Information Design in Allocation with Costly Verification

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Abstract

A principal who values an object allocates it to one or more agents. Agents always want the object. Agents hold private signals about the allocation payoff to the principal; and an information designer can influence agents' signal distributions. Based upon the designed information, the principal designs a mechanism to maximizes her payoff; monetary transfer is not available but the principal can verify any agent's private signal at a cost. We find the agent-optimal information and the principal-optimal information. Moreover, with a single agent, any agent-optimal information is principal-worst. The latter facilitates the discovery of optimal robust mechanisms in general settings.

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Contents

1 Introduction

Example 1: Consider an asset management firm which chooses a drug company from a set to fund, for launching a new plant. Every candidate drug company wants the investment. The asset management firm cares about drug efficacy. Drug efficacy is reflected in trials data; and the format and informativeness of trials data are influenced by a drug regulator through clinical trials guidances.[2] The asset management firm needs to make the investment decision based upon its ex ante understanding about drug companies as well as detailed information reported in their prospectuses, where the latter is possibly misreported and is costly to verify.

Example 2: Consider a city government which has an indivisible plot of land to allocate among promising enterprises. Every candidate company wants the land. The city government cares about the ESG (i.e., Environmental, Social and Governance) features and restrict attention to listed companies. Those ESG features are private information of listed companies; and the format and informativeness of such information is influenced by the exchange where the companies are listed, through information disclosure rules, auditing requirements and etc.[3] The city government needs to make the allocation decision based upon its ex ante understanding about candidate companies as well as detailed information

[2] The US Food and Drug Administration (FDA) provides detailed guidance documents for clinical trials; see https://www.fda.gov/regulatory-information/search-fda-guidance-documents/clinical-trials-guidance-documents (accessed on July 5, 2023).

[3] Shanghai Stock Exchange revised its stock listing rules in January 2022, which incorporates new auditing requirements and requires companies to disclosure finer ESG information. See http://www.sse.com.cn/lawandrules/sselawsrules/repeal/rules/c/c_20230217_5716486.shtml and http://www.sse.com.cn/lawandrules/sselawsrules/repeal/rules/c/c_20220107_5679292.shtml for the 2022 version and the 2020 version, respectively (accessed both on July 5, 2023).

reported in their application documents, where the latter is possibly misreported and is costly to verify.

The problem above can be formulated as mechanism design with prior information design. Particularly, a principal who values an object allocates it to one or more agents. Agents always want the object. They hold private signals about the allocation payoff to the principal; the information designer can influence the agents' signal distributions.[4] The principal maximizes her payoff based upon signal distributions as well as signals reported by agents. Monetary transfer is not available but the principal can verify any agent's private signal at a cost (Ben-Porath et al., 2014).

In this environment, the information designer (regulator/exchange) is a third party other than the principal (asset management firm/city government) and the agents (drug companies/listed companies). Moreover, the designing of information that would be feasible in the allocation problem is orthogonal to how the allocation is done and whether the principal verifies the information presented by agents or not. Those two features follow from the perspective of Roesler and Szentes (2017), where information design is orthogonal to how the seller designs a mechanism to sell her good to the buyer.

The information designer may either favor the principal or favor the agents. To

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[4] Information designers that are similar to drug regulators include stock exchanges such as Nasdaq (which influence feasible information about listed companies through auditing requirements and information disclosure rules), China Accounting Standards Committee, or CASC (which influence feasible financial information about companies through accounting standards design), testing agencies such as Educational Testing Service, or ETS (which influence feasible information about students' language/reasoning ability through subjects and difficulty design), colleges and universities (which influence feasible information about their students' performance through transcript design and Grade Point Average, or GPA calculating rules), trading platforms such as Amazon (which influence feasible information about sellers' products through comments disclosing rules), and etc.

analyze different objectives, we first consider the case with a single agent which can already show many key insights. In this case, we characterize the agent-optimal information and the principal-optimal information.[5] For concrete examples, an agent-optimal information pools information above a cutoff signal and fully reveals information below the cutoff. To see the intuition, we take the efficient allocation rule as a benchmark (i.e., the agent gets the good whenever his signal is greater than the principal's reservation value).[6] Pooling information at the top allows the agent to obtain the good even if his signal falls below the principal's reservation value, which thus favors the agent. Moreover, information design beneath the cutoff signal does not affect agent-optimality, leading to multiple solutions. In contrast to the agent-optimal information, making the signal distribution the most informative is principal-optimal. Intuitively, finner signal distributions enhance the principal's flexibility in designing mechanisms, which thus favors the principal.

Technically, some features of the agent-optimal information appear in the general characterization results of Dworczak and Martini (2019) and Kleiner et al. (2021): some agent-optimal information partitions signals into intervals and is either pooling or fully revealing for different intervals. However, unlike their characterizations without explicit solutions, we fully characterize the solutions in a concrete allocation context. Our direct and elementary proof also facilitates economic interpretations and the study of other research questions, such as the one we discuss now.

By characterizing the agent-optimal information as well as the principal-worst infor-

[5] We use two terms "information" and "signal distribution" interchangeably.

[6] It is legitimate to compare the agent's signal with the principal's reservation value because it is without loss of generality to focus on signals that are unbiased estimators of allocation payoffs to the principal; see Roesler and Szentes (2017).

mation[7], we find that any agent-optimal information is principal-worst. This implication is somewhat surprising for two reasons: First, the objectives of the principal and the agent are not directly comparable; namely, the principal maximizes her net allocation payoff whereas the agent maximizes the probability of obtaining the good. Second, even if we spell out the intensity of the agents' preferences, without monetary transfer the model lacks a natural link between the principal's payoff and the agents' payoffs. Actually, the two parties' interests are even overlapped, despite the opposition in information-design solutions.

When there are more agents, while making the signal distribution the most informative remains to be principal-optimal, agent-optimal informations exhibit new features. Particularly, an agent-optimal information maximizes the total probability of agents' obtaining the good.[8] Compared with the information designer's prior distribution, under some agent-optimal information, all agents can be better off; while under some other agent-optimal information, some agents get worse off. Moreover, agent-optimal informations may deliver different payoffs to the principal, which implies that an agent-optimal information needs *not* be principal-worst.

The principal's payoff under a principal-worst information provides an upper bound for the payoff that can be achieved by a "robust" mechanism which does not depend on details of the agent's type distribution. We find a robust mechanism that does achieve such an upper bound payoff, which is therefore an optimal robust mechanism. More precisely, the optimal robust mechanism simply ignores agents' reported signals and allocates the good to the agent who has the highest expected signal.

[7] Designing information to hurt the principal, although itself lacks motivation, facilitates the study of robust mechanism design (Brooks and Du, 2021); see Section 5.3 for detailed analysis.

[8] This is only *one* way of defining agent optimality; see Section 5.2 for conceptual discussions.

This result is more general than it appears in at least three aspects. First, in similar models of allocation without transfer (Mylovanov and Zapechelnyuk, 2017; Li, 2020), the aforementioned simple mechanism is still an optimal robust mechanism regardless of incentive providing tools in different models, because the mechanism circumvents the incentive issue. Second, it is straightforward to extend the optimal robust mechanism to the multiple-unit setting (Ben-Porath et al., 2019; Chua et al., 2023): it is robustly optimal to allocate n goods to the n agents who have the n highest expected signals. Finally, allowing for the information designer to design correlated signal distributions does not affect the result.

The rest of this section reviews the literature. In Section 2, we set out the model, the allocation problem and the associated information design problems. In Section 3, we present a solution to the benchmark allocation problem, which we attribute to Ben-Porath et al. (2014). Section 4 characterizes the agent-optimal information and the principal-optimal information, and explores the implications of those characterizations. In Section 5, we extend our model to allow for multiple agents and study the robust mechanism design problem from an information design perspective. Section 6 concludes. All omitted proofs are in Appendix A.

1.1 Related literature

Our paper is related to three streams of literature: allocation with costly verification, information design, and robust mechanism design.

The literature on costly state verification is initiated by Townsend (1979) which studies optimal debt contracts; see also Gale and Hellwig (1985) and Mookherjee and Png (1989). Unlike those earlier papers, Ben-Porath et al. (2014) studies the role of costly verification in allocation problems *without* monetary transfer. Since then, their model has been modified or extended to different directions; see, e.g., Mylovanov and Zapechelnyuk

(2017), Halac and Yared (2020), Erlanson and Kleiner (2020), Li (2020), Epitropou and Vohra (2019), Kattwinkel and Knoepfle (2023), and Chua et al. (2023), among many others.[9] We depart from Ben-Porath et al. (2014) by introducing an information designer who can influence the prevailing information in the allocation problem.

We formulate the information design problems à la Roesler and Szentes (2017) in allocation problems with costly verification. Although the analysis of information design advances rapidly in the auction literature (see, e.g., Bergemann et al. 2017, Yang 2019, Yang 2021, Chen and Yang 2020) and in other areas (see Bergemann and Morris 2019 for a comprehensive survey), its counterpart in the aforementioned allocation setting is rarely studied. One exception is Kattwinkel and Knoepfle (2023), which shows in a single-agent model that a principal who privately observes a signal correlated with the agent's type does not profit from persuading the agent to reveal his information with any form of information design. Information design in their paper pertains to the disclosure of the principal's private signal to the agent who knows his own type. However, in our setting, there is an independent information designer who designs the entire uncertainty in the underlying allocation problem, perceived by both the principal and the agent.[10]

Our result that any agent-optimal information is principal-worst resembles the equivalence between the agent-optimal and the principal-worst information in Roesler and Szentes (2017). However, both the problem formulations and the results are different.

[9] Ben-Porath et al. (2019) studies a general model of mechanism design with evidence, which differs significantly from the allocation problem of Ben-Porath et al. (2014). However, it turns out that the results in their 2019 paper can be used to solve allocation problems in, say, their 2014 paper, Erlanson and Kleiner (2020) and Chua et al. (2023).

[10] Another difference is that they focus on the correlation between the principal's signal and the agent's type, whereas correlation is not our focus. Nevertheless, both their results and ours convey the (loosely) similar idea that making information more precise is better for the principal.

In Roesler and Szentes (2017), the buyer and the seller are roughly playing a zero-sum game: the monetary surplus of trading is split between them and payoffs are perfectly transferable. Yet in our paper, the objectives of the principal and the agent are not directly comparable as we discussed in the introduction. Moreover, the two parties' interests are even overlapped, which makes the opposition in information-design solutions somewhat surprising. In terms of results, Roesler and Szentes (2017) offers an equivalence as a consequence of the formulation, whereas our connection between the agent-optimal and principal-worst information is only one directional: some principal-worst information distributions are not agent-optimal.

