies an arbitrarily small tax of $\epsilon > 0$ on the citizen and contributes the tax revenue to public good ω. The citizen, after re-optimization, will continues to contribute χ^* to public good θ and reduces his private consumption to $1 - \chi^* - \epsilon$. ■

The result of no crowding-out effect is admittedly extreme, and relies on the linearity of $u(\cdot)$. If $u(\cdot)$ is instead strictly concave, as assumed in Section 3, there will again be some crowding-out effect, but still less than dollar-by-dollar.

More interestingly, it takes only a tiny little step to go from an example with no crowding-out effect to one with a crowding-*in* effect. Modify the above example slightly so that $H(\cdot)$ is no longer the identity function but a strictly increasing and strictly convex function instead. This in effect makes the public goods θ and ω complements. An increase in government contributions to public good ω will then increase the citizen's marginal gain in contributing to public good θ and crowd-*in* his private contribution to public goods. Econometricians who observe a mixture of these crowding-in and crowding-out instances will document an average crowding-out effect far weaker than what classical models predict.

Example 3: Specifically, suppose $H(\cdot)$ is only "slightly" convex in the sense that $H'(0)$ and $H'(A \cdot g(1))$ remains close enough to 1 so that the following modified version of condition (2) remains true:

$$H'(0) \cdot A \cdot g'(0) > 1 > H'(A \cdot g(1)) \cdot A \cdot g'(1). \tag{5}$$

Suppose a remain sufficiently small so that the following modified version of condition (3) holds:

$$1 \geq H'(A \cdot g(1)) \cdot a \cdot g'(0), \tag{6}$$

which guarantees that it never pays for the citizen to contribute to public good ω.

Then, in the absence of the government, the citizen will contribute an interior amount $\chi^* \in (0, 1)$ to public good θ and spend the remaining $1 - \chi^*$ on private consumption. Suppose the government levies an arbitrarily small tax of $\epsilon > 0$ on the citizen and contributes the tax revenue to public good ω. Any contribution $\chi < \chi^*$ to public good θ remains suboptimal for the citizen because

$$\begin{aligned}
[(1-\epsilon) - \chi] + H(A \cdot g(\chi) + a \cdot g(\epsilon)) &= (1-\chi) + H(A \cdot g(\chi)) - \epsilon + [H(A \cdot g(\chi) + a \cdot g(\epsilon)) - H(A \cdot g(\chi))] \\
&< (1-\chi^*) + H(A \cdot g(\chi^*)) - \epsilon + [H(A \cdot g(\chi^*) + a \cdot g(\epsilon)) - H(A \cdot g(\chi^*))] \\
&= [(1-\epsilon) - \chi^*] + H(A \cdot g(\chi^*) + a \cdot g(\epsilon)),
\end{aligned}$$

where the inequality follows from the suboptimality of χ in the absence of the government and from the strict convexity of $H(\cdot)$. Meanwhile, the citizen's marginal gain from contributing to public good θ at $\chi = \chi^*$ becomes

$$H'(A \cdot g(\chi^*) + a \cdot g(\epsilon)) \cdot A \cdot g'(\chi^*) > 1$$

due to the strict convexity of $H(\cdot)$. As a result, he increases his contribution χ to public good θ beyond χ^*, and we witness a strictly positive crowding-in effect.

Note that, since $H(\cdot)$ is strictly convex, the composite function $H(A \cdot g(\cdot))$ needs not be concave even though $g(\cdot)$ is strictly concave. As a result, the first order condition

$$H'(G_\theta) \cdot A \cdot g'(\chi) = 1$$

is necessary but not sufficient for optimality. Nevertheless, the crowding-in effect still follows from the standard argument in monotone comparative statics. ■

Do citizens regard different charitable activities as substitutes or as complements? Our own introspection suggests that we, as examples of citizens, do regard many different charitable activities as complements. Many of us, to the extent affordable, would like to move into a "beautiful community", where air is clean, fellow citizens are caring, and civic culture is vibrant. When some of these components are missing, even if we cannot afford to move out, we would start to feel detached, so much so that we decrease our civic engagement and our contributions to charitable activities. To the extent that this is more than our own personal experiences but rather a general phenomenon, the crowding-*in* effect is real, and can go a long way in explaining the empirical puzzle of why crowding out is not complete.

Incidentally, prominent economists making plans for the reconstruction of Ukraine's war-torn economy also seem to believe that the crowding-in effect is real. Lawrence H. Summers and colleagues, for example, suggested on *Foreign Affairs* that "[a]lthough much assistance could eventually come from private investors, private money will follow or be secured only by very large grants of public funds."[5]

Ultimately, whether they are correct is an empirical question that we may have to leave for future empirical studies. This paper can be viewed as laying down the ground work for these empirical studies by deriving the testable implications of a model of differentiated public goods.

5 Government Policies

One may observe that, in all three examples in Section 4, government contributions to public goods are either irrelevant or counterproductive—had they been good for the citizen, he would have already done it by himself in the absence of the government. This dismal result is of course an artefact of a one-citizen economy, where no meaningful "public" good exists. When there are two or more citizens, public good provision is typically suboptimal due to the free-rider problem. Government contributions to public goods, especially if they crowd *in* instead of crowd out private giving, can easily improve welfare.

In classical models, where there is only one public good, there is not much room for government policies, except for different ways to tax citizens in order to finance the

[5]Lawrence H. Summers, Philip Zelikow, and Robert B. Zoellick, "The Other Counteroffensive to Save Ukraine", *Foreign Affairs*, June 15, 2023.

government's contribution to that single public good. Bergstrom, Blume, and Varian (1986) argued that, if the government wants to boost the total public good contribution, it may need to tax the poor instead of the rich. The reason is that the rich are making positive contributions to the public good, and hence taxing them would only reduce these citizens' contributions dollar-by-dollar. Taxing the poor would not have this drawback, because their contributions to the public good is already at the lower bound of zero. Andreoni (1989), after introducing "impure altruism" and "warm glow", argued instead that the government should tax those who enjoy more from donating. These are exactly the citizens who would not scale back their donations by too much after being taxed.

A model of differentiated public goods allows us to study many more government policies. In particular, instead of merely choosing who to tax, the government can also choose how to allocate a given budget to the many differentiated public goods. One obvious principle is that the government should allocate the budget to those public goods that (1) few citizens care the most about, and (2) many citizens consider as complementary to what they care the most about. This helps avoid running into the crowding-out effect and helps unleash the crowding-in effect. But there is also a less obvious tradeoff between allocating the budget to a public good that is more plagued by the free-rider problem, and allocating it to a public good that is less plagued but is considered by many as complementary to another public good that is plagued by the free-rider problem. We illustrate this tradeoff in an example in Appendix B. We, however, are not able to obtain general principles regarding how to balance this tradeoff. The optimal policy seems to be sensitive to model parameters, and we leave its exploration to future research.

