Rank-preserving Multidimensional Mechanisms*

Sushil Bikhchandani† and Debasis Mishra‡

September 15, 2023

Abstract

We show that the mechanism-design problem for a monopolist selling multiple, heterogeneous objects to a buyer with ex ante symmetric and additive values is equivalent to the mechanism-design problem for a monopolist selling identical objects to a buyer with decreasing marginal values. Symmetric and incentive-compatible mechanisms for heterogeneous objects are *rank preserving*, i.e., higher-valued objects are assigned with a higher probability. In the identical-objects model, every mechanism is rank preserving. This facilitates the equivalence, which we use in three applications.

JEL Classification number: D82

Keywords: rank-preserving mechanism, revenue maximization, multidimensional mechanism design

*We are grateful to Gabriel Carroll, Rahul Deb, Takuro Yamashita and to seminar participants at Academia Sinica, Ashoka University, Bar-Ilan University, UCLA, 2021 Conference of the Society for Advancement of Economic Theory (SAET), 2021 BRICS NU Conference, and 2022 Conference on Mechanism Design, Singapore for thoughtful comments. Debasis Mishra acknowledges financial support from the Science and Engineering Research Board (SERB Grant No. SERB/CRG/2021/003099) of India.

†Anderson School at UCLA, Los Angeles (sbikhcha@anderson.ucla.edu).

‡Indian Statistical Institute, Delhi (dmishra@isid.ac.in).

1 Introduction

We prove an equivalence result between two models for selling multiple, indivisible objects to a buyer – an identical-objects model and a heterogeneous-objects model. The seller chooses an incentive compatible (IC) and individually rational (IR) mechanism with the goal of maximizing expected revenue. The buyer's type is multidimensional, a (marginal) value for each object. With identical objects, the buyer's value for k objects is the sum of the marginal values for these objects. With heterogeneous objects, the buyer's value for a bundle of objects is also additive over the values of objects in the bundle. The seller knows the distribution of buyer values.

We show that any identical-objects model with decreasing marginal values is equivalent to a heterogeneous-objects model in the following sense. There is a one-to-one mapping between the set of IC and IR mechanisms in the identical-objects model and the set of symmetric,[1] IC and IR mechanisms in the heterogeneous-objects model. If the distribution of buyer values in the heterogeneous-objects model is exchangeable[2] then the expected revenue of an IC and IR mechanism in the identical-objects model is equal to the expected revenue of its equivalent mechanism in the heterogeneous-objects model. In the heterogeneous-objects model, there exists an optimal mechanism that is symmetric. Hence, the optimal revenues in the two models are equal.

To establish this equivalence, we show that any IC and IR mechanism in the identical-objects model can be extended to a symmetric mechanism in the heterogeneous-objects model, while preserving IC and IR. In the other direction, a complication is that the restriction of an IC and IR mechanism in the heterogeneous-objects model to the domain of identical objects with decreasing marginal values need not yield a mechanism that is feasible in the identical-objects model. This is because in order to allocate the $(i+1)^{\text{st}}$ unit in the identical-objects model, the i^{th} unit must also be allocated; thus, feasibility requires that the i^{th} unit is allocated with a (weakly) greater probability than the $(i+1)^{\text{st}}$ unit. There is no such feasibility restriction in the heterogeneous-objects model.

The property of rank preserving plays a key role in showing that a symmetric, IC and IR

[1] A mechanism is *symmetric* if a permutation of the allocation probabilities (of objects) at a buyer type is equal to the allocation probabilities at the same permutation of the buyer type.

[2] A distribution is exchangeable if its density function is symmetric.

mechanism in the heterogeneous-objects model maps into a feasible, IC and IR mechanism in the identical-objects model. A mechanism is *rank preserving* if whenever the (buyer's) value for object i is greater than the value for object j, the probability that object i is allocated to the buyer is at least as large as the probability that object j is allocated.

In the identical-objects model, decreasing marginal values implies that any feasible mechanism is rank preserving. In the heterogeneous-objects case, however, there exist feasible, IC and IR mechanisms that are not rank preserving. We show that if a mechanism for allocating heterogeneous objects is IC, then symmetry implies that the mechanism must be rank preserving. This is critical in establishing equivalence.

In the heterogeneous-objects model, the average of all permutations of an IC and IR asymmetric mechanism is a symmetric IC and IR mechanism. Consequently, linearity of the revenue functional and exchangeability implies that for every asymmetric mechanism in the heterogeneous-objects model there exists a symmetric mechanism which yields the same expected revenue. Thus, in the heterogeneous-objects model, there exists an optimal mechanism that is symmetric and therefore, rank-preserving; its equivalent mechanism in the identical-objects model is optimal in that model.

Much of the multidimensional screening literature has focused on the sale of heterogeneous objects (see, for instance, McAfee and McMillan (1988), Manelli and Vincent (2006), Hart and Reny (2015), Carroll (2017)). Papers that investigate the sale of identical objects include Malakhov and Vohra (2009), Devanur et al. (2020), and Bikhchandani and Mishra (2022). A general solution to the optimal mechanism design problem for the sale of multiple objects is unknown.

Our equivalence result is useful in adapting known results for heterogeneous objects to identical objects. With n identical objects, allocation rules induce probability distributions with $n+1$ outcomes (number of objects) whereas with n heterogeneous objects, allocations rules induce probability distributions with 2^n outcomes (bundles of objects). Therefore, as it has a smaller allocation space, the identical-objects model is a more tractable setting than the heterogeneous-objects model for the discovery of new results (as we demonstrate in our two applications). These results can be adapted to the exchangeable, heterogeneous-objects model via the equivalence.

While the assumption of exchangeability is strong as it entails a presumption of ex ante symmetric buyer values, it is plausible when the seller is somewhat uninformed about buyer preferences. Moreover, exchangeability is a weaker assumption than an i.i.d. distribution of buyer values, which is often assumed in the literature.

We provide three applications to demonstrate the usefulness of the equivalence. These applications establish novel results in the identical-objects model and use our equivalence result to find new results in the heterogeneous-objects model. The first application weakens the notion of incentive compatibility as follows: instead of requiring IC to hold for all pairs of types, the weakened notion requires IC only for pairs (v, v') if $v \geq v'$ or $v' \geq v$. We call this weaker notion of IC upper-set IC (UIC). We show that under a mild condition,[3] every deterministic UIC mechanism is IC in the identical-objects model. Our equivalence result implies that every symmetric, deterministic, UIC mechanism is IC in the heterogeneous-objects model. This leads to a simplification of the design problem for deterministic mechanisms, which we plan to investigate in future research.

In a second application, we show that in the heterogeneous-objects model with i.i.d. distribution of values, any optimal mechanism in the class of symmetric and deterministic mechanisms cannot be supermodular. This is established by first showing that in the identical-objects model, prices of units cannot be increasing if the marginal distribution of units satisfy the hazard-rate order.

In a third application, we obtain new results on revenue monotonicity. Hart and Reny (2015) established that the optimal revenue need not be monotone in the distribution of values. We obtain a new sufficient condition, *majorization monotonicity*, for a mechanism to be revenue monotone in the identical-objects model and therefore also in the heterogeneous-objects model with an exchangeable distribution. We show that if an optimal mechanism is symmetric and almost deterministic (i.e., in each outcome in the range of the mechanism, there is randomization over at most one object), then it satisfies majorization monotonicity; consequently, the optimal revenue is monotone.

The rest of the paper is organized as follows. We present the two models in Section 2 and establish the connection between rank preserving and symmetry in Section 3. The equivalence of the two models is obtained in Section 3.2. The applications are in Section 4.

[3] This condition, which we call *object non-bossiness*, requires that if the allocation of object i is the same for types (v_i, v_{-i}) and (v_i', v_{-i}), then the entire allocation vector to these two types is the same. We show that object non-bossiness is generically satisfied by any IC mechanism.

We end with a discussion in Section 5. All the proofs are in an appendix.

2 Two Models of Selling Multiple Objects

In both models, the set of objects is denoted by $N = \{1, \ldots, n\}$ and the type of an agent (the buyer) is a vector of valuations $v := (v_1, \ldots, v_n)$, where each $v_i \in [\underline{v}, \overline{v}]$, $0 \leq \underline{v} < \overline{v} < \infty$. The seller is the mechanism designer.[4]

In the **heterogeneous-objects model**, the type space is

$$\overline{D}^{\text{H}} := [\underline{v}, \overline{v}]^n$$

For any type v, v_i denotes the agent's value for object i and the value for a bundle of objects $S \subseteq N$ is additive: $\sum_{i \in S} v_i$. As the n objects may be distinct, there is no restriction on values across objects, i.e., both $v_i > v_j$ or $v_j < v_i$ are possible. The values $v_1, \ldots, v_n$ are jointly distributed with cumulative distribution function (cdf) F^{H} and density function f^{H} with support in $\overline{D}^{\text{H}}$. An alternate interpretation is that there is a unit of mass of buyers distributed in $\overline{D}^{\text{H}}$ with density f^{H}.

In the **identical-objects model**, all objects are identical and v_i denotes the (marginal) value of consuming the i^{th} unit of the object. We assume that marginal values are decreasing. The type space is

$$\overline{D}^{\text{I}} := \left\{ v \in [\underline{v}, \overline{v}]^n \,\middle|\, v_1 \geq v_2 \geq \ldots \geq v_n \right\}$$

The values $v_1, \ldots, v_n$ are jointly distributed with cdf F^{I} and density function f^{I} with support in $\overline{D}^{\text{I}}$.

We refer to the heterogeneous-objects model as $\mathcal{M}^{\text{H}} := (N, \overline{D}^{\text{H}}, f^{\text{H}})$. Similarly, the identical-objects model is denoted as $\mathcal{M}^{\text{I}} := (N, \overline{D}^{\text{I}}, f^{\text{I}})$.

In either model, a **mechanism** is an allocation probability vector $q : \overline{D}^{\text{M}} \to [0,1]^n$ and a payment $t : \overline{D}^{\text{M}} \to \Re$, $M = H$ or I.[5] An agent with (reported) type v is allocated object i with probability $q_i(v)$, $i = 1, 2, \ldots, n$ and makes a payment of $t(v)$. Thus, the expected

[4] We assume that the seller's cost for selling each object is not more than $\underline{v}$.

[5] To simplify notation, we do not attach superscript H or I to q and t, except when the model is not clear from the context.

utility of an agent of type v from mechanism (q,t) is

$$u(v) := v \cdot q(v) - t(v)$$

In the identical-objects model, $q_i(v)$ denotes the probability of getting the i^{th} unit of the object, which happens whenever at least i units are allocated. In other words, the $(i+1)^{\text{st}}$ unit can be allocated only if the i^{th} is also allocated. Thus, a feasibility restriction of

$$q_i(v) \geq q_{i+1}(v) \qquad \forall v \in \overline{D}^{\text{I}},\ \forall i \in \{1,\ldots,n-1\} \tag{1}$$

is imposed.[6] There are no such restrictions on the allocation probabilities of a mechanism in the heterogeneous-objects model. In either model, a mechanism (q,t) is **deterministic** if $q_i(v) \in \{0,1\}$ for all v and all i.

A mechanism (q,t) is **incentive compatible (IC)** if for every $v, v' \in \overline{D}^{\text{M}}$, we have

$$u(v) \geq v \cdot q(v') - t(v') = u(v') + (v - v') \cdot q(v')$$

A mechanism (q,t) is **individually rational (IR)** if for every $v \in \overline{D}^{\text{M}}$, $u(v) \geq 0$. If (q,t) is IC, it is IR if and only if $u(\underline{v},\ldots,\underline{v}) \geq 0$.

We assume that every mechanism (q,t) satisfies $u(\underline{v}) = 0$. This is without loss of generality as the seller is interested in maximizing expected revenue. Then IC implies that $0 = u(\underline{v}) \geq \underline{v} \cdot q(v) - t(v)$ or $t(v) \geq \underline{v} \cdot q(v)$. Note that IR implies $t(v) \leq v \cdot q(v)$. Since the domain of types is bounded, this implies that for any IC and IR mechanism (q,t), $u(v)$ and $t(v)$ are bounded above and below for every v.

3 Symmetric and Rank-preserving Mechanisms

We formally define symmetric and rank-preserving IC mechanisms in model $\mathcal{M}^{\text{H}}$, and show that these properties are closely related.

A type vector v is **strict** if $v_i \neq v_j$ for all $i,j \in N$. Let D^{H} and D^{I} denote the **set of all strict types** in $\overline{D}^{\text{H}}$ and $\overline{D}^{\text{I}}$, respectively.[7]

[6] If we denote the probability of getting exactly k units by $Q_k(v)$, then $q_i(v) = \sum_{k=i}^{n} Q_k(v)$. This immediately shows $q_i(v) = Q_i(v) + q_{i+1}(v) \geq q_{i+1}(v)$.

[7] Thus, D^{I} is the largest subset of $\overline{D}^{\text{I}}$ with strictly decreasing types.

Lemma 1 *Let (q,t) be an IC and IR mechanism defined on D^M, $M = H$ or I. There exists an IC and IR mechanism $(\bar{q},\bar{t})$ defined on $\overline{D}^M$ such that*

$$(\bar{q}(v),\bar{t}(v)) = (q(v),t(v)) \qquad \forall v \in D^M$$

Throughout, we assume that the probability distribution of types has a density function. Thus, the set of non-strict types has zero probability. Consequently, the expected revenue from $(\bar{q},\bar{t})$ is the same as the expected revenue from (q,t). Hence, Lemma 1 allows us to define mechanisms on the set of strict types, i.e., on D^{M}, and then extend them to $\overline{D}^{\mathrm{M}}$. This results in a simplification of the proofs.

Let σ represent a permutation of the set N. The identity permutation is $\sigma^{\mathrm{I}} := (1,\ldots,n)$. This is a slight abuse of notation as I also refers to the identical-objects model. The set of all permutations of N is denoted by Σ. We partition the set of strict types in the heterogeneous-objects model, D^{H}, using permutations in Σ. For any permutation $\sigma \in \Sigma$, let

$$D(\sigma) = \{v \in D^{\mathrm{H}} : v_{\sigma(1)} > v_{\sigma(2)} > \ldots > v_{\sigma(n)}\} \tag{2}$$

Note that $D^{\mathrm{H}} \equiv \cup_{\sigma\in\Sigma} D(\sigma)$ and $D(\sigma) \cap D(\sigma') = \emptyset$ if $\sigma \neq \sigma'$. Also, $D^{\mathrm{I}} = D(\sigma^{\mathrm{I}})$.

Every type in D^{H} can be mapped to a type in $D(\sigma^{\mathrm{I}})$. To see this, take any $v \in D^{\mathrm{H}}$. There exists a unique σ such that $v \in D(\sigma) \subset D^{\mathrm{H}}$. Let v^σ denote the permuted type of v, i.e., $v^\sigma_j = v_{\sigma(j)}$ for all $j \in N$. Eq. (2) implies that $v^\sigma \in D(\sigma^{\mathrm{I}})$. More generally, for an arbitrary type $v \in D^{\mathrm{H}}$ and a permutation σ, $v^\sigma \in D(\sigma^{\mathrm{I}})$ if and only if $v \in D(\sigma)$.

We start with a mechanism defined on $\overline{D}^{\mathrm{H}}$ and assume that it satisfies the properties of symmetry and rank preserving, defined below, on the subset D^{H}. As $\overline{D}^{\mathrm{H}} \setminus D^{\mathrm{H}}$ has zero measure, these properties are satisfied for almost all $v \in \overline{D}^{\mathrm{H}}$.

Definition 1 *In model $\mathcal{M}^H$, a mechanism (q,t) is* **symmetric** *if for every $v \in D^H$ and for every $\sigma \in \Sigma$,*

$$\begin{aligned} q_i(v^\sigma) &= q_{\sigma(i)}(v) \qquad \forall\, i \in N \\ t(v^\sigma) &= t(v) \end{aligned}$$

In a symmetric mechanism, the allocation probabilities at a permutation of type v are the permutation of allocation probabilities at v, while the payment function is invariant to permutations of v.[8] Later, we show in Theorem 2 that in an exchangeable environment, it is without loss of generality to consider symmetric mechanisms.

To construct a symmetric mechanism, it is enough to define the mechanism on $D(\sigma^{\text{I}})$, say, and then extend it to D^{H} symmetrically (as made precise later in Definition 3 and Lemma 2). The following property plays a crucial role in maintaining incentive compatibility in such symmetric extensions.