Our exercise of robust mechanism design follows the convention of, e.g., Bergemann et al. (2016), Du (2018), Koçyiğit et al. (2020), Brooks and Du (2021) and He and Li (2022). Focusing on the allocation problem of Ben-Porath et al. (2014), our paper is most closely related to Bayrak et al. (2017) and Bayrak et al. (2022). Instead of solving the principal-worst information design problem first, as we do, Bayrak et al. (2017) directly works on the mechanism design problem with uncertain type distributions. Particularly, they adopt the linear programming approach initiated by Vohra (2012), which necessitates the assumption of discrete (and finitely many) types that we do not impose.[11] Contemporary with our paper, Bayrak et al. (2022) studies a similar robust mechanism design problem with respect to two sets of distributions, where the second is more related to our analysis. The Markov ambiguity set, as they call it, is parametrized by the lower and upper bounds on the

[11] Our paper also differs from Bayrak et al. (2017) in the following two aspects: First, they assume i.i.d. distributions of agent types whereas we allow for arbitrary asymmetry among agents; see Section 5. Second, their robust mechanism is with respect to a set of distributions that is notably different from ours, i.e., they either assume first-order stochastic dominance within the set of possible distributions or restrict attention to a binary set, whereas we work with a general class of sets of distributions that have the same mean without imposing other structures.

expected type for each agent. When the lower and upper bounds coincide for all agents, this special case of the general Markov ambiguity set coincides with a special case of our general set of possible distributions. Therefore, in terms of robust mechanism design, the two papers complement each other in both methods and results.

2 The model

2.1 Preliminaries

We slightly modify the setup of Ben-Porath et al. (2014) first, to open up a window for information design. Namely, a principal *who values an indivisible good* is to decide whether or not to allocate it to *an* agent; the case of multiple agents will be studied in Section 5.

The net value to the principal of allocating the good is $t-R$, where R is the principal's reserve value and t is the agent's type. R is fixed and is common knowledge among the agents and the principal. In contrast, there is an underlying distribution for t that is over the interval $T := [\underline{t}, \bar{t}]$, where $0 \leq \underline{t} < R < \bar{t} < \infty$. We denote by F the distribution function of t and f the density; assume $f(t) > 0$ for all $t \in T$. Unlike in Ben-Porath et al. (2014), we assume that neither the principal nor the agent knows the distribution F or its realization t, but they may receive related information from an *information designer*.

Following Roesler and Szentes (2017), there is an information designer who can influence the "private information" that the agent receives and the "distributional information" that the principal perceives. More formally, the information designer knows the underlying distribution F (but not its realization). She can design a statistical experiment which reveals some information about t to the agent and to the principal. Generally, this is through a joint distribution between t and another random variable, which is called "signal" and denoted by s, subject to the constraint that the marginal distribution of t is still F. The agent privately learns the realized signal s and the principal only learns the distribution of s.

According to Roesler and Szentes (2017), without loss of generality, we restrict attention to *unbiased signals* such that $\mathbb{E}(t|s) = s$, where s naturally belongs to T. Furthermore, the payoffs of both the principal and the agent are determined by the marginal distribution of the signal s (since $t - R$ is linear in t and signals are unbiased estimations of t). Thus, we further restrict our attention to the *marginal distributions of unbiased signals*. Let $\mathcal{G}$ be the set of all cumulative distribution functions defined over T and $\mathcal{G}_F \subseteq \mathcal{G}$ the set of marginal distributions of unbiased signals. Based upon the characterization of Blackwell (1953), the set $\mathcal{G}_F$ is exactly the set of *mean-preserving contractions (MPC)* of F, i.e.

$$\mathcal{G}_F = \left\{ G \in \mathcal{G} : \int_{\underline{t}}^{x} G(s)\mathrm{d}s \leq \int_{\underline{t}}^{x} F(t)\mathrm{d}t \quad \forall x \in T \quad \text{and} \quad \int_{\underline{t}}^{\bar{t}} G(s)\mathrm{d}s = \int_{\underline{t}}^{\bar{t}} F(t)\mathrm{d}t \right\}.$$

For example, $F \in \mathcal{G}_F$, which is often referred to as the *full information* distribution. Let $\mu = \mathbb{E}(t)$. Then the degenerate distribution $G = \delta(\mu)$ that assigns probability one to the atom $s = \mu$ is also in $\mathcal{G}_F$; we refer to $\delta(\mu)$ as the *null information*. In what follows, the information designer has the flexibility to choose a marginal distribution from $\mathcal{G}_F$.

The principal can *check* the agent's private signal s at a cost $c > 0$. We interpret checking as obtaining information (e.g. by requesting documentation, interviewing the agent, or hiring outside evaluators) which perfectly reveals the signal of the agent. The cost to the agent of providing information is assumed to be zero. To avoid the trivial case, let us assume $R + c \leq \max_{s \in T} s = \bar{t}$; otherwise the principal never checks the agent.

We assume that the agent strictly prefers receiving the good to not receiving it. Consequently, we can take the agent's payoff to be the probability with which he receives the good. The intensity of the agent's preference plays no role in the analysis, so it is omitted. We also assume that the agent's reservation utility is less than or equal to his utility from not receiving the good. Since monetary transfers are not allowed, not receiving a good delivers the worst payoff to an agent. Consequently, individual rationality constraints do not bind and so are disregarded throughout.

2.2 Direct mechanisms

The mechanism design part is identical to that of Ben-Porath et al. (2014), except that the type t is replaced by signal s. A general mechanism in this setting is a game that specifies (i) what messages the agent can report, (ii) which of those messages lead to checking (and with which probability), and (iii) depending on the reported message and the checking outcome, whether the agent receives the good (and with which probability). We can apply the same argument as in the online appendix of Ben-Porath et al. (2014), up to terminology replacement, to establish a revelation principle and to obtain the following two necessary conditions for any direct mechanism to be optimal:

1. If the agent is checked and found lying, then the conditional probability of him receiving the good has to be zero.
2. If the agent is checked and found truth-telling, then the conditional probability of him receiving the good has to be one.

Hence, we restrict our attention to the "(simplified) direct mechanisms."

Formally, a *direct mechanism* consists of (i) a checking rule q that maps each reported signal $s \in T$ to a checking probability $q(s) \in [0, 1]$, which is also the checking-and-assigning probability since the agent is truthful on equilibrium path, and (ii) a total allocation rule p that maps each reported signal $s \in T$ to a total assignment probability $p(s) \in [0, 1]$. Naturally, $p(s) \geq q(s)$ for all $s \in T$ as the former also accommodates the assignment probability conditional on not being checked. The artificial game tree in Figure 1 illustrates the simplified direct mechanism.

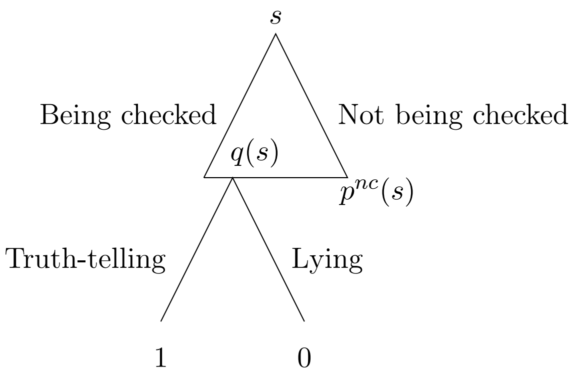

Figure 1: The simplified direct mechanism, where $p^{nc}(s)$ is the assignment probability conditional on no checking and $p(s) = q(s) \cdot 1 + (1 - q(s)) \cdot p^{nc}(s)$.

2.3 Mechanism design and information design

Given a signal distribution (interchangeably, designed information) $G \in \mathcal{G}_F$, the principal selects a mechanism (p, q) to maximize her expected net payoff:

$$\max_{p,q} \quad \mathbb{E}_G \left[p(s)(t - R) - q(s)c \right] \tag{1}$$

$$\text{subject to} \quad p(s) \in [0, 1], \quad \forall s \in T, \tag{2}$$

$$q(s) \in [0, 1], \quad \forall s \in T, \tag{3}$$

$$q(s) \leq p(s), \quad \forall s \in T, \tag{4}$$

$$p(s) \geq p(s') - q(s'), \quad \forall s, s' \in T. \tag{5}$$

(1) is the objective of the principal; sometimes we also refer to the entire constrained maximization as problem (1). Constraints (2)-(4) are the feasibility constraints. Constraint (5) is the incentive compatibility constraint to ensure that the agent always prefers truth-telling to lying.

Problem (1) here is a simplification of the allocation problem in Ben-Porath et al. (2014): Instead of having multiple agents with private information, we only have one agent with private information and another "naive agent"—the principal—who has a constant type R. Therefore, the general results in Ben-Porath et al. (2014) fully characterize the solution to problem (1), which is the benchmark of our analysis and we will review it shortly in Section 3.

The main focus of our paper is the information design problems associated with problem (1) and its multiple-agent counterpart. We will study information design from both the agent's perspective and the principal's perspective and set up optimization problems in corresponding sections.

3 Benchmark optimal mechanism for a given G

This section presents the solution to the benchmark mechanism design problem (1), which we adopt from Ben-Porath et al. (2014).

The optimal mechanism is essentially unique and is determined by a single threshold s^*.[12] Particularly, s^* is defined implicitly by the following equation

$$\mathbb{E}_G(s) = \mathbb{E}_G(\max\{s, s^*\}) - c. \tag{6}$$

Since the equation is equivalent to $\int_{\underline{t}}^{s^*} G(s)ds = c$, it has a unique solution unless $c = 0$. If $c = 0$, any $s^* \le \min supp(G)$ solves the equation, but it is natural to take $s^* = \min supp(G)$ since it is the limit value of s^* as c goes to 0 from above. Overall, there is a unique solution to (6) in $[\min supp(G), \infty)$. We usually suppress the dependence of s^* on G when there is no ambiguity. For notational convenience, we denote by t^* the threshold defined by equation (6) but using the underlying distribution F instead of G.

If we view the principal as another "agent 0," then the principal has a constant type

[12] The word "essential" means that the optimal mechanism is unique up to manipulation on a set of measure zero. For example, if a mechanism is optimal, then the alternative mechanism that retains the good more frequently on a measure zero set is also optimal. See Ben-Porath et al. (2014) for a formal notion of equivalence between different mechanisms.

R and we have a similar equation to define a threshold t_0^* for the principal:

$$\mathbb{E}_{G_0}(s_0) = \mathbb{E}_{G_0}(\max\{s_0, s_0^*\}) - c_0,$$

where $s_0 \equiv R$ and $c_0 = 0$. Obviously, any $s_0^* \leq R = \min supp(G_0)$ solves the equation.

Proposition 1 (Ben-Porath et al.). *The following mechanism is essentially the unique optimal mechanism:*

1. *If $s^* - c \geq R$, then the agent receives the good without being checked.*
2. *If $s^* - c < R$, then proceed as follows:*
   (a) *If $s - c < R$, then the principal retains the good.*
   (b) *If $s - c \geq R$, then the agent is checked (truth-telling on equilibrium path) and allocated the good.*

For completeness and easy reference, a proof of Proposition 1 is included in Appendix A; readers who are familiar with Ben-Porath et al. (2014) would immediately see that the mechanism in Proposition 1 is simply a special case of their optimal favored-agent mechanism for multiple agents.