6 Diversity and Polarization

While there is a common understanding of what it means by a homogeneous community, the notion of a heterogeneous community often invokes two mutually contradictory connotations. Some think of it as a diverse society, while others think of it as a polarized one. The former is often associated with a vibrant and harmonious society, while the latter not. A model of differentiated public goods can help reconcile this seeming contradiction by associating a diverse and a polarized communities with two different kinds of heterogeneous communities. A diverse community is a heterogeneous community of the kind we have been describing so far. Here, heterogeneity helps alleviate the free-rider problem, and move the equilibrium level of public good provision closer to the first best.

A polarized community can be thought of as a heterogeneous community of a different kind. Imagine that there are two groups of differentiated public goods. The first group, Ω, is regarded by Democrats as public goods, but regarded by Republicans as public *bads*. One possible example is the protection of abortion right. Symmetrically, the second group, $\widehat{\Omega}$, is regarded by Republicans as public goods, but regarded by Democrats as public bads. One possible example is the protection of the right to bear arms. Suppose higher total contributions to public goods in $\widehat{\Omega}$ reduces Democrats' marginal utilities of public goods in Ω, and likewise higher total contributions to public goods in Ω reduces Republicans' marginal utilities of public goods in $\widehat{\Omega}$. Then, in a polarized community roughly equally divided into Democrats and Republicans, each group's contributions to their favorite public goods will depress the other group's enthusiasm in contributing to theirs, and vice versa. This explains why communities with more polarized citizens tend to donate less to charity (Sullivan, 2018).

To illustrate, consider a society with two citizens and two groups of differentiated public goods: $\Omega = \{\theta_1, \theta_2\}$ and $\widehat{\Omega} = \left\{\widehat{\theta}_1, \widehat{\theta}_2\right\}$. The utility function of a type-θ_1 citizen is

$$U = c + H\left(G_{\theta_1}\right)$$

with

$$G_{\theta_1} = A \cdot g\left(x\left(\theta_1\right)\right) + a \cdot g\left(x\left(\theta_2\right)\right) - b \cdot \sum_{\widehat{\theta} \in \widehat{\Omega}} g\left(x\left(\widehat{\theta}\right)\right),$$

where $H(\cdot)$ is strictly increasing and strictly convex, and $A > a, b > 0$. For this citizen, any $\widehat{\theta} \in \widehat{\Omega}$ is more a public *bad* than a public good. Likewise for the utility function of a citizen with type θ_2, $\widehat{\theta}_1$, or $\widehat{\theta}_2$.

In this setting, a homogeneous society can be thought of a society where both citizens are of the same type, say θ_1. A diverse society is one where one citizen is of type θ_1 and the other is of type θ_2. A polarized society is one where one citizen is of type θ_1 and the other is of type $\widehat{\theta}_1$. We shall study each of these societies in turn.

Note that, anticipating what may happen in a polarized society, G_{θ_1} can be as negative as $-b \cdot g(1)$. Therefore, $H(\cdot)$ needs to be a function defined over $[-b \cdot g(1), \infty)$.

Let's continue to assume condition (5), which we replicate here as for easy reference:

$$H'(0) \cdot A \cdot g'(0) > 1 > H'(A \cdot g(1)) \cdot A \cdot g'(1). \tag{7}$$

It says that, in a (fictitious) one-citizen economy, the single citizen would have contributed an interior amount $\chi^* \in (0,1)$ to his favorite public good. Let's also assume that the following modified version of condition (6) holds:

$$1 \geq H'(A \cdot g(2)) \cdot a \cdot g'(0), \tag{8}$$

which guarantees that it never pays for a type-θ_1 citizen to contribute to public good θ_2.

Assume also that the convexity of $H(\cdot)$ is mild relative to the concavity of $g(\cdot)$ so that the composite function $H(A \cdot g(x) + z)$ remains strictly concave in x for any $z \geq -b \cdot g(1)$. In Appendix C, we provide an example with $H(\cdot)$ and $g(\cdot)$ that jointly satisfy this condition, as well as conditions (7) and (8) above and condition (12) below.

Consider the homogeneous society first. In equilibrium, neither citizen will contribute to any public good other than θ_1. With a quasi-linear utility function, there are multiple equilibria, each with a different way the total contribution to public good θ_1 is split between the two citizens. But the total contribution $x(\theta_1)$ must be uniquely pinned down by the first order condition

$$H'(A \cdot g(x)) \cdot A \cdot g'(x) = 1 \tag{9}$$

("$\leq$" if $x = 0$, and "$\geq$" if $x = 2$), thanks to the strict convexity of the composite function $H(A \cdot g(x) + z)$ (with $z = 0$). By condition (7), the unique solution of the first order condition is some $x_h \in (0,1)$. In a symmetric equilibrium, each citizen contributes $x_h/2$ to public good θ_1, nothing to any other public good, and spends the remaining $1 - x_h/2$ on private consumption.

In the diverse society, each citizen contributes x_d to his favorite public good, where x_d is uniquely pinned down by the first order condition

$$H'((A + a) \cdot g(x)) \cdot A \cdot g'(x) = 1 \tag{10}$$

("$\leq$ if $x = 0$, and "$\geq$" if $x = 1$), again thanks to the strict convexity of the composite function $H(A \cdot g(x) + z)$ (with $z = a \cdot g(x_d)$). Comparing (10) against (9), we have $x_d > x_h$.

In the polarized society, each citizen contributes x_p to his favorite public good, where x_p is uniquely pinned down by the first order condition

$$H'((A-b)\cdot g(x))\cdot A\cdot g'(x)=1 \tag{11}$$

("$\leq$ if $x=0$, and "$\geq$" if $x=1$), again thanks to the strict convexity of the composite function $H(A\cdot g(x)+z)$ (with $z=-b\cdot g(x_p)$). Comparing (11) against (9), we have $x_p<x_h$.

The three societies are hence unambiguously ordered in terms of total contribution to public good θ_1: the diverse society has the most at x_d, the homogeneous society has the middle at x_h, and the polarized society has the least at x_p.

The same ordering holds in terms of the aggregate public good G_{θ_1} from the perspective of a type-θ_1 citizen: the diverse society has the most at $G_{\theta_1}=(A+a)\cdot g(x_d)$, the homogeneous society has the middle at $G_{\theta_1}=A\cdot g(x_h)$, and the polarized society has the least at $G_{\theta_1}=(A-b)\cdot g(x_p)$.

In terms of a type-θ_1 citizen's contribution to all public goods combined, which may be what econometricians observe, the diverse society still has the most at x_d, while the comparison between the homogeneous society (with $x_h/2$) and the polarized society (with x_p) is now ambiguous.

In terms of the welfare of a type-θ_1 citizen, the polarized society is unambiguously the worst. He fares worse than in a (fictitious) one-citizen economy, where he would have contributed x_h to public good θ_1 as well, as the first order condition is the same as that in the homogeneous society. In a polarized society, it is as if he is given a worse production function of public good θ_1.

The welfare comparison between the homogeneous society and the diverse society is ambiguous. From the perspective of a type-θ_1 citizen, the advantage of the homogeneous society is that he gets to have his fellow citizen paying $x_h/2$ to his favorite public good θ_1. In a diverse society, he will have to foot the whole bill himself. On the other hand, the advantage of the diverse society is that he gets to have his fellow citizen paying more than double of $x_h/2$ (as $x_d>x_h$), albeit to a public good that is not his favorite. Which society is better for him depends on how much he cares about public good θ_2; that is, it depends on the size of a.