DEFINITION **2** *In model $\mathcal{M}^{H}$, a mechanism (q,t) is* **rank preserving** *if for every $v \in D^{H}$ and every i, j, we have $q_i(v) \geq q_j(v)$ if $v_i > v_j$.*

In the identical-objects model, any feasible mechanism is rank preserving. To see this, note that for any $v \in D^{\text{I}}$, we have $v_i > v_{i+1}$. Moreover, by (1) we have $q_i(v) \geq q_{i+1}(v)$.

In the heterogeneous-objects model, an IC mechanism need not be rank preserving. For example, a mechanism that allocates some fixed object for zero payment to all types is IC and IR. Even an optimal mechanism need not be rank preserving as is clear from Proposition 3 in Hart and Reny (2015). These mechanisms are not symmetric.

As shown next, if in the heterogeneous-objects model a symmetric mechanism is IC then it must be rank preserving. Conversely, if a symmetric mechanism is rank preserving and IC on $D(\sigma^{\text{I}})$, then it is IC on D^{H}.

THEOREM **1** *Suppose that (q,t) is a symmetric mechanism in $\mathcal{M}^{H}$. Then, the following are equivalent:*

(i) (q,t) is IC on D^{H}.

(ii) (q,t) is rank preserving and (q,t) restricted to $D(\sigma^{I})$ is IC.

As noted earlier, asymmetric mechanisms need not be rank preserving. Thus, the assumption of symmetric mechanisms is essential for Theorem 1. The following example shows that rank preserving is also essential.

[8] Symmetry imposed on non-strict types implies additional restrictions. To see this, consider $n = 4$ and suppose $v \equiv (1, 0, 0, 1)$ and $v' \equiv (0, 1, 1, 0)$. The type v' can be obtained from v via two permutations: $\sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 4, \sigma(4) = 3$ and $\sigma'(1) = 3, \sigma'(2) = 4, \sigma'(3) = 1, \sigma'(4) = 2$. Hence, $v' = v^{\sigma} = v^{\sigma'}$. Applying symmetry here would mean $q_i(v^{\sigma}) = q_{\sigma(i)}(v) = q_i(v^{\sigma'}) = q_{\sigma'(i)}(v)$ for all i which implies $q_2(v) = q_3(v)$ and $q_1(v) = q_4(v)$. Such restrictions limit the set of mechanisms that can be considered. Consequently, we do not impose symmetry on non-strict types.

Example 1 Consider two objects with the buyer's valuation $v = (v_1, v_2)$ distributed on the unit square. Let $t(\cdot) \equiv 0$ and

$$q(v_1, v_2) = \begin{cases} (0,1), & \text{if } v_1 > v_2 \\ (1,0), & \text{if } v_1 < v_2 \end{cases}$$

This mechanism is symmetric but not rank preserving. Restricted to $D(\sigma^1) := \{(v_1, v_2) : v_1 > v_2\}$, this mechanism is IC. It is also IC when restricted to $D(\sigma^2) := \{(v_1, v_2) : v_1 < v_2\}$. But the mechanism is not IC as any type in $D(\sigma^1)$ benefits by reporting a type in $D(\sigma^2)$ and vice versa. ■

Is every rank preserving and IC mechanism symmetric? The answer is no as the following example illustrates.

Example 2 Suppose $n = 2$ and the type space is $[0, 1]^2$. Figure 1 describes a deterministic mechanism (q, t) for this type space. The mechanism (q, t) is clearly IC and rank preserving. But it is not symmetric. ■

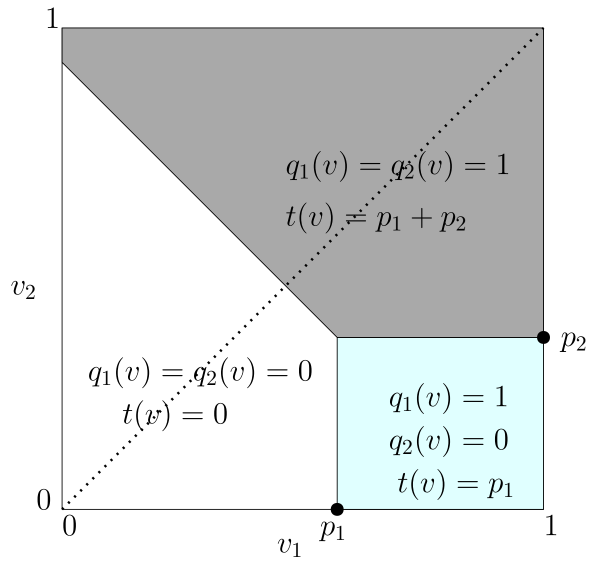

Figure 1: A rank-preserving IC mechanism that is not symmetric

We show next that under certain conditions on priors, there exists an optimal mechanism which is rank preserving.

3.1 Existence of a Rank-preserving Optimal Mechanism

The expected revenue from an IC and IR mechanism in $\mathcal{M}^{\text{M}}$, $M = H$ or I, is

$$\textsc{Rev}(q, t; f^{\text{M}}) := \int\limits_{\overline{D}^{\text{M}}} t(v) f^{\text{M}}(v) dv = \int\limits_{D^{\text{M}}} t(v) f^{\text{M}}(v) dv$$

Further,

$$\textsc{Rev}(q, t; f^{\text{H}}) = \sum_{\sigma \in \Sigma} \textsc{Rev}^{\sigma}(q, t; f^{\text{H}})$$

$$\text{where} \qquad \textsc{Rev}^{\sigma}(q, t; f^{\text{H}}) := \int\limits_{D(\sigma)} t(v) f^{\text{H}}(v) dv \tag{3}$$

We sometimes write $\textsc{Rev}(q, t; F^{\text{M}})$ instead of $\textsc{Rev}(q, t; f^{\text{M}})$.

A mechanism (q^*, t^*) is **optimal** for density function f^{H} if it is IC and IR and for any other IC and IR mechanism (q, t)

$$\textsc{Rev}(q^*, t^*; f^{\text{H}}) \geq \textsc{Rev}(q, t; f^{\text{H}})$$

In model $\mathcal{M}^{\text{H}}$, the joint density of values, f^{H}, is **exchangeable**[9] if

$$f^{\text{H}}(v) = f^{\text{H}}(v^{\sigma}) \qquad \forall\, v \in D^{\text{H}},\ \forall \sigma \in \Sigma$$

Exchangeability is satisfied if $v_1, v_2, \ldots, v_n$ are i.i.d. Exchangeable random variables may be positively correlated such as when $v_1, v_2, \ldots, v_n$ are distributed i.i.d. conditional on an underlying state variable.[10]

As shown next, an exchangeable distribution of buyer types in model $\mathcal{M}^{\text{H}}$ allows one to restrict attention to mechanisms that are symmetric and, therefore also rank preserving.

THEOREM **2** *Suppose that f^{H} in model $\mathcal{M}^{H}$ is exchangeable. Then, there exists an optimal mechanism which is symmetric and rank preserving.*

[9] Strictly speaking, the random variables $v_1, v_2, \ldots, v_n$ are *exchangeable*.

[10] As an example, consider an entity that sells "permits" for operating in n markets that are ex-ante identical. The seller might be a local government issuing licenses for liquor stores or a franchisor introducing its product in a new market via franchises. The buyer is knowledgeable about market conditions in the n markets. The value of market i to the buyer is $v_i = \eta\, m_i$, where η is the buyer's efficiency level and m_i is the size of market i. The buyer knows η and m_i. The seller has a distribution over η and has i.i.d. distributions over m_i. The random variables v_i are exchangeable from the seller's perspective.

In an exchangeable environment with two-dimensional types, Pavlov (2011a) notes (in the proof of his Corollary 1) that an optimal mechanism that is symmetric must exist.[11] For the sake of completeness, we provide a proof for n-dimensional types. The proof of Theorem 2 establishes that for any asymmetric mechanism in $\mathcal{M}^{\text{H}}$, there exists a symmetric mechanism with the same expected revenue.

3.2 An Equivalence Between the Two Models

Theorem 1 allows us to establish a formal equivalence between the identical objects and the heterogeneous-objects models. A mechanism defined on $D(\sigma^{\text{I}})$ may be extended symmetrically to D^{H} using the definition below. Recall that for every $v \in D(\sigma)$, we have $v^{\sigma} \in D(\sigma^{\text{I}})$.

Definition **3** *Let (q, t) be a mechanism defined on $D(\sigma^{I})$ (equivalently on D^{I}). The* **symmetric extension** *of (q, t) is a mechanism (q^s, t^s) on D^{H} such that for every $v \in D(\sigma)$ and for every $\sigma \in \Sigma$*

$$q^s_{\sigma(i)}(v) = q_i(v^{\sigma}) \qquad \forall i$$
$$t^s(v) = t(v^{\sigma})$$

A mechanism defined on $D(\sigma)$, where $\sigma \neq \sigma^{\text{I}}$, may also be extended symmetrically using Definition 3 after first relabeling the axes. A mechanism (q, t) on D^{I} for model $\mathcal{M}^{\text{I}}$ is rank preserving by definition. But an arbitrary mechanism (q, t) defined on $D(\sigma^{\text{I}}) \equiv D^{\text{I}}$ need not be rank preserving.[12]

Lemma **2** *Let (q, t) defined on $D(\sigma)$ be a rank-preserving, IC and IR mechanism [on $D(\sigma)$]. Then the symmetric extension of (q, t) to D^{H} is a rank-preserving, IC and IR mechanism.*

This leads to an equivalence between identical objects and heterogeneous-objects models:

[11] Maskin and Riley (1984) make a similar observation in a single object auction setting with ex ante symmetric bidders and one-dimensional types.

[12] Theorem 1 implies that the symmetric extension of a non-rank-preserving mechanism on D^{I} will not be IC on D^{H}. Example 1 illustrates this.

Proposition 1

(i) Any IC and IR mechanism in model $\mathcal{M}^I$ can be extended to a symmetric, IC, and IR mechanism in model $\mathcal{M}^H$.

(ii) The restriction of any symmetric, IC, and IR mechanism in model $\mathcal{M}^H$ to $D(\sigma^I)$ defines an IC and IR mechanism in model $\mathcal{M}^I$.

(iii) If the density f^H is exchangeable with $f^I(v) = n!f^H(v)\ \forall\ v \in \overline{D}^I$ then optimal mechanisms in models $\mathcal{M}^I$ and $\mathcal{M}^H$ generate the same expected revenue.

In general, the optimization problem for a seller of n heterogeneous objects is quite different from the optimization problem for a seller of n identical units of an object. Proposition 1 implies that if the density f^H in $\mathcal{M}^H$ is exchangeable then the seller's problem in these two settings is essentially the same.

The equivalence between models $\mathcal{M}^H$ and $\mathcal{M}^I$ relies on the decreasing marginal values assumption in the identical-objects model. Indeed, if marginal values are increasing, then $v_i \leq v_{i+1}$ for all v whereas, feasibility of a mechanism (q,t) requires that $q_i(v) \geq q_{i+1}(v)$. Thus, a feasible mechanism violates the rank-preserving property under increasing marginal values in the identical-objects model. Hence, Theorem 1 does not hold.

4 Applications

We provide three applications of the equivalence result for the sale of indivisible objects. These applications establish new results in the identical objects model, and then extend them to the heterogeneous-objects model using Theorem 1.

The applications require a new condition we call object non-bossiness, which is defined next.[13]

Definition 4 *A mechanism (q,t) satisfies* **object non-bossiness** *if for all i, for all v_{-i}, and for all v_i, v_i'*

$$\Big[q_i(v_i, v_{-i}) = q_i(v_i', v_{-i})\Big] \implies \Big[q_j(v_i, v_{-i}) = q_j(v_i', v_{-i})\ \forall\ j \in N\Big]$$

[13]Non-bossiness is assumed in the hypothesis of Theorem 3 and is used in the proof of Theorem 5.

In an object non-bossy mechanism, if the allocation probability of the i^{th} unit remains the same at types (v_i, v_{-i}) and (v_i', v_{-i}) then the allocation of every unit must remain the same at (v_i, v_{-i}) and (v_i', v_{-i}).[14]

Proposition 2 below shows that for any IC mechanism there exists a non-bossy and IC mechanism which is identical to the original mechanism almost everywhere. Thus, an assumption of object non-bossiness is without loss of generality in our environment. In the applications, we assume that two specific classes of mechanisms are non-bossy: deterministic and almost deterministic. Formally, an allocation rule q is **almost deterministic** if for every v there exists $k \in \{1, \dots, n\}$ such that $q_i(v) \in \{0, 1\}$ for all $i \neq k$. A mechanism (q, t) is (almost) deterministic if q is (almost) deterministic. Proposition 2 shows that if an IC mechanism is (almost) deterministic then its non-bossy version is also (almost) deterministic. We prove the proposition for an arbitrary type space D^* with decreasing marginal values, which can be finite or infinite. In particular, D^* can be D^{I} or $\bar{D}^{\text{I}}$.

Proposition **2** *Suppose (q, t) is an IC and IR mechanism in the identical objects model in a type space D^*. Then, there exists an IC, IR, and non-bossy mechanism $(q^\sharp, t^\sharp)$ such that:*

(i) $t^\sharp(v) \geq t(v)$ *and* $u^\sharp(v) = u(v)$ *for all* $v \in D^*$.

(ii) If D^* is convex, then $(q^\sharp(v), t^\sharp(v)) = (q(v), t(v))$ for almost all $v \in D^*$.

(iii) If (q, t) is deterministic (almost deterministic), then $(q^\sharp, t^\sharp)$ can be chosen to be deterministic (almost deterministic).

Hence, for any type space D^*, there is an optimal deterministic mechanism which is non-bossy. Note that no assumptions about the distribution of values are made in Proposition 2. While the proposition can be proved for (asymmetric mechanisms in) the heterogeneous object models, the restriction to identical objects is sufficient for our results.

The proof relies on making a non-bossy selection $(q^\sharp, t^\sharp)$ from the set of seller-favorable mechanisms (as defined in Hart and Reny (2015)) that are equal to (q, t) almost everywhere.[15] We illustrate this for a deterministic, and possibly bossy, mechanism next.

[14] The idea is similar to agent non-bossiness introduced by Satterthwaite and Sonnenschein (1981).

[15] While $(q^\sharp, t^\sharp)$ in Proposition 2 is seller favorable, in general non-bossy mechanisms need not be seller favorable.

Let (q,t) be a deterministic, IC mechanism in the identical-objects model with type space $D \in \{\overline{D}^{\mathrm{I}}, D^{\mathrm{I}}\}$. The buyer's utility function from this mechanism, u, is convex and $\partial u(v)$, the set of subgradients of u at any type v, is generically a singleton and contains $q(v)$. For each v, let $\partial^{\dagger} u(v)$ be the subset of $\partial u(v)$ that contains integer vectors only. Then, at any type v, $\partial^{\dagger} u(v)$ is finite (generically a singleton) and contains $q(v)$. Note that any $x \in \partial^{\dagger} u(v)$ must be of the form $x = (\overbrace{1,\ldots,1}^{k \text{ ones}}, \overbrace{0,\ldots,0}^{n-k \text{ zeroes}})$, where $0 \le k \le n$. Therefore, at each v there exists a unique largest element in $\partial^{\dagger} u(v)$ because either $\partial^{\dagger} u(v)$ is a singleton or if $x, x' \in \partial^{\dagger} u(v)$, $x \neq x'$, then either $x \le x'$ or $x' \le x$. Let this largest element be $q^{\sharp}(v)$. Define $t^{\sharp}(v) := v \cdot q^{\sharp}(v) - u(v)$ for all v.

Since $(q^{\sharp}, t^{\sharp})$ assigns an allocation from the subgradient correspondence $\partial^{\dagger} u(v)$ and $u^{\sharp} \equiv u$, it is a deterministic IC mechanism. Further, $(q^{\sharp}, t^{\sharp})$ coincides with (q,t) almost everywhere, except at v where $\partial^{\dagger} u(v)$ is not a singleton – at these points $t^{\sharp}(v) \ge t(v)$ as $q^{\sharp}(v) \ge q(v)$. To see that $(q^{\sharp}, t^{\sharp})$ is object non-bossy, pick $(v_i, v_{-i}), (v'_i, v_{-i})$ with $v_i > v'_i$ such that $q^{\sharp}_i(v_i, v_{-i}) = q^{\sharp}_i(v'_i, v_{-i})$. It may be verified that $q^{\sharp}(v'_i, v_{-i}) \in \partial^{\dagger} u(v)$ and $q^{\sharp}(v) \in \partial^{\dagger} u(v)$. Since $q^{\sharp}(v)$ is the unique largest element of $\partial^{\dagger} u(v)$, and $q^{\sharp}(v'_i, v_{-i})$ is the unique largest element of $\partial^{\dagger} u(v'_i, v_{-i})$, it must be that $q^{\dagger}(v'_i, v_{-i}) = q^{\dagger}(v)$, establishing object non-bossiness.

