Sharing sequentially triggered losses

Automatic conflict resolution through smart contracts

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Sharing sequentially triggered losses: Automated conflict resolution through smart contracts

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Abstract

When actions by one agent force another to deviate from their agreements with a third, "victim" turns into "injurer" in the chain's subsequent steps. Should the chain's initiator be responsible only for the direct harm they cause or also bear some of the indirect losses they trigger? Through an axiomatic approach, we characterize the class of fixed-fraction rules, which strike a balance between incentives for accident prevention on the one hand and fairness in terms of how liabilities are assigned on the other. Their simple design make the rules ideal for practical implementation through smart contracts, enabling automated conflict resolution.

Keywords: Sequential losses, fair allocation, smart contracts, cost allocation

JEL: K12, C7, D6

1. Introduction

Frequently, agents' actions initiate sequences of unanticipated events that affect other agents. For instance, suppose a construction company made mistakes when building a housing complex which the current owner now, years later, requires to be fixed. One of the tenants is a coffee shop. It seems intuitively appealing that the construction company should cover the costs of its mistake, but what about the losses incurred by the coffee shop due to reduced sales during the renovations? The construction company reckons that its obligations are towards the owners of the housing complex only, while the owners, facing compensation demands from the coffee shop, argue that there would be no losses whatsoever had the construction company

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been thorough to begin with. In this paper, we take an axiomatic approach to identify solutions to this and many other related disputes. We then show how the recent developments in blockchain technology provide the tools necessary for automated conflict resolution without interference by a third party.

Model. We study a stylized model in which agents have made decentralized bilateral agreements on actions to be taken. "Actions" are here interpreted quite broadly: in the example above, it could mean the construction quality or the tenants' renting conditions; in a supply chain, it could mean the delivery of an input at a particular point in time. By "decentralized" we mean that agents only are aware of the agreements that they themselves partake in. Hence, with a chain of agents 1, 2, ..., agent 1 is only aware of the agreement with agent 2; agent 2, on the other hand, knows also of the agreement with agent 3, and so on.

A problem now arises when the "initiator", agent 1, fails their agreement with agent 2. This is a breach of contract without which events would perhaps otherwise have transpired as planned. Once agent 1 fails to meet 2's requirements—say there is a flaw in the building construction or a delay in the supply chain—agent 2 no longer can satisfy the agreement with agent 3. In this way, agent 1's deviation from their intended action forces a deviation also by agent 2 (and later 3, 4, ...). These deviations lead to monetary losses. In this way, the initially incurred loss ℓ_1, caused by agent 1 on agent 2, is further reinforced by externalities triggered sequentially to other agents in a finite chain. Hence, it further causes a loss ℓ_2 to agent 3, ℓ_3 to agent 4, and so forth. In what follows, we abstract away the specific reasons for the losses as well as any efforts taken to prevent them. Indeed, in our interpretation, only the initiator is in position to prevent the cascade of losses: once the first harm is caused, the others follow. Hence, it is less a question of "Who is to blame?" but rather "How much liability should be assigned the initiator due to externalities unknown to them from the outset?" While our point of entry is *after* a disruption has occurred, the solution we propose is well-justified both in terms of how it fairly assigns liabilities and in terms of the incentives it induces for accident prevention.

We seek a systematic way to resolve these disputes (a "rule") that builds on normative foundations which account for the sequential nature of the losses. Our main contribution singles out a parametric class of rules in which each agent i pays a fixed fraction λ of the loss associated to her failed agreement with agent $i+1$ while the initiator, agent 1, covers the residual. Again, the unique role of agent 1 is due to 1's actions being the root cause for the loss chain. Every fraction λ corresponds to a different rule and the class of rules spans from agent 1 being liable for all losses ($\lambda = 0$) to each agent being liable for the loss directly associated with her ($\lambda = 1$). For the midpoint, $\lambda = 1/2$, the harm due to agent i's breach of contract with agent $i+1$ is shared equally between the initiator and i. This compromise takes into account agent 1's responsibility as the initiator as well as agent 1's ignorance of the externalities/relations between the agents further down the chain. It stresses that the loss inflicted by agent 1 is different from the other ones while the losses inflicted by agent 2 through n all are of "the same type": the initiator could have done something about the loss chain (though this is not modelled); the others could not. In this sense, fixed-fraction rules balance fairness and incentives: the initiator is incentivized to avoid starting the cascade by carrying the share $1 - \lambda \geq 0$ of every subsequent loss, but it is also acknowledged that the initiator does not control interdependencies further down the chain by having the share $\lambda \geq 0$ fall on the others. This type of rule has been studied extensively in the particular case where a supplier and a manufacturer together produce a good sold to a consumer. A "fixed share rate contract" then specifies that costs due to an "external failure", say a defective product requiring a costly recall (in our case, a loss experienced by the consumer), is shared between the supplier and the manufacturer. These contracts have been shown to incentivize improved product quality (see, e.g., Chao et al., 2009, and references therein).

Results. The first main result, Theorem 1, shows that the fixed-fraction rules have a solid normative foundation. Our point of departure is the well-known *additivity* axiom and an innocuous axiom dubbed *zero truncation*, which states that trailing zeros can be truncated without impacting the liability assigned to the remaining agents. Jointly, these two axioms allow us to restrict attention to so-called *elementary problems*, which have a simple structure and clear interpretation. Specifically, Proposition 1 states that, for *additive* rules that satisfy *zero truncation*, the solution to general loss problems can be found as the loss-weighted sum of solutions to elementary problems. We therefore formulate our next two axioms on elementary problems. The first of these is a version of the well-known *population monotonicity* axiom, stating that if more agents (causing zero losses) are added to the elementary loss chain, then no original agent should be worse off (as there now are more agents to shoulder the same loss). The final axiom, *merging proofness*, focuses on incentives. It asserts that, in a three-agent elementary chain, neither the first and the second agent, nor the second and third, should gain by internalizing their losses. Proposition 2 states that a *population-monotonic* and *merging-proof* rule must coincide with a fixed-fraction rule on elementary problems. Theorem 1 is now a direct consequence of Propositions 1 and 2 and thus characterizes fixed-fraction rules on general loss problems.

Our second main result, Theorem 2, presents a fairness argument in favor of a specific member of the class of fixed-fraction rules, namely the intermediate member $\varphi^{1/2}$. This is done through a novel take on the well-known principles of equal sacrifice and ideal points. For group decision problems, Yu (1973) identifies the highest utility u_i^* that an agent i can possibly obtain and then constructs the "utopia"/ideal point

$u^* = (u_1^*, \ldots, u_n^*)$. Typically u^* is not in the feasible set, so a compromise has to be identified. In particular, Yu suggests the point at minimum distance to the ideal point. For bargaining problems, Chun (1988) suggests the solution that equalizes losses/sacrifices compared to the corresponding ideal point. In this way, this approach entails finding one ideal point (outside the feasible set) with each coordinate representing one agent's ideal allocation, which then is used as a point of reference to "come close to". The novelty in Theorem 2 is to instead identify n ideal points (inside the feasible set), one for each agent, and then selecting the compromise that is at equal (Euclidean) distance to them. In general, such a compromise need not exist. However, Theorem 2 shows that, when losses are equal, $\ell_1 = \cdots = \ell_n$, there is an allocation that is equidistant to all ideal points, namely $\varphi^{1/2}(\ell)$.

Finally, our third main result, Theorem 3, presents a further argument in favor of $\varphi^{1/2}$ as well as a way to contrast it with the well-known *serial rule*. The *serial rule* holds agent 1 responsible for ℓ_1, shares the responsibility of ℓ_2 between agents 1 and 2, and, in general, holds agents 1 through i equally responsible for the loss ℓ_i that agent i causes.[1] In many linear network structures, the *serial rule* coincides with the Shapley value of an associated cooperative game (e.g. Littlechild and Owen, 1973; Gilles et al., 1992; Ni and Wang, 2007). We find a corresponding result in Theorem 3: the intermediate fixed-fraction rule assigns responsibility in a way that coincides with the Shapley value and the nucleolus of an associated liability game. In particular, this game is an instance of a "big boss" game (Muto et al., 1988).

Contribution. We offer the following interpretation of our contribution. Given that real-life loss chains may be very complex, it is arguably unlikely that one solution will be ideal in every circumstance. Indeed, high-valued problems will presumably be settled, case by case, in court. Still, this resembles the extensive literature on adjudicating conflicting claims (e.g. O'Neill, 1982; Thomson, 2019): even if there are unique aspects to every bankruptcy case, there has arguably been significant value in a systematic search for universally appealing properties. In this regard, our contribution then is to give a solid foundation for settling conflicts using fixed-fraction rules.