The optimal mechanism delivers the following payoff to the principal:

$$Y := \begin{cases} \int_{\underline{t}}^{\bar{t}} (s - R)\mathrm{d}G(s) & \text{if } s^* - c \geq R \\ \int_{R+c}^{\bar{t}} (s - c - R)\mathrm{d}G(s) & \text{if } s^* - c < R, \end{cases} \tag{7}$$

where the former can also be written as $\mathbb{E}_G(s) - R = \mathbb{E}_F(t) - R = \mu - R$.

We break ties in favor of the agent to facilitate the study of the agent-optimal information design problem in Section 4.1. Specifically, there are two kinds of ties to consider. First, when $s - c = R$, the principal is indifferent between retaining the good and allocating the good to the agent with checking. Second, from the proof of the proposition in Appendix A, we know that when $s^* - c = (resp. <)\ R$,

$$\begin{aligned} \mu - R &= \int_{\underline{t}}^{\bar{t}} (s - R) \mathrm{d}G(s) \\ &= (resp. <) \int_{R+c}^{\bar{t}} (s - c - R) \mathrm{d}G(s), \end{aligned} \tag{8}$$

i.e. the principal is indifferent between allocating the good without checking and allocating the good only if $s - c \geq R$ and the agent tells the truth. In both cases, we opt to let the agent receive the good. Of course, the principal is always indifferent between tie-breaking rules.[13]

4 Information design with a single agent

We characterize the agent-optimal information and the principal-optimal information in this section, and also show that any agent-optimal information is principal-worst.

4.1 Agent-optimal information design

The agent wants to obtain the good with the largest probability regardless of his signal, given that the principal employs an optimal mechanism to maximize her expected net payoff.

Before we set up the agent-optimal information design problem, we first observe that mean preserving contraction leads to a larger threshold. Recall that s^* is uniquely defined on $[\min supp(G), \infty)$. Moreover, if $c > 0$, then $s^* > \underline{t}$.

Lemma 1. *$s^* \geq t^*$ for any $G \in \mathcal{G}_F$.*

Since the agent obtains the good with probability one at all types if $s^* - c \geq R$, and with probability one only when $s - c \geq R$ if $s^* - c < R$ (Proposition 1), the *agent-optimal information design* problem concerns two scenarios:

[13] One may break ties the other way around if interested in the agent-worst information design problem, which, however, is not the focus of the current paper.

1. If there exists $G \in \mathcal{G}_F$ such that $s^* - c \geq R$, then any such G maximizes the agent's probability of obtaining the good.

2. If for all $G \in \mathcal{G}_F$ we have $s^* - c < R$, then we need to solve

$$\max_{G \in \mathcal{G}_F} \quad 1 - G^-(R + c), \tag{9}$$

where $G^-(R+c) := \lim_{s \uparrow R+c} G(s)$. (9) becomes trivial if $R + c > \bar{t}$, since $G^-(R+c) \geq G(\bar{t}) = 1$ implies $G^-(R+c) = 1$ for any $G \in \mathcal{G}_F$. This is why we have assumed $R + c \leq \bar{t}$ in Section 2.1.

Proposition 2. *Given R, F and c:*

1. *If $\mu \geq R$, then there exists a distribution $G \in \mathcal{G}_F$ such that $s^* - c \geq R$, and any such G maximizes the agent's payoff. In particular, the null information $\delta(\mu)$ satisfies $s^* - c = \mathbb{E}_G(s) = \mu \geq R$ and is agent-optimal.*

2. *If $\mu < R$, then an information $G \in \mathcal{G}_F$ is agent-optimal if and only if G has an atom $R + c$ with probability $1 - F(s^\dagger)$, where the cutoff $s^\dagger$ is determined by*

$$R + c = \mathbb{E}_F \left(t \middle| t \in \left[s^\dagger, \bar{t} \right] \right).[14] \tag{10}$$

In particular, the following information is agent-optimal:

$$\hat{G}(s) = \begin{cases} F(s) & \text{for all } s \in [\underline{t}, s^\dagger) \\ F(s^\dagger) & \text{for all } s \in [s^\dagger, R + c) \\ 1 & \text{for all } s \in [R + c, \bar{t}]. \end{cases} \tag{11}$$

The proposition provides a full characterization of the agent-optimal information. The particular distribution $\hat{G}$ in part 2 is derived from F by concentrating the probability mass of F over $[s^\dagger, \bar{t}]$ onto a single point $R + c$, as required by the MPC constraint, and maintaining F on $[\underline{t}, s^\dagger)$. This is illustrated in the left panel of Figure 2. Intuitively, (9)

[14] The cutoff $s^\dagger$ uniquely exists and satisfies $s^\dagger < R + c \leq \bar{t}$, since $f(t) > 0$ for all $t \in T$.

says that reducing the probability mass on $[\underline{t}, R+c)$ and increasing the probability mass on $[R+c, \bar{t}]$ would benefit the agent. This can be done to the largest extent while maintaining the MPC constraint $G \in \mathcal{G}_F$ if we concentrate as much probability mass as possible to the single point $R+c$. The endogenously determined $s^\dagger$ captures such a limit.

However, $\hat{G}$ is not the unique agent-optimal information. For example, $\tilde{G}$ in the right panel of Figure 2 is also agent-optimal. Generally, varying the lower part of an agent-optimal information does not affect its optimality as long as the MPC constraint is maintained.

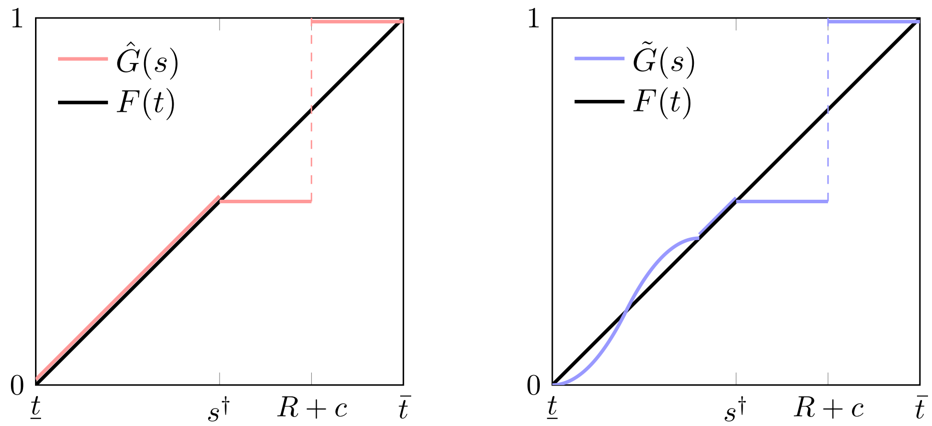

Figure 2: Two examples of agent-optimal information.

Technically, some features of an agent-optimal information are implied by the results of Dworczak and Martini (2019) and Kleiner et al. (2021). In particular, given an information distribution, the equilibrium outcomes of both the principal and the agent under the optimal mechanism in Proposition 1 can be reformulated as a special case of the principal's expected payoff and a special case of the agent's expected utility à la Dworczak and Martini (2019) and Kleiner et al. (2021). Thus, applying Theorem 3 of the former or Proposition 2 of the latter, we know that some agent-optimal information partitions signals into intervals and is either pooling or fully revealing for different intervals. Unlike their characterizations without explicit solutions, we fully characterize the solutions in a concrete context. Moreover, our direct and elementary proof facilitates economic interpretations and the study of other research questions, such as the one to be answered in Proposition 5.

Now we turn to the principal's payoff. Given any information $G \in \mathcal{G}_F$, the principal's payoff from an optimal mechanism is (7). Since the two cases in Proposition 2 are comparing μ and R, whereas the two cases in (7) are comparing $s^* - c$ and R, we shall clarify the connection to facilitate the calculation of principal payoff. The following lemma does so and it is also used in the proof of Proposition 2.

Lemma 2. *If $\mu < R$, then for any $G \in \mathcal{G}_F$, we have $s^* - c < R$.*

Using the particular information $\hat{G}$ such that $\hat{G}(s) = F(s)$ for all $s \in [\underline{t}, s^\dagger)$, we can easily calculate the principal's payoff under an agent-optimal information:

$$Y^{AO} := \begin{cases} \mu - R & \text{if } \mu \geq R \\ 0 & \text{if } \mu < R. \end{cases} \tag{12}$$

As we will see shortly, all agent-optimal informations deliver the same payoff (12) to the principal.

We close this section by noticing a welfare consequence of information design. As documented by Roesler and Szentes (2017), the buyer-optimal information structure can generate efficient trade which generally does not occur without information design. Here in our context of allocation with costly verification, we have a similar observation that information design *may* make allocation more efficient. For example, when $\mu < R$ (which implies $t^* - c < R$), the optimal mechanism under the prior distribution F allocates the good if and only if $t - c \geq R$ (Proposition 1). Note that it is ex-post more efficient to allocate the good whenever $t \geq R$. Therefore, slightly pooling information around $R + c$ via mean-preserving contraction would improve allocation efficiency.

4.2 Principal-optimal information design

The principal-optimal information design problem is as follows:

$$\max_{G \in \mathcal{G}_F} \quad \max_{p,q} \quad \mathbb{E}_G\left[p(s)(s-R) - q(s)c\right] \tag{13}$$

$$\text{subject to } (2) - (5).$$

PROPOSITION 3. *Given R, F and c:*

1. *If $t^* - c \geq R$, the principal's payoff is independent of information design. In particular, full information $F \in \mathcal{G}_F$ is principal-optimal.*
2. *If $t^* - c < R$, an information $G \in \mathcal{G}_F$ is principal-optimal if and only if $G^-(R+c) = F(R+c)$ and $\mathbb{E}_G(s \big| [R+c, \bar{t}]) = \mathbb{E}_F(s \big| [R+c, \bar{t}])$. In particular, full information $F \in \mathcal{G}_F$ is principal-optimal.*

The characterization in part 2 simply says that, relative to F, G does not pool information across the point $R + c$. Using full information, we know that the principal's payoff under principal-optimal information is given by (7), with s replaced by t and G replaced by F.

The agent's payoff does not vary within principal-optimal information. To see it, we still consider two cases. If $t^* - c \geq R$, then the agent receives the good with probability one, whatever the principal-optimal information is. If $t^* - c < R$, then according to the proof of Proposition 3, we know that any principal-optimal information G must maintain $s^* - c < R$. Thus, under the optimal mechanism in Proposition 1, the agent receives the good if and only if $s - c \geq R$, with probability $1 - G^-(R + c)$. But part 2 of Proposition 3 says that this probability is always $1 - F(R + c)$ and is independent of G.

If we think of the principal and the information designer in our model as the same person, then the spirit of Proposition 3 echoes a key message of Kattwinkel and Knoepfle (2023). Namely, Proposition 3 says that the principal who knows F does not profit from any form of information design, e.g., choosing a signal distribution G that is a coarsening of F. Somewhat similarly, in a principal-agent model where the principal privately observes a signal that is correlated with the agent's type, Kattwinkel and Knoepfle (2023) show that the principal does not profit from any form of information design, e.g., releasing parts of her information to the agent to manipulate his beliefs. Nevertheless, the difference is drastic. Information design in Kattwinkel and Knoepfle (2023) pertains to the disclosure of the principal's private signal to the agent who knows his own type, whereas in our setting, the information designer designs the entire uncertainty in the underlying allocation problem, perceived by both the principal and the agent.