4.1 Upper-set Incentive Compatibility

We establish that only upper-set incentive constraints (i.e., incentive constraints between v and v' whenever $v \ge v'$ or $v' \ge v$) are binding in deterministic mechanisms in the identical-objects model. We then use Theorem 1 to show that this relaxation of the set of IC constraints applies to symmetric, rank-preserving and deterministic mechanisms in the heterogeneous-objects model (whether or not the distribution of types is exchangeable).

Theorem 3 below, which shows that upper-set IC constraints are sufficient for IC for deterministic mechanisms in model $\mathcal{M}^{\mathrm{I}}$, applies to a broad class of type spaces which include $\overline{D}^{\mathrm{I}}$ and D^{I}. Let D be an arbitrary type space in the identical-objects model. Denote the IC constraint of a mechanism where type v does not gain by misreporting type v' as $v \to v'$. For every $v \in D$, the **upper set** of v is defined as

$$T(v) = \{v' \in D : v'_i \ge v_i \;\forall\; i \in N\}$$

Definition 5 *A mechanism (q,t) for the identical-objects model defined on D is* **upper-set**

incentive compatible (UIC) *if for every* $v \in D$ *and every* $v' \in T(v)$*, the IC constraints* $v \to v'$ *and* $v' \to v$ *hold.*

Consider the following assumption on the domain of types D:

Definition **6** *A domain of buyer types* D *satisfies the* **strong-lattice property** *if for every* $v, v' \in D$*, for every* $k \in \{1, \dots, n\}$ *the types* $\tilde{v}$ *and* $\check{v}$*, defined below, belong to the domain:*

$$\tilde{v}_i = \begin{cases} v_i & \text{if } i < k \\ \min(v_i, v'_i) & \text{if } i \geq k \end{cases} \tag{4}$$

$$\check{v}_i = \begin{cases} \max(v_i, v'_i) & \text{if } i \leq k \\ v_i & \text{if } i > k \end{cases} \tag{5}$$

The strong-lattice property is equivalent to the lattice property under an assumption on the richness of domain of types defined next.

Definition **7** *A domain of buyer types* D *is* **rich** *if for every* $v, v' \in D$ *such that* $v_k \geq v'_{k+1}$*, the type* $v'' := (v_1, \dots, v_k, v'_{k+1}, \dots, v'_n) \in D$*.*

Lemma **3** *A rich domain of types satisfies the strong-lattice property if and only if it satisfies the lattice property w.r.t. with* min *and* max*.*

As the domains $\overline{D}^{\mathrm{I}}$ and D^{I} are rich and satisfy the lattice property, Lemma 3 implies that these domains satisfy the strong-lattice property.

Theorem **3** *Consider an identical-objects model in which the domain of buyer types satisfies the strong-lattice property. In this setting, every deterministic, object non-bossy, and upper-set incentive compatible mechanism is incentive compatible.*

Remark **1** We contrast the sufficiency of UIC for IC from previous work that has shown the sufficiency of a reduced set of incentive constraints in multidimensional environments. A mechanism satisfies *local IC constraints* if for every v, there exists an $\epsilon > 0$ such that for all v' in an ϵ-ball around v, the IC constraints $v \to v'$ and $v' \to v$ hold. Carroll (2012) shows that local IC constraints imply all IC constraints in convex type spaces; this holds for random mechanisms also. This result has been extended to non-convex type spaces by Mishra et al. (2016); Kumar and Roy (2021). UIC does not correspond to any notion of locality, and hence, the set of redundant IC constraints identified by UIC is quite different.[16]

REMARK **2** In mechanism-design problems, whenever a subset of IC constraints are dropped, the set of mechanisms satisfying the smaller set of IC constraints is usually larger. This is called a *relaxed program*. The aim is then to figure out conditions on primitives of the problem when the optimal of the relaxed program is also the optimal of the original problem (with all the IC constraints). Armstrong (2000) uses this approach in an auction with binary types and Bayesian IC constraints.

In contrast, we show that UIC constraints imply all IC constraints for any deterministic and object non-bossy mechanism. That is, the set of deterministic and object non-bossy mechanisms satisfying UIC constraints is not larger in the relaxed program. Thus, for any optimization program in this setting involving IC constraints (be it revenue maximization or some other objective function), it is without loss of generality to consider a strictly smaller set of IC constraints, the set of UIC constraints.

EXAMPLE **3** We present examples to show that none of the three sufficient conditions in Theorem 3 can be dropped. In these examples, there are two identical objects with decreasing marginal values.

The type space is $\overline{D}^{\text{I}}$ for $n = 2$. The mechanism is specified as follows, with buyer types identified by the number of units they receive:

$$(q(v),\, t(v)) = \begin{cases} ((0,0),\, 0), & \text{if } v_1 \leq 0.5 \text{ and } v_1 + v_2 < 0.75, \quad \textbf{Type 0} \\ ((1,0),\, 0.5), & \text{if } v_1 > 0.5 \text{ and } v_2 < 0.25, \quad \textbf{Type 1} \\ ((1,1),\, 0.75), & \text{if } v_1 + v_2 > 0.75 \text{ and } v_2 \geq 0.25, \quad \textbf{Type 2} \\ ((\tilde{q},0),\, \tilde{q}v_1), & \text{if } v_1 + v_2 = 0.75 \text{ and } v_1 \leq 0.5,\ v_2 \geq 0.25, \quad \textbf{Type } \tilde{\mathbf{q}} \end{cases}$$

[16] In fact, we can use Carroll (2012) to strengthen our result. The upper set of any type v is a convex set if D is convex. Hence, satisfying all IC constraints in the upper set of v is equivalent to satisfying all local IC constraints in the upper set of v provided D is convex. Similarly, if $v \geq v' \geq v''$, and IC constraints $v \leftrightarrow v', v' \leftrightarrow v''$ hold, then a straightforward argument shows that the IC constraint $v \leftrightarrow v''$ holds. Thus, it is possible to define a (weaker) local notion of upper set IC, which implies IC.

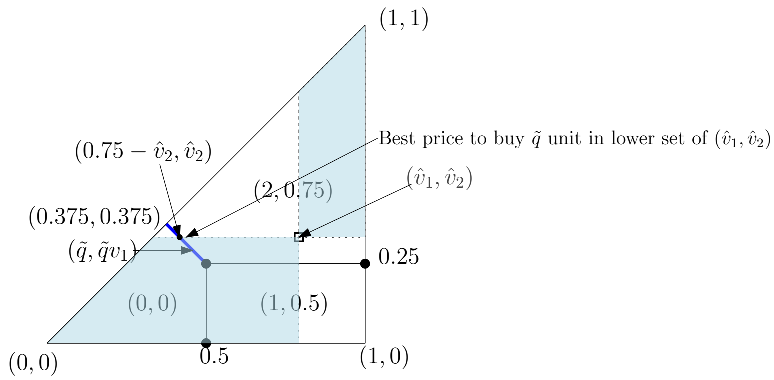

Figure 2: UIC, deterministic, bossy but not IC

Type $\tilde{q}$ buyers are on the thick blue line-segment in Figure 2. Although each type $\tilde{q}$ buyer receives the same allocation, $\tilde{q} \in (0, 1]$, the transfer $\tilde{q}v_1$ is a function of the buyer's value for the first unit. Thus, each type $\tilde{q}$ buyer has a payoff of zero.

We verify that this mechanism satisfies UIC. First, note that a type 0 cannot get a positive payoff by reporting their type as $\tilde{q}$, 1, or 2. By UIC, type 1 buyers can misreport their type as 0 and 2 buyers only; it is easily verified that a misreport to one of these two types is not profitable.

Next, the upper (or lower) set of any type $\tilde{q}$ buyer does not include any another type $\tilde{q}$ buyer or a type 1 buyer. Under truthful reporting, a type $\tilde{q}$ gets payoff zero which is the same payoff it would obtain with a misreport to a type 2 buyer, as $v_1 + v_2 = 0.75$ for any type $\tilde{q}$ buyer. Its payoff remains zero by misreporting as type 0. Hence, type $\tilde{q}$ buyers also satisfy UIC.

Finally, consider a type 2 buyer with value $(\hat{v}_1, \hat{v}_2)$. If it misreports a type $\tilde{q}$ within its UIC constraint set, it gets $\tilde{q} \in (0, 1]$ units. The lowest possible price it pays with such a misreport is $\tilde{q}\max(0.375, 0.75 - \hat{v}_2)$ (see Figure 2). This deviation is not profitable as

$$\begin{aligned}\tilde{q}\hat{v}_1 - \tilde{q}\max(0.375, 0.75 - \hat{v}_2) &\leq \tilde{q}[\hat{v}_1 + \hat{v}_2 - 0.75] \\ &\leq \hat{v}_1 + \hat{v}_2 - 0.75\end{aligned}$$

The other UIC constraints (unprofitability of misrepresentations to a type 0 or 1) hold for a type 2 buyer.

However, any type $\tilde{q}$ buyer with $v_1 > 0.375$ profits from misreporting their type as (0.375.0.375). Thus IC, is not satisfied. In addition, IC is not satisfied for a positive measure of type 2 buyers. Take any type 2 buyer whose values satisfy $0.75 < v_1 + v_2 \leq 0.75 + \epsilon_1$, $v_1 \geq 0.5 - \epsilon_2$. Under truthful reporting, this buyer has a payoff less than ϵ_1 in the mechanism. If this type misreports its type as $(0.375, 0.375)$ its obtains a payoff of at least $\tilde{q}(0.125 - \epsilon_2)$. This deviation is profitable when $\epsilon_1 < \tilde{q}(0.125 - \epsilon_2)$.

In this example, for any $\tilde{q} \in [0, 1)$, the mechanism is UIC but not IC. If $\tilde{q} \in (0, 1)$, this mechanism is non-bossy. As the domain is $\overline{D}^{\text{I}}$, the strong-lattice property is satisfied. Hence, if $\tilde{q} \in (0, 1)$, this example shows that a stochastic, non-bossy, and UIC mechanism in a strong lattice type space need not be IC.

If, instead, $\tilde{q} = 1$ then the mechanism is deterministic but bossy. To see this, note that the allocation to a type $\tilde{q}$ buyer with valuation (v_1, v_2) is $q(v_1, v_2) = (1, 0)$ and the allocation to a type 2 buyer with value $(v_1 + \epsilon, v_2)$ is $q(v_1, v_2) = (1, 1)$. Thus, in going from (v_1, v_2) to $(v_1 + \epsilon, v_2)$, only v_1 changes and only q_1 changes which is bossiness.

Finally, consider another example in which the domain is the set of $\tilde{q}$ types in the previous example (the blue types in Figure 2). The domain does not satisfy the lattice property and the upper or lower set of each type is empty. Every mechanism in this type space is UIC. Clearly, not every mechanism is IC.

Thus, none of the assumptions of Theorem 3 can be dropped. □

We now apply Theorem 1 to derive an analog of Theorem 3 for model $\mathcal{M}^{\text{H}}$. For model $\mathcal{M}^{\text{H}}$, we consider strict types only, i.e., consider the type space D^{H}. UIC is defined slightly differently in that it is sufficient to define it for $v \in D(\sigma^{\text{I}})$ and $v' \in D(\sigma^{\text{I}})$ that are in the upper set of v.

Definition 8 *A symmetric and rank-preserving mechanism (q, t) for the heterogeneous-objects model defined on type space D^H is* **upper-set incentive compatible (UIC)** *if for every $v \in D(\sigma^I)$ and for every $v' \in T(v) \cap D(\sigma^I)$, the IC constraints $v \to v'$ and $v' \to v$ hold.*

For a mechanism in the heterogeneous-objects model, we use object non-bossiness for types in $D(\sigma^{\text{I}})$ only, i.e., a mechanism (q, t) for the heterogeneous-objects model is object non-bossy if $q_i(v_i, v_{-i}) = q_i(v'_i, v_{-i})$ implies $q(v_i, v_{-i}) = q(v'_i, v_{-i})$ for every $(v_i, v_{-i}), (v'_i, v_{-i}) \in D(\sigma^{\text{I}})$. As we only consider symmetric mechanisms in the heterogeneous-objects model, this definition of object non-bossiness suffices.

Theorem 3 and Theorem 1 imply the following:

Corollary **1** *Every symmetric, deterministic, object non-bossy, rank preserving and upper-set incentive compatible mechanism for a heterogeneous-objects model defined on D^{H} is incentive compatible.*

Since the definition of symmetry only applies to strict types, Corollary 1 does not have a counterpart for $\overline{D}^{\mathrm{H}}$. However, if (q,t) is a symmetric, rank-preserving, deterministic, object non-bossy, and UIC mechanism defined on $\overline{D}^{\mathrm{H}}$, then its restriction to D^{H} is IC by Corollary 1. Using Lemma 1, there exists another IC mechanism (q',t') defined on $\overline{D}^{\mathrm{H}}$ that coincides with (q,t) almost everywhere, and hence, generates the same expected revenue as (q,t).

4.2 Pricing Mechanisms

In this section, we continue our focus on deterministic mechanisms. We first show that a deterministic mechanism can be equivalently expressed as a *pricing* mechanism. We start our analysis for the identical objects model, and then use our equivalence result to extend it to the heterogeneous objects model.

Definition **9** *A deterministic mechanism (q,t) for the identical objects model defined on the type space D^{I} is a* **pricing mechanism** *if there exists prices $p_0 = 0, p_1, \ldots, p_n \in [0, \overline{v}]$, such that for all $v \in D^{I}$,*

$$t(v) = \sum_{k=0}^{k(v)} p_k$$

where $k(v) := \sum_{i=1}^{n} q_i(v)$.

Suppose that (q,t) is an deterministic, IC, mechanism with full range, i.e., for each $k = 0, 1, \ldots, n$ there exists a type v^k that is allocated k units. Then the pricing mechanism implementation of (q,t) has prices $p_k = t(v^k) - t(v^{k-1})$. The next result shows that pricing mechanism implementations exist for deterministic mechanisms without full range.

Proposition **3** *If (q,t) is a deterministic IC and IR mechanism, then it is a pricing mechanism with prices $p_0 = 0, p_1, \ldots, p_n \in [0, \overline{v}]$ such that*

$$u(v) \geq \sum_{i=0}^{k} v_i - \sum_{i=0}^{k} p_k \qquad \forall\ v \in D^{I}\ \forall\ k \in \{0, 1, \ldots, n\} \tag{6}$$

where we use $v_0 = 0$.

The proof of Proposition 3 is in Appendix B. Notice that the IC constraints in (6) are required to hold for $k \notin R(q)$ also. Hence, Proposition 3 implies that even if the range of a mechanism does not contain all the units, any IC mechanism is equivalent to defining prices for *every* unit and letting the buyer choose the payoff maximizing units. Hart and Reny (2015) show a similar result for the heterogeneous objects model with an unbounded type space. Due to unbounded type space, prices of objects in their model can be infinite also.

Our main result in this section provides a sufficient condition on F, the probability distribution of buyer values, under which prices in the identical objects model cannot be increasing in an optimal pricing mechanism. For distribution F, let F_k denote the marginal distribution of the value for the k-th unit.

Theorem 4 *Suppose F is a probability distribution over buyer values in model $\mathcal{M}^I$ such that F_k hazard-rate dominates F_{k+1} for all $k \in \{1, \ldots, n-1\}$. Then, in the identical objects model, there is no optimal mechanism with respect to F with prices $p_0, p_1, \ldots, p_n$ such that*

$$p_1 \leq p_2 \leq \ldots \leq p_n$$

with one inequality holding strict.