It should also be noted that not all cases warrant lengthy negotiations and costly adjudications. Especially in smaller-scale problems and situations that are always resolved in the same way (leaving no room for negotiations), there is value in automated conflict resolution. For this purpose, in Section 5, we show how to implement the fixed-fraction rules in practice through a smart contract running on a blockchain. Besides

[1] In contrast, fixed-fraction rules center responsibility for ℓ_i on the agent causing the loss chain and the agent causing the particular loss, namely agents 1 and i. In terms of Theorem 1, the *serial rule* fails *merging proofness*.

their appealing fairness guarantees and the induced incentives to protect against contract breach, the rules have a very simple structure. In smart contracts, every computation comes with a cost; the simple design of the rules therefore help to further reduce transaction costs.

Literature review. Our study relates to several strands of literature. Economically optimal assignment of liability has been analyzed within the law and economics literature at least since Coase (1960). This literature studies agents' (here injurers' and victims') incentives to prevent losses from occurring (e.g., Brown, 1973; Marchand and Russell, 1973; Diamond and Mirrlees, 1975; Green, 1976; Emons and Sobel, 1991). A typical aim of these studies is to analyze how different types of liability rules, in different economic environments, affect socially optimal resource allocation in terms of accident avoidance. In particular, the main concerns of liability rules are whether the injurer or the victim should be held responsible as well as whether, and how, negligence on both sides should be taken into account (see e.g., Shavell, 1980; Landes and Posner, 1987; Shavell, 2007).

In the context of incentives, our paper is also related to the literature on supply chain liability. Potentially questionable behavior by an upstream producer in the supply chain can seriously affect downstream firms. It is often observed that product liability is shared among supply chain members (Fan et al., 2020). For example, in 2007, Sanyo and Lenovo shared the cost of recalling laptop batteries because of a faculty design by Sanyo. As shown by Chao et al. (2009), product recall cost sharing can lead to improved product quality and increased supply chain profits (see also, e.g., Balachandran and Radhakrishnan, 2005; Lim, 2001; Reyniers and Tapiero, 1995a,b).

But our focus is broader than just incentives. We want to balance incentives with distributional fairness, which places our contribution squarely into the literature on fair division (see e.g., Moulin, 1988, 2004; Thomson, 2016). There is a large literature on cost allocation in networks (surveyed in Hougaard, 2018), a considerable share focused on chain structures. This includes the airport problem (Littlechild and Owen, 1973), river sharing (Ambec and Sprumont, 2002; Ni and Wang, 2007), games with permission structures (Gilles et al., 1992), peer-group games (Brânzei et al., 2002), and revenue sharing in hierarchies (Hougaard et al., 2017). While similar in terms of mathematical structure, there are important conceptual differences between our framework and this literature. This is nicely illustrated by comparing with the model of Dehez and Ferey (2013), as we will do next.

Dehez and Ferey (2013) also analyze liability sharing using cooperative game theory, but with a different interpretation. In Dehez and Ferey (2013), a sequence of agents cause injury to a lone victim; in our model, each intermediate agent takes on both the role as injurer and victim. This leads to a difference in the cooperative games that best capture the problems: while we in Subsection 4.2 derive an instance of a "big boss" game, Dehez and Ferey (2013) study a dual airport problem. When applying the Shapley value, this difference leads them to a variation on the *serial rule* and us to the intermediate fixed-fraction rule. However, the *serial rule* is less appealing in our case due to the special role of intermediate agents. Oishi et al. (2022) generalize the linear structure of Dehez and Ferey (2013) to "rooted trees" in which an agent can harm multiple agents. They consider a different cooperative game (akin to a game with conjunctive permission structure, see Gilles et al., 1992) and axiomatically characterize the nucleolus on this class of games. Recently, Juarez et al. (2018) also analyze a model of sharing sequentially generated values but their focus is on how to select a value-generating path within a network as well as how to share value along the path. Hougaard et al. (2022) consider optimal reallocation of generated values in an evolving chain structure where agents, contrary to our case, exert individual efforts in sequentially extending a value-generating process with random success.

Moreover, our axiomatic approach, that involves conditioning the liabilities on the different characteristics the agents have (distinguishing the initiator from the others), resembles that taken by Gimènez-Gòmez and Osòrio (2015) in the context of bankruptcy problems. In addition, Proposition 1, which shows that the solution to general loss problems can be found as the loss-weighted sum of solutions to elementary problems, provides an alternative interpretation along the lines of recent sequential approaches taken in the claims literature (see e.g. Estévez-Fernández et al., 2021; Sanchez-Soriano, 2021). However, it is important to emphasize that despite a certain structural resemblance at first glance, our problem is qualitatively different from a standard bankruptcy problem, and sequential versions of it, due to the intricate externalities between the agents.

As already indicated, our model has potential applications in many fields. For instance, this includes contagion in financial networks (e.g., Elliott et al., 2014; Acemoglu et al., 2015; Demange, 2018; Csoka and Herings, 2018). Typically, these papers study how network topology influences cascades of failures among interdependent financial organizations based on a general liability matrix. We also study cascades through interdependency but in the simple linear structure of a chain of inflicted losses. Like Csoka and Herings (2018), we focus on the outcome of cascades in terms of payments and agents claims on each other. A somewhat similar type of application concerns rare events and their contagion effect among countries in dynamic global game model as in Chen and Suen (2016). Another application relates to environmental, social and corporate governance (ESG) in supply chains. As consumers and investors become more socially conscious, they hold more favorable views for companies with good ESG (Servaes and Tamayo, 2013; Albuquerque et al., 2019). One notable example is the boycott of Nestlé in 2010. One of their suppliers, Sinar Mas, cut down the rainforest to increase farmland for palm oil. While Nestlé is not directly responsible for the unsustainable conduct, consumers still hold the focal firm accountable (Hartmann and Moeller, 2014). Beyond economic consequences directly from the market, recent court cases suggest that there are also legal consequences for a downstream firm from an upstream firm's behavior, sometimes referred to as "supply chain liability" (Terwindt et al., 2017; Ulfbeck and Ehlers, 2019).

Outline. The paper is structured as follows. In Section 2, we introduce the model. The axiomatic analysis and the characterization of the fixed-fraction rules are in Section 3. In Section 4, we provide two further arguments in favor of the midpoint among the fixed-fraction rules. In Section 5, we show how to implement our suggested rules through smart contracts. We close with a discussion of model extensions in Section 6. Proofs and technical details are postponed to the Appendix.

2. Model

In this section, we introduce our model of sequentially triggered losses with interdependencies following a chain structure. Let $N = \{1, \ldots, n\}$ be the set of agents, representing a loss chain in which agent 1 initiates the chain and agents 2 through $n-1$ are "intermediate" agents. Agent n is the final agent: the loss she causes to agent $n+1$ triggers no further losses. An ***n*-loss problem** is a vector $\ell = (\ell_1, \ldots, \ell_n)$ in which agent i causes loss $\ell_i \geq 0$ to agent $i+1$. Let $\mathcal{L}^n \equiv \mathbb{R}^n_{\geq 0}$ denote the set of loss vectors of length n.

An **allocation** $x = (x_1, \ldots, x_n)$ specifies each agent i's liability $x_i \geq 0$ and is such that $x_1 \geq \ell_1$, $x_1 + x_2 \geq \ell_1 + \ell_2$, $\ldots$, $x_1 + \cdots + x_{n-1} \geq \ell_1 + \cdots + \ell_{n-1}$, and $x_1 + \cdots + x_n = \ell_1 + \cdots + \ell_n \equiv L$. That is, agents "later in the chain" bear no responsibility for losses caused by "earlier" agents and there is a balance between liabilities $\sum x_i$ and losses $\sum \ell_i$. Let $X(\ell)$ denote the set of allocations. A **rule** φ maps to each problem $\ell \in \mathcal{L}^n$ an allocation $\varphi(\ell) = (\varphi_1(\ell), \ldots, \varphi_n(\ell)) \in X(\ell)$.[2] Throughout, we restrict attention to continuous rules.[3]

We will devise rules that build on the notion of *strict liability*—that an injurer is responsible for the losses she causes—from the literature on law and economics (Shavell, 2007). Here, in particular, agent $n+1$ incurs a loss but causes none. Therefore, we take as given that she is free of liability and leave her out of the analysis. For the other agents, however, strict liability leaves open many interpretations: is for instance

[2] As will become apparent, the number of agents may vary. Strictly speaking, we therefore assume that there is an infinite set of potential agents out of which N is a generic finite subset.