4.3 Principal-worst information design and conflict of interests

We follow Roesler and Szentes (2017) to study the principal-worst information design problem. Designing information to hurt the principal, although itself lacks motivation, facilitates the study of robust mechanism design (Brooks and Du, 2021); see Section 5.3 for detailed analysis.

$$\min_{G \in \mathcal{G}_F} \quad \max_{p,q} \quad \mathbb{E}_G\left[p(s)(s-R) - q(s)c\right] \tag{14}$$

$$\text{subject to } (2) - (5)$$

and its solution is characterized below.

Proposition 4. *Given R, F and c:*

1. *If $\mu \geq R$, an information $G \in \mathcal{G}_F$ is principal-worst if and only if $s^* - c \geq R$. In particular, the null information $\delta(\mu)$ is principal-worst.*
2. *If $\mu < R$, an information $G \in \mathcal{G}_F$ is principal-worst if and only if $G(R + c) = 1$. In particular, the null information $\delta(\mu)$ is principal-worst.*

Using null information, together with (7), we can easily calculate the principal's worst-case payoff:

$$Y^{PW} := \begin{cases} \mu - R & \text{if } \mu \geq R \\ 0 & \text{if } \mu < R. \end{cases} \tag{15}$$

Comparing (12) and (15), we immediately see that the principal's payoff is the same under the agent-optimal information and the principal-worst information, i.e. $Y^{PW} = Y^{AO}$. This is surprising in the sense that the objectives of the two information design problems are very different: The agent-optimal problem concerns the *probability* that the agent receives the good, whereas the principal-worst problem concerns the net *payoff* to the principal.

Here is the reason for such coincidence: (a) When $\mu \geq R$, the agent-optimal information leads to probability-one allocation without checking, which obviously achieves the principal's lower-bound payoff $\mu - R$. (b) When $\mu < R$, since $s^* - c < R$ for all $G \in \mathcal{G}_F$ (Lemma 2), allocating the good to the agent necessarily incurs checking for any $G \in \mathcal{G}_F$ and the principal's net payoff is $\int_{R+c}^{\bar{t}}(s - c - R)\mathrm{d}G(s)$. We can rewrite the net payoff as follows:

$$\int_{R+c}^{\bar{t}} s\mathrm{d}G(s) - [1 - G^{-}(R+c)](R+c).$$

So intuitively, maximizing $[1 - G^{-}(R+c)]$ (agent-optimal) at least partially coincides with minimizing the principal's payoff.

Formally, we have the following proposition:

Proposition 5. *Any agent-optimal information is principal-worst.*

Two remarks are in order. First, the converse of Proposition 5 is not true. For example, null information is principal-worst, but it is usually not agent-optimal. Second, an agent-optimal distribution, say $\hat{G}$ in (11), may be more informative than the null distribution in the sense that null distribution comes from completely coarsening the underlying distribution. However, the proposition says that both of them are principal-worst, i.e., the additional informativeness of $\hat{G}$ does not help the principal to generate a higher payoff.

Proposition 5 resembles the equivalence between buyer-optimal and seller-worst information in Roesler and Szentes (2017). However, both the problem formulations and the results are different. In Roesler and Szentes (2017), the buyer and the seller are roughly playing a zero-sum game: the monetary surplus of trading is split between them and payoffs are perfectly transferable. Yet in our model, the agent's and the principal's interests are largely overlapped, which makes the opposition in information-design solutions surprising at the first glance. In terms of results, Roesler and Szentes (2017) offers an equivalence as a consequence of the formulation, whereas our result is only one-directional: some principal-worst information distributions are not agent-optimal.

5 Multiple agents and robust mechanism design

5.1 General model and the benchmark optima mechanism

When there are multiple agents, we denote the set of agents by $\mathcal{I} = \{1, \ldots, I\}$. Let $\hat{\mathcal{I}} := \{0, 1, \ldots, I\}$, where 0 stands for the principal. A generic agent or the principal is denoted by i, and we abuse the terminology "agent" to always say agent i. The distribution of agent i's type is F_i over $T_i = [\underline{t}_i, \bar{t}_i]$. Assume for all i that $0 < \underline{t}_i < R < \bar{t}_i < \infty$, $R + c \leq \bar{t}_i$ and $f_i(t_i) > 0$ for all t_i. In the special case of $i = 0$, $t_i \equiv R$. Define profiles $\mathbf{t} := (t_1, \ldots, t_I)$ and define $\mathbf{t}_{-i}$, $\mathbf{T}$ and $\mathbf{T}_{-i}$ in the usual manner.

Let $\mathcal{G}_{F_i}$ be the set of marginal distributions of unbiased signals for agent i. The cost to verify agent i's private signal is c_i; in the special case of $i = 0$, $c_i = 0$. Each agent i, including the principal, has a critical type s_i^* that is defined in the manner of (6) using distribution $G_i \in \mathcal{G}_{F_i}$.

The information designer now has the flexibility to simultaneously choose $G_i \in \mathcal{G}_{F_i}$

for every $i \in \mathcal{I}$. Assume that the information designer's choices of G_i's are independent for now, which we will revisit near the end of Section 5.3. Define $\mathbf{G} := (G_1, \ldots, G_I)$ and define $\mathbf{G}_{-i}$, $\mathbf{s}$ and $\mathbf{s}_{-i}$ accordingly.

Given $\mathbf{G}$, the principal's mechanism design problem is as follows:

$$\max_{\{p_i,q_i\}_{i\in\mathcal{I}}} \quad \mathbb{E}_{\mathbf{G}}\left\{\sum_{i=1}^{I}\left[p_i(\mathbf{s})(s_i - R) - q_i(\mathbf{s})c_i\right]\right\} \tag{16}$$

$$\text{subject to} \quad 0 \leq q_i(\mathbf{s}) \leq p_i(\mathbf{s}) \leq 1, \quad \forall \mathbf{s} \in \mathbf{T}, \quad \forall i \in \mathcal{I}, \tag{17}$$

$$\sum_{i=1}^{I} p_i(\mathbf{s}) \leq 1, \quad \forall \mathbf{s} \in \mathbf{T}, \tag{18}$$

$$\mathbb{E}_{\mathbf{G}_{-i}}\left[p_i(s_i, \mathbf{s}_{-i})\right] \geq \mathbb{E}_{\mathbf{G}_{-i}}\left[p_i(s_i', \mathbf{s}_{-i})\right] - \mathbb{E}_{\mathbf{G}_{-i}}\left[q_i(s_i', \mathbf{s}_{-i})\right], \quad \forall s_i, s_i' \in T_i, \quad \forall i \in \mathcal{I}. \tag{19}$$

We first recall the solution to this problem due to Ben-Porath et al. (2014). As defined in their paper, a *favored-agent mechanism* specifies a *favored agent* $i^* \in \hat{\mathcal{I}}$ (we treat the principal as an "agent" here) and a *threshold* $v^* \in \mathbb{R}$ such that:

1. If $s_j - c_j < v^*$ for all $j \neq i^*$, then

$$p_{i^*}(\mathbf{s}) = 1, \quad p_j(\mathbf{s}) = 0, \forall j \neq i^*; \quad \text{and} \quad q_j(\mathbf{s}) = 0, \forall j.$$

2. If there exists $j \neq i^*$ such that $s_j - c_j > v^*$ and $s_i - c_i > \max_{k\neq i} s_k - c_k$, then

$$p_i(\mathbf{s}) = q_i(\mathbf{s}) = 1, \quad \text{and} \quad p_k(\mathbf{s}) = q_k(\mathbf{s}) = 0, \forall k \neq i.$$

Proposition 6 (Ben-Porath et al.). *The mechanism in each case below is essentially the unique optimal mechanism*[15]*:*

1. If $\max_{i\in\mathcal{I}}\{s_i^* - c_i\} \geq R$*, then implement the favored-agent mechanism with a favored*

[15] We display two cases here to keep consistency with Proposition 1. If one regards the principal as an "agent", then the favored-agent mechanism that favors the "agent" who has the largest $s^* - c$ (R for the principal) and adopts the threshold of such $s^* - c$, is essentially the unique optimal mechanism, exactly as in Ben-Porath et al. (2014).

agent $i^* \in \arg\max_{i \in \mathcal{I}} \{s_i^* - c_i\}$ *and threshold* $v^* = s_{i^*}^* - c_{i^*}$*.*

2. *If* $\max_{i \in \mathcal{I}} \{s_i^* - c_i\} < R$*, then implement the favored-agent mechanism with the principal being favored and threshold* $v^* = R$*.*

5.2 Agent-optimal and principal-optimal information design

Now we revisit the agent-optimal information design problem in a multi-agent setting. In this setting, the information designer wants to maximize the agents' *aggregate* payoff, i.e. the total allocation probability.[16] The idea to solve this problem is much similar to the single-agent case characterized in Proposition 2. Particularly, if there exists some agent i who can be a favored agent ($s_i^* - c \geq R$), then in any case, the principal will allocate this good out. If there does not exist such a favored agent, then for all agents, the information designer will put as much as possible mass on $R + c$ to maximize the total allocation probability. We summarize this in Proposition 7.

Proposition 7. *Given* R, $\{F_i\}_{i \in \mathcal{I}}$ *and* c*:*

1. *If there exists some* $i \in \mathcal{I}$*, such that* $\mathbb{E}_{F_i}(t_i) \geq R$*, then for agent* i*, there exists a distribution* $G_i \in \mathcal{G}_{F_i}$ *such that* $s_i^* - c \geq R$*. Any profile of signal distributions that has such a component is aggregate agent-optimal.*

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[16] Such an aggregate agent-optimal information is also a *Pareto* agent-optimal information in the sense that no other information design can make some agent strictly better off without making any agent worse off. We provide a counterexample in Appendix B saying that the converse is not true, i.e. a Pareto agent-optimal information needs not be aggregate agent-optimal. If we consider the special case where the prior distributions are identical, i.e. $F_i = F$, and require that the information designer can only choose identical signal distributions, i.e. $G_i = G \in \mathcal{G}_F$, then obviously Pareto agent optimality is equivalent to aggregate agent optimality.

We also explore the situation where the agents have the power to acquire the signals by themselves in Appendix C. We actually study a strategic information acquisition game among agents where those agents themselves are "information designers".