Let $(q^{\text{H}}, t^{\text{H}})$ be a deterministic, symmetric, IC and IR mechanism in the heterogeneous objects model $\mathcal{M}^{\text{H}}$. By Proposition 1, its restriction to $\mathcal{D}^{\text{I}}$ defines an IC and IR mechanism $(q^{\text{I}}, t^{\text{I}})$ in model $\mathcal{M}^{\text{I}}$. By Proposition 3, $(q^{\text{I}}, t^{\text{I}})$ is a pricing mechanism with prices $p_0 = 0, p_1, \ldots, p_n$. Let $v \in \mathcal{D}^{\text{H}}$ be a type such that $q^{\text{H}}(v)$ allocates bundle S in $\mathcal{M}^{\text{H}}$. There exists a permutation σ such that $v^\sigma \in \mathcal{D}^{\text{I}}$. By symmetry, $t^{\text{H}}(v) = t^{\text{H}}(v^\sigma)$ and $q^{\text{H}}(v^\sigma)$ allocates the permuted bundles S^σ. But $q^{\text{H}}(v^\sigma) = q^{\text{I}}(v^\sigma)$ and since $(q^{\text{I}}, t^{\text{I}})$ is a pricing mechanism, we have

$$t^{\text{H}}(v^\sigma) = t^{\text{I}}(v^\sigma) = \sum_{j \in S^\sigma} p_j = \sum_{j \in S} p_j = t^{\text{H}}(v)$$

Hence, the *prices* of a symmetric, deterministic, IC and IR mechanism in model $\mathcal{M}^{\text{H}}$ can be described by prices $p_0 = 0, p_1, \ldots, p_n$. For notational convenience, let $P(S) \equiv \sum_{j \in S} p_j$ be the price of bundle S.

Definition 10 *A deterministic mechanism (q^H, t^H) with prices $\{P(S)\}_S$ in model $\mathcal{M}^H$ is* **supermodular** *if*

$$P(S \cup \{k\}) - P(S) \leq P(T \cup \{k\}) - P(T), \qquad \forall S \subsetneq T \subseteq N,\ \forall k \notin T$$

A mechanism is strictly supermodular if at least one of the above inequalities is strict.

Supermodularity is well-defined for deterministic, symmetric mechanisms as $P(S) := \sum_{j \in S} p_j$ is defined for every bundle $S \subseteq N$, even if no type is allocated S. Note that supermodularity is equivalent to $p_0 = 0 \leq p_1 \leq \ldots \leq p_n$ and strict supermodularity is equivalent to requiring one of these inequalities to be strict.

A corollary of Theorem 4 is that if values of objects are distributed independently and identically in the heterogeneous objects model, then the prices in the optimal mechanism cannot be supermodular.

Corollary 2 *Consider a heterogeneous objects model $\mathcal{M}^{\text{H}}$ in which the values of the objects are distributed i.i.d. Then there is no optimal, deterministic, symmetric mechanism which is strictly supermodular.*

Babaioff et al. (2018) showed that even if F is i.i.d. in model $\mathcal{M}^{\text{H}}$, then the optimal deterministic mechanism need not be symmetric when there are three or more objects for sale. Identifying conditions on i.i.d. F under which the optimal, deterministic mechanism is symmetric is an open question.

4.3 Revenue Monotonicity

The optimal revenue from the sale of n objects is monotone if the optimal revenue increases when the distribution of the buyer's values increases in the sense of first-order stochastic dominance. Monotonicity of the optimal revenue is a desirable property as it provides an incentive for the seller to improve her products. It is satisfied in the optimal mechanism for the sale of a single object. However, as Hart and Reny (2015) show, the optimal revenue may not be monotone in the heterogeneous objects model. They also show that if the optimal mechanism is symmetric and deterministic or if the optimal payment function is submodular, then the optimal revenue is monotone in the heterogeneous-objects model.[17] We provide other sufficient conditions on a mechanism that guarantee that the expected revenue from

[17] In a recent paper, Ben Moshe et al. (2022) show that a restriction to monotone mechanisms can severely reduce expected revenue for some classes of distributions.

the mechanism is monotone in the identical-objects model and the heterogeneous-objects model.

Consider the following definition.

Definition 11 *A mechanism* (q,t) *is* **revenue monotone** *if for every cdf* F *and every cdf* $\widetilde{F}$*, where* $\widetilde{F}$ *first-order stochastic dominates* F*, we have*

$$\textsc{Rev}(q,t;\widetilde{F}) \geq \textsc{Rev}(q,t;F)$$

The definition applies to models $\mathcal{M}^{\text{H}}$ and $\mathcal{M}^{\text{I}}$, where either F and $\widetilde{F}$ both have support in $\overline{D}^{\text{H}}$ or both have support in $\overline{D}^{\text{I}}$.

If a mechanism (q,t) satisfies[18]

$$t(\hat{v}) \geq t(v) \qquad \forall\ \hat{v} > v \tag{7}$$

then it satisfies revenue monotonicity, as its expected revenue under a cdf $\widetilde{F}$ is greater than equal to its expected revenue under a first-order stochastically-dominated cdf F.[19] Thus, if an optimal mechanism satisfies (7) then it is revenue monotone.

Fix an IC mechanism (q,t) and a pair of types v, v'. Are there sufficient conditions on $q(v)$ and $q(v')$ that imply $t(v) \geq t(v')$? We show that one such condition takes the form of majorization in model $\mathcal{M}^{\text{I}}$. We use this to derive new sufficient conditions for revenue monotonicity in both the models.

For any allocation probability vector $q = (q_1, q_2, \dots, q_n)$, let $q_{[i]}$ be the i^{th} highest element of q. That is, $q_{[1]} \geq q_{[2]} \geq \dots \geq q_{[n]}$.[20] If, for two allocation probability vectors $\hat{q}$, q,

$$\sum_{i=1}^{j} \hat{q}_{[i]} \geq \sum_{i=1}^{j} q_{[i]} \qquad \forall\ j \in \{1,\dots,n\}$$

then $\hat{q}$ **weakly majorizes** q, denoted $\hat{q} \succ_w q$.[21] If each of the inequalities above is satisfied with equality, then $\hat{q} \succ_w q$ and $q \succ_w \hat{q}$; in this case, either $q = \hat{q}$ or q is a permutation of $\hat{q}$. The $\succ_w$ relation is transitive and incomplete.

[18]As we assume the existence of densities, if (7) holds for almost all $\hat{v} > v$ then revenue monotonicity is satisfied.

[19]Note that IC and IR constraints do not involve the distribution of values; therefore, if (q,t) is IC and IR under F then it is IC and IR under $\widetilde{F}$.

[20]In model $\mathcal{M}^{\text{I}}$, $q_{[i]} = q_i$, for all $i \in N$.

[21]If, in addition, $\sum_{i=1}^{n} q_{[i]} = \sum_{i=1}^{n} \hat{q}_{[i]}$ then $\hat{q}$ **majorizes** q. The condition $\sum_{i=1}^{n} q_{[i]} = \sum_{i=1}^{n} \hat{q}_{[i]}$ is not usually satisfied by mechanisms in our setting.

In $\mathcal{M}^{\text{I}}$, since $q_{[i]} = q_i$ for each i, a sufficient condition for $\hat{q} \succ_w q$ is that (the cumulative probability distribution function induced by) $\hat{q}$ dominates q by second-order stochastic dominance – for a formal proof see Lemma 4 in Appendix A.2.

Proposition 4 *Let (q, t) be an IC mechanism which is either (i) in model $\mathcal{M}^{\text{I}}$ or (ii) in model $\mathcal{M}^{\text{H}}$ and is symmetric. Then, for almost all $v, \hat{v}$,*

$$q(\hat{v}) \succ_w q(v) \implies t(\hat{v}) \geq t(v) \tag{8}$$

The intuition behind Proposition 4 derives from the following inequality which is implied by IC:

$$t(\hat{v}) - t(v) \geq v \cdot q(\hat{v}) - v \cdot q(v)$$

For a mechanism (q, t) in model $\mathcal{M}^{\text{I}}$, we have $q_i(v) \geq q_{i+1}(v)$ and $v_i \geq v_{i+1}$. Thus, if $q(\hat{v}) \succ_w q(v)$ then the probabilities of acquiring the most valuable bundles are greater at $q(\hat{v})$ than at $q(v)$. Hence, the expected value of the allocation under $q(\hat{v})$ is at least as high as the expected value of the allocation under $q(v)$. In consequence, the right-hand expression in the above inequality is non-negative and $t(\hat{v}) \geq t(v)$.[22]

Consider the following property for an allocation rule.

Definition 12 *An allocation rule q satisfies* **majorization monotonicity** *if for all $(v_i, v_{-i}), (\hat{v}_i, v_{-i}) \in D^{\text{I}}$,*

$$q_i(\hat{v}_i, v_{-i}) > q_i(v_i, v_{-i}) \implies q(\hat{v}_i, v_{-i}) \succ_w q(v_i, v_{-i})$$

Note that if (q, t) is IC, then $q_i(\hat{v}_i, v_{-i}) > q_i(v_i, v_{-i})$ implies $\hat{v}_i > v_i$. Hence, majorization monotonicity (for an IC mechanism) is weaker than requring $\hat{v}_i > v_i$ implies $q(\hat{v}_i, v_{-i}) \succ_w q(v_i, v_{-i})$. Theorem 5 below establishes that majorization monotonicity is sufficient for revenue monotonicity. We also provide a sufficient condition for majorization monotonicity (and, hence, for revenue monotonicity).

[22] Kleiner et al. (2021) study monotone functions in $\Re$ which majorize or are majorized by a given monotone function. They characterize the extreme points of such functions and apply their result to several economic problems. Our results do not follow from their characterization.

THEOREM **5** *Suppose* (q,t) *is an IC mechanism in model* $\mathcal{M}^{I}$ *or a symmetric IC mechanism in model* $\mathcal{M}^{H}$.

(a) If q *satisfies majorization monotonicity then* (q,t) *is revenue monotone.*

(b) If q *is almost deterministic, then it satisfies majorization monotonicity, and hence,* (q,t) *is revenue monotone.*

REMARK **3** By Theorem 2, if the distribution is exchangeable then there exists a symmetric mechanism that is optimal in the heterogeneous-objects model. Hence, such an optimal mechanism is revenue monotone if it is either (a) majorization monotone or (b) almost deterministic. In other words, if F^{H} is an exchangeable distribution, then for every $\widetilde{F}^{\text{H}}$ that first-order stochastic dominates F^{H}, the optimal revenue under $\widetilde{F}^{\text{H}}$ is no less than the optimal revenue under F^{H}. Note that $\widetilde{F}^{\text{H}}$ need not be an exchangeable distribution.

Theorem 5(ii) strengthens the result in Hart and Reny (2015), who showed that the optimal mechanism in model $\mathcal{M}^{\text{H}}$ is revenue monotone if it is symmetric and deterministic. It is difficult to find sufficient conditions on the primitives of the model that guarantee existence of an optimal mechanism which is symmetric and deterministic. On the other hand, when $n = 2$, there is a simple condition on distribution that guarantees that the optimal mechanism is almost deterministic.

Consider the following condition on the density of buyer tpes, which was introduced by McAfee and McMillan (1988):

$$3f^{\text{M}}(v) + v \cdot \nabla f^{\text{M}}(v) \geq 0 \qquad \forall\, v \in \overline{D}^{\text{M}},\; M = H \text{ or } I \tag{9}$$

The uniform family of distributions, the truncated exponential distribution, and a family of Beta distributions satisfy condition (9). As shown in Proposition 1 of Pavlov (2011b) for model $\mathcal{M}^{\text{H}}$ and in Proposition 1 of Bikhchandani and Mishra (2022) for model $\mathcal{M}^{\text{I}}$, if there are two objects and $\underline{v} = 0$, then, in both the models, (9) is sufficient for the existence of an optimal mechanism which is almost deterministic. Thus, we have the following result.

COROLLARY **3** *Suppose that* $n = 2$, $\underline{v} = 0$, *(9) is satisfied, and* f^{M} *is continuously differentiable and positive for* $M = H$ *or* I*. Then*

(a) An optimal mechanism in model $\mathcal{M}^{I}$ *is revenue monotone.*

(b) *Further, if f^{H} is exchangeable, then an optimal mechanism in model $\mathcal{M}^{\text{H}}$ is revenue monotone.*

Note that under the hypothesis of Corollary 3, revenue increases with any distribution f' that dominates f^{M} by first-order stochastic dominance, whether or not f' satisfies (9).

5 Discussion

We have provided new results in the heterogeneous-objects model by first obtaining new results in the identical-objects model and then using our equivalence result. One can also translate known results in the identical-objects setting to the heterogeneous-objects setting. In a model with two identical objects, Bikhchandani and Mishra (2022) obtain a sufficient condition for the existence of an optimal mechanism that is deterministic. This result can be directly applied to selling two heterogeneous objects with an exchangeable distribution of values.

It is possible to go in the other direction, i.e., convert results in the heterogeneous-objects model to identical objects. For instance, the revenue monotonicity of symmetric and deterministic mechanisms for selling heterogeneous objects established by Hart and Reny (2015) implies revenue monotonicity of deterministic mechanisms for selling identical objects. Similarly, approximate expected revenue maximization results for i.i.d. priors in Hart and Nisan (2017) imply analogous approximate expected revenue maximization results in model $\mathcal{M}^{\text{I}}$ for a class of priors.

Finally, while our presentation is in terms of selling indivisible objects, the results apply to broader settings. For example, consider the following two models:

1. Model I. A seller offers one durable product which, depending on its quality level, may be consumed for up to n periods. An object with quality level i lasts i periods. The seller sells (at most) one object to a buyer at the beginning of the first period at one of the n quality levels; no sales take place at any later time period. If a buyer purchases an object of quality i at the beginning of the first period, then she consumes it in each of the periods $1, 2, \ldots, i$. The value of consuming the product in the n periods is $(v_1, \ldots, v_n)$. Owing to discounting, which need not be same across periods,

$v_1 \geq v_2 \geq \ldots \geq v_n$. So, v_i is the marginal value of increasing the quality level from $i-1$ to i. Note that the product can be consumed in period i only if it is consumed in period $i-1$ and hence $q_{i-1}(\cdot) \geq q_i(\cdot)$, where $q_i(\cdot)$ is the probability of consuming the product in period i.

2. Model H. This is a one-period model in which a seller offers n different products, each of which lasts one period. So, v_i denotes the value for product i to the buyer. The buyer has additive values over any subset of the products.

As long as the products in Model H are *ex-ante symmetric*, the results of Section 3 imply that the two models are equivalent.

Appendix A Omitted Proofs

A.1 Proofs of Section 3

Proof of Lemma 1: Let $(\bar{q}(v), \bar{t}(v)) \equiv (q(v), t(v)),\ \forall v \in D^{\mathrm{M}}$. For any $v \in \overline{D}^{\mathrm{M}} \setminus D^{\mathrm{M}}$, take a sequence $\{v^k\}_k$ in D^{M} that converges to v. (As D^{M} is dense in $\overline{D}^{\mathrm{M}}$, for each $v \in \overline{D}^{\mathrm{M}} \setminus D^{\mathrm{M}}$, there exists such a sequence.) Note that $q_i(v^k) \in [0,1]$ for each i and $t(v^k) \geq \underline{v} \cdot q(\underline{v})$ is bounded above due to IR and bounded $\overline{D}^{\mathrm{M}}$. Thus, $\{q(v^k), t(v^k)\}_k$ is a bounded sequence and hence, it has an accumulation point. Set $(\bar{q}(v), \bar{t}(v))$ equal to an accumulation point of this sequence. For every $v \in \overline{D}^{\mathrm{M}} \setminus D^{\mathrm{M}}$, $(\bar{q}(v), \bar{t}(v))$ is an accumulation point of outcomes of a sequence of types in D^{M}. Therefore, as (q,t) is IC and IR on D^{M}, and the buyer's payoff function is continuous in v, it follows that $(\bar{q}, \bar{t})$ is IC and IR on $\overline{D}^{\mathrm{M}}$. ■

Proof of Theorem 1: Let (q,t) be a symmetric mechanism in $\mathcal{M}^{\mathrm{H}}$.
(i) ⇒ (ii): Assume that (q,t) is IC on D^{H}. Fix i and j. Let $v \in D^{\mathrm{H}}$ and let σ be the permutation such that $\sigma(i) = j, \sigma(j) = i$ and $\sigma(k) = k$ for all $k \notin \{i,j\}$. We have

$$\begin{aligned} 0 &= t(v) - t(v^\sigma) && \text{(by symmetry of } (q,t)) \\ &\leq v \cdot (q(v) - q(v^\sigma)) && \text{(by IC of } (q,t)) \\ &= v_i(q_i(v) - q_i(v^\sigma)) + v_j(q_j(v) - q_j(v^\sigma)) \\ &= (v_i - v_j)(q_i(v) - q_j(v)) \end{aligned}$$

where the second equality follows from $q_k(v) = q_k(v^\sigma)$ for all $k \notin \{i,j\}$, and the last equality follows from symmetry. Thus, if $v_i > v_j$, then $q_i(v) \geq q_j(v)$. Hence, (q,t) is rank-preserving.