[3] In particular, we invoke a minimal version of continuity that applies only to proportional loss changes. A rule φ is assumed to be such that, for each $n \in \mathbb{N}$ and $\ell \in \mathcal{L}^n$, $\lim_{a \to 1} \varphi(a\ell) = \lim_{a \to 1} \varphi(a\ell_1, \ldots, a\ell_n) = \varphi(\ell)$.

the initiator liable only for the direct loss she causes or also for parts of the indirect losses triggered further down the chain? Taken to their respective extremes, these interpretations lead us in two opposite directions. On the one hand, the **direct liability** rule φ^1 assigns full liability to each agent for the loss associated to her, $\varphi^1(\ell) = \ell$. On the other, if an agent can shift blame for the loss on those who appear before her in the loss chain, then responsibility ultimately falls squarely on the initiator. This is the **indirect liability** rule φ^0 for which $\varphi^0(\ell) = (L, 0, \ldots, 0)$.[4]

3. Fixed-fraction rules

In this section, we characterize the class of *fixed-fraction rules*, which can be viewed as compromises between φ^0 and φ^1. Each member of the class is associated with a parameter λ. In particular, every agent i is held accountable for the fraction λ of the loss experienced by agent $i+1$. Agent 1, whose actions are the root cause as they initiated the loss chain, holds a special position and is held liable for the residual $(1-\lambda)L$. To ensure non-negative liabilities, the solutions are defined only for $\lambda \in [0, 1]$. For $\lambda = 1$, we obtain the direct liability rule; for $\lambda = 0$, we obtain the indirect liability rule.[5] Formally,

Definition 1 (The fixed-fraction rule parameterized by λ)**.** For each $\ell \in \mathcal{L}^n$,

$$
\begin{aligned}
\varphi_1^\lambda(\ell) &= \lambda \ell_1 + (1-\lambda)L \\
\varphi_2^\lambda(\ell) &= \lambda \ell_2 \\
&\vdots \\
\varphi_n^\lambda(\ell) &= \lambda \ell_n.
\end{aligned}
$$

To single out the fixed-fraction rules on normative grounds, we first introduce the axioms *additivity* and *zero truncation*. The former is a staple in the axiomatic literature relating the joint solution of separate problems to the solution of a joint problem; the latter is an innocuous axiom asserting that trailing zeros can be truncated from the loss chain without impact on the liability assignment. Proposition 1 shows that *additivity* and *zero truncation* together extend solutions on so called *elementary* problems to all problems.

[4]These extreme cases have been used in related contexts as well. See for instance "local responsibility sharing" (Ni and Wang, 2007), the "full transfer rule" (Hougaard et al., 2017), and "top value" (van den Brink et al., 2017).

[5]These solutions have been examined extensively in the particular case of $n = 2$. This arises for instance when a supplier, agent 1, and a buyer, agent 2, produce and sell a good to a consumer. When the sold good turns out to be defective, the fixed-fraction rules entail that the buyer passes on some of the customer dissatisfaction cost ℓ_2 to the supplier on top of the replacement cost ℓ_1 of the supplier's part (compare, e.g., Balachandran and Radhakrishnan, 2005, page 1270).

That is, we can focus attention on identifying desirable solutions to elementary problems, and then use *additivity* and *zero truncation* to generalize these principles to all problems.

For this reason, we thereafter hone in on elementary problems. The third axiom, *population monotonicity*, is a very compelling solidarity property. It asserts that, when there are more agents to shoulder the same losses, no agent should be worse off. Our final axiom, *merging proofness*, pertains to the strategic incentives for consecutive agents to internalize their losses. However, we recognize that a merger requires coordinated efforts by multiple parties, having to account for potential mergers by other agents as well. For this reason, *merging proofness* disincentivizes only mergers in the most basic setting, namely for two-loss elementary problems. Proposition 2 shows that *population monotonicity* and *merging proofness* together imply that the rule has to coincide with a fixed-fraction rule on elementary problems. Theorem 1 then follows from Propositions 1 and 2, characterizing the class of fixed-fraction rules for general loss problems.

3.1. Additivity and zero truncation

When an agent inflicts a loss on another, the loss can sometimes be decomposed into different types of losses. In the case of a polluted river, where upstream pollution negatively affects downstream water quality, a factory may leak multiple pollutants; one may consider it as several different cases, each pertaining to a specific pollutant, or one may combine them into a single case of multiple pollutants. To avoid unnecessary legal complications, it is desirable that the determined liability is independent of the method of decomposition. This is captured by *additivity*, tracing back to Shapley (1953), which equates the joint solutions of separate problems to the solution of the joint problem (see also e.g. Moulin, 2002; Bergantiños and Moreno-Ternero, 2020).

Axiom 1 (Additivity)**.** For each $n \in \mathbb{N}$ and $\{\ell, \ell'\} \subseteq \mathcal{L}^n$,

$$\varphi(\ell) + \varphi(\ell') = (\varphi_1(\ell) + \varphi_1(\ell'), \ldots, \varphi_n(\ell) + \varphi_n(\ell')) = \varphi(\ell_1 + \ell'_1, \ldots, \ell_n + \ell'_n) = \varphi(\ell + \ell').$$

Next, *zero truncation* asserts that trailing zeroes in the loss chain can be truncated without affecting how the losses are assigned. If instead such trailing zeros were to influence liabilities, it would create an immediate way for agents to manipulate the outcome.[6]

[6]A stronger form of this principle is used for instance in O'Neill (1982, Theorem C.1, condition A.4).

Axiom 2 (Zero truncation)**.** For each $n \in \mathbb{N}$, $\ell \in \mathcal{L}^n$, and $i \in \{1, \ldots, n\}$,

$$\varphi_i(\ell) = \varphi_i(\ell, 0) = \varphi_i(\ell_1, \ldots, \ell_n, 0).$$

Proposition 1 shows that the two axioms imply that the solution to any $\ell \in \mathcal{L}^n$ can be obtained by decomposing ℓ into "elementary" problems. The **elementary** n-loss problem is given by $e_n \equiv (0, \ldots, 0, 1) \in \mathcal{L}^n$. The simplest non-trivial elementary problem is $e_2 = (0, 1) \in \mathcal{L}^2$, in which agent 1's breach of contract has no immediate effect on agent 2, but triggers 2's breach of contract that is harmful to agent 3. These problems approximate, for instance, supply chain disruptions. In the iterated refinement of the good, each step adds a small, incremental value to the final product, which eventually should reach the consumers. A disruption at the start therefore leads to several small losses, incurred throughout the supply chain, followed by a relatively large loss experienced by the consumers.

Proposition 1. *A rule φ satisfies additivity and zero truncation if and only if, for each $n \in \mathbb{N}$, $\ell \in \mathcal{L}^n$, and $i \in \{1, \ldots, n\}$,*

$$\varphi_i(\ell) = \sum_{j \geq i} \ell_j \cdot \varphi_i(e_j).$$

Still, *additivity* and *zero truncation* are silent on how to actually solve the elementary problems. *Additive* rules that satisfy *zero truncation* can share the unit loss in $e_n = (0, \ldots, 0, 1)$ in any way, be it mainly assigned the initiating agent, shared between everyone, or perhaps solely assigned the last agent. The purpose of our two remaining axioms is therefore to pin down desirable solutions to elementary problems.

3.2. Population monotonicity

Next, we impose a notion of population monotonicity to capture that the addition of "new" agents should affect all "old" agents in the same direction (see e.g. Thomson, 2016, 2019). Here, as agents are ordered, this has to be done with care. Take for instance $\ell = (1, 1) \in \mathcal{L}^2$ as the starting point and contrast it with the situation in which there is an additional middleman, $\ell' = (1, \underline{1}, 1) \in \mathcal{L}^3$. One can argue that the initiator and the last agent should be affected differently here: the initiator may pay more by taking on some of the new loss, while there are more predecessors to share the last agent's loss.

However, this is a non-issue when the new loss is zero: then there simply are more agents among whom to share the same losses. Whether the "new" agent should carry some of the burden is not immediately clear, but we hold as desirable that at least no "old" agent should be worse off. Formally, this is expressed as follows.

Take a problem $\ell \in \mathcal{L}^n$ and augment it after agent $1 < j < n$ to define $\ell' = (\ell_1, \ldots, \ell_j, 0, \ell_{j+1}, \ldots, \ell_n) \in \mathcal{L}^{n+1}$. For each agent $i \leq j$ and $k > j$,

$$\varphi_i(\ell) \geq \varphi_i(\ell') \text{ and } \varphi_k(\ell) \geq \varphi_{k+1}(\ell').$$

Again, we focus here on elementary problems only; the condition then reduces to *population monotonicity* as defined next.