Apparently, if a is arbitrarily small, he cares very little about public good θ_2, and hence the homogeneous society will bring him higher welfare. On the other hand, a sufficient condition for the diverse society to have a higher welfare than the homogeneous society is

$$a \geq A/2, \tag{12}$$

which we prove in the following proposition.

Proposition 4 *In the two-citizen model in this section, the diverse society has the highest total contribution to a single public good, the highest aggregate public good from the perspective of a single citizen, and the highest public good contribution from a single citizen. The polarized society has the lowest total contribution to a single public good, the lowest aggregate public good from the perspective of a single citizen, and the lowest welfare. The homogeneous society has a higher welfare than the diverse society when a is small enough. The diverse society has a higher welfare than the homogeneous society when $a \geq A/2$.*

Proof: We shall prove only the last sentence of the proposition. The proof has two steps.

We first prove that, starting from the equilibrium in the homogeneous society, a type-θ_1 citizen is better off when he doubles his contribution to public good θ_1 from $x_h/2$ to x_h, while his fellow citizen also doubles his contribution from $x_h/2$ to x_h but switches it from public good θ_1 to θ_2. Formally, we prove that the indirect utility function

$$U(\Delta) = [1 - x_h/2 - \Delta] + H(A \cdot g(x_h) + a \cdot g(2\Delta))$$

is increasing in Δ for any $\Delta \in (0, x_h/2)$. This is true because

$$\begin{aligned} U'(\Delta) &= H'\left(A \cdot g\left(x_h\right) + a \cdot g(2\Delta)\right) \cdot 2a \cdot g'(2\Delta) - 1 \\ &> H'\left(A \cdot g\left(x_h\right)\right) \cdot A \cdot g'\left(x_h\right) - 1 \\ &= 0, \end{aligned}$$

where the inequality follows from the strict convexity of $H(\cdot)$, from $a \geq A/2$, and from the strict concavity of $g(\cdot)$, and the last equality follows from the first order condition (9).

We next prove that the type-θ_1 citizen is better off when both his and his fellow citizen's contributions to public goods θ_1 and θ_2, respectively, further increase from x_h to x_d. This is true because

$$[1 - x_d] + H(A \cdot g(x_d) + a \cdot g(x_d)) > [1 - x_h] + H(A \cdot g(x_h) + a \cdot g(x_d))$$
$$> [1 - x_h] + H(A \cdot g(x_h) + a \cdot g(x_h)),$$

where the first inequality follows from the fact that x_d is the citizen's unique best response, and the second inequality follows from $x_d > x_h$. ■

7 Designing a Public Good

Our usual concept of public goods has been heavily shaped by classical models. In classical models, where there is only one public good, it is implicitly assumed that the single public good comes exogenously instead of being invented by any entrepreneur. This stands in sharp contrast to models of differentiated private goods such as Dixit and Stiglitz (1977), where new varieties are invented by new entrepreneurs entering the market. In reality, differentiated public goods are not unlike differentiated private goods, and are often inventions of entrepreneurial activists. Entrepreneurial activists—we may call them activist entrepreneurs—invent new varieties of public goods, market them to the public, and solicit donations so that they can scale up and reach more beneficiaries. A new variety can be a new product whose target users are too poor to pay for it (such as a purification straw, targeting users who have no access to treated water and have to drink directly from the river), or a new service whose provision was previously thought impossible (such as humanitarian medical care in conflict zones, as professionally provided by Médecins Sans Frontières).

Just like a new variety of private good, a new variety of public good often comes with several different components. Its inventor needs to make a non-trivial product-design decision on whether to bundle these components and market the composite good as a whole, or to market each component separately. The former gives the inventor more control on the composition within the bundle, whereas the latter allows the inventor to tap into the public's heterogeneous preferences over separate components and collect more donations. We provide a preliminary exploration of this tradeoff in this section.

Consider a society with two citizens, an activist entrepreneur, and $N \geq 2$ public goods:

$\Omega = \{\theta_1, \theta_2, \ldots, \theta_N\}$. We may alternatively interpret these θ's as N different components of a single public good. For example, we can think of them as N different conflict zones where Médecins Sans Frontières operates. Among them, θ_1 and θ_2 are conflict zones that potential donors are more familiar with (say Gaza and Ukraine), and hence each is the favorite public good of one of the citizens. All other θ's are conflict zones that rarely appear in the news, and hence none is the favorite public good of any citizen.

Specifically, each citizen i, $i = 1, 2$, has a budget of 1, is of type θ_i, and has a utility function of

$$U_i = c_i + A \cdot g\left(x\left(\theta_i\right)\right) + a \cdot \sum_{\theta \neq \theta_i} g(x(\theta)),$$

where A, a and $g(\cdot)$ satisfy conditions (2) and (3).

The activist entrepreneur has a budget of 0, but is uniquely capable of transforming citizens' donations into the N public goods. Unlike the citizens, she cares about each conflict zone equally. Her favorite allocation of resources among different conflict zones, however, depends on where humanitarian crises arise, which cannot be known *ex ante* at the time when donations are solicited. Specifically, for each $i = 1, 2, \ldots, N$, her utility function will be

$$U_e = x\left(\theta_i\right)$$

with probability $1/N$.

We first consider the case where the activist entrepreneur markets and solicits donations for the N public goods separately. Under condition (3), citizen i will not contribute to any public good $\theta \neq \theta_i$, and will contribute to public good θ_i up to χ^*, where χ^* solves the first order condition of

$$A \cdot g'(\chi) = 1.$$

The activist entrepreneur's expected utility is hence $U_e^s = 2\chi^*/N$ (with the superscript s denoting "separating").

Next consider the alternative case where the activist entrepreneur bundles the N public goods and market Ω as a whole. Let X be the total donations she collects. For simplicity, let's also assume that the activist entrepreneur does not commit to any specific allocation of X among the N pubic goods, and preserves the discretion to allocate the whole of X to whichever public good that needs resources the most *ex post*. The two citizens hence play the public-good contribution game anticipating this subsequent behavior of the activist entrepreneur.

If $X > 0$, one of the citizens must be contributing a strictly positive amount. The marginal gain of his last dollar of contribution is $[(A + (N - 1)a)/N] \cdot g'(X)$, while the marginal cost is 1. Therefore, he is willing to contribute that last dollar only if

$$\frac{A + (N-1)a}{N} \cdot g'(X) \geq 1, \tag{13}$$

with "=" if his contribution is strictly less than 1. By condition (2),

$$\frac{A + (N-1)a}{N} \cdot g'(1) < A \cdot g'(1) < 1,$$

and hence we have $X < 1$ and (13) holds with equality. On the other hand, if $X = 0$, then neither citizen is willing to contribute the first dollar, and hence we must have

$$\frac{A + (N-1)a}{N} \cdot g'(0) \leq 1.$$

In summary, the activist entrepreneur's expected utility is $U_e^b = X^*$ (with the superscript b denoting "bundling"), where X^* solves the first order condition of

$$\frac{A + (N-1)a}{N} \cdot g'(X) \leq 1, \tag{14}$$

with "=" if $X^* > 0$.