2. If $\mathbb{E}_{F_i}(t_i) < R$ *for all* $i \in \mathcal{I}$*, in which case we let* $s_i^\dagger$ *solve*

$$R + c = \mathbb{E}_{F_i}\left[t_i \middle| t_i \in \left[s_i^\dagger, \bar{t}_i\right]\right]$$

for each i*, then a profile of signal distributions* $\{G_j\}_{j\in\mathcal{I}}$ *is agent-optimal if and only if for each* i*,* G_i *has an atom* $R + c$ *with probability* $1 - F_i(s_i^\dagger)$*. In particular, the following profile of signal distributions is aggregate agent-optimal:*

$$\forall j \in \mathcal{I}, \quad \hat{G}_j(s_j) = \begin{cases} F_j(s_j) & \textit{for all } s_j \in [\underline{t}_j, s_j^\dagger) \\ F_j(s_j^\dagger) & \textit{for all } s_j \in [s_j^\dagger, R + c) \\ 1 & \textit{for all } s_j \in [R + c, \bar{t}_j]. \end{cases}$$

Below we use an example to show that *aggregate* agent-optimal information design may affect *individual* agents' payoffs in diverse ways: under some aggregate agent-optimal information, all agents can be better off; while under some other aggregate agent-optimal information, one agent gets better off at the cost of the other's payoff.

Example 1. Suppose there are two agents who have the same underlying type distribution F which is uniform on $[0, 1]$ and a common checking cost $c = 0.08$. Suppose the principal's reservation value is $R = 0.4$. In this case, it is straightforward to verify that $t^* = 0.4$ so that $t^* - c < R$. Therefore, under the benchmark optimal mechanism, the principal retains the good with probability 0.48^2 and assigns the good to the two agents with probability $\left(1 - 0.48^2\right) \approx 0.76$ in total, and 0.38 for each.

Both agents are better off. Consider the designed signal distribution profile such that $\hat{G}_1 = \delta(\mathbb{E}_F(t)) = \delta(0.5)$ and

$$\hat{G}_2(s) = \begin{cases} s & \text{for all } s \in [0, 0.5) \\ 0.5 & \text{for all } s \in [0.5, 0.55) \\ 0.6 & \text{for all } s \in [0.55, 0.6) \\ s & \text{for all } s \in [0.6, 1]. \end{cases}$$

Namely, $\hat{G}_2$ is derived from F by pooling the probability mass over $(0.5, 0.6)$ at its mean 0.55. Then $s_1^* = 0.58$ and $s_2^* = 0.4$. Since $s_1^* - c > R > s_2^* - c$, the optimal mechanism favors agent 1 and the designed information is aggregate agent-optimal. Under this favored-agent mechanism, agent 1 receives the good with probability 0.6 and agent 2 receives the good with probability 0.4.

One is better off and the other is worse off. Consider another designed signal distribution profile that $\tilde{G}_1 = \delta(\mathbb{E}_F(t)) = \delta(0.5)$ and

$$\tilde{G}_2(s) = \begin{cases} s & \text{for all } s \in [0, 0.3) \\ 0.3 & \text{for all } s \in [0.3, 0.55) \\ 0.8 & \text{for all } s \in [0.55, 0.8) \\ s & \text{for all } s \in [0.8, 1]. \end{cases}$$

Namely, $\tilde{G}_2$ is derived from F by pooling the probability mass over $(0.3, 0.8)$ at its mean 0.55. Then $s_1^* = 0.58$ and $s_2^* \approx 0.42$. Since $s_1^* - c > R > s_2^* - c$, the optimal mechanism favors agent 1 and the designed information is aggregate agent-optimal. Under this favored-agent mechanism, agent 1 receives the good with probability 0.8 and agent 2 receives the good with probability 0.2.

Example 1 also serves as a counterexample, showing that the counterpart of Proposition 5 does not hold anymore. More specifically, $\left(\hat{G}_1, \hat{G}_2\right)$ and $\left(\tilde{G}_1, \tilde{G}_2\right)$ in Example 1 are both aggregate agent-optimal. The former leads to a gross principal payoff of

$$0.6 \cdot 0.5 + \int_{0.6}^{1} (s_2 - 0.08)\mathrm{d}s_2 = 0.588,$$

whereas the latter leads to a gross principal payoff of

$$0.8 \cdot 0.5 + \int_{0.8}^{1} (s_2 - 0.08)\mathrm{d}s_2 = 0.564.$$

Since the two principal payoffs are different, at least one of them is not principal-worst, invalidating the counterpart of Proposition 5. The intuition of this counterexample is as follows: Given that agent 1 is favored with $s_1^* = 0.58$ in both cases, the principal's payoff depends purely on how frequently agent 2 can have a signal larger than 0.58 (generally, how severe the competition between agents are). $\left(\tilde{G}_1, \tilde{G}_2\right)$ pools more information on 0.55 (generally, reduces agents' competition to a larger extent) and thus delivers lowers payoff to the principal.

Nevertheless, some agent-optimal information is indeed principal-worst. For example, we will see shortly in the next subsection that $(G_1, G_2) = (\delta(0.5), \delta(0.5))$, which is also aggregate agent-optimal, is a principal-worst information and the corresponding gross principal payoff is 0.5. In general, it is straightforward to see that requiring symmetry, i.e., $F_i = F$ and $G_i = G$ for all i, would lead to a situation similar to the single-agent case, which would then restore Proposition 5.

We close this subsection by an extension of Proposition 3.

Proposition 8. *Full information is principal-optimal.*

5.3 Robust mechanism design via principal-worst information design

We solve the robust-mechanism design problem from an information design perspective. To come straight to the point, we formulate the information design problem and a robust mechanism design problem together: Let $\mathcal{G}_i$ be the set of all cumulative distribution functions defined over $[\underline{t}_i, \bar{t}_i]$ (then $\mathcal{G}_{F_i} \subseteq \mathcal{G}_i$) and let $\mu_i = \mathbb{E}_{F_i}(t_i)$ be the mean type for agent i, which is the only information that the principal can rely on in designing robust mechanisms. The principal-worst information design problem and the robust mechanism design problem are as follows, respectively:

$$(20) \qquad \min_{G_i \in \mathcal{G}_{F_i}, \forall i \in \mathcal{I}} \quad \max_{\{p_i, q_i\}_{i \in \mathcal{I}}} \quad \mathbb{E}_{\mathbf{G}} \left\{ \sum_{i=1}^{I} [p_i(\mathbf{s})(s_i - R) - q_i(\mathbf{s}) c_i] \right\}$$

$$\text{subject to } (17) - (19)$$

$$(21) \qquad \max_{\{p_i, q_i\}_{i \in \mathcal{I}}} \quad \min_{\substack{G_i \in \mathcal{G}_i, \forall i \in \mathcal{I} \\ \mathbb{E}_{G_i}(s_i) = \mu_i, \forall i \in \mathcal{I}}} \quad \mathbb{E}_{\mathbf{G}} \left\{ \sum_{i=1}^{I} [p_i(\mathbf{s})(s_i - R) - q_i(\mathbf{s}) c_i] \right\}$$

$$\text{subject to } (17) - (19).$$

Obviously, the principal-worst information design problem (20) is a min-max problem, which has a natural connection with (21) via the max-min inequality. To wit, since (omitting the identical objective function and constraints)

$$\begin{aligned} & \max_{\{p_i, q_i\}_{i \in \mathcal{I}}} \quad \min_{\substack{G_i \in \mathcal{G}_i, \forall i \in \mathcal{I} \\ \mathbb{E}_{G_i}(s_i) = \mu_i, \forall i \in \mathcal{I}}} \\ \leq & \min_{\substack{G_i \in \mathcal{G}_i, \forall i \in \mathcal{I} \\ \mathbb{E}_{G_i}(s_i) = \mu_i, \forall i \in \mathcal{I}}} \quad \max_{\{p_i, q_i\}_{i \in \mathcal{I}}} \quad \text{(Max-min inequality)} \\ \leq & \min_{G_i \in \mathcal{G}_{F_i}, \forall i \in \mathcal{I}} \quad \max_{\{p_i, q_i\}_{i \in \mathcal{I}}} \quad (\mathcal{G}_{F_i} \subseteq \{G_i \in \mathcal{G}_i : \mathbb{E}_{G_i}(s_i) = \mu_i\}), \end{aligned}$$

the optimized value of problem (20) provides an upper bound for the optimized value of problem (21). As a result, if a robust mechanism can achieve this upper bound, then it has to be an optimal robust mechanism.

Our formal result deals with ambiguity sets that are more general than mean-constrained $\mathcal{G}_i$'s. More precisely, for each i, we consider an arbitrary ambiguity set $\tilde{\mathcal{G}}_i$ of signal distributions on $[\underline{t}_i, \bar{t}_i]$ such that

(1) $\tilde{\mathcal{G}}_i$ contains the null information, i.e., $\delta(\mu_i) \in \tilde{\mathcal{G}}_i$, and

(2) distributions in $\tilde{\mathcal{G}}_i$ have the same mean, i.e., $\mathbb{E}_{G_i}(s_i) = \mu_i$ for all $G_i \in \tilde{\mathcal{G}}_i$.

Proposition 9. *The following two statements are true:*

1. *The principal's payoff under the principal-worst information is*

$$Y^M := \max \{R, \mathbb{E}_{F_1}(t_1), \ldots, \mathbb{E}_{F_I}(t_I)\}. \tag{22}$$

   Particularly, the profile of null information for every agent is principal-worst.

2. *For any profile of ambiguity sets $\left\{\tilde{\mathcal{G}}_i\right\}_{i\in\mathcal{I}}$, where each $\tilde{\mathcal{G}}_i$ satisfies (1) and (2), the robust mechanism that allocates the good to the agent (possibly the principal) who has the highest expected value is optimal within robust mechanisms.*

The first statement is analogous to Proposition 4.

Proposition 9 is more general than it appears in at least three aspects. First, in the model of Mylovanov and Zapechelnyuk (2017) which assumes costless ex-post verification and limited punishment on lying agents or the model of Li (2020) which accommodates both costly verification and limited punishment, the mechanism in part 2 of Proposition 9 is still an optimal robust mechanism—when the incentive issue is circumvented by the robust mechanism, how severe the punishment is or whether verification is costly becomes irrelevant.

Second, the problem of allocating multiple homogeneous goods has a similar formulation and a similar solution as the single-good allocation problem; see Ben-Porath et al. (2019) and Chua et al. (2023). Suppose there are n goods to be allocated among $I(> n)$ agents. It is straightforward to see that the idea of Proposition 9 applies to this multiple-good setting, which leads to an optimal robust mechanism that allocates the n goods to the n agents (including the principal) who have the n highest expected types.

Third, one may consider the possible correlation among G_i's. In short, allowing for correlated information design does not affect Proposition 9. Note that, since independent information design is always feasible, allowing for correlated design weakly reduces the principal's payoff under the principal-worst information. Thus, we have a weakly smaller upper bound for the payoffs that can be achieved by robust mechanisms. However, the upper bound (22) is indeed achievable by the robust mechanism in Proposition 9, which in turn implies that the principal-worst payoff cannot be strictly reduced. Therefore, both the principal-worst payoff and the payoff under the optimal robust mechanism are unchanged, i.e., Proposition 9 still holds.[17]

6 Concluding remarks

In this paper, we examine the information design problems in the allocation setting of Ben-Porath et al. (2014), as well as the implications of their solutions. Many of our insights in the single-agent case extend to the multiple-agent case.