(ii) ⇒ (i): Pick any $v \in D(\sigma)$ and $\hat{v} \in D(\hat{\sigma})$. These map to $v^\sigma, \hat{v}^{\hat{\sigma}} \in D(\sigma^{\mathrm{I}})$ such that for every i,

$$v_i^\sigma = v_{\sigma(i)}, \qquad \hat{v}_i^{\hat{\sigma}} = \hat{v}_{\hat{\sigma}(i)} \tag{10}$$

We know that

$$\begin{aligned} \sum_{i=1}^n v_i q_i(v) - t(v) &= \sum_{i=1}^n v_{\sigma(i)} q_{\sigma(i)}(v) - t(v) \\ &= \sum_{i=1}^n v_i^\sigma q_i(v^\sigma) - t(v^\sigma) && \text{(by symmetry of } (q,t) \text{ and (10))} \\ &\geq \sum_{i=1}^n v_i^\sigma q_i(\hat{v}^{\hat{\sigma}}) - t(\hat{v}^{\hat{\sigma}}) \end{aligned}$$

$$= \sum_{i=1}^{n} v_i^{\sigma} q_{\hat{\sigma}(i)}(\hat{v}) - t(\hat{v}), \qquad \text{(by symmetry of } (q,t)) \tag{11}$$

where the inequality follows as $v^{\sigma}, \hat{v}^{\hat{\sigma}} \in D(\sigma^{\mathrm{I}})$ and (q,t) restricted to $D(\sigma^{\mathrm{I}})$ is IC.

Note that

$$v_1^{\sigma} > v_2^{\sigma} > \ldots > v_n^{\sigma} \qquad \text{(since } v^{\sigma} \in D(\sigma^{\mathrm{I}}))$$
$$q_{\hat{\sigma}(1)}(\hat{v}) \geq q_{\hat{\sigma}(2)}(\hat{v}) \geq \ldots \geq q_{\hat{\sigma}(n)}(\hat{v}) \qquad \text{(since } (q,t) \text{ is rank-preserving, } \hat{v} \in D(\hat{\sigma}), \text{ and } (2))$$

As $(v_{\hat{\sigma}(1)}, v_{\hat{\sigma}(2)}, \ldots, v_{\hat{\sigma}(n)})$ is a permutation of $(v_1^{\sigma}, v_2^{\sigma}, \ldots, v_n^{\sigma})$, these inequalities imply that[23]

$$\sum_{i=1}^{n} v_i^{\sigma} q_{\hat{\sigma}(i)}(\hat{v}) \geq \sum_{i=1}^{n} v_{\hat{\sigma}(i)} q_{\hat{\sigma}(i)}(\hat{v}) \tag{12}$$

Using (11) and (12), we have

$$\begin{aligned} \sum_{i=1}^{n} v_i q_i(v) - t(v) &\geq \sum_{i=1}^{n} v_{\hat{\sigma}(i)} q_{\hat{\sigma}(i)}(\hat{v}) - t(\hat{v}) \\ &= \sum_{i=1}^{n} v_i q_i(\hat{v}) - t(\hat{v}), \end{aligned}$$

which is the desired IC constraint. ■

Proof of Theorem 2: Suppose (q,t) is an asymmetric, IC, and IR mechanism in model $\mathcal{M}^{\mathrm{H}}$. From (q,t) we construct another IC and IR mechanism (q^*, t^*) which is symmetric and has the same expected revenue as (q,t). Consequently, there exists an optimal mechanism which is symmetric.

For any $\sigma \in \Sigma$, let $\sigma^{-1} \in \Sigma$ be such that $\sigma\sigma^{-1} = \sigma^{-1}\sigma = \sigma^{\mathrm{I}}$. For all $v \in \overline{D}^{\mathrm{H}}$, define

$$\begin{aligned} \hat{q}(v;\sigma) &:= q^{\sigma^{-1}}(v^{\sigma}) = (q_{\sigma^{-1}(1)}(v^{\sigma}), \ldots, q_{\sigma^{-1}(n)}(v^{\sigma})) \\ \hat{t}(v;\sigma) &:= t(v^{\sigma}) \end{aligned}$$

Then for any $v, \check{v} \in \overline{D}^{\mathrm{H}}$

$$\begin{aligned} v \cdot \hat{q}(v;\sigma) - \hat{t}(v;\sigma) &= v \cdot q^{\sigma^{-1}}(v^{\sigma}) - t(v^{\sigma}) \\ &= v^{\sigma} \cdot q(v^{\sigma}) - t(v^{\sigma}) \end{aligned}$$

[23] See also the rearrangement inequality in Theorem 368 of Hardy et al. (1952).

$$
\begin{aligned}
&\geq v^{\sigma} \cdot q(\check{v}^{\sigma}) - t(\check{v}^{\sigma}) && \text{(since } (q,t) \text{ is IC)}\\
&= v \cdot q^{\sigma^{-1}}(\check{v}^{\sigma}) - t(\check{v}^{\sigma})\\
&= v \cdot \hat{q}(\check{v};\sigma) - t(\check{v};\sigma)
\end{aligned}
$$

Hence $(\hat{q}(\cdot\,;\sigma), \hat{t}(\cdot\,;\sigma))$ is IC. That $(\hat{q}(\cdot\,;\sigma), \hat{t}(\cdot\,;\sigma))$ is IR follows from IR of (q,t).

For all $v \in \overline{D}^{\mathrm{H}}$, define,

$$
\begin{aligned}
q^*(v) &:= \frac{1}{n!}\sum_{\sigma\in\Sigma} \hat{q}(v;\sigma)\\
t^*(v) &:= \frac{1}{n!}\sum_{\sigma\in\Sigma} \hat{t}(v;\sigma)
\end{aligned}
$$

The mechanism (q^*, t^*) is IC and IR as it is a convex combination of IC and IR mechanisms. To see that (q^*, t^*) is a symmetric mechanism, note that for any fixed permutation $\check{\sigma}$,[24]

$$
\begin{aligned}
t^*(v^{\check{\sigma}}) &= \frac{1}{n!}\sum_{\sigma\in\Sigma} \hat{t}(v^{\check{\sigma}};\sigma) = \frac{1}{n!}\sum_{\sigma\in\Sigma} t(v^{\check{\sigma}\sigma}) = \frac{1}{n!}\sum_{\sigma'\in\Sigma} t(v^{\sigma'})\\
&= \frac{1}{n!}\sum_{\sigma'\in\Sigma} \hat{t}(v;\sigma') = t^*(v)
\end{aligned}
$$

$$
\begin{aligned}
q^*(v^{\check{\sigma}}) &= \frac{1}{n!}\sum_{\sigma\in\Sigma} \hat{q}(v^{\check{\sigma}};\sigma) = \frac{1}{n!}\sum_{\sigma\in\Sigma} q^{\sigma^{-1}}(v^{\check{\sigma}\sigma}) = \frac{1}{n!}\sum_{\sigma\in\Sigma} q^{\check{\sigma}\check{\sigma}^{-1}\sigma^{-1}}(v^{\check{\sigma}\sigma})\\
&= \frac{1}{n!}\sum_{\sigma'\in\Sigma} q^{\check{\sigma}(\sigma')^{-1}}(v^{\sigma'}) = \frac{1}{n!}\sum_{\sigma'\in\Sigma} \hat{q}^{\check{\sigma}}(v;\sigma') = (q^*)^{\check{\sigma}}(v)
\end{aligned}
$$

where $\sigma' = \check{\sigma}\sigma$.

Finally, the expected revenue from (q^*, t^*) is

$$
\begin{aligned}
\textsc{Rev}(q^*, t^*; f^{\mathrm{H}}) &= \int_{\overline{D}^{\mathrm{H}}} t^*(v) f^{\mathrm{H}}(v)dv = \int_{\overline{D}^{\mathrm{H}}} \frac{1}{n!}\Big(\sum_{\sigma} \hat{t}(v;\sigma)\Big) f^{\mathrm{H}}(v)dv = \int_{\overline{D}^{\mathrm{H}}} \frac{1}{n!}\Big(\sum_{\sigma} t(v^{\sigma}) f^{\mathrm{H}}(v^{\sigma})\Big)dv\\
&= \frac{1}{n!}\sum_{\sigma}\int_{\overline{D}^{\mathrm{H}}} t(v^{\sigma}) f^{\mathrm{H}}(v^{\sigma})dv = \frac{1}{n!}\sum_{\sigma}\int_{\overline{D}^{\mathrm{H}}} t(v) f^{\mathrm{H}}(v)dv = \textsc{Rev}(q,t;f^{\mathrm{H}}),
\end{aligned}
$$

where we used exchangeability of f^{H} in the third and fifth equalities. Hence, (q^*, t^*) is a symmetric, IC and IR mechanism. By Theorem 1, (q^*, t^*) is a rank-preserving mechanism.

[24]Note that even if the mechanism (q,t) is deterministic (and asymmetric), the mechanism (q^*, t^*) may be random.

Thus, for every IC and IR mechanism, there exists a symmetric and rank-preserving IC and IR mechanism that generates the same expected revenue. Hence, there exists a symmetric and rank-preserving optimal mechanism. ■

Proof of Lemma 2: Relabelling the objects if necessary, assume that (q, t) is defined on $D(\sigma^{\text{I}})$. Let (q^s, t^s) be the symmetric extension of (q, t). As (q, t) is rank preserving on $D(\sigma^{\text{I}})$, (q^s, t^s) is rank preserving on D^{H}. As $(q^s, t^s) = (q, t)$ on $D(\sigma^{\text{I}})$ and (q, t) is IC on $D(\sigma^{\text{I}})$, we conclude that (q^s, t^s) is IC on D^{H} (by Theorem 1).

For any $v \in D(\sigma^{\text{I}})$, $(q^s(v), t^s(v)) = (q(v), t(v))$. Thus, (q^s, t^s) is IR on $D(\sigma^{\text{I}})$. That (q^s, t^s) is IR follows from the fact that the payoff of any type $v \in D(\sigma)$ is the same as the payoff of type $v^\sigma \in D(\sigma^{\text{I}})$. ■

Proof of Proposition 1:
(i) & (ii) Note that any mechanism in model $\mathcal{M}^{\text{I}}$ is rank preserving due to feasibility restriction (1). Thus, by Lemma 2 the symmetric extension of an IC and IR mechanism in model $\mathcal{M}^{\text{I}}$ is a rank-preserving, IC and IR mechanism on D^{H}, which can be extended to $\overline{D}^{\text{H}}$, i.e., to model $\mathcal{M}^{\text{H}}$, by Lemma 1. Conversely, if (q, t) is a symmetric, IC, and IR mechanism in model $\mathcal{M}^{\text{H}}$, then it is rank preserving by Theorem 1. Hence, the restriction of (q, t) to $D(\sigma^{\text{I}})$ defines an IC and IR mechanism for model $\mathcal{M}^{\text{I}}$ (since rank preserving implies that the feasibility restriction (1) holds).

(iii) As f^{H} is exchangeable, Theorem 2 implies that there exists a symmetric and rank-preserving mechanism that is optimal in model $\mathcal{M}^{\text{H}}$. Let $(q^{\text{H}}, t^{\text{H}})$ be this optimal mechanism in model $\mathcal{M}^{\text{H}}$ and let $(q^{\text{I}}, t^{\text{I}})$ be the corresponding IC and IR mechanism for model $\mathcal{M}^{\text{I}}$ obtained by restricting $(q^{\text{H}}, t^{\text{H}})$ to $D(\sigma^{\text{I}})$. As $(q^{\text{H}}, t^{\text{H}})$ is symmetric, we have

$$\textsc{Rev}(q^{\text{H}}, t^{\text{H}}; f^{\text{H}}) = n! \int_{D^{\text{I}}} t^{\text{H}}(v) f^{\text{H}}(v) dv = \int_{D^{\text{I}}} t^{\text{I}}(v) f^{\text{I}}(v) dv = \textsc{Rev}(q^{\text{I}}, t^{\text{I}}; f^{\text{I}}) \tag{13}$$

As $(q^{\text{H}}, t^{\text{H}})$ is optimal in $\mathcal{M}^{\text{H}}$, $(q^{\text{I}}, t^{\text{I}})$ must be optimal in $\mathcal{M}^{\text{I}}$. To see this, suppose that some other mechanism $(q^{\text{I}'}, t^{\text{I}'})$ yields a strictly higher revenue than $(q^{\text{I}}, t^{\text{I}})$ in $\mathcal{M}^{\text{I}}$. Let $(q^{\text{H}'}, t^{\text{H}'})$ be the symmetric extension of $(q^{\text{I}'}, t^{\text{I}'})$ to $\mathcal{M}^{\text{H}}$. Then we have

$$\textsc{Rev}(q^{\text{H}}, t^{\text{H}}; f^{\text{H}}) = \textsc{Rev}(q^{\text{I}}, t^{\text{I}}; f^{\text{I}}) < \textsc{Rev}(q^{\text{I}'}, t^{\text{I}'}; f^{\text{I}}) = \textsc{Rev}(q^{\text{H}'}, t^{\text{H}'}; f^{\text{H}})$$

where the equalities follow from (13). But this contradicts the assumption that $(q^{\text{H}}, t^{\text{H}})$ is optimal in $\mathcal{M}^{\text{H}}$. Thus, optimal mechanisms in the two models yield the same expected revenue. ■

A.2 Proofs of Section 4

PROOF OF PROPOSITION 2: We provide a proof of (i) and (ii) for an arbitrary IC mechanism, and indicate the changes needed for the proof to work for deterministic and almost deterministic mechanisms, i.e., for part (iii) of the proposition.

(i): Let (q, t) be an IC mechanism defined on D^* with corresponding utility function u. An $x \in [0, 1]^n$ with $x_1 \geq \ldots \geq x_n$ is a *subgradient*[25] of u at $v \in D^*$ if for every $v' \in D^*$

$$u(v') \geq u(v) + (v' - v) \cdot x \qquad \forall\, v' \in D^* \tag{14}$$

We know that for every $v \in D^*$, $q(v)$ is a subgradient of u at v due to IC. Let $\partial u(v)$ denote the set of all subgradients of u at v. Since $q(v) \in \partial u(v)$ for all $v \in D^*$, we know that $\partial u(v)$ is non-empty.

Define a strict linear order $\succ$ on the set of all feasible allocations in the identical objects model, i.e., on the set $X := \{x \in [0, 1]^n : x_1 \geq \ldots \geq x_n\}$. An example of such an ordering is the following *lexicographic order*: for any $x, y \in X$, $x \succ y$ if either $x_1 > y_1$ or $x_1 = y_1, x_2 > y_2$ or $x_i = y_i$ for $i \in \{1, 2\}$, $x_3 > y_3$, or etc. If x and y are deterministic allocations (i.e., $0-1$ vectors) and $\succ$ is the lexicographic order, then $x \succ y$ if and only if $\sum_i x_i > \sum_i y_i$, i.e., we allocate more units in x than in y.

For every $v \in D^*$, let

$$X(v) := \{x \in \partial u(v) : v \cdot x \geq v \cdot y \ \forall\, y \in \partial u(v)\}$$

Hart and Reny (2015) call a mechanism (q, t) *seller favorable* if $q(v) \in X(v)$ for every v in the domain. If D^* is convex, IC implies u is convex. For convex u defined on convex D^*, Rockafellar (Theorem 23.2) characterizes $v \cdot x$ for each $x \in X(v)$ (note that $v \cdot x$ is the same for all $x \in X(v)$) as the *directional derivative* of u in the direction v.

We make a single-valued selection from $X(v)$ at every v in a consistent manner to achieve non-bossiness. This is done by picking the maximal element in $X(v)$ with respect to $\succ$.

Let $x(v)$ denote the maximal vector in $X(v)$ with respect to $\succ$.[26]

[25] For the proof of deterministic and almost deterministic mechanisms, only consider subgradients which are deterministic and almost deterministic respectively.

[26] For the proof of (iii), $x(v)$ will choose a deterministic vector and an almost deterministic vector respectively.