We can now readily see that, when every public good is the favorite of some citizens (i.e., when $N = 2$), separating is always better than bundling, because

$$U_e^s = \chi^* = (g')^{-1}\left(\frac{1}{A}\right) > (g')^{-1}\left(\frac{2}{A + a}\right) = X^* = U_e^b,$$

where the strict inequality follows from $(g')^{-1}(1/A) > 0$ by condition (2).[6] Although bundling allows the activist entrepreneur to move the total donations X^* to whichever public good that needs resources the most *ex post*, this flexibility lowers the citizens' incentive to donate. If the activist entrepreneur markets the two public goods separately, not only that the elimination of *ex post* flexibility makes citizens more eager to donate

[6]We adopt the convention that $(g')^{-1}(z) = 0$ if $z \geq g'(0)$.

(resulting in $\chi^* > X^*$), citizens' heterogeneity also leads them to channel this higher eagerness towards different public goods. When every public good is the favorite of some citizens, all public goods are adequately "covered" in this way, with none being left in the cold, and hence *ex post* flexibility is not valuable.

This is no longer true when some public goods are not the favorite of any citizen (i.e., when $N > 2$). In this case, separating will leave only θ_1 and θ_2 "covered", and leave all other public goods in the cold. This in itself does not imply the superiority of bundling. As before, bundling comes with *ex post* flexibility, which depresses citizens' incentives to donate, and if anything even more so when N increases.

Mathematically, as N increases beyond 2, both U_e^s and U_e^b decrease. U_e^s decreases because it becomes more and more likely that the public good that needs resources the most *ex post* does not get funded. U_e^b decreases because citizens gets more and more discouraged to contribute, as it is less and less likely that the total donations will be allocated to their favorite public goods. Without further assumptions on A, a, and $g(\cdot)$, we do not know which of these two will decrease faster. As Example 4 below shows, it is possible that bundling is strictly better than separating (i.e., $U_e^b > U_e^s$) for some $N > 2$.

Even when bundling beats separating, it is strictly dominated by yet another strategy, which we may call "separate bundles" (SB). Let $M = N/2$ if N is even, and $M = (N-1)/2$ if N is odd. Suppose the activist entrepreneur partition Ω into bundles $\Omega_1 = \{\theta_1, \theta_3, \ldots, \theta_{2M-1}\}$ and $\Omega_2 = \{\theta_2, \theta_4, \ldots, \theta_{2M}, \theta_N\}$.[7] Suppose she markets each bundle separately, promising that the total donations X_i to bundle Ω_i will stay within the bundle. Assume that, if humanitarian crises arise in any conflict zone $\theta \notin \Omega_i$, the activist entrepreneur will break her indifference and allocate the total donations X_i to θ_i (to the delight of citizen i).

By almost the same argument as above, citizen i will contribute only to bundle Ω_i, and his contribution χ_i to Ω_i solves the first order conditions of

$$\frac{[N - (|\Omega_i| - 1)]A + (|\Omega_i| - 1)a}{N} \cdot g'(\chi) \leq 1, \tag{15}$$

with "=" if $\chi_i > 0$. Comparing (14) and (15), we have $\chi_i \geq X^*$, with ">" if $\chi_i > 0$. The

[7] If N is even, we have $|\Omega_1| = M = |\Omega_2|$. If N is odd, we have $|\Omega_1| = M < M + 1 = |\Omega_2|$.

activist entrepreneur's expected utility is hence

$$U_e^{sb} = \frac{|\Omega_1|}{N}\chi_1 + \frac{|\Omega_2|}{N}\chi_2 \geq X^* = U_e^b,$$

with ">" if $\chi_1 > 0$.[8] Note that bundling beats separating only if X^* is strictly positive, and hence whenever that happens we must have U_e^{sb} strictly higher than U_e^b.

Proposition 5 *Consider the model in this section. Marketing all public goods in a single bundle is never optimal for the activist entrepreneur, and is dominated by marketing separate bundles, with each bundle containing at most one public good that is the favorite of some citizens.*

Can the activist entrepreneur do better than both separating and SB? There are at least three possible ways to improve upon them. The first is to consider a bundle $\widehat{\Omega}_i$ that is in between (in set-inclusion sense) $\{\theta_i\}$ and Ω_i. Intuitively, the smaller is $\widehat{\Omega}_i$, the stronger is citizen i's incentives to contribute, but there will also be more conflict zones being left in the cold. The second is to restrict the activist entrepreneur's discretion and impose a lower bound on how much the total contributions χ_i to Ω_i must be allocated to conflict zone θ_i. This will increase citizen i's incentives to contribute to bundle Ω_i, but will also reduce the activist entrepreneur's *ex post* utility if humanitarian crises arise in a different conflict zone within Ω_i. The third is to include both θ_1 and θ_2 into a single bundle that is strictly small than Ω (i.e., $\{\theta_1, \theta_2\} \subset \widehat{\Omega} \subsetneq \Omega$), together with some rule to allocate the total donations $\widehat{X}$ within $\widehat{\Omega}$ in case humanitarian crises arise elsewhere. We have not yet explored these and other possible improvements, and shall leave the characterization of the activist entrepreneur's optimal strategy for future research.

Example 4: Assume $A > 1 = a$, and

$$g(x) = \begin{cases} x - x^2/2 & \text{if } 0 \leq x \leq 1 \\ 1/2 & \text{if } x > 1 \end{cases}. \tag{16}$$

It can be readily check that both (2) and (3) are satisfied.

[8]Note that $\chi_1 \geq \chi_2$, and hence $\chi_2 > 0$ implies $\chi_1 > 0$ as well.

Given assumption (16), it can be readily calculated that

$$\chi^* = \frac{A-1}{A},$$
$$X^* = \frac{A-1}{A+(N-1)},$$
$$U_e^s = \frac{2(A-1)}{NA}, \quad \text{and}$$
$$U_e^b = \frac{A-1}{A+(N-1)}.$$

Therefore, we have $U_e^s < U_e^b$ iff $N(A-2) > 2(A-1)$. This is apparently impossible if $A \leq 2$. On the other hand, if $A > 2$, then we have $U_e^s < U_e^b$ iff

$$N > \frac{2(A-1)}{A-2} > 2.$$

If the activist entrepreneur markets separate bundles, then

$$\chi_1 = \chi_2 = \frac{A-1}{A+\frac{M-1}{M+1}},$$

assuming for simplicity that N is even (and hence $M = N/2$). Therefore,

$$U_e^{sb} = \frac{M}{N}\chi_1 + \frac{M}{N}\chi_2 = \frac{A-1}{A+\frac{M-1}{M+1}} > \frac{A-1}{A+(N-1)} = U_e^b,$$

and hence SB is always strictly better than bundling. ■

8 Concluding Remarks

This paper has presented a model of differentiated public goods, which can help resolve all three empirical puzzles documented in Andreoni's (1988) "The Limits of Altruism". Government contributions to charitable activities can even crowd in, instead of crowd out, private giving. A diverse society is conceptually different from a polarized one, with more (less) charity donations in the former (latter) than in a homogeneous society. Like any profit-seeking entrepreneurs, activist entrepreneurs also face a product-design problem of how to package differentiated public goods into bundles to maximize donations.