Third directions deserve further investigation. The first one is to study the information acquisition game that we briefly discussed in Footnote 16, though not in the scope of information design as modeled in the current paper. Second, from one agent to multiple agents, when studying principal-optimal information design, we assume independent design across agents. The discussion of possible correlation shall rely on the form of the principal's payoff under the optimal mechanism, which itself, to the best of our knowledge, is an open question. Third, in the multiple-agent case, we only provide one solution to the principal-worst and principal-optimal information design problems. The characterization of the set of all solutions may be of further interest.

[17] See He and Li (2022) for a study of correlation robust mechanism design in the auction setting. Although their model differs drastically from ours, their idea of using max-min inequality is similar.

Appendix A Proofs

Proof of Proposition 1. There are three steps. The first step simplifies the problem. The second step restricts our attention to a special class of mechanisms called threshold mechanisms. The third step finds the optimal mechanisms within the class of threshold mechanisms.

Step 1. Getting rid of $q(s)$ to simplify the problem.

From the incentive compatibility constraint (5), we know that

$$\inf_{s \in T} p(s) \geq p(s') - q(s') \quad \text{for all } s, s' \in T.$$

Let $\varphi := \inf_{s \in T} p(s)$. Then we have

$$\varphi \geq p(s') - q(s') \quad \text{for all } s' \in T. \tag{23}$$

Since checking is costly, (23) must hold with equality for all $s' \in T$. Therefore,

$$\varphi = p(s) - q(s) \quad \text{for all } s \in T. \tag{24}$$

Plug (24) into the objective function (1) to eliminate $q(s)$. Then we have a simplified problem:

$$\max_{p} \quad \mathbb{E}_G\left[p(s)(s - R) - (p(s) - \varphi)c\right] = \mathbb{E}_G\left[p(s)(s - c - R)\right] + \varphi c \tag{25}$$

$$\text{subject to} \quad p(s) \in [0, 1], \quad \forall s \in T, \tag{26}$$

$$\varphi = \inf_{s \in T} p(s), \quad \forall s \in T. \tag{27}$$

Problem (25)-(27) is equivalent to the following problem, where we refer to the inner maximization as the *relaxed problem*:

$$\begin{aligned} \max_{\varphi} \quad \max_{p} \quad & \mathbb{E}_G\left[p(s)(s - c - R)\right] \quad + \varphi c \\ \text{subject to} \quad & p(s) \in [0, 1], \quad \forall s \in T, \\ & p(s) \geq \varphi, \quad \forall s \in T. \end{aligned} \tag{28}$$

In what follows, we study the solution to the relaxed problem first and then the original one. Due to the simplification above, we also call $p(\cdot)$ a mechanism.

Step 2. The solution to the relaxed problem is a *threshold mechanism.*

Definition 1. *A mechanism p is a* ***threshold mechanism*** *with parameter φ if there exists a threshold v^* such that the following two requirements hold for almost all $s \in T$:*

1. *If $s - c < v^*$, then $p(s) = \varphi$.*
2. *If $s - c \geq v^*$, then*

$$p(s) = \begin{cases} 1 & \text{if } s - c \geq R, \\ 0 & \text{if } s - c < R. \end{cases}$$

When $\varphi = 1$, the only feasible mechanism is $p(s) = 1$ for all $s \in T$, which has to be the solution to the relaxed problem. It is straightforward to see that this constant mechanism is a threshold mechanism with, say, $v^* = \bar{t} - c$.

Suppose $\varphi < 1$. Let p be an optimal mechanism in the sense that it solves the relaxed problem. We define a candidate threshold v^*:

$$v^* := \inf\{s - c : \quad p(s') = 1 \text{ for almost all } s' \geq s\}$$

and proceed to argue that p is a threshold mechanism with threshold v^*.

We claim that $v^* \geq R$, so that p satisfies the second requirement of the threshold mechanism. If $v^* < R$, then for $s \in T$ such that $v^* \leq s - c < R$, which has a strictly positive measure, we have $s - c - R < 0$. By the definition of v^*, $p(s) = 1$ for almost all $s \in T$ such that $v^* \leq s - c < R$. Since the objective function is decreasing in p when $s - c - R < 0$, the principal's payoff improves if p is reduced. More precisely, we consider the following modification of p:

$$p'(s) = \begin{cases} \varphi & \text{if } v^* \leq s - c < R, \\ p(s) & \text{otherwise.} \end{cases}$$

Obviously, p' is feasible. Since $\varphi < 1$, p' delivers a strictly higher payoff to the principal than p, which is a contradiction to the optimality of p. Therefore, $v^* \geq R$.

To see that p also satisfies the first requirement of threshold mechanism, i.e. $p(s) = \varphi$ for almost all $s - c < v^*$, we consider two cases: (a) $v^* = R$ and (b) $v^* > R$.

(a) If $v^* = R$, then $s - c < v^*$ means $s - c < R$. Thus, any mechanism with $\int_{\{s-c<v^*:p(s)>\varphi\}} \mathrm{d}G(s) > 0$ can be strictly improved by reducing $p(s)$ to φ on the set $\{s - c \leq v^* : p(s) > \varphi\}$, contradicting the optimality of p. Thus, in this case, $p(s) = \varphi$ for almost all $s - c < v^*$.

(b) Now suppose $v^* > R$. And suppose to the contrary that there is a positive measure set $D \subseteq [\underline{t}, v^* + c)$ (equivalently, $s - c < v^*$ for all $s \in D$) such that $\varphi < p(s)$ for all $s \in D$. Without loss of generality, there exists an $\epsilon > 0$ such that $\varphi + \epsilon < p(s)$ for all $s \in D$. By the definition of v^*, for an arbitrarily small $\delta > 0$, particularly $\delta < v^* - R$, there exists a positive measure set $E \subseteq (v^* + c - \delta, v^* + c)$ such that $p(s) < 1$ for all $s \in E$. Without loss of generality, there exists a small $\gamma > 0$ such that $p(s) < 1 - \gamma$ for all $s \in E$. Without loss of generality, we assume that $D \cap E = \emptyset$ and $\max D < \min E$.[18] Consider two numbers $\eta > 0$ and $\xi > 0$ such that $\eta\mu(D) = \xi\mu(E)$, and consider the following mechanism:

$$p'(s) = \begin{cases} p(s) - \eta & \text{for all } s \in D, \\ p(s) + \xi & \text{for all } s \in E, \\ p(s) & \text{otherwise.} \end{cases}$$

We can choose η and ξ to satisfy $\eta, \xi < \min\{\epsilon, \gamma\}$. Then the new mechanism p' is feasible.

[18] We can make this true by choosing a small enough δ and re-defining D to exclude E.

Since $\max D < \min E$, we have $s' - c - R < s - c - R$ for all $s \in E$ and all $s' \in D$. Therefore, p' improves on p, which contradicts the optimality of p. Hence, $p(s) = \varphi$ for almost all $s - c < v^*$.

Step 3. The solution to the original problem.

Based on Step 2, the principal's objective function can be written as follows:

$$\begin{aligned}
&\int_{\underline{t}}^{\bar{t}} p(s)(s-c-R)\mathrm{d}G(s) + \varphi c \\
=&\int_{\underline{t}}^{v^*+c} \varphi(s-c-R)\mathrm{d}G(s) + \int_{v^*+c}^{\bar{t}} (s-c-R)\mathrm{d}G(s) + \varphi c \qquad \text{(Threshold Mechanism)} \\
=&\varphi\left[\int_{\underline{t}}^{v^*+c}(s-c-R)\mathrm{d}G(s) + c\right] + \int_{v^*+c}^{\bar{t}}(s-c-R)\mathrm{d}G(s).
\end{aligned}$$

We need to find the optimal v^* and φ. Obviously, the optimal mechanism must satisfy

$$\begin{cases} \varphi = 1 & \text{if } \int_{\underline{t}}^{v^*+c}(s-c-R)\mathrm{d}G(s) + c \geq 0, \\ \varphi = 0 & \text{if } \int_{\underline{t}}^{v^*+c}(s-c-R)\mathrm{d}G(s) + c < 0. \end{cases}$$

When $\varphi = 1$, the principal's objective function becomes $\int_{\underline{t}}^{\bar{t}}(s-R)\mathrm{d}G(s)$ and the optimal v^* does not matter (but there may be some restriction). When $\varphi = 0$, the principal's objective function becomes $\int_{v^*+c}^{\bar{t}}(s-c-R)\mathrm{d}G(s)$ and the optimal $v^* = R$.

The former mechanism is optimal if and only if

$$\int_{\underline{t}}^{\bar{t}}(s-R)\mathrm{d}G(s) \geq \int_{R+c}^{\bar{t}}(s-c-R)\mathrm{d}G(s), \tag{29}$$

which is equivalent to saying that

$$\mathbb{E}(s) \geq \mathbb{E}(\max\{t, R+c\}) - c,$$

which in turn is equivalent to saying that

$$s^* - c \geq R.$$

One can easily verify that $v^* = s^* - c$ is an eligible threshold such that $\int_{\underline{t}}^{v^*+c}(s - c - R)\mathrm{d}G(s) + c \geq 0$ holds in this case (by checking (29)).

Similarly, the latter mechanism is optimal if and only if

$$\int_{\underline{t}}^{\bar{t}}(s-R)\mathrm{d}G(s) < \int_{R+c}^{\bar{t}}(s-c-R)\mathrm{d}G(s), \tag{30}$$

which is equivalent to saying that

$$\mathbb{E}(s) < \mathbb{E}(\max\{t, R+c\}) - c,$$

which in turn is equivalent to saying that

$$s^* - c < R,$$

where s^* is defined in (6). One can easily verify that $\int_{\underline{t}}^{R+c}(s-c-R)\mathrm{d}G(s)+c<0$ holds in this case (by checking (30)).

It is straightforward to see that $\varphi = 1$, $v^* = s^* - c$ and $s^* - c \geq R$ correspond to the first scenario in the statement of the proposition, and $\varphi = 0$, $v^* = R$ and $s^* - c < R$ to the second. This completes the proof. □

Proof of Lemma 1. Note that

$$\begin{aligned}
\mathbb{E}_G(\max\{s, s^*\}) &= \mathbb{E}_G(s) + c \quad \text{(Definition of } s^*\text{)} \\
&= \mathbb{E}_F(t) + c \quad (G \in \mathcal{G}_F) \\
&= \mathbb{E}_F(\max\{t, t^*\}) \quad \text{(Definition of } t^*\text{)} \\
&\geq \mathbb{E}_G(\max\{s, t^*\}). \quad (G \text{ is a MPC of } F \text{ and } \max\{\cdot, t^*\} \text{ is convex})
\end{aligned}$$

Suppose $s^* < t^*$ on the contrary. Since $\mathbb{E}_G(\max\{s, \cdot\})$ is weakly increasing, we have

$$\mathbb{E}_G(\max\{s, s^*\}) < \mathbb{E}_G(\max\{s, t^*\}),$$

where the inequality holds strictly because s^* is defined as the greatest threshold that achieves $S := \mathbb{E}_G(\max\{s, s^*\})$. Hence, we have a contradiction. □

Proof of Lemma 2. Suppose to the contrary that $s^* \geq R + c$. Then we would have

$$\begin{aligned}
\mathbb{E}_G(s) &= \mathbb{E}_G(\max\{s, s^*\}) - c \quad \text{(Definition of } s^*\text{)} \\
&\geq \mathbb{E}_G(\max\{s, R+c\}) - c \quad (\mathbb{E}_G(\max\{s, \cdot\}) \text{ is weakly increasing}) \\
&\geq R + c - c \\
&= R.
\end{aligned}$$

Since $G \in \mathcal{G}_F$, we have $\mathbb{E}_G(s) = \mu$. Therefore, $\mu \geq R$, a contradiction. □

Proof of Proposition 2. Suppose $\mu \geq R$. Consider the degenerate signal distribution $G = \delta(\mu)$ which assigns probability one to the atom $s = \mu$; $G \in \mathcal{G}_F$. Then,

$$\mathbb{E}_G(\max\{s, s^*\}) - c = \max\{\mathbb{E}_G(s), s^*\} - c.$$

By definition of s^*, i.e. (6), we have $\mathbb{E}_G(s) = \max\{\mathbb{E}_G(s), s^*\} - c$, which implies $\mathbb{E}_G(s) = s^* - c$. Since $\mathbb{E}_G(s) = \mu$ for any $G \in \mathcal{G}_F$, we have $\mu = s^* - c$. Therefore, $s^* - c \geq R$ and, consequently, G maximizes the agent's payoff.