Axiom 3 (Population monotonicity)**.** For each $n \in \mathbb{N}$, $1 < j < n$, and $1 \leq i \leq j < k \leq n$,

$$\varphi_i(e_n) \geq \varphi_i(e_{n+1}) \text{ and } \varphi_k(e_n) \geq \varphi_{k+1}(e_{n+1}).$$

By applying the axiom twice, for $j = 2$ and $j = n - 1$, we arrive at the following alternative conditions:

$$\varphi_1(e_n) \geq \varphi_1(e_{n+1})$$
$$\varphi_i(e_n) \geq \varphi_i(e_{n+1}) \text{ and } \varphi_i(e_n) \geq \varphi_{i+1}(e_{n+1}) \text{ for each } 1 < i < n$$
$$\varphi_n(e_n) \geq \varphi_{n+1}(e_{n+1}).$$

An immediate example of a *population-monotonic* rule is the *serial rule.* Indeed, this is also *additive* and satisfies *zero truncation.* For elementary problems, the *serial rule* shares the unit loss equally among the agents, $\varphi_i(e_n) = 1/n$; in this way, the more agents there are, the lower each agent's liability. Other *population-monotonic* rules include the *geometric rules* with parameter $\lambda \in [0, 1]$, which set $\varphi_i(e_n) = \lambda(1 - \lambda)^{n-i}$ (Hougaard et al., 2017).[7] Further examples include, say, "staircase" rules with $|\varphi_{i+1}(e_n) - \varphi_i(e_n)| = \lambda$. In contrast to the fixed-fraction rules, the rules above intuitively depend on the size of the problem (number of agents/losses). We will argue next that this can be a significant drawback if agents are, say, companies that can file claims on various aggregation levels to thereby manipulate the problem size.

3.3. Merging proofness

In general, if a group of agents choose to internalize their losses, it is beneficial if this does not affect agents outside the group. This allows the group to settle their liabilities in private without changing the remaining agents' liabilities; the group can come to an agreement without having to cross-check with the

[7] This if for $i = 2, \ldots, n$; we then set $\varphi_1(e_n)$ such that the liabilities add to 1.

"outsiders" whether they, also, agree to it. At the very least, it is important from a strategic point of view that groups should not be able to collude to reduce their (joint) liability. While it is arguably difficult for agents who appear in very distant "positions" in the loss chain to orchestrate this, the situation is different for consecutive agents, and especially so if only few agents are involved. For this reason, *merging proofness* requires only that, in the setting most favorable to a merger manipulation, it should not be beneficial. Starting from $e_3 = (0, 0, 1)$, agents 1 and 2 (and 2 and 3) can coordinate to $e_2 = (0 + 0, 1) = (0, 0 + 1)$. *Merging-proof* rules are resilient to such manipulations—the joint liability from acting independently should be no greater than that when colluding. The axiom has appeared frequently in the literature in the stronger form of an invariance axiom (with equality rather than inequality; see e.g. Ju et al., 2007; de Frutos, 1999; Chun, 1988).

Axiom 4 (Merging proofness)**.** For each $i \in \{1, 2\}$,

$$\varphi_i(e_3) + \varphi_{i+1}(e_3) \leq \varphi_i(e_2).$$

Analogous to Proposition 1, we next examine the implication of the two axioms on elementary problems. Proposition 2 shows that a *population-monotonic* and *merging-proof* rule must coincide with a fixed-fraction rule for elementary problems. Theorem 1 thereafter follows immediately from Propositions 1 and 2. Independence of the axioms is shown in the Appendix.

Proposition 2. *A rule φ satisfies population monotonicity and merging proofness if and only if there is $\lambda \in [0, 1]$ such that, for each $n \in \mathbb{N}$,*

$$\varphi(e_n) = \varphi^{\lambda}(e_n).$$

Theorem 1. *A rule φ satisfies additivity, zero truncation, merging proofness, and population monotonicity if and only if there is $\lambda \in [0, 1]$ such that $\varphi = \varphi^{\lambda}$.*

Beyond protecting against opportunistic mergers, fixed-fraction rules also provide incentives against creating artificial losses. If agent i adds an artificial zero loss, say to go from ℓ to $(\ell_1, \ldots, \ell_i, 0, \ell_{i+1}, \ldots, \ell_n)$, this has no impact on the liability assignment of a fixed-fraction rule. In the context of Section 5, where potentially anonymous parties interact through a smart contract, it is arguably very easy to artificially inflate the loss chain (known as a "Sybil attack" in the computer science literature, e.g. Douceur, 2002); ensuring that such manipulations are not beneficial is thus a highly desirable feature of the fixed-fraction rules.

4. The midpoint of the fixed-fraction rules

The class of fixed-fraction rules still provides many possible rules to choose from. For that purpose, in this section, we will argue for a particular one among these, namely the midpoint of the class, $\varphi^{1/2}$. The first argument is inspired by the principles of equal sacrifice and ideal points that has been suggested in the literature on bankruptcy, taxation, and bargaining problems (Yu, 1973; Aumann and Maschler, 1985; Young, 1987; Chun, 1988; Ju and Moreno-Ternero, 2017). The conventional approach is to identify an "ideal point", with each coordinate representing one agent's ideal outcome, and then selecting an allocation "close to" the ideal point. Our take is novel: rather than one ideal point, we instead identify n ideal allocations, one for each agent. Thereafter, Theorem 2 shows that the selection of $\varphi^{1/2}$ is at equal (Euclidean) distance to all ideal allocations when losses are equal. Such problems capture, for instance, how a one-week delay at the start of the supply chain may shift each subsequent step by one week. Thereafter, we highlight a connection to well-known solution concepts from cooperative game theory by showing that $\varphi^{1/2}$ selects the allocation that corresponds to the Shapley value of the convex "liability game" associated to the problem (Theorem 3).

4.1. Equidistance to ideal liability assignments

To build up to Theorem 2, we consider first a problem $\ell \in \mathcal{L}^2$ with $\ell_1 = \ell_2$. For every rule φ, we have $\varphi_1(\ell) \geq \varphi_2(\ell)$. We suggest to capture the difference between the agents—that agent 2's harm is a consequence of agent 1's actions—in this inequality. Subject to this inequality, we identify each agent's "ideal" assignment of liability. Here, agent 1's ideal allocation is $\ell = (\ell_1, \ell_2) = (L/2, L/2)$; for agent 2, it is instead $(\ell_1 + \ell_2, 0) = (L, 0)$. We argue then that the difference between the agents already has been accounted for (through the inequality, leading to different ideal allocations), and that we therefore now should treat them equally. We thus recommend to select the midpoint, $(3L/4, L/4)$.

It is immediate that this corresponds to the selection of $\varphi^{1/2}$. However, it is less clear whether this principle extends to larger instances. For that purpose, consider now $\ell \in \mathcal{L}^n$ with $\ell_1 = \cdots = \ell_n$ and capture the difference between the agents by the constraints $\varphi_1(\ell) \geq \cdots \geq \varphi_n(\ell)$; let $\bar{X}(\ell) = \{x \in X(\ell) \mid x_1 \geq \cdots \geq x_n\}$ denote the associated allocations. Intuitively, for each pair of agents i and j, it is desirable if i cannot argue that j receives "better" treatment. A way to make this concrete is to say that, for any allocation $x \in \bar{X}(\ell)$ that i can suggest, we can point to an allocation $y \in \bar{X}(\ell)$ that we can argue is ideal for j (either $y_j = 0$ or, for $j = 1$, $y_1 = \ell_1$) for which $\varphi(\ell)$ is no closer to y than it is to x. In particular, for $k \in \mathbb{N}$, define

$$c_k = \left(\frac{L}{k}, \ldots, \frac{L}{k}, 0, \ldots, 0 \right).$$

As argued before, $c_1 = (L, 0, \dots, 0)$ is agent 2's ideal allocation and $c_n = \ell = (L/n, \dots, L/n)$ is agent 1's. For agent k, the allocation c_{k-1} is ideal for k as it minimizes k's liability (while maximizing $(k-1)$'s). From a technical standpoint, the points c_1 through c_n form the extreme points of $\bar{X}(\ell)$, that is, $\bar{X}(\ell)$ is the convex hull of $\{c_1, \dots, c_n\}$. With this generalized notion of ideal allocations, the question is whether it is still possible to apply the principle of equalizing distances to ideal allocations. Theorem 2 answers this in the affirmative: for $\ell \in \mathcal{L}^n$ with $\ell_1 = \dots = \ell_n$, there is a unique allocation at equal (Euclidean) distance to every ideal allocation c_k, and that is $\varphi^{1/2}(\ell)$. Figure 1 provides a graphical illustration for $n = 3$.