So far we have treated as exogenous how diverse/polarized a community is. This is perhaps acceptable when we have in mind the "textbook examples" of public goods such as national defence and public health. The provision of these public goods directly affects our wellbeing regardless of whether we are aware of them. However, we have already alluded to some examples of public goods whose values are predicated on our awareness. For example, one cannot care about cultural preservation unless he is aware of the beauty of his cultural heritage. Likewise, he cannot be upset by the protection of gay rights unless he is aware that some men can be gay. In the terminology of this paper, whether $x(\omega)$ increases/decreases or has no effect on a type-θ citizen's perceived G_θ depends on whether he is aware of public good ω in the first place.

To the extent that the increasing polarization of our society (which many believe is true) is partly due to our increasing awareness of our fellow citizens' peculiar behavior that some may find disagreeable, and to the extent that the expansion of our awareness is irreversible, the increased polarization of our society may never be reversed completely. The damage may be mitigated, however, if we can invest in uncovering more social causes that many, once becoming aware of them, will care about. We leave the exploration of endogenous diversity/polarization to future research.

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Appendix A: Omitted Proofs in Section 3

Proof of Lemma 1:

According to condition (1), there exists $\epsilon > 0$ small enough such that

$$(h-\epsilon)\cdot A\cdot g'(0) > u'(1) > (h+\epsilon)\cdot a\cdot g'(0).$$

Let $\underline{x}$ be the solution of the equality

$$u'(1-x) = (h-\epsilon)\cdot A\cdot g'(x).$$

Note that the LHS is strictly increasing, and the RHS is strictly decreasing, in x. Therefore, $\underline{x}$, if exists, is unique. By our choice of ϵ, we have $\underline{x} > 0$. If the LHS is strictly small than the RHS for all $x \in [0,1]$, let $\underline{x} = 1$.

Let $\overline{x}$ be the solution of the equality

$$u'(1) = (h+\epsilon)\cdot A\cdot g'(x).$$

Note that the RHS is strictly decreasing, is strictly larger than $u'(1)$ when $x = 0$, and goes to 0 when x goes to infinity, Therefore, $\overline{x}$ exists, is unique, and is strictly bigger than $\underline{x}$ because

$$(h+\epsilon)\cdot A\cdot g'\left(\underline{x}\right) \geq u'\left(1-\underline{x}\right) > u'(1),$$

where the last inequality follows from $\underline{x} > 0$.

Pick $\overline{G}$ large enough such that, $\forall G > \overline{G}$,

$$|H'(G)-h| < \epsilon \qquad \text{and} \qquad |H''(G)| < \frac{h-\epsilon}{A\cdot\left[g'\left(1+\overline{x}\right)\right]^2}\cdot \min_{x\in\left[0,1+\overline{x}\right]} |g''(x)|. \tag{17}$$

Pick M large enough such that $(M-1)\cdot a\cdot g\left(\underline{x}\right) > \overline{G}$. Fix any society with (n_k, m_k) such that $m_k > M$. Let's drop the subscript k from now on to ease notations.

Let i be a type-θ citizen. Suppose, for every $\theta' \in \Theta$ with $\theta' \neq \theta$,

1. public good θ' receives a common total contribution $x(\theta') \equiv x \in \left[\underline{x}, \overline{x}\right]$, and

2. every type-θ' citizen contributes only to public good θ' but not to any other public

good.

Suppose every other type-θ citizen contributes a common amount of $z \in [0, \overline{x}/(t-1)]$ to public good θ. Then i's optimization problem is

$$\max_{\chi \in [0,1]} U = u(1-\chi) + H(G_\theta),$$

where

$$G_\theta = A \cdot g(\chi + (t-1)z) + (m-1) \cdot a \cdot g(x) \geq (m-1) \cdot a \cdot g\left(\underline{x}\right) > \overline{G}$$

and hence $|H'(G_\theta) - h| < \epsilon$ by (17).

The first derivative wrt χ is $U' = -u' + H' \cdot A \cdot g'$. Note that u' is strictly increasing in χ. Meanwhile, $H' \cdot A \cdot g'$ is strictly decreasing in χ because its derivative is

$$\begin{aligned} H'' \cdot [A \cdot g']^2 - H' \cdot A \cdot |g''| &= [A \cdot g']^2 \cdot \left\{ H'' - \frac{H' \cdot |g''|}{A \cdot [g']^2} \right\} \\ &< [A \cdot g']^2 \cdot \left\{ H'' - \frac{h - \epsilon}{A \cdot [g'(1+\overline{x})]^2} \cdot \min_{x \in [0, 1+\overline{x}]} |g''(x)| \right\} \\ &< 0, \end{aligned}$$

where the first inequality follows from $\chi + (t-1)z \leq 1 + \overline{x}$ and (17). Therefore, U' is strictly decreasing in χ, and hence a unique best response $\chi(z)$ to z exists. We note for future reference that this unique best response $\chi(z)$ is continuous in x by the Maximum Theorem.

A similar argument also establishes that $H' \cdot A \cdot g'$ is strictly decreasing in z, and hence $\chi(z)$ is weakly decreasing in z (strictly if $\chi(z)$ is interior). Moreover, $\chi(z)$ is continuous in z by the Maximum Theorem.

Note that $\chi(z) \geq \max\left\{\underline{x} - (t-1)z, 0\right\}$ because

$$\begin{aligned} u'\left(1 - \max\left\{\underline{x} - (t-1)z, 0\right\}\right) &\leq u'\left(1 - \underline{x}\right) \\ &\leq (h - \epsilon) \cdot A \cdot g'\left(\underline{x}\right) \\ &< H'(G_\theta) \cdot A \cdot g'\left(\max\left\{\underline{x} - (t-1)z, 0\right\} + (t-1)z\right); \end{aligned}$$

in particular, we have $\chi(0) > 0$.

Note also that $\chi(z) < \overline{x} - (t-1)z$ for any $z < \overline{x}/(t-1)$ because

$$u'(1) = (h + \epsilon) \cdot A \cdot g'(\overline{x}) > H'(G_\theta) \cdot A \cdot g'(\overline{x});$$

in particular, we have $\chi(\overline{x}/(t-1)) = 0$.

There is hence a unique fixed point $z^* \in (0, \overline{x}/(t-1))$ such that $\chi(z^*) = z^*$. It is hence a symmetric equilibrium within the subgroup of type-θ citizens to each contribute z^* to public good θ *taking the strategies of citizens in other subgroups as given*. Moreover, no type-θ citizen has any incentives to contribute to any public good $\omega \neq \theta$ because

$$u'(1) > (h + \epsilon) \cdot a \cdot g'(0) > H'(G_\theta) \cdot a \cdot g'(x(\omega));$$

i.e., the marginal utility of private consumption is higher than the marginal gain from contributing to any public good $\omega \neq \theta$.