Now suppose $\mu < R$. We prove the "if" part first. By Lemma 2, we know that for any $G \in \mathcal{G}_F$, $s^* < R + c$. Then for any $G \in \mathcal{G}_F$, we need to solve (9), i.e.

$$\max_{G \in \mathcal{G}_F} \quad 1 - G^-(R + c).$$

In other words, we need to make $G^-(R + c)$ as small as possible. We claim that for any $G \in \mathcal{G}_F$, $G^-(R + c) \geq F(s^\dagger)$ or, equivalently,

$$1 - G^-(R + c) \leq 1 - F(s^\dagger).$$

Then if some distribution $G \in \mathcal{G}_F$ could put probability mass $1 - F(s^\dagger)$ at the atom $R + c$, then it must be agent-optimal, which would complete the proof for the "if" part.

Claim 1. *For any $G \in \mathcal{G}_F$, $G^-(R + c) \geq F(s^\dagger)$.*

Proof. We will use $\hat{G}$ in (11) as an intermediate distribution to obtain the claim.

Suppose to the contrary that for $G^-(R+c) < F(s^\dagger)$ for some $G \in \mathcal{G}_F$. Since G is a mean-preserving contraction of F, we know that for any $s' \in [\underline{t}, \bar{t}]$,

$$\int_{\underline{t}}^{s'} G(s)\mathrm{d}s \leq \int_{\underline{t}}^{s'} F(s)\mathrm{d}s = \int_{\underline{t}}^{s'} \hat{G}(s)\mathrm{d}s.$$

Particularly, when $s' = s^\dagger$, we have

$$\int_{\underline{t}}^{s^\dagger} G(s)\mathrm{d}s \leq \int_{\underline{t}}^{s^\dagger} \hat{G}(s)\mathrm{d}s.$$

Since $G^-(R+c) < F(s^\dagger) = \hat{G}(s^\dagger)$, we know that for every $s \in (s^\dagger, R+c)$,

$$\begin{aligned} G(s) &\leq G^-(R+c) \quad (G \text{ is weakly increasing}) \\ &< \hat{G}(s^\dagger) \\ &\leq \hat{G}(s). \quad (\hat{G} \text{ is weakly increasing}) \end{aligned}$$

Therefore,

$$\int_{\underline{t}}^{(R+c)^-} G(s)\mathrm{d}s < \int_{\underline{t}}^{(R+c)^-} \hat{G}(s)\mathrm{d}s.$$

Finally, for every $s \in [R+c, \bar{t}]$, we have $G(s) \leq 1 = \hat{G}(R+c)$ and thus

$$\int_{\underline{t}}^{\bar{t}} G(s)\mathrm{d}s < \int_{\underline{t}}^{\bar{t}} \hat{G}(s)\mathrm{d}s,$$

which contradicts the fact that G and $\hat{G}$ are both in $\mathcal{G}_F$. Hence, $G^-(R+c) \geq F(s^\dagger)$. $\square$

Obviously, the particular information $\hat{G} \in \mathcal{G}_F$ in (11) puts probability mass $1 - F(s^\dagger)$ at the atom $R+c$, as desired.

Now we prove the "only-if" part for the case of $\mu < R$. Since $\hat{G}$ solves (9) and satisfies $\hat{G}^-(R+c) = F(s^\dagger)$, any solution $G \in \mathcal{G}_F$ to (9) must satisfy $G^-(R+c) = F(s^\dagger)$ as well. We claim that

$$\int_{\underline{t}}^{(R+c)^-} s\mathrm{d}G(s) \geq \int_{\underline{t}}^{s^\dagger} s\mathrm{d}F(s). \tag{31}$$

Since $G \in \mathcal{G}_F$, we know that $\mathbb{E}_G(s) = \mathbb{E}_F(s)$. The claim would imply

$$\int_{R+c}^{\bar{t}} s\mathrm{d}G(s) \leq \int_{s^\dagger}^{\bar{t}} s\mathrm{d}F(s).$$

Using $G^-(R+c) = F(s^\dagger)$ again, we have

$$\begin{aligned}\frac{1}{1-G^-(R+c)}\int_{R+c}^{\bar{t}} s\mathrm{d}G(s) &\leq \frac{1}{1-F(s^\dagger)}\int_{s^\dagger}^{\bar{t}} s\mathrm{d}F(s) \\ &= R+c. \quad \text{(The definition of } s^\dagger\text{)}\end{aligned}$$

This can occur only if G puts all the probability mass $1 - G^-(R+c) = 1 - F(s^\dagger)$ on the atom $R+c$, in which case the inequality holds with equality. This would complete the "only if" part. The rest of the proof verifies the claim (31):

$$\begin{aligned}\int_{\underline{t}}^{(R+c)^-} s\mathrm{d}G(s) &= \left[(R+c)G^-(R+c) - 0\right] - \int_{\underline{t}}^{(R+c)^-} G(s)\mathrm{d}s \\ &= \left[s^\dagger G^-(R+c) - 0\right] + \left[(R+c) - s^\dagger\right] G^-(R+c) - \int_{\underline{t}}^{(R+c)^-} G(s)\mathrm{d}s \\ &= \left[s^\dagger G^-(R+c) - 0\right] + \int_{(s^\dagger)^+}^{(R+c)^-} G^-(R+c)\mathrm{d}s - \int_{\underline{t}}^{(R+c)^-} G(s)\mathrm{d}s \\ &\geq \left[s^\dagger G^-(R+c) - 0\right] - \int_{\underline{t}}^{s^\dagger} G(s)\mathrm{d}s \quad (G \text{ is increasing}) \\ &= \left[s^\dagger F(s^\dagger) - 0\right] - \int_{\underline{t}}^{s^\dagger} G(s)\mathrm{d}s \quad (G^-(R+c) = F(s^\dagger)) \\ &\geq \left[s^\dagger F(s^\dagger) - 0\right] - \int_{\underline{t}}^{s^\dagger} F(s)\mathrm{d}s \quad (G \text{ is a MPC of } F) \\ &= \int_{\underline{t}}^{s^\dagger} s\mathrm{d}F(s).\end{aligned}$$

□

Proof of Proposition 3. If $t^* - c \geq R$, then we know from Lemma 1 that $s^* - c \geq R$. In this case, the principal's payoff is $\mu - R$, independent of information design.

Now suppose $t^* - c < R$. The principal's payoff is again independent of the specific information G as long as the induced s^* satisfies $s^* - c \geq R$, which is $\mu - R$. It remains to search for the principal-optimal information within

$$\{G \in \mathcal{G}_F : s^* - c < R\},$$

where the principal's payoff is

$$\int_{R+c}^{\bar{t}} (s - c - R)\mathrm{d}G(s).$$

In what follows, we first find out one principal-optimal information, and then the characterization follows immediately.

Consider the relaxed problem where the constraint $s^* - c < R$ is dropped:

$$\max_{G \in \mathcal{G}_F} \quad \int_{R+c}^{\bar{t}} (s - c - R) \mathrm{d}G(s)$$

First, for any $G \in \mathcal{G}_F$, the support of G is a subset of $[\underline{t}, \bar{t}]$. Without loss of generality, we assume G is defined on $[\underline{t}, \bar{t}]$. Note that

$$\int_{R+c}^{\bar{t}} (s - c - R) \mathrm{d}G(s) = \int_{\underline{t}}^{\bar{t}} \max\{0, s - c - R\} \mathrm{d}G(s).$$

Since G is a MPC of F and $\max\{0, \cdot\}$ is a convex function, we know that

$$\int_{\underline{t}}^{\bar{t}} \max\{0, s - c - R\} \mathrm{d}F(s) \geq \int_{\underline{t}}^{\bar{t}} \max\{0, s - c - R\} \mathrm{d}G(s). \tag{32}$$

That is, $F \in \mathcal{G}_F$ solves the relaxed problem. Since $t^* - c < R$, the constraint is also satisfied. Hence, F itself is principal-optimal within $\{G \in \mathcal{G}_F : s^* - c < R\}$.

Overall, when $t^* - c < R$, the principal's maximal payoff under designed information is

$$\max\left\{\mu - R, \int_{R+c}^{\bar{t}} (t - c - R) \mathrm{d}F(t)\right\}.$$

By (8), we know that the latter is greater, which means that F is principal-optimal when $t^* - c < R$.

Now we turn to the characterization of the principal-optimal information. Since $\max\{0, s - c - R\}$ is piecewise linear in s and has a unique kink at $s = R + c$, any non-trivial mean-preserving contraction of F across $R + c$ makes (32) strict. Therefore, for any principal-optimal information $G \in \mathcal{G}_F$, we must have $G^-(R + c) = F(R + c)$. In this case, by rewriting the principal's payoff as

$$\int_{R+c}^{\bar{t}} (s - c - R) \mathrm{d}G(s) = G^-(R + c)(R + c) + [1 - G^-(R + c)]\mathbb{E}_G(s \big| [R + c, \bar{t}]) - (R + c),$$

we know that an information $G \in \mathcal{G}_F$ such that $G^-(R + c) = F(R + c)$ is principal-optimal if and only if

$$\mathbb{E}_G(s \big| [R+c, \bar{t}]) \quad = \quad \mathbb{E}_F(s \big| [R+c, \bar{t}]).$$

This completes the characterization and thus the entire proof. □

Proof of Proposition 4. Suppose $\mu \geq R$. To examine (7), we first note that when $s^* - c =$ (*resp.* $<$) R, we have

$$\int_{R+c}^{\bar{s}} (s - c - R)\mathrm{d}G(s) \quad = (\mathit{resp.} >) \quad \mu - R. \tag{33}$$

Therefore, the principal's payoff is bounded from below by $\mu - R$, which is independent of the information G to be designed.