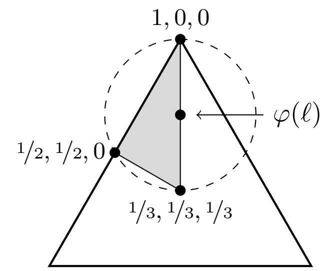

Figure 1: Illustration of Theorem 2 for $n = 3$ and $\ell = (1/3, 1/3, 1/3)$. The gray area in the simplex is the set of ordered allocations $\bar{X}(\ell)$, the convex hull of $c_1 = (1, 0, 0)$, $c_2 = (1/2, 1/2, 0)$, and $c_3 = (1/3, 1/3, 1/3)$. The allocation $\varphi^{1/2}(\ell) = (4/6, 1/6, 1/6)$ is equidistant to c_1, c_2, and c_3.

Theorem 2. *For each $n \in \mathbb{N}$ and $\ell \in \mathcal{L}^n$ with $\ell_1 = \dots = \ell_n$, the allocation $\varphi^{1/2}(\ell)$ is at equal (Euclidean) distance from the ideal allocations $c_1, \dots, c_n$.*

If we apply this reasoning on the basis of the elementary problems $e_n = (0, \dots, 0, 1)$ rather than the constant-loss problems, the corresponding ideal allocations instead take the form $c_k = (0, \dots, 0, 1, 0, \dots, 0)$. In this case, it is instead the *serial rule* that is equidistant to the ideal allocations.

4.2. Core and Shapley value of liability game

We continue our analysis of the fixed-fraction rules by highlighting a connection between well-known solution concepts from cooperative game theory (see e.g., Peleg and Sudhölter, 2007) and $\varphi^{1/2}$. Fix a problem $\ell \in \mathcal{L}^n$. A way to capture the special role played by the initiator is by defining the associated *liability game* v_ℓ as follows. For every coalition $S \subseteq N$, let

$$v_\ell(S) = \begin{cases} \sum_{i \in S} \ell_i & \text{if } 1 \in S \\ 0 & \text{otherwise.} \end{cases}$$

That is, only coalitions that include the initiator are considered as liable, and their liability equals the total (direct) losses caused by its members. Coalitions that do not include the initiator are not considered liable for any losses: none of its members would have caused any losses had it not been for the actions of the initiator. In particular, N is liable for the total loss, $v_\ell(N) = L$. As such, the liability game v_ℓ is an instance of a so-called "big boss" game (Muto et al., 1988). We prove that the liability game is convex, so its core is non-empty and contains the game's Shapley value (Shapley, 1953). In addition, the core is the unique stable set (Muto et al., 1988, Theorem 3.5). In particular, Theorem 3 shows that $\varphi^{1/2}(\ell)$ coincides with the Shapley value and the nucleolus (Schmeidler, 1969) of the associated liability game v_ℓ.[8]

Theorem 3. *For each $\ell \in \mathcal{L}^n$, $\varphi^{1/2}(\ell)$ coincides with the Shapley value and the nucleolus of v_ℓ.*

Armed with a characterization of the class of fixed-fraction rules and a suggestion for a particular member of the class, we next turn to the problem of practically implementing the solutions.

5. Automated conflict resolution through smart contracts

To argue that the fixed-fraction rules are well suited for practical implementation, we first provide a simplified introduction to the relevant technology, namely blockchains and smart contracts.[9] Smart contracts pose an ideal decentralized replacement for the social planners, auctioneers, and centralized clearinghouses prevalent in economic theory. In the present context, tasks one otherwise might assign a trusted third party, such as receiving deposits and transferring funds from injurer to victim, can be automated through the contract.

5.1. Introduction to smart contracts

A *smart contract* is a piece of code that governs a set of variables and provides functions to modify these variables. The code is publicly available and can be inspected by all parties before use to ensure that it works as intended. Interactions with the contract occur through *transactions*, which may specify functions (in the contract) to run as well as inputs to run them on. An elementary feature is that a transaction may transfer value between accounts through an associated cryptocurrency. This can for instance be from the user to the contract (say as a deposit) or the other way around (say by calling a "refund" function within the contract

[8] Chun and Hokari (2007) also show coincidence between the Shapley value and the nucleolus but in the context of queuing problems.

[9] Many excellent sources cover these topics in greater detail; we refer the interested reader to Nakamoto (2008), Ferguson et al. (2010), Katz and Lindell (2014), Damgård et al. (2020), and `http://ethereum.org`.

that returns the deposit from the contract's account).[10] For efficiency purposes, transactions are grouped together and ran sequentially in *blocks*. The blocks are cryptographically chained in the sense that each block contains a pointer to the block it extends on. This permits a consistent, global view of the current state of the contract: anyone can rerun all transactions from the contract's inception to the most recent block and thereby determine the current values of the contract's variables. Once the contract is deployed on the blockchain, it obtains a unique address and its code cannot be altered. In this way, users are safe in knowing that no one can "override" the contract and make it do something beyond its intended functionalities—no one can for instance empty the contract's balance unless there is a function specifically for this purpose.

5.2. Practical implementation of the fixed-fraction rules

A simple implementation of automated conflict resolution can be designed with a few basic functions in a smart contract. Below, we will illustrate the case of a supply chain and the immediate generalization of the fixed-fraction rules from a linear to a tree structure (that is, one firm may supply many others).[11] We take the interpretation that the supply chain emerges dynamically: agent i decides to supply another agent $i+1$, who, at a later point in time, strikes a deal with a third agent $i+2$, and so on. We acknowledge that high-value disruptions in supply chains likely will be settled in court rather than through an automated smart contract; for that reason, our proposal is intended first and foremost for small-scale agreements.

Figure 2 illustrates a 7-agent loss tree. Losses are indicated in the left part, where agent j experiences loss ℓ_i^j from a failed agreement with agent i. In the right part, liabilities are given for the case that agent 2 fails their agreement with agent 5 for $\lambda = 1/2$; out of the total loss $\ell_2^5 + \ell_5^6 + \ell_5^7 = 9 + 6 + 4 = 19$, agent 2 pays $9 + 3 + 2 = 14$ and agent 5 pays the remaining $3 + 2 = 5$. Agents 1, 3, and 4 are unaffected, so their transactions go through as intended. Note that disruptions can occur at any point in the tree. In this way, any agent can become the initiator of their respective "loss tree".

Each bilateral negotiation addresses several factors such as the terms of the deal (quality requirements, deadlines) and the assessed harm caused if the agreement fails. The agents themselves register these details in the smart contract. To enable fully automated conflict resolution, each agent also makes a large-enough deposit to cover her potential liability in case she fails her agreement and is found liable for the triggered losses as per the fixed-fraction rule in place. (Given that we have small-scale problems in mind, also deposits

[10] This provides a simple way to incentivize users: all may be required to make a deposit at the outset, but only those who act as intended get refunded in the end (compare, e.g., the mechanism suggested by Gerber and Wichardt, 2009).

[11] A tree structure also emerges when agents i and $i+1$ make several agreements. In the context of a supply chain, the agents could make one deal that covers a potential one-week delay and another for a two-week delay.

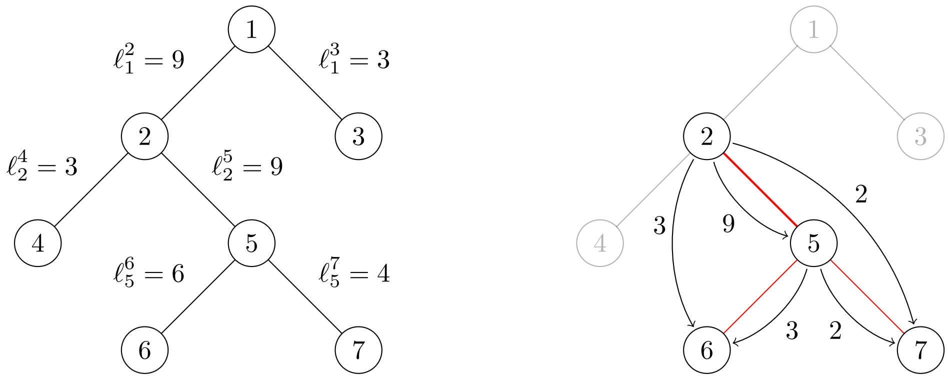

Figure 2: Left: Seven-agent loss tree in which connected agents have bilateral agreements. Agent i causes loss ℓ_i^j to agent j if unsuccessful. Right: Agent 2 fails their agreement with agent 5. Compensations from agents 2 and 5 to agents 5, 6, and 7 as computed by the fixed-fraction rule with $\lambda = 1/2$ are indicated along the respective arcs.

will be small.) The deposits are returned once the agent fulfills their requirements. As a final component of the agreement, the agents may also name the price paid by the buyer to the seller in case the requirements are fulfilled.

Next, we detail three basic functions to be used in the smart contract. (In practice, it is likely desirable to add functionalities to the contract to offer a better user experience, say the option for two agents, if both agree, to cancel a registered agreement between themselves.) The contract will keep track of two variables. The first stores each agent's balance (deposits made). The second stores the information of the evolving loss tree, such as on the agents involved, the terms of the bilateral agreements, and so on. The parameter λ for the fixed-fraction rule is set when the smart contract is initially deployed on the blockchain.