Let $y = tz^*$ be the total contribution to public good θ. Note that $y = \chi(z^*) + (t-1)z^* \in \left[\underline{x}, \overline{x}\right]$ as we noted earlier.

We have now constructed a continuous mapping from $x \in \left[\underline{x}, \overline{x}\right]$ to $y \in \left[\underline{x}, \overline{x}\right]$, where continuity follows from the continuity of $\chi(z)$ in x as we noted earlier. The existence of an equilibrium satisfying the properties stated in Lemma 1 then follows from Brouwer's Fixed Point Theorem.

As we noted in Section 3.3, Lemma 1 continues to hold in a model with heterogeneous wealth. We highlight here only the part of the proof that needs adaptation.

Suppose the t citizens within every subgroup (of citizens with the same type) have wealths $1 = w_1 \geq w_2 \geq \cdots \geq w_t \geq 0$. Fix any $x \in \left[\underline{x}, \overline{x}\right]$ (the common total contribution to every public good $\theta' \in \Theta$ with $\theta' \neq \theta$). For any $z \in \left[\underline{x}, \overline{x}\right]$, let $\chi_i(z) \in [0, w_i]$, $i \in \{1, 2, \ldots, t\}$, be the solution of the equality

$$u'(w_i - \chi) = H'(G_\theta) \cdot A \cdot g'(z), \tag{18}$$

where

$$G_\theta = A \cdot g(z) + (m-1) \cdot a \cdot g(x) \geq (m-1) \cdot a \cdot g\left(\underline{x}\right) > \overline{G}$$

and hence $|H'(G_\theta) - h| < \epsilon$ by (17). If the LHS is strictly bigger (smaller) than the RHS for all $\chi \in [0, w_i]$, let $\chi_i(z) = 0$ ($\chi_i(z) = w_i$). Note that $\chi_i(z) \geq \chi_{i+1}(z)$ for every $i \in \{1, 2, \ldots, t-1\}$

and any $z \in \left[\underline{x}, \overline{x}\right]$. Note also that $\chi_1(z) \geq \underline{x}$ for any $z \in \left[\underline{x}, \overline{x}\right]$ because

$$u'\left(1 - \underline{x}\right) \leq (h - \epsilon) \cdot A \cdot g'\left(\underline{x}\right)$$

and that $\chi_1(\overline{x}) = 0$ because

$$(h - \epsilon) \cdot A \cdot g'(0) > u'(1) > (h + \epsilon) \cdot A \cdot g'(\overline{x}).$$

By exactly the same argument as the one used earlier, $H' \cdot A \cdot g'$ is strictly decreasing in z, and hence $\chi_i(z)$ is weakly decreasing in z (strictly if $\chi_i(z)$ is interior). Moreover, $\chi_i(z)$ is continuous in both z and x because $H' \cdot A \cdot g'$ is continuous in both.

Let $y(z) = \sum_i \chi_i(z)$. Then $y(z)$ is continuously decreasing in z as well. Moreover, $y\left(\underline{x}\right) \geq \chi_1\left(\underline{x}\right) \geq \underline{x}$ and $y(\overline{x}) = \sum_i 0 = 0$. Therefore, there exists a fixed point z^* such that $y(z^*) = z^*$. It is hence an equilibrium within the subgroup of type-θ citizens to each contribute $\chi_i(z^*)$ to public good θ *taking the strategies of citizens in other subgroups as given*. That they will not contribute to any public good $\omega \neq \theta$ follows the same argument as before.

We have now constructed a continuous mapping from $x \in \left[\underline{x}, \overline{x}\right]$ to $z^* \in \left[\underline{x}, \overline{x}\right]$, where continuity follows from the continuity of $\chi(z)$ in x as we noted earlier. The existence of an equilibrium satisfying the properties stated in Lemma 1 then follows from Brouwer's Fixed Point Theorem. ■

Proof of Proposition 1:

Suppose there exists an $\epsilon > 0$ (which is independent of k) such that, for every K, there exists a society $k > K$ such that there exists an equilibrium where every citizen contributes to public goods a share of his budget larger than ϵ.[9]

Fix any such k and any such equilibrium. Let $x(\omega)$ be the equilibrium total contribution to public good $\omega \in \Omega$. Let $\widehat{\Omega}_k = \{\omega \notin \Theta_k \,|\, x(\omega) > 0\}$. By symmetry, $x(\theta) = x(\theta') = x_1$ for all $\theta, \theta' \in \Theta_k$, and $x(\omega) = x(\omega') = x_2$ for every $\omega, \omega' \in \widehat{\Omega}_k$. Since $A \cdot g'(x_1) > a \cdot g'(x_1)$, no type-$\theta$ citizen will contribute to public good $\theta' \neq \theta$.

The "Part of the Proof" in the main text has already covered the case of $\widehat{\Omega}_k = \emptyset$. So

[9] Recall that we are focusing on symmetric equilibria. Therefore, if one citizen contributes to public goods a share of his budget larger than ϵ in an equilibrium, every citizen does in that equilibrium.

let's suppose $\widehat{\Omega}_k \neq \emptyset$. Let $\widehat{m}_k = |\widehat{\Omega}_k|$.

If $x_1 \leq x_2$, then $A \cdot g'(x_1) > a \cdot g'(x_2)$ and no citizen will contribute to any $\omega \in \widehat{\Omega}_k$, contradicting $\widehat{\Omega}_k \neq \emptyset$. We hence have $x_1 > x_2 > 0$. Therefore, for any $\theta \in \Theta_k$,

$$\begin{aligned} G_\theta &= (A + (m_k - 1) \cdot a) \cdot g(x_1) + \widehat{m}_k \cdot a \cdot g(x_2) \\ &> m_k \cdot a \cdot g(x_1) + \widehat{m}_k \cdot a \cdot g(x_2) \\ &> m_k \cdot a \cdot g\left(x_1 + \frac{\widehat{m}_k x_2}{m_k}\right) \\ &\geq m_k \cdot a \cdot g(t_k \epsilon), \end{aligned}$$

where the second inequality follows from $x_1 > x_2 > 0$ and the strict concavity of $g(\cdot)$.

As K and hence the corresponding k go to infinity, G_θ (for any type $\theta \in \Theta_k$) will also go to infinity, and hence $H'(G_\theta)$ will go to h. But then, by condition (1), no type-θ citizen will contribute to any public good $\omega \neq \theta$, contradicting $\widehat{\Omega}_k \neq \emptyset$. ■

Appendix B: Omitted Details of Section 5

This appendix provides an example illustrating the tradeoff a government may face between contributing to a public good that is more plagued by the free-rider problem, and contributing to a public good that is less plagued but is considered by many as complementary to another public good that is plagued by the free-rider problem.