Definition 13 *For an IC mechanism (q,t) defined on D^*, define its* **maximal extension (with respect to $\succ$)** $(q^\sharp, t^\sharp)$ *as*

$$q^\sharp(v) = x(v) \qquad \forall\ v \in D^*$$
$$t^\sharp(v) = v \cdot q^\sharp(v) - u(v) \qquad \forall\ v \in D^*$$

Hence, $(q^\sharp, t^\sharp)$ is a seller-favorable mechanism as in Hart and Reny (2015). However, an arbitrary seller-favorable mechanism need not be non-bossy.[27]

First, $u^\sharp(v) = v \cdot q^\sharp - t^\sharp(v) = u(v)$ for all $v \in D^*$. Also, $t^\sharp(v) = v \cdot x(v) - u(v) \geq v \cdot q(v) - u(v) = t(v)$ for all $v \in D^*$, where we used $v \cdot x(v) \geq v \cdot y$ for all $y \in \partial u(v)$ and $q(v) \in \partial u(v)$. For every v^*, we know that $x(v) \in \partial u(v)$, and hence, (14) holds. As a result, $(q^\sharp, t^\sharp)$ defines an IC mechanism.

To complete the proof of (i), we show that $(q^\sharp, t^\sharp)$ is non-bossy. Take $v = (v_i, v_{-i}) \in D^*$ and $(v'_i, v_{-i}) \in D^*$. Suppose that $q^\sharp_i(v) = q^\sharp_i(v'_i, v_{-i})$. IC constraints between v and (v_i, v_{-i}) are

$$u^\sharp(v) - u^\sharp(v'_i, v_{-i}) \geq (v_i - v'_i)q^\sharp_i(v_i, v'_{-i}) = (v_i - v'_i)q^\sharp_i(v)$$
$$u^\sharp(v'_i, v_{-i}) - u^\sharp(v) \geq (v'_i - v_i)q^\sharp_i(v)$$

where we use the fact that $q^\sharp_i(v) = q^\sharp_i(v'_i, v_{-i})$ in the equality above. Combining these

$$u^\sharp(v) - u^\sharp(v'_i, v_{-i}) = (v_i - v'_i)q^\sharp_i(v) \tag{15}$$

We show that $q^\sharp(v) \in \partial u^\sharp(v'_i, v_{-i})$. Pick any $\hat{v} \in D^*$. The IC constraint from $\hat{v}$ to v is

$$\begin{aligned} u^\sharp(\hat{v}) &\geq u^\sharp(v) + (\hat{v} - v) \cdot q^\sharp_j(v) \\ &= u^\sharp(v'_i, v_{-i}) + (v_i - v'_i)q^\sharp_i(v) + (\hat{v}_i - v_i)q^\sharp_i(v) + \sum_{j \neq i}(\hat{v}_j - v_j)q^\sharp_j(v) \qquad \text{(by eq. 15)} \\ &= u^\sharp(v'_i, v_{-i}) + (\hat{v}_i - v'_i)q^\sharp_i(v) + \sum_{j \neq i}(\hat{v}_j - v_j)q^\sharp_j(v) \end{aligned}$$

Thus, $q^\sharp(v) \in \partial u^\sharp(v'_i, v_{-i})$. An identical proof shows $q^\sharp(v'_i, v_{-i}) \in \partial u^\sharp(v)$. So, $q^\sharp(v), q^\sharp(v'_i, v_{-i}) \in \partial u^\sharp(v) \cap \partial u^\sharp(v'_i, v_{-i})$. Since $(q^\sharp, t^\sharp)$ is the maximal extension of (q,t) with respect to same order $\succ$, we must have $q^\sharp(v) = q^\sharp(v'_i, v_{-i})$. This implies that $q^\sharp$ is non-bossy.

(ii): If D^* is convex, u is convex and $\partial u(v)$ is singleton almost everywhere (Theorems 25.1 and 25.3 in Rockafellar). Consequently, $(q(v), t(v)) = (q^\sharp(v), t^\sharp(v))$ almost everywhere. ■

[27]Moroever, there exist non-bossy mechanisms that are not seller favorable.

Proof of Lemma 3: Suppose that the domain of types D satisfies the strong-lattice property. Taking $k = 1$ in Definition 6, we see that the lattice property w.r.t. with min and max is implied by the strong-lattice property.

Next, we prove that in a rich domain the lattice property implies the strong-lattice property. Take any $v, v' \in D$. The two types $v^\dagger = \min(v, v')$ and $v^\ddagger = \max(v, v')$ belong to D by the lattice property. We show $\tilde{v}$ as defined in (4) belongs to D. Fix any k. If $k = 1$, then $\tilde{v} = v^\dagger$ and we are done. If $k > 1$, then $\tilde{v} = (v_1, \ldots, v_{k-1}, v_k^\dagger, \ldots, v_n^\dagger)$. By definition, $v_{k-1} \geq v_k \geq v_k^\dagger$. Hence, richness implies that $\tilde{v} \in D$.

To show $\check{v}$ as defined in (5) belongs to D, fix k. If $k = n$, then $\check{v} = v^\ddagger$, and we are done. If $k < n$, then $\check{v} = (v_1^\ddagger, \ldots, v_k^\ddagger, v_{k+1}, \ldots, v_n)$. Since $v_k^\ddagger \geq v_k \geq v_{k+1}$, richness implies that $\check{v} \in D$. ■

Proof of Theorem 3: Let (q, t) be a deterministic, object non-bossy, and UIC mechanism defined on type space D that satisfies the strong-lattice property. We start with some preliminary results. Let $k(v) := \sum_{i=1}^{n} q_i(v)$, $k(v') := \sum_{i=1}^{n} q_i(v')$, etc.

Claim 1 *Suppose (q, t) is UIC. If $v \geq v'$ then the following are true:*

(i) $\sum_{i=1}^{k(v)} (v_i - v_i') \geq \sum_{i=1}^{k(v')} (v_i - v_i')$

(ii) either $k(v) \geq k(v')$ or $v_i = v_i'$ for all $i \in \{k(v) + 1, \ldots, k(v')\}$.

(iii) either $k(v) \geq k(v')$ or the IC constraints between v and v' bind.

Proof: If $v \geq v'$ or $v' \geq v$, then UIC between v and v' implies:

$$\sum_{i=1}^{k(v)} v_i - \sum_{i=1}^{k(v')} v_i \geq t(v) - t(v') \geq \sum_{i=1}^{k(v)} v_i' - \sum_{i=1}^{k(v')} v_i' \tag{16}$$

(i) Follows directly from (16).

(ii) If $k(v') > k(v)$, then (i) implies $\sum_{i=k(v)+1}^{k(v')} (v_i - v_i') \leq 0$. Since $v \geq v'$, this is possible only if $v_i = v_i'$ for all $i \in \{k(v) + 1, \ldots, k(v')\}$.

(iii) If $k(v') > k(v)$, the IC constraints in (16) are equivalent to

$$\sum_{i=k(v)+1}^{k(v')} v_i' \geq t(v') - t(v) \geq \sum_{i=k(v)+1}^{k(v')} v_i$$

Using (ii), the lower bound and upper bound of $t(v') - t(v)$ is the same. Hence, the IC constraints between v and v' bind. $\square$

CLAIM **2** *Suppose (q, t) is UIC and q is object non-bossy. Then, the following hold.*

(i) If $v \geq v'$ and $v_i = v'_i$ for all $i \leq \max(k(v), k(v'))$, then, $q(v) = q(v')$ and $t(v) = t(v')$.

(ii) If $v \leq v'$ and $v_i = v'_i$ for all $i > \min(k(v), k(v'))$. Then, $q(v) = q(v')$ and $t(v) = t(v')$.

PROOF: Assume without loss of generality v and v' differ in only one dimension, say, dimension j.[28]

(i): Let $v' = (v'_j, v_{-j})$, where $v'_j < v_j$. Since $v_i = v'_i$ for all $i \leq \max(k(v'), k(v))$, we have $j > \max(k(v'), k(v))$. Hence, $q_j(v') = q_j(v) = 0$. By non-bossiness, $q(v') = q(v)$. By UIC, we have $t(v) = t(v')$.

(ii): Let $v' \equiv (v'_j, v_{-j})$ where $v'_j > v_j$. By assumption $j \leq \min(k(v), k(v'))$. Hence, $q_j(v) = q_j(v') = 1$. By non-bossiness, $q(v) = q(v')$. By UIC, we have $t(v) = t(v')$. $\square$

CLAIM **3** *Suppose (q, t) is UIC and q is object non-bossy. Then, q is increasing, i.e., for all $v \geq v'$, $\kappa(v) \geq \kappa(v')$.*

PROOF: Let $v \geq v'$. Without loss of generality assume that $v' \equiv (v'_j, v_{-j})$ for some j with $v'_j < v_j$. If $k(v') > k(v)$, then $j \notin \{k(v) + 1, \dots, k(v')\}$ since $v_i = v'_i$ for all $i \in \{k(v) + 1, \dots, k(v')\}$ by Claim 1. If $j \leq k(v)$, then $q_j(v) = q_j(v') = 1$, and non-bossiness implies $q(v) = q(v')$. If $j > k(v')$, then $q_j(v) = q_j(v') = 0$, and non-bossiness implies $q(v) = q(v')$. This gives us $k(v) = k(v')$, a contradiction to our assumption that $k(v') > k(v)$. $\square$

The next two claims prove Theorem 3.

CLAIM **4** *Suppose (q, t) is a UIC mechanism and q is object non-bossy. Let v and v' be such that $v \not\geq v'$, $v' \not\geq v$ and $k(v) \geq k(v')$ Then, the incentive constraint $v' \to v$ is satisfied.*

PROOF: We consider two cases.

[28] If v and v' differ in more than one dimension, we can use this argument repeatedly.

Case 1: $k(v) = n$. By Claim 3, the type $v^\dagger = \max(v, v')$ (by the strong-lattice property, $v^\dagger \in D$) satisfies $k(v^\dagger) \geq k(v) = n$. Hence, $k(v^\dagger) = n$. As n units are allocated at v and at $v^\dagger$, we have $t(v) = t(v^\dagger)$ by UIC. As UIC between v' and $v^\dagger$ holds, we have

$$\sum_{i=1}^{k(v')} v'_i - t(v') \geq \sum_{i=1}^{n} v'_i - t(v^\dagger)$$
$$= \sum_{i=1}^{k(v)} v'_i - t(v)$$

Hence, $v' \to v$ is satisfied.

Case 2: $k(v) < n$. Define $\tilde{v}$ and $\tilde{v}'$ as follows.

$$\tilde{v}_i = \begin{cases} v_i & \text{if } i \leq k(v) \\ \min(v_i, v'_i) & \text{if } i > k(v) \end{cases}$$

$$\tilde{v}'_i = \begin{cases} v'_i & \text{if } i \leq k(v) \\ \min(v_i, v'_i) & \text{if } i > k(v) \end{cases}$$

By the strong-lattice property $\tilde{v}, \tilde{v}' \in D$. Since $\tilde{v} \leq v$, by Claim 3 we have $k(\tilde{v}) \leq k(v)$. As $\tilde{v}_i = v_i$ for all $i \leq k(v) = \max(k(\tilde{v}), k(v))$, by Claim 2($i$), $q(\tilde{v}) = q(v)$ and $t(\tilde{v}) = t(v)$.

Similarly, as $\tilde{v}' \leq v'$, by Claim 3 we have $k(\tilde{v}') \leq k(v') \leq k(v)$. Since $\tilde{v}'_i = v'_i$ for all $i \leq k(v)$ and $k(v') = \max(k(\tilde{v}'), k(v')) \leq k(v)$, by Claim 2($i$), $q(\tilde{v}') = q(v')$ and $t(\tilde{v}') = t(v')$.

Define a new type $\hat{v}$ as follows:

$$\hat{v}_i = \max(\tilde{v}_i, \tilde{v}'_i) \qquad \forall\, i \in \{1, \ldots, n\}$$

By the strong-lattice property, $\hat{v} \in D$. Note that for all $i > k(v)$, $\hat{v}_i = \tilde{v}_i = \tilde{v}'_i = \min(v_i, v'_i)$. Since $\hat{v} \geq \tilde{v}$, Claim 3 implies $k(\hat{v}) \geq k(v)$.

Considering UIC from $\tilde{v}'$ to $\hat{v}$ and $\hat{v}$ to $\tilde{v}$ (and using $v'_i = \tilde{v}'_i$ for all $i \leq k(v)$ and $k(v) \geq k(v')$, and $t(v') = t(\tilde{v}')$, $t(\tilde{v}) = t(v)$), we get

$$\sum_{i=1}^{k(v')} v'_i - t(v') = \sum_{i=1}^{k(v')} \tilde{v}'_i - t(\tilde{v}') \geq \sum_{i=1}^{k(\hat{v})} \tilde{v}'_i - t(\hat{v})$$

$$\sum_{i=1}^{k(\hat{v})} \hat{v}_i - t(\hat{v}) \geq \sum_{i=1}^{k(v)} \hat{v}_i - t(\tilde{v}) = \sum_{i=1}^{k(v)} \hat{v}_i - t(v)$$

Adding these constraints

$$\sum_{i=1}^{k(v')} v'_i - t(v') \geq \sum_{i=1}^{k(v)} \hat{v}_i - \sum_{i=1}^{k(\hat{v})} \hat{v}_i + \sum_{i=1}^{k(\hat{v})} \tilde{v}'_i - t(v) \tag{17}$$

If $k(\hat{v}) = k(v)$, then (17) reduces to

$$\sum_{i=1}^{k(v')} v'_i - t(v') \geq \sum_{i=1}^{k(\hat{v})} \tilde{v}'_i - t(v) = \sum_{i=1}^{k(v)} v'_i - t(v)$$

where we used the fact that $v'_i = \tilde{v}'_i$ for all $i \leq k(v)$. Hence, $v' \to v$ holds.

If $k(\hat{v}) > k(v)$, then (17) reduces to

$$\begin{aligned}
\sum_{i=1}^{k(v')} v'_i - t(v') &\geq \sum_{i=1}^{k(\hat{v})} \tilde{v}'_i - \sum_{i=k(v)+1}^{k(\hat{v})} \hat{v}_i - t(v) \\
&= \sum_{i=1}^{k(\hat{v})} \tilde{v}'_i - \sum_{i=k(v)+1}^{k(\hat{v})} \tilde{v}'_i - t(v) \qquad (\text{using } \hat{v}_i = \tilde{v}_i = \tilde{v}'_i \text{ for all } i > k(v)) \\
&= \sum_{i=1}^{k(v)} \tilde{v}'_i - t(v) \\
&= \sum_{i=1}^{k(v)} v'_i - t(v) \qquad (\text{using } v'_i = \tilde{v}_i \text{ for all } i \leq k(v))
\end{aligned}$$

Hence, $v' \to v$ is satisfied. □

CLAIM **5** *Suppose (q, t) is a UIC mechanism and q is object non-bossy. Let v and v' be such that $v \ngeq v'$, $v' \ngeq v$ and $k(v) > k(v')$. Then, the incentive constraint $v \to v'$ is satisfied.*

PROOF: We consider two cases.

CASE 1: $k(v') = 0$. As 0 units are allocated at v', Claim 3 implies that 0 units are allocated at $v^\dagger = \min(v, v')$ (by the strong-lattice property $v^\dagger \in D$). Thus, UIC implies that $t(v') = t(v^\dagger)$. As UIC between v and $v^\dagger$ holds, we have

$$\sum_{i=1}^{k(v)} v_i - t(v) \geq -t(v^\dagger) = -t(v')$$

We conclude that $v \to v'$.

Case 2: $k(v') > 0$. Define $\tilde{v}$ and $\tilde{v}'$ as follows.

$$\tilde{v}_i = \begin{cases} \max(v_i, v_i') & \text{if } i \le k(v') \\ v_i & \text{if } i > k(v') \end{cases}$$

$$\tilde{v}_i' = \begin{cases} \max(v_i, v_i') & \text{if } i \le k(v') \\ v_i' & \text{if } i > k(v') \end{cases}$$

By the strong-lattice property, $\tilde{v}, \tilde{v}' \in D$. Since $\tilde{v} \ge v$, by Claim 3, $k(\tilde{v}) \ge k(v) \ge k(v')$. Hence, $\tilde{v}_i = v_i$ for all $i > k(v')$ implies $\tilde{v}_i = v_i$ for all $i > \min(k(\tilde{v}), k(v))$. By Claim 2(ii), we have $q(v) = q(\tilde{v})$, i.e., $k(v) = k(\tilde{v})$, and $t(v) = t(\tilde{v})$.