Fallback: The smart contract defaults to this function and it will be used for deposits. When an agent transfers some amount to the contract's address, it gets added to the agent's balance.

Extend (predecessor, successor, conditions, loss, price): This function lets agent i name her predecessor/supplier $i-1$ and add her agreement with her successor $i+1$. This gets added to the variable tracking the emerging loss tree. (To start a new branch, one may set the predecessor to the smart contract's own address.) To permit fully automated conflict resolution, the agent provides verifiable conditions regarding the agreement. For instance, this could specify a location and a deadline; the goods can then be equipped with an IoT sensor that publishes the package's location to the blockchain for easy verification (see e.g. Christidis and Devetsikiotis, 2016). In addition, the agent specifies the loss (to be covered by i) and the price (to be paid by $i+1$), which will be transferred once the Evaluate function is run depending on the outcome of the agreement.

Evaluate: This function iterates through the entries of the tree, returning deposits when conditions are met and executing the fixed-fraction rule where needed. It can be run at any time by any agent.

This completes the description of the contract's basic functions. In practice, the contract can be deployed by anyone. Then, over time as agents form bilateral relations, they deposit funds to the contract and add the terms of their agreements. For small-scale agreements, only small deposits are required, and once an agent has verifiable proof of having completed their task, they can call the EVALUATE function to have the deposits returned. Alternatively, they can reuse their old deposit to register a new agreement, either by extending the existing tree or initiating a new one. Analogously, if an agent has proof that an agreement was not met (causing the agent to fail her agreement with another), the EVALUATE function can again be used, now to execute the fixed-fraction rule to share the losses. In this way, the contract evolves dynamically as old agreements automatically get resolved and new agreements get added.

6. Concluding remarks

Our analysis of sequentially triggered losses has been centered on allocational fairness and characterizations of normatively desirable allocation rules. The solutions that we derive can be applied also in more general settings. We conclude with some remarks on possible extensions and avenues for future research.

It is immediate that fixed-fraction rules can be applied in any setting that distinguishes the initiator from the other agents. For instance, as illustrated in Figure 2, we can extend the model to a tree structure in which one agent interacts with many (compare e.g., Oishi et al., 2022). The axioms generalize readily and we obtain an analogue characterization of the fixed-fraction rules.[12] On the other hand, more challenging may be to permit multiple initiators; we then require a tie-breaking rule among the initiators to specify how they share the residuals. See Hougaard et al. (2017) for a similar extension when sharing revenues in hierarchical organizations.

An interesting related setting is where the individual losses ℓ_i are unknown/unobservable or contested among the agents. That is to say, one may relax the assumption that there is implicit agreement on the individual losses and assume instead that there is agreement only on the total loss L. For instance, all may agree that agents 1 and n are the most and least liable ones—but they disagree on *how* liable everyone is. Such an ordering may arise due to a legal responsibility structure or a hierarchical power structure. Thus, the problem then would be to allocate the loss for the group as a whole while respecting the liability order.

[12] More precisely, we conjecture that an additional axiom will be needed. Specifically, for "star networks", each "branch" should be solved symmetrically (i.e., with the same parameter λ corresponding to the same fixed-fraction rule).

Without access to individual losses, the rules developed in Section 3 cannot be applied. Gudmundsson et al. (2020) examine an alternative approach for this setting.

7. Acknowledgements

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Appendix A. Proofs

Proposition 1. *A rule φ satisfies additivity and zero truncation if and only if, for each $n \in \mathbb{N}$, $\ell \in \mathcal{L}^n$, and $i \in \{1, \ldots, n\}$,*

$$\varphi_i(\ell) = \sum_{j \geq i} \ell_j \cdot \varphi_i(e_j).$$

Proof. ($\Longleftarrow$) As each such rule φ is linear, it is immediate that it satisfies *additivity*. Moreover, for each $i \in \{1, \ldots, n\}$,

$$\varphi_i(\ell, 0) = \sum_{j=i}^{n+1} \ell_j \cdot \varphi_i(e_j) = \sum_{j=i}^{n} \ell_j \cdot \varphi_i(e_j) + 0 \cdot \varphi_i(e_{n+1}) = \varphi_i(\ell),$$

so φ satisfies *zero truncation*.

($\Longrightarrow$) Let $n \in \mathbb{N}$, $\ell \in \mathcal{L}^n$, and $i \in \{1, \ldots, n\}$. By repeatedly appealing to *additivity*,

$$\varphi(\ell) = \varphi(\ell_1, 0, \ldots, 0) + \varphi(0, \ell_2, 0, \ldots, 0) + \cdots + \varphi(0, \ldots, 0, \ell_n),$$

where each $(0, \ldots, 0, \ell_j, 0, \ldots, 0) \in \mathcal{L}^n$. By *zero truncation*, agents "after" the loss are not liable; that is,

$$\varphi_i(\ell) = \sum_{h<i} \underbrace{\varphi_i(0, \ldots, 0, \ell_h, 0, \ldots, 0)}_{0} + \sum_{j \geq i} \varphi_i(0, \ldots, 0, \ell_j, 0, \ldots, 0).$$

Take now an arbitrary $j \geq i$. By *zero truncation*,

$$\varphi_i(0, \ldots, 0, \ell_j, 0, \ldots, 0) = \varphi_i(0, \ldots, 0, \ell_j).$$

Next, by considering three cases, we will conclude that $\varphi_i(0,\ldots,0,\ell_j) = \ell_j \cdot \varphi_i(0,\ldots,0,1) = \ell_j \cdot \varphi_i(e_j)$.

1. (Integer) If $\ell_j \in \mathbb{N}$, then, by repeatedly applying *additivity*,

$$\varphi(0,\ldots,0,\ell_j) = \varphi(0,\ldots,0,1) + \cdots + \varphi(0,\ldots,0,1) = \ell_j \cdot \varphi(e_j).$$

2. (Rational) If $\ell_j = (p/q) \in \mathbb{Q} \setminus \mathbb{N}$ for $p, q \in \mathbb{N}$, then, by repeatedly applying *additivity*,

$$q \cdot \varphi(0,\ldots,0,\ell_j) = \varphi(0,\ldots,0,p) = p \cdot \varphi(e_j),$$

so

$$\varphi(0,\ldots,0,\ell_j) = \frac{p}{q} \cdot \varphi(e_j) = \ell_j \cdot \varphi(e_j).$$

3. (Real) If $\ell_j \in \mathbb{R} \setminus \mathbb{Q}$, let $(a_1, a_2, \ldots) \in \mathbb{Q}^\infty$ be a rational sequence that converges to ℓ_j. At each step $k = 1, 2, \ldots$, by case 2 above, $\varphi(0,\ldots,0,a_k) = a_k \cdot \varphi(e_j)$. As φ is continuous,

$$\varphi(0,\ldots,0,\ell_j) = \lim_{k\to\infty} \varphi(0,\ldots,0,a_k) = \lim_{k\to\infty} a_k \cdot \varphi(e_j) = \ell_j \cdot \varphi(e_j).$$

The statement of Proposition 1 now follows. □

Proposition 2. *A rule φ satisfies population monotonicity and merging proofness if and only if there is $\lambda \in [0,1]$ such that, for each $n \in \mathbb{N}$,*

$$\varphi(e_n) = \varphi^\lambda(e_n).$$

Proof. ($\Longleftarrow$) It is immediate that each fixed-fraction rule satisfies *population monotonicity* and *merging proofness*.