Consider a society with two citizens, 1 and 2, and three public goods, θ, ω_1, and ω_2. The two citizens' utility functions are

$$U_1 = c_1 + H\left[e \cdot g(x(\theta)) + a \cdot x(\omega_1)\right] + b \cdot x(\omega_2)$$

and

$$U_2 = c_2 + B \cdot x(\theta) + b \cdot x(\omega_2),$$

respectively, where e is Euler's number, and

$$H(G) = \frac{1}{r} \cdot \left[G + \frac{e^{-G} - 1}{e + 1}\right]$$
$$\text{and} \quad g(x) = \left(1 - e^{-rx}\right),$$

with $r \in (0, 1)$ being a positive constant to be determined below.

Note that public good ω_1 is more a private good of citizen 1, and hence is not plagued by the free-rider problem at all. Meanwhile, public good ω_2 is plagued by the free-rider problem: if $1 > b > 1/2$, neither citizen will contribute to public good ω_2, even though it is socially efficient to do so.

Assume that $a = e \cdot g'(1)$. Then citizen 1 will never contribute to public good ω_1 because

$$e \cdot g'(\chi) > e \cdot g'(1) = a$$

for any $\chi < 1$. Also assume that $B < 1$, and hence citizen 2 will never contribute to public good θ either.

Note that $H(\cdot)$ is strictly increasing and strictly convex, as for any $G > 0$,

$$\begin{aligned} H'(G) &= \frac{1}{r} \cdot \left[1 - \frac{e^{-G}}{e+1}\right] > 0 \\ \text{and} \quad H''(G) &= \frac{1}{r} \cdot \frac{e^{-G}}{e+1} > 0, \end{aligned}$$

and hence citizen 1 considers public goods θ and ω_1 as complements.

Meanwhile, $g(\cdot)$ is strictly increasing and strictly concave, as for any $x > 0$,

$$\begin{aligned} g'(x) &= r \cdot e^{-rx} > 0 \\ \text{and} \quad g''(x) &= -r^2 \cdot e^{-rx} < 0. \end{aligned}$$

The composite function $H(e \cdot g(x) + z)$ is strictly increasing and strictly concave in x for any $z \geq 0$, as for any $x > 0$,

$$\begin{aligned} \frac{d}{dx}H(e \cdot g(x) + z) &= H'(e \cdot g(x) + z) \cdot e \cdot g'(x) \\ &= \frac{1}{r} \cdot \left[1 - \frac{e^{-(e \cdot g(x)+z)}}{e+1}\right] \cdot e \cdot r \cdot e^{-rx} \\ &= \cdot \left[1 - \frac{e^{-(e \cdot g(x)+z)}}{e+1}\right] \cdot e \cdot e^{-rx} > 0 \\ \text{and} \quad \frac{d^2}{dx^2}H(e \cdot g(x) + z) &= \left[\frac{e^{-(e \cdot g(x)+z)}}{e+1} \cdot e \cdot g'(x)\right] \cdot e \cdot e^{-rx} - \left[1 - \frac{e^{-(e \cdot g(x)+z)}}{e+1}\right] \cdot e \cdot r \cdot e^{-rx} \\ &\propto \left[\frac{e^{-(e \cdot g(x)+z)}}{e+1} \cdot e \cdot g'(x)\right] - \left[1 - \frac{e^{-(e \cdot g(x)+z)}}{e+1}\right] \cdot r \\ &= \frac{e^{-(e \cdot g(x)+z)}}{e+1} \cdot [e \cdot g'(x) + r] - r \\ &< \frac{1}{e+1} \cdot [e \cdot r \cdot e^{-rx} + r] - r \\ &< \frac{1}{e+1} \cdot [e \cdot r + r] - r = 0. \end{aligned}$$

Therefore, citizen 1's contribution χ to public good θ is uniquely pinned down by the first order condition

$$H'[e \cdot g(\chi)] \cdot e \cdot g'(\chi) = 1.$$

Note that $\chi > 0$ because

$$H'(0) \cdot e \cdot g'(0) = \frac{1}{r} \cdot \frac{e}{e+1} \cdot e \cdot r = \frac{e^2}{e+1} > 1.$$

On the other hand,

$$\begin{aligned} H'(e \cdot g(1)) \cdot e \cdot g'(1) &= \frac{1}{r} \cdot \left[1 - \frac{e^{-(e \cdot g(1))}}{e+1}\right] \cdot e \cdot r \cdot e^{-r} \\ &= \left[1 - \frac{e^{-e \cdot (1-e^{-r})}}{e+1}\right] \cdot e^{1-r}, \end{aligned}$$

which approaches $e^2/(e+1) > 1$ when $r \searrow 0$, approaches $1 - e^{-(e-1)}/(e+1) < 1$ when $r \nearrow 1$, and is strictly decreasing in r because

$$\begin{aligned} \frac{d}{dr}\left[1 - \frac{e^{-e \cdot (1-e^{-r})}}{e+1}\right] \cdot e^{1-r} &= \left[\frac{e^{-e \cdot (1-e^{-r})} \cdot e^{1-r}}{e+1}\right] \cdot e^{1-r} - \left[1 - \frac{e^{-e \cdot (1-e^{-r})}}{e+1}\right] \cdot e^{1-r} \\ &\propto \left[\frac{e^{-e \cdot (1-e^{-r})} \cdot e^{1-r}}{e+1}\right] - \left[1 - \frac{e^{-e \cdot (1-e^{-r})}}{e+1}\right] \\ &= e^{-e \cdot (1-e^{-r})} \cdot \frac{e^{1-r}+1}{e+1} - 1 \\ &< 1 \cdot 1 - 1 = 0. \end{aligned}$$

Therefore, there exists $\underline{r} \in (0,1)$ such that $H'(e \cdot g(1)) \cdot e \cdot g'(1)$ is strictly smaller than 1 for any $r \in \left(\underline{r}, 1\right)$. For any such r, we have $\chi \in (0,1)$. As $r \searrow \underline{r}$, we have $\chi \nearrow 1$.

Suppose the government can levy a small tax $\Delta > 0$ on both citizens, and contribute the tax revenue 2Δ to one of the three public goods. Contributing to either public good θ clearly will have no effect on social welfare because of the classical crowding-out effect. Citizen 1 will simply reduce his contribution to θ by 2Δ, maintaining the total contribution to θ at the original level of χ, because χ_1 continues to maximize

$$[(1-\Delta) - \chi] + H[e \cdot g(\chi)]$$

for small enough Δ. In the end, citizen 1's private consumption increases to $1 - \chi_1 + \Delta$, whereas that of citizen 2 decreases to $1 - \chi_2 - \Delta$. What the government achieves is a pure transfer of Δ from citizen 2 to citizen 1, and has no effect on social welfare.

Contributing to public good ω_2 does increase social welfare. As we noted earlier, public good ω_2 is plagued by the free-rider problem, with private contributions being suboptimal at the 0 lower bound. The per-dollar increase in social welfare is $2b - 1 > 0$.