Similarly, $\tilde{v}' \ge v'$, and Claim 3 imply $k(\tilde{v}') \ge k(v')$. Hence, $\tilde{v}_i' = v_i'$ for all $i > k(v')$ implies $\tilde{v}_i' = v_i'$ for all $i > \min(k(\tilde{v}'), k(v'))$. By Claim 2(ii), we have $q(v') = q(\tilde{v}')$, i.e., $k(v') = k(\tilde{v}')$, and $t(v') = t(\tilde{v}')$.

Since $\tilde{v} \ge v$, by UIC, $t(v) = t(\tilde{v})$. Similarly, $t(v') = t(\tilde{v}')$. Now, define $\hat{v}$ as follows:

$$\hat{v}_i = \min(\tilde{v}_i, \tilde{v}_i') \qquad \forall\, i \in \{1, \ldots, n\}$$

By the strong-lattice property, $\hat{v} \in D$. By Claim 3, $q(\hat{v}) \le q(v)$ and $q(\hat{v}) \le q(v')$. Hence, $k(\hat{v}) \le k(v') \le k(v)$. Applying UIC constraints $\tilde{v} \to \hat{v}$ and $\hat{v} \to \tilde{v}'$, and recalling that $k(v) = k(\tilde{v})$, $t(v) = t(\tilde{v})$, and $k(v') = k(\tilde{v}')$, $t(v') = t(\tilde{v}')$, we have

$$\sum_{i=1}^{k(\tilde{v})} \tilde{v}_i - t(\tilde{v}) = \sum_{i=1}^{k(v)} \tilde{v}_i - t(v) \ge \sum_{i=1}^{k(\hat{v})} \tilde{v}_i - t(\hat{v})$$
$$\sum_{i=1}^{k(\hat{v})} \hat{v}_i - t(\hat{v}) \ge \sum_{i=1}^{k(\tilde{v}')} \hat{v}_i - t(\tilde{v}') = \sum_{i=1}^{k(v')} \hat{v}_i - t(v')$$

Adding the two constraints, we get

$$\sum_{i=1}^{k(v)} \tilde{v}_i - t(v) \ge \sum_{i=1}^{k(\hat{v})} \tilde{v}_i + \sum_{i=1}^{k(v')} \hat{v}_i - \sum_{i=1}^{k(\hat{v})} \hat{v}_i - t(v') \tag{18}$$

If $k(\hat{v}) = k(v')$, then (18) reduces to

$$\sum_{i=1}^{k(v)} \tilde{v}_i - t(v) \ge \sum_{i=1}^{k(v')} \tilde{v}_i - t(v') \tag{19}$$

If, instead, $k(\hat{v}) < k(v')$, then (18) reduces to

$$\begin{aligned}\sum_{i=1}^{k(v)} \tilde{v}_i - t(v) &\geq \sum_{i=1}^{k(\hat{v})} \tilde{v}_i + \sum_{i=k(\hat{v})+1}^{k(v')} \hat{v}_i - t(v') \\ &= \sum_{i=1}^{k(\hat{v})} \tilde{v}_i + \sum_{i=k(\hat{v})+1}^{k(v')} \tilde{v}_i - t(v') \qquad (\text{since } \tilde{v}_i = \hat{v}_i \text{ for all } i \leq k(v')) \\ &= \sum_{i=1}^{k(v')} \tilde{v}_i - t(v')\end{aligned}$$

This is the same equation as (19).

Next, we show that (19) implies $v \to v'$. Since $k(v') < k(v)$, (19) reduces to

$$\sum_{i=k(v')+1}^{k(v)} \tilde{v}_i - t(v) \geq -t(v')$$

Using $\tilde{v}_i = v_i$ for all $i > k(v')$ and adding $\sum_{i=1}^{k(v')} v_i$ on both sides, we get

$$\sum_{i=1}^{k(v)} v_i - t(v) \geq \sum_{i=1}^{k(v')} v_i - t(v')$$

Hence, $v \to v'$. □

Claims 4 and 5 prove Theorem 3. ■

Proof of Corollary 1: Let (q, t) be a symmetric, rank-preserving, deterministic, object non-bossy, and UIC mechanism defined on D^{H}. Since (q, t) is rank preserving, (q, t) restricted to $D(\sigma^{\text{I}})$ defines a feasible, object non-bossy, and UIC mechanism for the identical-objects model. By Theorem 3, such a mechanism is IC. Hence, (q, t) restricted to $D(\sigma^{\text{I}})$ is an IC and rank-preserving mechanism. By Theorem 1, (q, t) is an IC mechanism. ■

Proof of Theorem 4: Suppose (q, t) is an optimal (pricing) mechanism with respect to prices $p_0 = 0, p_1, \ldots, p_n$. By Proposition 3, we know that such a mechanism exists.

Suppose this optimal mechanism satisfies $p_1 \leq \ldots \leq p_n$ with one strict inequality. WLOG choose an optimal mechanism with the least number of strict inequalities. Given prices $p_1, \ldots, p_n$, the regions where exactly k units are allocated must satisfy:

$$\sum_{i=1}^{k} v_i - \sum_{i=1}^{k} p_i \geq \sum_{i=1}^{j} v_i - \sum_{i=1}^{j} p_i \qquad \forall\ j \neq k$$

Hence, the closure of the set of types where exactly k units is sold is

$$R_k(p) = \{v \in D : \sum_{i=k+1}^{j} v_i \leq \sum_{i=k+1}^{j} p_i \text{ for all } j > k, \ \sum_{i=j+1}^{k} v_i \geq \sum_{i=j+1}^{k} p_i \text{ for all } j < k\}$$

If $v_k \geq p_k \geq p_{k-1} \geq \ldots \geq p_1$, then using $v_1 \geq v_2 \geq \ldots \geq v_k$, we have

$$\sum_{i=j+1}^{k} v_i \geq \sum_{i=j+1}^{k} p_i \qquad \forall j < k$$

Similarly, if $v_{k+1} \leq p_{k+1} \leq p_{k+2} \leq \ldots \leq p_n$, using $v_{k+1} \geq v_{k+2} \geq \ldots \geq v_n$, we have

$$\sum_{i=k+1}^{j} v_i \leq \sum_{i=k+1}^{j} p_i \qquad \forall j > k$$

Hence, $v \in R_k(p)$ if and only if $v_k \geq p_k$ and $v_{k+1} \leq p_{k+1}$. Thus, $R_k(p)$ is characterized as follows when $p_1 \leq p_2 \leq \ldots \leq p_n$:

$$R_k(p) = \{v \in D : v_{k+1} \leq p_{k+1}, v_k \geq p_k\}$$

Consequently, the closure of the set of types where *at least* k units are sold is

$$\begin{aligned} D_k(p) = \cup_{i=k}^{n} R_k(p) &= \{v \in D : v_k \geq p_k\} \\ \implies \qquad \Pr[v \in D_k(p)] &= 1 - F_k(p_k) \end{aligned}$$

where F_k is the marginal distribution of the value of the k^{th} unit. Hence, the expected revenue from the mechanism is

$$\textsc{Rev}(p) = \sum_{k=1}^{n} p_k(1 - F_k(p_k))$$

Suppose the inequalities $p_1 \leq p_2 \leq \ldots \leq p_n$ are strict at j, i.e., $p_j < p_{j+1}$. Consider two modifications of this pricing mechanism $\hat{p}$ and $\check{p}$, both with increasing prices, and the expected revenue they generate:

$$\begin{aligned} \hat{p} &= (p_0, p_1, \ldots, p_{j-1}, p_j, p_j, p_{j+2}, \ldots, p_n) \\ \textsc{Rev}(\hat{p}) &= \sum_{k \neq j+1} p_k(1 - F_k(p_k)) + p_j(1 - F_{j+1}(p_j)) \\ \check{p} &= (p_0, p_1, \ldots, p_{j-1}, p_{j+1}, p_{j+1}, p_{j+2}, \ldots, p_n) \end{aligned}$$

$$\textsc{Rev}(\check{p}) = \sum_{k \neq j} p_k(1 - F_k(p_k)) + p_{j+1}(1 - F_j(p_{j+1}))$$

By optimality,

$$\sum_{k=1}^{n} p_k(1 - F_k(p_k)) > \sum_{k \neq (j+1)} p_k(1 - F_k(p_k)) + p_j(1 - F_{j+1}(p_j))$$
$$\sum_{k=1}^{n} p_k(1 - F_k(p_k)) > \sum_{k \neq j} p_k(1 - F_k(p_k)) + p_{j+1}(1 - F_j(p_{j+1}))$$

These inequalities are strict since the new mechanisms have one less strict inequality than the original mechanism. The above inequalities are equivalent to

$$p_{j+1}(1 - F_{j+1}(p_{j+1})) > p_j(1 - F_{j+1}(p_j))$$
$$p_j(1 - F_j(p_j)) > p_{j+1}(1 - F_j(p_{j+1}))$$
$$\implies \quad \frac{1 - F_{j+1}(p_{j+1})}{1 - F_{j+1}(p_j)} > \frac{p_j}{p_{j+1}} > \frac{1 - F_j(p_{j+1})}{1 - F_j(p_j)}$$
$$\implies \quad \frac{1 - F_j(p_j)}{1 - F_{j+1}(p_j)} > \frac{1 - F_j(p_{j+1})}{1 - F_{j+1}(p_{j+1})}$$

But as $p_j < p_{j+1}$, this contradicts the assumption that F_j hazard rate dominates F_{j+1}. ■

Proof of Corollary 2: Let g be the (marginal) density of the value of each object in model $\mathcal{M}^{\text{H}}$. Define

$$f(v_1, v_2, \ldots, v_n) = n!\, g(v_1)g(v_2)\ldots g(v_n) \qquad 1 \geq v_1 \geq v_2 \geq \ldots v_n$$

An identical objects model, $\mathcal{M}^{\text{I}}$, with joint density f is equivalent to the heterogeneous objects model, $\mathcal{M}^{\text{H}}$, in the sense of Proposition 1.[29] Let F_k be the marginal distribution of the k^{th} unit. By Theorem 1.B.26 in Shaked and Shanthikumar (2007), F_k hazard-rate dominates F_{k+1}. Hence, by Theorem 4, in any optimal pricing mechanism p^* if $p_1^* \leq p_2^* \leq \ldots \leq p_n^*$ then $p_1^* = p_n^*$.

Let $(q^{\text{H}}, t^{\text{H}})$ be an optimal deterministic symmetric mechanism for model $\mathcal{M}^{\text{H}}$ with prices $p_0^* = 0, p_1^*, \ldots, p_n^*$, where price of bundles S is $\sum_{j \in S} p_j^*$. By Proposition 1, its restriction to $D(\sigma^{\text{I}})$ defines an IC and IR mechanism for model $\mathcal{M}^{\text{I}}$ – denote this mechanism as $(q^{\text{I}}, t^{\text{I}})$.

[29] The identical objects model is an ordered decreasing values model as defined in Bikhchandani and Mishra (2022).

By construction, the expected revenue of $(q^{\text{I}}, t^{\text{I}})$ with density f in model $\mathcal{M}^{\text{H}}$ is equal to the expected revenue of $(q^{\text{H}}, t^{\text{H}})$ with i.i.d. draws from g in model $\mathcal{M}^{\text{H}}$. By Proposition 1, $(q^{\text{I}}, t^{\text{I}})$ is an optimal mechanism for model $\mathcal{M}^{\text{I}}$. Also, this is a pricing mechanism with prices $p_0^* = 0, p_1^*, \ldots, p_n^*$. By Theorem 4, there is no optimal mechanism with prices $p_0^* = 0 \leq p_1^* \leq \ldots \leq p_n^*$ with one inequality strict. Thus, means $(q^{\text{H}}, t^{\text{H}})$ is not strictly supermodular. ■

Weak Majorization and Second-order Stochastic Dominance

In Lemma 4 below, we show the equivalence between weak majorization and second-order stochastic dominance of allocation rules in model $\mathcal{M}^{\text{I}}$.

Let $x \equiv (x_0, x_1, x_2, \ldots, x_n = 1)$ and $y \equiv (y_0, y_1, \ldots, y_n = 1)$ be two cumulative distribution functions (cdfs) over $\{0, 1, \ldots, n\}$. Note that since x and y are cdfs, $1 = x_n \geq x_{n-1} \geq \ldots \geq x_0$ and $1 = y_n \geq y_{n-1} \geq \ldots \geq y_0$.

Definition **14** *The cdf x* **second-order stochastically dominates (SOSD)** *cdf y, denoted $x \succ_{SOSD} y$, if*

$$\sum_{i=0}^{k} x_i \leq \sum_{i=0}^{k} y_i \qquad \forall\ k \in \{0, 1, \ldots, n\}$$

Take two allocation probability vectors $q, q' \in \mathcal{M}^{\text{I}}$. The pdf over $\{0, 1, \ldots, n\}$ induced by q is

$$(1 - q_1,\ q_1 - q_2,\ q_2 - q_3, \ldots, q_n)$$

and the cdf over $\{0, 1, \ldots, n\}$

$$F(q) := (1 - q_1,\ 1 - q_2,\ 1 - q_3, \ldots, 1)$$

Lemma **4** *The following are equivalent for any pair of allocation probability vectors $q, q' \in \mathcal{M}^{I}$.*

$$\left[q \succ_w q'\right] \Longleftrightarrow \left[F(q) \succ_{SOSD} F(q')\right]$$

Proof:

$$q \succ_w q' \Leftrightarrow \sum_{i=1}^{k} q_k \geq \sum_{i=1}^{k} q_i' \quad \forall\ k \in \{1, \ldots, n\}$$

$$\iff \quad \sum_{i=0}^{k}\big(1-F_i(q)\big) \geq \sum_{i=0}^{k}\big(1-F_i(q')\big) \quad \forall\ k \in \{0,\dots,n\}$$

$$\iff \quad \sum_{i=0}^{k} F_i(q) \leq \sum_{i=0}^{k} F_i(q') \quad \forall\ k \in \{0,\dots,n\}$$

$$\iff \quad F(q) \succ_{SOSD} F(q') \qquad \blacksquare$$

Proof of Proposition 4: We provide a proof (i) for all $v, \hat{v} \in \overline{D}^{\text{I}}$ and (ii) for all $v, \hat{v} \in D^{\text{H}}$. Thus, (8) is satisfied for all $v, \hat{v} \in \mathcal{M}^{\text{I}}$ and almost all $v, \hat{v} \in \mathcal{M}^{\text{H}}$.

Take $v, \hat{v} \in \overline{D}^{\text{I}}$. By IC,

$$t(\hat{v}) - t(v) \geq \sum_{j=1}^{n} v_j q_j(\hat{v}) - \sum_{j=1}^{n} v_j q_j(v) \tag{20}$$

Let $\Delta_j(v) := v_j - v_{j+1}$ for all $j \in \{1,\dots,n\}$, where $v_{n+1} := 0$. As $\hat{v}, v \in \overline{D}^{\text{I}}$, $\Delta_j(v) \geq 0$ for all j. So,

$$\sum_{j=1}^{n} v_j q_j(\hat{v}) = \sum_{j=1}^{n} q_j(\hat{v})\Big(\sum_{k=j}^{n}\Delta_k(v)\Big) = \sum_{k=1}^{n}\Delta_k(v)\Big(\sum_{j=1}^{k} q_j(\hat{v})\Big) \tag{21}$$

Using (20) with (21), we have

$$t(\hat{v}) - t(v) \geq \sum_{k=1}^{n}\Delta_k(v)\sum_{j=1}^{k}\big(q_j(\hat{v}) - q_j(v)\big) \tag{22}$$

If $q(\hat{v}) \succ_w q(v)$, then the RHS of (22) is non-negative. As a result, $t(\hat{v}) \geq t(v)$. This completes the proof for (q,t) defined on $\overline{D}^{\text{I}}$.