($\Longrightarrow$) Define $\lambda \in [0,1]$ through $\varphi(e_2) = \varphi^\lambda(e_2) = (1-\lambda, \lambda)$. By construction,

$$\varphi_1(e_3) + \varphi_2(e_3) + \varphi_3(e_3) = \varphi_1(e_2) + \varphi_2(e_2).$$

By *merging proofness*,

$$\varphi_1(e_2)+\varphi_2(e_2)\geq(\varphi_1(e_3)+\varphi_2(e_3))+(\varphi_2(e_3)+\varphi_3(e_3))=\varphi_1(e_3)+2\varphi_2(e_3)+\varphi_3(e_3),$$

so $\varphi_2(e_3)=\varphi_2^\lambda(e_3)=0$. Consider now $n\geq 3$ and $1<i<n$. By construction, $\varphi_i(e_n)\geq 0$. By repeatedly applying *population monotonicity*, first extending e_2 with zeros "after" agent 2 and thereafter "before" agent 2,

$$0=\varphi_2(e_3)\geq\varphi_2(e_4)\geq\cdots\geq\varphi_2(e_{n-i+2})\geq\varphi_3(e_{n-i+3})\geq\cdots\geq\varphi_i(e_n).$$

Hence, $\varphi_i(e_n)=\varphi_i^\lambda(e_n)=0$. By balance, $\varphi_1(e_n)+\varphi_n(e_n)=1=\varphi_1(e_2)+\varphi_2(e_2)$. By repeatedly applying *population monotonicity*, $\varphi_1(e_2)\geq\varphi_1(e_3)\geq\cdots\geq\varphi_1(e_n)$ and $\varphi_2(e_2)\geq\varphi_3(e_3)\geq\cdots\geq\varphi_n(e_n)$. But then $\varphi_1(e_n)=\varphi_1(e_2)=\varphi_1^\lambda(e_n)$ and $\varphi_n(e_n)=\varphi_2(e_2)=\varphi_n^\lambda(e_n)$, so $\varphi=\varphi^\lambda$ for elementary problems. □

Theorem 2. *For each $n\in\mathbb{N}$ and $\ell\in\mathcal{L}^n$ with $\ell_1=\cdots=\ell_n$, the allocation $\varphi^{1/2}(\ell)$ is at equal (Euclidean) distance from the ideal allocations $c_1,\ldots,c_n$.*

Proof. The (squared) distance between $x\in\bar{X}(\ell)$ and an arbitrary ideal allocation $c_k=(L/k,\ldots,L/k,0,\ldots,0)$ is as follows:

$$\sum_{i=1}^{k}(x_i-L/k)^2+\sum_{i=k+1}^{n}(x_i-0)^2=\sum_{i=1}^{k}x_i^2-\sum_{i=1}^{k}\frac{2Lx_i}{k}+\sum_{i=1}^{k}\frac{L^2}{k^2}+\sum_{i=k+1}^{n}x_i^2=\sum_{i=1}^{n}x_i^2-\frac{2L}{k}\sum_{i=1}^{k}x_i+\frac{L^2}{k}.$$

We claim that there is an allocation x equidistant to all ideal allocation c_k, that is, an allocation x for which the above expression is constant in k. This is already satisfied for the first terms (the sum of squares), so we can ignore these henceforth. Hence, we wish to find $\alpha\in\mathbb{R}$ such that, for $k=1,\ldots,n$,

$$\frac{2L}{k}\sum_{i=1}^{k}x_i-\frac{L^2}{k}=2\alpha L.$$

The right-hand side is chosen in a way that will be convenient later; the key is that it is constant in k. Equivalently,

$$\sum_{i=1}^{k}x_i-\frac{L}{2}=\alpha k.$$

Rearranging,

$$x_k = \alpha k + \frac{L}{2} - \sum_{i=1}^{k-1} x_i.$$

For $k = 1$, this reduces to $x_1 = \alpha + L/2$. For $k = 2, \ldots, n$, we have $x_k - x_{k-1} = \alpha - x_{k-1}$, so $x_k = \alpha$. That is, $x_2 = \cdots = x_n = \alpha$. We then determine α through $x_1 + \cdots + x_n = 1$:

$$n\alpha + \frac{L}{2} = L \iff \alpha = \frac{L}{2n} = \varphi_2^{1/2}(\ell) = \cdots = \varphi_n^{1/2}(\ell).$$

Finally, $x_1 = L/(2n) + L/2 = \varphi_1^{1/2}(\ell)$. □

Theorem 3. *For each $\ell \in \mathcal{L}^n$, $\varphi^{1/2}(\ell)$ coincides with the Shapley value and the nucleolus of the associated liability game v_ℓ.*

Proof. The result can be derived from Muto et al. (1988) as follows.

For any coalition S such that $1 \notin S$, we have $v_\ell(S) = 0$ and $v_\ell(S \cup \{1\}) = \sum_{i \in S} \ell_i + \ell_1$. As each $\ell_i \geq 0$, for each $S \subseteq T$, we have $v_\ell(S \cup \{1\}) - v_\ell(S) \leq v_\ell(T \cup \{1\}) - v_\ell(T)$. Moreover, for each agent $i \neq 1$, we have $v_\ell(S \cup \{i\}) - v_\ell(S) = v_\ell(T \cup \{i\}) - v_\ell(T)$ if $1 \in S$ or $1 \notin T$, while $v_\ell(S \cup \{i\}) - v_\ell(S) \leq v_\ell(T \cup \{i\}) - v_\ell(T)$ otherwise. Hence, v_ℓ is convex.

The agents' contributions to the grand coalition are $M_1(v_\ell) = v_\ell(N) - v_\ell(N \setminus \{1\}) = L$ and, for each $i \neq 1$, $M_i(v_\ell) = v_\ell(N) - v_\ell(N \setminus \{i\}) = \ell_i$. By Muto et al. (1988, Theorem 4.2), the nucleolus, $\nu(v_\ell)$, is such that, for each $i \neq 1$, $\nu_i(v_\ell) = M_i(v_\ell)/2 = \ell_i/2 = \varphi_i^{1/2}(\ell)$. By balance, we also have $\nu_1(v_\ell) = \varphi_1^{1/2}(\ell)$. Finally, by Muto et al. (1988, Theorem 4.5), as v_ℓ is convex, the Shapley value and nucleolus of v_ℓ coincide. □

Appendix B. Independence of axioms

We show that the axioms imposed in Theorems 1 are independent. For each axiom, we identify a rule that is not a fixed-fraction rule yet satisfies all other axioms.

Proposition 1 describes all *additive* rules satisfying *zero truncation*. A rule that in addition is *population monotonic* is the *serial rule* φ^A. For each $n \in \mathbb{N}$, $\ell \in \mathcal{L}^n$, and $i \in \{1, \ldots, n\}$,

$$\varphi_i^A(\ell) = \sum_{j \geq i} \frac{\ell_j}{j}.$$

The rule is not *merging proof* as $\varphi_1^A(e_3) + \varphi_2^A(e_3) = 2/3 > 1/2 = \varphi_1^A(e_2)$.

With a slight adjustment, the *serial rule* can be made *merging proof*. Define the rule φ^B to coincide with the *serial rule* on all elementary problems except for $\varphi^B(e_3) = (1/2, 0, 1/2) \neq (1/3, 1/3, 1/3) = \varphi^A(e_3)$. Using the construction in Proposition 1, we can extend this to define φ^B on every problem. The rule is not *population monotonic* as $\varphi_2^B(e_3) = 0 < 1/4 = \varphi_2^B(e_4)$.

Proposition 2 shows that a *population monotonic* and *merging proof* rule must coincide with a fixed-fraction rule for elementary problems. A rule that in addition is *additive* is φ^C constructed next. For each $n \in \mathbb{N}$ and $\ell \in \mathcal{L}^n$,

$$\varphi^C(\ell) = (\ell_1 + \ell_n, \ell_2, \ldots, \ell_{n-1}, 0).$$

The rule does not satisfy *zero truncation* as $\varphi_1^C(0,1) = 1 \neq 0 = \varphi_1^C(0,1,0)$.

Finally, a rule that satisfies all axioms except *additivity* is φ^D. It is defined similar to a fixed-fraction rule but the parameter λ depends on the losses. For each $n \in \mathbb{N}$ and $\ell \in \mathcal{L}^n$,

$$\varphi^D(\ell) = (\lambda\ell_1 + (1-\lambda)L, \lambda\ell_2, \ldots, \lambda\ell_n).$$

where $\lambda = (L - \ell_1)/L$. The rule is not *additive* as $\varphi_1^D(0,1) + \varphi_1^D(1,0) = 0 + 1 \neq 3/2 = \varphi_1^D(0+1, 1+0)$.

References

Acemoglu, D., Ozdaglar, A., Tahbaz-Salehi, A., 2015. Systemic risk and stability in financial networks. American Economic Review 105, 564–608.

Albuquerque, R., Koskinen, Y., Zhang, C., 2019. Corporate Social Responsibility and Firm Risk: Theory and Empirical Evidence. Management Science 65, 4451–4469.

Ambec, S., Sprumont, Y., 2002. Sharing a river. Journal of Economic Theory 107, 453–462.

Aumann, R., Maschler, M., 1985. Game theoretic analysis of a bankruptcy problem from the Talmud. Journal of Economic Theory 36, 195–213.

Balachandran, K.R., Radhakrishnan, S., 2005. Quality Implications of Warranties in a Supply Chain. Management Science 51, 1266–1277.

Bergantiños, G., Moreno-Ternero, J.D., 2020. Sharing the Revenues from Broadcasting Sport Events. Management Science 66, 2417–2431.