As we also noted earlier, public good ω_1 is not plagued by the free-rider problem at all. But contributing to it can also increase social welfare, because it is consider by citizen 1 as complementary to public good θ, which is plagued by the free-rider problem. For a small $\Delta > 0$, the per-dollar increase in social welfare is approximately

$$\frac{dW}{d\Delta}\bigg|_{\Delta=0} = H'[e \cdot g(\chi)] \cdot a + \{H'[e \cdot g(\chi)] \cdot e \cdot g'(\chi) + B\} \cdot \frac{d\chi}{d\Delta}\bigg|_{\Delta=0} - 1, \tag{19}$$

where χ (as a function of Δ) is uniquely pinned down by the first order condition

$$H'[e \cdot g(\chi) + a \cdot \Delta] \cdot e \cdot g'(\chi) = 1$$

thanks to the strict concavity of the composite function $H[e \cdot g(x) + z]$ (with $z = a \cdot \Delta$) established earlier. Using $a = e \cdot g'(1)$ and the first order condition at $\Delta = 0$, we can simplify (19) as

$$\frac{dW}{d\Delta}\bigg|_{\Delta=0} = \frac{e \cdot g'(1)}{e \cdot g'(\chi)} + (1 + B) \cdot \frac{d\chi}{d\Delta}\bigg|_{\Delta=0} - 1.$$

By the Implicit Function Theorem,

$$\frac{d\chi}{d\Delta}\bigg|_{\Delta=0} = -\frac{H''[e \cdot g(\chi)] \cdot a \cdot e \cdot g'(\chi)}{H''[e \cdot g(\chi)] \cdot [e \cdot g'(\chi)]^2 + H'[e \cdot g(\chi)] \cdot e \cdot g''(\chi)}, \tag{20}$$

which we know is strictly positive because the denominator is strictly negative by the second order condition.

After some algebra, we can simplify (20) as

$$\frac{d\chi}{d\Delta}\bigg|_{\Delta=0} = -\frac{a \cdot \left[\frac{1}{r} \cdot e \cdot g'(\chi) - 1\right]}{\frac{1}{r} \cdot [e \cdot g'(\chi)]^2 - e \cdot g'(\chi) - r}.$$

As $r \searrow \underline{r}$, we have $\chi \nearrow 1$, and hence both a $(= e \cdot g'(1))$ and $e \cdot g'(\chi)$ converge to $\underline{r} \cdot e^{1-\underline{r}}$. Therefore,

$$\lim_{h \searrow \underline{r}} \frac{d\chi}{d\Delta}\bigg|_{\Delta=0} = -\frac{\left[e^{1-\underline{r}}\right]^2 - e^{1-\underline{r}}}{[e^{1-\underline{r}}]^2 - e^{1-\underline{r}} - 1} =: K$$

for some constant $K \in (0, 1)$.

Therefore

$$\lim_{r \searrow \underline{r}} \frac{dW}{d\Delta}\bigg|_{\Delta=0} = 1 + (1+B) \cdot K - 1 = (1+B) \cdot K > 0.$$

Proposition 6 *In the two-citizen model in this appendix, if r is close enough to $\underline{r}$ and b close enough to $1/2$, then the government should contribute to public good ω_1 instead of ω_2, even though the latter is plagued by the free-rider problem while the former is not.*

Appendix C: Omitted Details of Section 6

In Section 6, we assume that the strictly increasing and strictly convex function $H(\cdot)$ and the strictly increasing and strictly concave function $g(\cdot)$ jointly satisfy the assumption that the composite function $H(A \cdot g(x) + z)$ is strictly concave in x for any $z \geq -b \cdot g(1)$. In this appendix, we provide an example with such $H(\cdot)$ and $g(\cdot)$.

For any $x \geq 0$, let

$$g(x) := 1 - e^{-x},$$

which can be readily verified as a strictly increasing and strictly concave, as

$$\begin{aligned} & g'(x) = e^{-x} > 0 \\ \text{and} \quad & g''(x) = -e^{-x} < 0. \end{aligned}$$

For any $G \geq -b \cdot g(1)$, let

$$H(G) := G + e^{-K}\left(e^{-G} - 1\right)$$

with some $K > \ln(A+1) + b \cdot g(1) > 0$. It can be readily check that $H(\cdot)$ is strictly increasing and strictly convex, as

$$\begin{aligned} & H'(G) = 1 - e^{-K}e^{-G} > 0 \\ \text{and} \quad & H''(G) = e^{-K}e^{-G} > 0, \end{aligned}$$

where the first inequality follows from $K + G > [\ln(A+1) + b \cdot g(1)] - b \cdot g(1) = \ln(A+1) > 0$.

For any $z \geq -b \cdot g(1)$, we have

$$\begin{aligned}
\frac{d}{dx}H(A \cdot g(x) + z) &= H'(A \cdot g(x) + z) \cdot A \cdot g'(x) \\
&= \left[1 - e^{-K}e^{-(A \cdot g(x)+z)}\right] \cdot A \cdot e^{-x}
\end{aligned}$$

$$\begin{aligned}
\text{and} \quad \frac{d^2}{dx^2}H(A \cdot g(x) + z) &= \left[e^{-K}e^{-(A \cdot g(x)+z)} \cdot A \cdot g'(x)\right] \cdot A \cdot e^{-x} - \left[1 - e^{-K}e^{-(A \cdot g(x)+z)}\right] \cdot A \cdot e^{-x} \\
&\propto \left[e^{-K}e^{-(A \cdot g(x)+z)} \cdot A \cdot g'(x)\right] - \left[1 - e^{-K}e^{-(A \cdot g(x)+z)}\right] \\
&= e^{-K}e^{-(A \cdot g(x)+z)} \cdot [A \cdot g'(x) + 1] - 1 \\
&\leq e^{-K}e^{-z} \cdot [A \cdot g'(0) + 1] - 1 \\
&< e^{-(\ln(A+1)+b \cdot g(1))}e^{b \cdot g(1)} \cdot [A \cdot 1 + 1] - 1 \\
&= e^{-\ln(A+1)} \cdot [A + 1] - 1 \\
&= 0,
\end{aligned}$$

and hence the composite function $H(A \cdot g(x) + z)$ is strictly concave in x for any $z \geq -b \cdot g(1)$.

To further satisfy condition (7), let $A = 2$. Then we have

$$\begin{aligned}
H'(0) \cdot A \cdot g'(0) &= \left(1 - e^{-K}\right) \cdot A \cdot 1 \\
&> \left(1 - e^{-\ln(A+1)}\right) \cdot A \\
&= \left(1 - (A + 1)^{-1}\right) \cdot A \\
&= 4/3 \\
&> 1
\end{aligned}$$

and

$$\begin{aligned}
1 &> \left[1 - e^{-K}e^{-(A \cdot g(1))}\right] \cdot \left[2 \cdot e^{-1}\right] \\
&= H'(A \cdot g(1)) \cdot A \cdot g'(1),
\end{aligned}$$

and hence condition (7) is satisfied. Finally, condition (8) can be satisfied by picking $a = 1$, as

$$
\begin{aligned}
1 &> \left[1 - e^{-K} e^{-(A \cdot g(2))}\right] \cdot 1 \cdot 1 \\
&= H'(A \cdot g(2)) \cdot a \cdot g'(0).
\end{aligned}
$$

Finally, we note that $A = 2$ and $a = 1$ also satisfy condition (12) with equality.