Next, consider (q,t) defined on domain D^{H}. As (q,t) is symmetric, IC and IR, Theorem 1 implies that it is rank preserving. Thus, (q,t) on D^{H} is the symmetric extension of (q,t) on $D(\sigma^{\text{I}}) = D^{\text{I}}$. For any $\acute{v} \in D(\acute{\sigma})$, $\grave{v} \in D(\grave{\sigma})$, we have $\acute{v}^{\acute{\sigma}}, \grave{v}^{\grave{\sigma}} \in D(\sigma^{\text{I}})$. By symmetry, $t(\acute{v}) = t(\acute{v}^{\acute{\sigma}})$, $t(\grave{v}) = t(\grave{v}^{\grave{\sigma}})$, $q^{\acute{\sigma}}(\acute{v}) = q(\acute{v}^{\acute{\sigma}})$, and $q^{\grave{\sigma}}(\grave{v}) = q(\grave{v}^{\grave{\sigma}})$. As weak majorization is invariant to permutations of vectors, and (q,t) is symmetric,

$$q(\grave{v}) \succ_w q(\acute{v}) \quad \iff \quad q^{\grave{\sigma}}(\grave{v}) \succ_w q^{\acute{\sigma}}(\acute{v}) \quad \iff \quad q(\grave{v}^{\grave{\sigma}}) \succ_w q(\acute{v}^{\acute{\sigma}})$$

Thus, the fact that (8) for holds types in $D(\sigma^{\text{I}})$ implies that (8) holds for D^{H}. $\blacksquare$

Proof of Theorem 5:

(a) Let (q,t) be an IC and IR mechanism that satisfies majorization monotonicity in model $\mathcal{M}^{\text{I}}$. By Proposition 2, there exists a non-bossy mechanism $(q^\sharp(v), t^\sharp(v))$ such that $(q(v), t(v)) = (q^\sharp(v), t^\sharp(v))$ almost everywhere. To be precise, there exists a set $\check{D}^{\text{I}} \subseteq D^{\text{I}}$, where $D^{\text{I}} \setminus \check{D}^{\text{I}}$ has zero measure and $(q(v), t(v)) = (q^\sharp(v), t^\sharp(v))$ for all $v \in \check{D}^{\text{I}}$. Also, $\text{Rev}(q, t; f^{\text{I}}) = \text{Rev}(q^\sharp, t^\sharp; f^{\text{I}})$. Moreover, $(q^\sharp, t^\sharp)$ satisfies majorization monotonicity on the set $\check{D}^{\text{I}}$ and is rank preserving.

Let $(v_i, v_{-i}), (\hat{v}_i, v_{-i}) \in \check{D}^{\text{I}}$, with $\hat{v}_i > v_i$. By IC, $q_i^\sharp(\hat{v}_i, v_{-i}) \geq q_i^\sharp(v_i, v_{-i})$. If $q_i^\sharp(\hat{v}_i, v_{-i}) = q_i^\sharp(v_i, v_{-i})$, then object non-bossiness implies $q^\sharp(v_i, v_{-i}) = q^\sharp(\hat{v}_i, v_{-i})$, and IC implies $t^\sharp(v_i, v_{-i}) = t^\sharp(\hat{v}_i, v_{-i})$. If, instead, $q_i^\sharp(\hat{v}_i, v_{-i}) > q_i^\sharp(v_i, v_{-i})$, then majorization monotonicity of $(q^\sharp, t^\sharp)$ on $\check{D}^{\text{I}}$ implies that $q^\sharp(\hat{v}_i, v_{-i}) \succ_w q^\sharp(v_i, v_{-i})$. By Proposition 4, we have $t^\sharp(\hat{v}_i, v_{-i}) \geq t^\sharp(v_i, v_{-i})$. Thus, for all $(v_i, v_{-i}), (\hat{v}_i, v_{-i}) \in \check{D}^{\text{I}}$, we have $\hat{v}_i > v_i$ implies $t^\sharp(\hat{v}_i, v_{-i}) \geq t^\sharp(v_i, v_{-i})$. This in turn implies that for all $\hat{v}, v \in \check{D}^{\text{I}}$, if $\hat{v} > v$ then $t^\sharp(\hat{v}) \geq t^\sharp(v)$. Since $D^{\text{I}} \setminus \check{D}^{\text{I}}$ has zero measure, for any pair of distributions F^{I} with density f^{I} and $\widetilde{F}^{\text{I}}$ with density $\tilde{f}^{\text{I}}$ such that $\widetilde{F}^{\text{I}}$ first-order stochastically dominates F^{I}, we have

$$\begin{aligned}\text{Rev}(q, t; F^{\text{I}}) = \text{Rev}(q^\sharp, t^\sharp; F^{\text{I}}) = \int_{D^{\text{I}}} t^\sharp(v) f^{\text{I}}(v) dv &\leq \int_{D^{\text{I}}} t^\sharp(v) \tilde{f}^{\text{I}}(v) dv \\ &= \text{Rev}(q^\sharp, t^\sharp; \widetilde{F}^{\text{I}}) = \text{Rev}(q, t; \widetilde{F}^{\text{I}})\end{aligned}$$

This establishes revenue monotonicity of (q,t).

Next, let (q,t) be a symmetric, IC, and IR mechanism that satisfies majorization monotonicity in model $\mathcal{M}^{\text{H}}$. By Theorem 1, its restriction to $D(\sigma^{\text{I}})$ defines an IC, IR, and rank-preserving mechanism; clearly, it satisfies majorization monotonicity. Hence, this is a mechanism for model $\mathcal{M}^{\text{I}}$ on type space D^{I}. By the argument above, there exists another IC and IR mechanism $(q^\sharp, t^\sharp)$ which coincides with the restriction of (q,t) to $D(\sigma^{\text{I}})$ almost everywhere and is object non-bossy. By our earlier argument, majorization monotonicity implies that if $\hat{v} \geq v$ and $\hat{v}, v \in D(\sigma^{\text{I}})$, then $t^\sharp(\hat{v}) \geq t^\sharp(v)$. Consequently, $t(\hat{v}) \geq t(v)$ for almost all $\hat{v}, v \in D(\sigma^{\text{I}})$. Since (q,t) is a symmetric mechanism, for almost all $v, \hat{v} \in \overline{D}^{\text{H}}$, with $\hat{v} \geq v$, we have $t(\hat{v}) \geq t(v)$. This in turn implies revenue monotonicity.

(b) In model $\mathcal{M}^{\text{I}}$, let (q,t) be an IC and IR mechanism which is almost deterministic. WLOG, we assume that q is non-bossy.[30] Let $(\hat{v}_i, v_{-i})$ and (v_i, v_{-i}) be two type profiles with $\hat{v}_i > v_i$. By IC, $q_i(\hat{v}_i, v_{-i}) \geq q_i(v_i, v_{-i})$. By non-bossiness, if $q_i(\hat{v}_i, v_{-i}) = q_i(v_i, v_{-i})$,

[30]If (q,t) is bossy then by Proposition 2, there exists another mechanism $(q^\sharp, t^\sharp)$ which is non-bossy, almost

we have $q(\hat{v}_i, v_{-i}) = q(v_i, v_{-i})$. Suppose, instead, that $q_i(\hat{v}_i, v_{-i}) > q_i(v_i, v_{-i})$. Since q is almost deterministic, for all $k < i$, $q_k(\hat{v}_i, v_{-i}) = 1 \geq q_k(v_i, v_{-i})$. Further, for all $k > i$, $q_k(v_i, v_{-i}) = 0 \leq q_k(\hat{v}_i, v_{-i})$. Hence, $q(\hat{v}_i, v_{-i}) \succ_w q(v_i, v_{-i})$. This shows that q satisfies majorization monotonicity. From part (a), (q, t) is revenue monotone.

The proof for model $\mathcal{M}^{\text{H}}$ is similar to the proof of part (a). ■

deterministic, and rank preserving which agrees with (q, t) except on a set of measure zero (just as in the proof of Theorem 5). Hence, the expected revenue from $(q^\sharp, t^\sharp)$ and (q, t) is equal, and (q, t) is revenue monotone if and only if $(q^\sharp, t^\sharp)$ is revenue monotone.

References

Armstrong, M. (2000): "Optimal Multi-object Auctions," *The Review of Economic Studies*, 67, 455–481.

Babaioff, M., N. Nisan, and A. Rubinstein (2018): "Optimal Deterministic Mechanisms for an Additive Buyer," in *Proceedings of the 2018 ACM Conference on Economics and Computation*, 429–429.

Ben Moshe, R., S. Hart, and N. Nisan (2022): "Monotonic Mechanisms for Selling Multiple Goods," ArXiv preprint arXiv:2210.17150.

Bikhchandani, S. and D. Mishra (2022): "Selling Two Identical Objects," *Journal of Economic Theory*, 200.

Carroll, G. (2012): "When are local incentive constraints sufficient?" *Econometrica*, 80, 661–686.

——— (2017): "Robustness and separation in multidimensional screening," *Econometrica*, 85, 453–488.

Devanur, N. R., N. Haghpanah, and A. Psomas (2020): "Optimal Multi-unit Mechanisms with Private Demands," *Games and Economic Behavior*, 121, 482–505.

Hardy, G. H., J. E. Littlewood, and G. Pólya (1952): *Inequalities*, Cambridge University Press.

Hart, S. and N. Nisan (2017): "Approximate Revenue Maximization with Multiple Items," *Journal of Economic Theory*, 172, 313–347.

Hart, S. and P. J. Reny (2015): "Maximal Revenue with Multiple Goods: Nonmonotonicity and other Observations," *Theoretical Economics*, 10, 893–922.

Kleiner, A., B. Moldovanu, and P. Strack (2021): "Extreme points and majorization: Economic applications," *Econometrica*, 89, 1557–1593.

Kumar, U. and S. Roy (2021): "Local incentive compatibility in non-convex type-spaces," Tech. rep., University Library of Munich, Germany.

Malakhov, A. and R. V. Vohra (2009): "An Optimal Auction for Capacity Constrained Bidders: A Network Perspective," *Economic Theory*, 39, 113–128.

Manelli, A. M. and D. R. Vincent (2006): "Bundling as an Optimal Selling Mechanism for a Multiple-good Monopolist," *Journal of Economic Theory*, 127, 1–35.

Maskin, E. and J. Riley (1984): "Optimal auctions with risk averse buyers," *Econometrica*, 1473–1518.

McAfee, R. P. and J. McMillan (1988): "Multidimensional Incentive Compatibility and Mechanism Design," *Journal of Economic Theory*, 46, 335–354.

Mishra, D., A. Pramanik, and S. Roy (2016): "Local incentive compatibility with transfers," *Games and Economic Behavior*, 100, 149–165.

Pavlov, G. (2011a): "Optimal Mechanism for Selling Two Goods," *The BE Journal of Theoretical Economics*, 11.

——— (2011b): "A Property of Solutions to Linear Monopoly Problems," *The BE Journal of Theoretical Economics*, 11.

Satterthwaite, M. A. and H. Sonnenschein (1981): "Strategy-proof allocation mechanisms at differentiable points," *The Review of Economic Studies*, 48, 587–597.

Shaked, M. and J. G. Shanthikumar (2007): *Stochastic Orders*, Springer.

Appendix B Online Appendix

Proof of Proposition 3: Let (q, t) be a deterministic, IC, and IR mechanism. Let the range of q be $R(q) := \{k(v) : v \in D^{\mathrm{I}}\}$. By IC, if $k(v) = k(v')$, we have $t(v) = t(v')$. For every $k \in R(q)$, define $P(k) := t(v)$ for some $v \in D^{\mathrm{I}}$ with $k(v) = k$.

We now define P for every $k \notin R(q)$. If $k = 0 \notin R(q)$, let $P(0) := 0$. Consider $k \geq 1$, $k \notin R(q)$. For every $k' \in R(q)$, define[31]

$$d(k', k) := \sup_{v \in D: k(v) = k'} \Big[\sum_{i=0}^{k} v_i - \sum_{i=0}^{k'} v_i \Big] \tag{23}$$

and

$$P(k) := \max_{k' \in R(q)} \Big[P(k') + d(k', k) \Big] \tag{24}$$

This completes the definition of P. We now proceed in several steps.

Step 1. We show that $P(0) = 0$. If $0 \notin R(q)$, we have $P(0) = 0$ by definition. If $0 \in R(q)$, then IC implies that $q(\underline{v}) = 0$. Thus, $0 = u(\underline{v}) = -P(0)$ implies that $P(0) = 0$.

Step 2. We show the IC constraints hold with respect to the prices P, i.e., for all $v \in D^{\mathrm{I}}$,

$$u(v) \geq \sum_{i=0}^{k'} v_i - P(k') \qquad \forall\ k' \in \{0, 1, \ldots, n\} \tag{25}$$

If $k' = 0$, then the above inequality holds because $P(0) = 0$ and IR of (q, t). Consider $k' \geq 1$. If $k' \in R(q)$, then (25) follows from IC of (q, t) and the definition of P. Suppose, instead, that $k' \notin R(q)$. Take any $v \in D^{\mathrm{I}}$ and suppose that $k(v) = k \in R(q)$. Then, by the definition of $P(k')$, we see that

$$\begin{aligned} P(k') \geq P(k) + d(k, k') \geq P(k) + \sum_{i=1}^{k'} v_i - \sum_{i=1}^{k} v_i \\ \Longrightarrow \qquad u(v) = \sum_{i=1}^{k} v_i - P(k) \geq \sum_{i=1}^{k'} v_i - P(k') \end{aligned}$$

[31] We use the convention that $v_0 \equiv 0$.

Step 3. Next, we show that P is monotone. Pick $k' > k$. If $k \in R(q)$, by IC constraint (25) for some $v \in D^{\mathrm{I}}$ with $k(v) = k$, we have

$$P(k') - P(k) \geq \sum_{i=0}^{k'} v_i - \sum_{i=0}^{k} v_i \geq 0$$

If $k \notin R(q)$, we consider two cases. If $k = 0$, then by IC constraint (25) of type $\underline{v}$, $u(\underline{v}) = 0 \geq \sum_{i=0}^{k'} \underline{v}_i - P(k')$. Hence, $P(k') \geq 0 = P(0)$, where $P(0) = 0$ follows from Step 1.

If $k \notin R(q)$ and $k \geq 1$, from (24) we know that there exists a $k'' \in R(q)$ such that

$$P(k) = P(k'') + d(k'', k) \tag{26}$$

Pick any type v such that $k(v) = k''$. IC constraint (25) implies

$$\begin{aligned} & \sum_{i=0}^{k''} v_i - P(k'') \geq \sum_{i=0}^{k'} v_i - P(k') \\ \iff \quad & P(k') - P(k'') \geq \sum_{i=0}^{k'} v_i - \sum_{i=0}^{k''} v_i = \sum_{i=0}^{k'} v_i - \sum_{i=0}^{k} v_i + \sum_{i=0}^{k} v_i - \sum_{i=0}^{k''} v_i \geq \sum_{i=0}^{k} v_i - \sum_{i=0}^{k''} v_i \end{aligned}$$

where the last inequality is implied by $k' > k$. As this holds for all v with $k(v) = k''$, we get

$$P(k') - P(k'') \geq \sup_{v:k(v)=k''} \left[\sum_{i=0}^{k} v_i - \sum_{i=0}^{k''} v_i \right] = d(k'', k) = P(k) - P(k'')$$

where the last equality follows from (26). Hence, we get $P(k') \geq P(k)$. Thus, each $p_k \geq 0$.

Step 4. Next, we show that $p_k \leq \overline{v}$ for all $k \geq 1$. If $k \in R(q)$, by IC constraints (25), for some $v \in D$ with $k(v) = k$, we have

$$p_k = P(k) - P(k-1) \leq \sum_{i=1}^{k} v_i - \sum_{i=1}^{k-1} v_i = v_k \leq \overline{v}$$

If $k \notin R(q)$, by definition of $P(k)$, there is k' such that

$$P(k) = P(k') + d(k', k) \tag{27}$$

By IC constraints (25), for any v with $k(v) = k'$, we have

$$\sum_{i=1}^{k'} v_i - P(k') \geq \sum_{i=1}^{k-1} v_i - P(k-1)$$

$$\begin{aligned}
\iff \quad P(k') - P(k-1) &\leq \sum_{i=1}^{k'} v_i - \sum_{i=1}^{k-1} v_i \\
&= \sum_{i=1}^{k'} v_i - \sum_{i=1}^{k} v_i + \sum_{i=1}^{k} v_i - \sum_{i=1}^{k-1} v_i \\
&= \sum_{i=1}^{k'} v_i - \sum_{i=1}^{k} v_i + v_k \\
&\leq \sum_{i=1}^{k'} v_i - \sum_{i=1}^{k} v_i + \overline{v}
\end{aligned}$$

Thus,

$$\overline{v} + P(k-1) \geq P(k') + \sup_{v:k(v)=k'} \Big[\sum_{i=1}^{k} v_i - \sum_{i=1}^{k'} v_i \Big] = P(k') + d(k', k) = P(k)$$

where the last equality is from (27). Hence, we get $p_k = P(k) - P(k-1) \leq \overline{v}$. ■