Brânzei, R., Fragnelli, V., Tijs, S.H., 2002. Tree-connected peer group situations and peer group games. Mathematical Methods of Operations Research 55, 93–106.

van den Brink, R., Dietz, C., van der Laan, G., Xu, G., 2017. Comparable characterizations of four solutions for permission tree games. Economic Theory 63, 903–923.

Brown, J.P., 1973. Toward an economic theory of liability. The Journal of Legal Studies 2, 323–349.

Chao, G.H., Iravani, S.M.R., Savaskan, R.C., 2009. Quality Improvement Incentives and Product Recall Cost Sharing Contracts. Management Science 55, 1122–1138.

Chen, H., Suen, W., 2016. Falling Dominoes: A Theory of Rare Events and Crisis Contagion. American Economic Journal: Microeconomics 8, 228–255.

Christidis, K., Devetsikiotis, M., 2016. Blockchains and Smart Contracts for the Internet of Things. IEEE Access 4, 2292–2303.

Chun, Y., 1988. The equal-loss principle for bargaining problems. Economics Letters 26, 103–106.

Chun, Y., Hokari, T., 2007. On the Coincidence of the Shapley Value and the Nucleolus in Queueing Problems. Seoul Journal of Economics 20, 223–237.

Coase, R.H., 1960. The problem of social cost. Journal of Law and Economics 3, 1–44.

Csoka, P., Herings, J.J., 2018. Decentralized clearing in financial networks. Management Science 643, 4681–4699.

Damgård, I., Nielsen, J.B., Orlandi, C., 2020. Distributed Systems and Security. `https://cs.au.dk/~orlandi/dsikdist`. Accessed 2020-11-04.

Dehez, P., Ferey, S., 2013. How to share joint liability: A cooperative game approach. Mathematical Social Sciences 66, 44–50.

Demange, G., 2018. Contagion in financial networks: A threat index. Management Science 64, 955–970.

Diamond, P.A., Mirrlees, J.A., 1975. On the assignment of liability: The uniform case. The Bell Journal of Economics , 487–516.

Douceur, J.R., 2002. The sybil attack, in: International workshop on peer-to-peer systems, pp. 251–260.

Elliott, M., Golub, B., Jackson, M., 2014. Financial networks and contagion. American Economic Review 104, 3115–3153.

Emons, W., Sobel, J., 1991. On the Effectiveness of Liability Rules when Agents are not Identical. Review of Economic Studies 58, 375–390.

Estévez-Fernández, A., Giménez-Gómez, J.M., Solís-Baltodano, M.J., 2021. Sequential bankruptcy problems. European Journal of Operational Research 292, 388–395.

Fan, J., Ni, D., Fang, X., 2020. Liability cost sharing, product quality choice, and coordination in two-echelon supply chains. European Journal of Operational Research 284, 514–537.

Ferguson, N., Schneier, B., Kohno, T., 2010. Cryptography Engineering: Design Principles and Practical Applications. John Wiley & Sons.

de Frutos, M.A., 1999. Coalitional manipulations in a bankruptcy problem. Review of Economic Design 4, 255–272.

Gerber, A., Wichardt, P.C., 2009. Providing public goods in the absence of strong institutions. Journal of Public Economics 93, 429–439.

Gilles, R.P., Owen, G., van den Brink, R., 1992. Games with permission structures: The conjunctive approach. International Journal of Game Theory 20, 277–293.

Gimènez-Gòmez, J.M., Osòrio, A., 2015. Why and how to differentiate in claims problems? An axiomatic approach. European Journal of Operational Research 241, 842–850.

Green, J., 1976. On the optimal structure of liability laws. The Bell Journal of Economics , 553–574.

Gudmundsson, J., Hougaard, J.L., Ko, C.Y., 2020. Sharing sequentially triggered losses. Working paper 2020/05. University of Copenhagen.

Hartmann, J., Moeller, S., 2014. Chain liability in multitier supply chains? Responsibility attributions for unsustainable supplier behavior. Journal of Operations Management 32, 281–294.

Hougaard, J.L., 2018. Allocation in Networks. MIT Press, Cambridge, MA.

Hougaard, J.L., Moreno-Ternero, J.D., Østerdal, L.P., 2022. Optimal Management of Evolving Hierarchies. Management Science 68, 6024–6038.

Hougaard, J.L., Moreno-Ternero, J.D., Tvede, M., Østerdal, L.P., 2017. Sharing the proceeds from a hierarchical venture. Games and Economic Behavior 102, 98–110.

Ju, B.G., Miyagawa, E., Sakai, T., 2007. Non-manipulable division rules in claim problems and generalizations. Journal of Economic Theory 132, 1–26.

Ju, B.G., Moreno-Ternero, J.D., 2017. Fair Allocation of Disputed Properties. International Economic Review 58, 1279–1301.

Juarez, R., Ko, C.Y., Xue, J., 2018. Sharing sequential values in a network. Journal of Economic Theory 177, 734–779.

Katz, J., Lindell, Y., 2014. Introduction to Modern Cryptography. 2 ed., Chapman & Hall.

Landes, W.M., Posner, R.A., 1987. The economic structure of tort law. Harvard University Press.

Lim, W.S., 2001. Producer-Supplier Contracts with Incomplete Information. Management Science 47, 709–715.

Littlechild, S.C., Owen, G., 1973. A simple expression for the shapley value in a special case. Management Science 20, 370–372.

Marchand, J.R., Russell, K.P., 1973. Externalities, liability, separability, and resource allocation. The American Economic Review 63, 611–620.

Moulin, H., 1988. Axioms of Cooperative Decision Making. Cambridge University Press.

Moulin, H., 2002. Axiomatic cost and surplus sharing, in: Arrow, K.J., Sen, A.K., Suzumura, K. (Eds.), Handbook of Social Choice and Welfare. volume 1. chapter 6, pp. 289–357.

Moulin, H., 2004. Fair division and collective welfare. MIT Press.

Muto, S., Nakayama, M., Potters, J., Tijs, S., 1988. On big boss games. The Economics Studies Quarterly 39, 303–321.

Nakamoto, S., 2008. Bitcoin: A Peer-to-Peer Electronic Cash System. `https://bitcoin.org/bitcoin.pdf`. Accessed 2020-11-04.

Ni, D., Wang, Y., 2007. Sharing a Polluted River. Games and Economic Behavior 60, 176–186.

Oishi, T., van der Laan, G., van den Brink, R., 2022. Axiomatic analysis of liability problems with rooted-tree networks in tort law. Economic Theory .

O'Neill, B., 1982. A problem of rights arbitration from the Talmud. Mathematical Social Sciences 2, 345–371.

Peleg, B., Sudhölter, P., 2007. Introduction to the Theory of Cooperative Games. Kluwer Academic Publishers.

Reyniers, D.J., Tapiero, C.S., 1995a. Contract design and the control of quality in a conflictual environment. European Journal of Operational Research 82, 373–382.

Reyniers, D.J., Tapiero, C.S., 1995b. The Delivery and Control of Quality in Supplier-Producer Contracts. Management Science 41, 1581–1589.

Sanchez-Soriano, J., 2021. Families of sequential priority rules and random arrival rules with withdrawal limits. Mathematical Social Sciences 113, 136–148.

Schmeidler, D., 1969. The nucleolus of a characteristic function game. SIAM Journal on Applied Mathematics 17, 1163–1170.

Servaes, H., Tamayo, A., 2013. The Impact of Corporate Social Responsibility on Firm Value: The Role of Customer Awareness. Management Science 59, 1045–1061.

Shapley, L.S., 1953. A Value for n-Person Games, in: Kuhn, H., Tucker, A. (Eds.), Contributions to the Theory of Games II. Princeton University Press, Princeton, pp. 307–317.

Shavell, S., 1980. Strict Liability versus Negligence. Journal of Legal Studies 9, 1–25.

Shavell, S., 2007. Economic Analysis of Accident Law. Harvard University Press.

Terwindt, C., Leader, S., Yilmaz-Vastardis, A., Wright, J., 2017. Supply Chain Liability: Pushing the Boundaries of the Common Law? Journal of European Tort Law 8, 261–296.

Thomson, W., 2016. Fair allocation, in: Adler, M.D., Fleurbaey, M. (Eds.), The Oxford Handbook of Well-Being and Public Policy. Oxford University Press. chapter 7, pp. 193–226.

Thomson, W., 2019. How to divide when there isn't enough: from Aristotle, the Talmud, and Maimonides to the axiomatics of resource allocation. Cambridge University Press.

Ulfbeck, V., Ehlers, A., 2019. Direct and vicarious liability in supply chains, in: Law and Responsible Supply Chain Management, pp. 91–110.

Young, P.H., 1987. Progressive taxation and the equal sacrifice principle. Journal of Public Economics 32, 203–214.

Yu, P.L., 1973. A Class of Solutions for Group Decision Problems. Management Science 19, 936–946.