If an information $G \in \mathcal{G}_F$ leads to a threshold s^* such that $s^* \geq R + c$, then the good is allocated to the agent without checking and the principal obtains the lower bound payoff $\mu - R$. Such a G would be principal-worst. So the "if" part holds.

To show the "only-if" part, we shall first notice that for the degenerate distribution $G = \delta(\mu)$, we have $s^* = \mu + c \geq R + c$. Therefore, it is a principal-worst information. If any other information $G \in \mathcal{G}_F$ is principal-worst, then it has to deliver the lower bound payoff $\mu - R$ to the principal. By (33), the principal's payoff is strictly higher than $\mu - R$ whenever $s^* - c < R$. Therefore, a principal-worst information must admit a threshold s^* such that $s^* \geq R + c$.

Now suppose $\mu < R$. We have seen in Lemma 2 that for any $G \in \mathcal{G}_F$, we have $s^* < R + c$. Therefore, according to (7), the payoff applicable here is $\int_{R+c}^{\bar{s}} (s - c - R)\mathrm{d}G(s)$. It is straightforward to see that

$$\int_{R+c}^{\bar{s}} (s - c - R)\mathrm{d}G(s) \geq 0. \tag{34}$$

Since the degenerate distribution $G = \delta(\mu)$ is in $\mathcal{G}_F$ and $\mu < R + c$, the lower bound payoff of zero is attainable by G and any information which attains the lower bound is principal-worst. Obviously, (34) holds with equality if and only if $G(R+c)=1$. $\square$

Proof of Proposition 5. The case of $\mu \geq R$ is trivial. When $\mu < R$, we claim that any agent-optimal information $G \in \mathcal{G}_F$ must satisfy $1-G(R+c)=0$. Then Proposition 4 would imply that G is also principal-worst.

Suppose to the contrary that $1-G(R+c)>0$. Then

$$\begin{aligned}
1-G^{-}(R+c) &= g(R+c)+[1-G(R+c)] \\
&= 1-F(s^{\dagger})+[1-G(R+c)] \quad \text{(Proposition 2)} \\
&> 1-F(s^{\dagger}) \\
&= 1-\hat{G}^{-}(R+c). \quad (11)
\end{aligned}$$

Therefore, the agent's payoff is strictly higher under G than under $\hat{G}$, which is a contradiction to the optimality of $\hat{G}$. This completes the proof. $\square$

Proof of Proposition 8. We need to consider two cases: (1) $\max_{i\in\mathcal{I}}\{t_i^*-c_i\} \geq R$ and (2) $\max_{i\in\mathcal{I}}\{t_i^*-c_i\} < R$.

If $\max_{i\in\mathcal{I}}\{t_i^*-c_i\} \geq R$, then by Lemma 1, we must have $\max_{i\in\mathcal{I}}\{s_i^*-c_i\} \geq R$ for any profile of designed signal distributions. According to Proposition 6, the optimal mechanism admits a favored-agent $i \in \mathcal{I}$ and has a threshold $v^* = s_i^* - c_i$. In this case, the principal's reservation value is irrelevant in the analysis. The rules of the favored-agent mechanism imply that the principal's payoff in this case is

$$\begin{aligned}
&(35) \quad \int_{\mathbf{s}:\max_{j\neq i}\{s_j-c_j\}\leq v^*} s_i \mathrm{d}\mathbf{G}(\mathbf{s}) + \int_{\mathbf{s}:\max_{j\neq i}\{s_j-c_j\}>v^*} \max_{k\in\mathcal{I}}\{s_k-c_k\}\,\mathrm{d}\mathbf{G}(\mathbf{s}) \\
&= \mathbb{E}_{\mathbf{G}}\left(s_i \middle| \max_{j\neq i}\{s_j-c_j\}\leq v^*\right)\prod_{j\neq i} G_j(v^*+c_j) \\
&(36) \quad + \mathbb{E}_{\mathbf{G}}\left(\max_{k\in\mathcal{I}}\{s_k-c_k\} \middle| \max_{j\neq i}\{s_j-c_j\}>v^*\right)\left[1-\prod_{j\neq i} G_j(v^*+c_j)\right].
\end{aligned}$$

Information design affects (36), as a weighted average, in two aspects: the weights and the two weighted terms. We first compare the two weighted terms:

Claim 2.

$$\mathbb{E}_{\mathbf{G}}\left(s_i \middle| \max_{j\neq i}\{s_j-c_j\}\leq v^*\right)\leq \mathbb{E}_{\mathbf{G}}\left(\max_{k\in\mathcal{I}}\{s_k-c_k\} \middle| \max_{j\neq i}\{s_j-c_j\}>v^*\right)$$

Proof. Rewrite the first term of (35) as follows:

$$\begin{aligned}
&\int_{\mathbf{s}:\max_{j\neq i}\{s_j-c_j\}\leq v^*} s_i \mathrm{d}\mathbf{G}(\mathbf{s}) \\
=&\int_{\mathbf{s}_{-i}:\max_{j\neq i}\{s_j-c_j\}\leq v^*}\int_{s_i} s_i \mathrm{d}G_i(s_i)\mathrm{d}\mathbf{G}_{-i}(\mathbf{s}_{-i}) \\
=&\int_{\mathbf{s}_{-i}:\max_{j\neq i}\{s_j-c_j\}\leq v^*} \mathbb{E}_{G_i}(s_i)\,\mathrm{d}\mathbf{G}_{-i}(\mathbf{s}_{-i}) \\
=&\int_{\mathbf{s}_{-i}:\max_{j\neq i}\{s_j-c_j\}\leq v^*} \mathbb{E}_{G_i}(\max\{s_i-c_i,v^*\})\,\mathrm{d}\mathbf{G}_{-i}(\mathbf{s}_{-i}) \quad (\text{Definition of } s_i^*)\ .
\end{aligned}$$

Then, we know that

$$\mathbb{E}_{\mathbf{G}}\left(s_i \middle| \max_{j\neq i}\{s_j-c_j\}\leq v^*\right)=\mathbb{E}_{\mathbf{G}}\left(\max\{s_i-c_i,v^*\} \middle| \max_{j\neq i}\{s_j-c_j\}\leq v^*\right).$$

Note that for any s_i, any $\mathbf{s}'=(s_i,\mathbf{s}'_{-i})\in\{\mathbf{s}:\max_{j\neq i}\{s_j-c_j\}\leq v^*\}$ and any $\mathbf{s}''=(s_i,\mathbf{s}''_{-i})\in\{\mathbf{s}:\max_{j\neq i}\{s_j-c_j\}>v^*\}$, we must have

$$\max\{s_i-c_i,v^*\}\leq\max\left\{s_i-c_i,\max_{j\neq i}\{s''_j-c_j\}\right\}=\max_{k\in\mathcal{I}}\{s''_k-c_k\},$$

which implies that

$$\mathbb{E}_{\mathbf{G}}\left(\max\{s_i-c_i,v^*\} \middle| \max_{j\neq i}\{s_j-c_j\}\leq v^*\right)\leq\mathbb{E}_{\mathbf{G}}\left(\max_{k\in\mathcal{I}}\{s_k-c_k\} \middle| \max_{j\neq i}\{s_j-c_j\}>v^*\right).$$

The claim follows immediately. □

We have two observations. First, since the weight for the smaller term in (36), i.e., $\prod_{j\neq i}G_j(v^*+c_j)$, is increasing in v^*/s_i^*, Claim 2 implies that the principal prefers a smaller s_i^* to a greater one.

The second observation is on the two weighted terms in (36). The first weighted term is simply $\mathbb{E}_{G_i}(s_i)=\mathbb{E}_{F_i}(t_i)$, which is independent of information design. The second weighted term is the conditional expectation of $\max_{k\in\mathcal{I}}\{s_k-c_k\}$, which is obviously a convex function of $\mathbf{s}$. Therefore, the second weighted term is decreasing in each G_j in the second-order stochastic dominance sense, i.e., the mean-preserving spread increases its value.

Overall, full information for every $j \in \mathcal{I}$, i.e., $G_j = F_j$, would (i) minimize the weight for the smaller weighted term in (36) by minimizing s_i^* (Lemma 1) and (ii) maximize the larger weighted term. Hence, such a design is principal-optimal.

Now we consider the case with $\max_{i\in\mathcal{I}} \{t_i^* - c_i\} < R$. If for some agent $i \in \mathcal{I}$, the designed signal distribution G_i induces an s_i^* such that $s_i^* - c_i \geq R$, then there will be a favored-agent and the principal's payoff takes the form of (36). However, since the principal prefers smaller s_i^* to a greater one and prefers mean-preserving spread to mean-preserving contraction, making $s_i^* - c_i \geq R$ (through a mean-preserving contraction) is never optimal. In contrast, if for all $i \in \mathcal{I}$, the designed signal distribution induces an s_i^* such that $s_i^* - c_i < R$, then the principal's payoff is

$$R \prod_{i\in\mathcal{I}} G_i(R+c) + \mathbb{E}_{\mathbf{G}} \left[\max_{k\in\mathcal{I}} \{s_k - c_k\} \,\middle|\, \max_{k\in\mathcal{I}} \{s_k - c_k\} > R \right] \left[1 - \prod_{i\in\mathcal{I}} G_i(R+c) \right].$$

By similar reasons as for (36), full information for every agent is principal-optimal. This completes the proof for the second case and thus the entire proposition. □

Proof of Proposition 9. The two statements are proved simultaneously, without figuring out a principal-worst information first.

First, the principal's payoff under the principal-worst information, denoted by Y^{PW}, is weakly worse than the payoff under the "uniform-null" information distributions (one feasible choice of information design since every $\tilde{\mathcal{G}}_i$ contains the null information $\delta(\mathbb{E}_{F_i}(t_i))$). That is,

$$\max \{R, \mathbb{E}_{F_1}(t_1), \ldots, \mathbb{E}_{F_I}(t_I)\} \geq Y^{PW}.$$

Second, by the max-min inequality, the principal's worst-case payoff is an upper bound for her payoff from the optimal robust mechanism, whatever it is. Therefore, we have $Y^{PW} \geq Y^{rbst}$, which implies that

$$\max\{R, \mathbb{E}_{F_1}(t_1), \ldots, \mathbb{E}_{F_I}(t_I)\} \geq Y^{rbst}.$$

Finally, since the mechanism that allocates the good to the agent (possibly the principal) who has the highest expected value is robust and it can achieve the upper bound payoff (22), we know that such a mechanism is optimal among robust mechanisms. As a byproduct, (22) is indeed the principal's payoff under the principal-worst information. □

Appendix B Pareto agent-optimal information

Example 2 (Pareto agent-optimal information needs not be aggregate agent-optimal). Consider an example with $c_i = c$ and for all i, $\mu_i < R$. Consider the following information structure $\{G_i\}_{i\in\mathcal{I}}$ that for some agent i, $G_i = \hat{G}_i$ defined in Proposition 8 (Section 6.1) and $G_i = \delta(\mu_j)$ for any $j \neq i$. Under $\{G_i\}_{i\in\mathcal{I}}$,

$$p_i = 1 - F_i(s_i^\dagger), \text{ and } p_j = 0 \quad \forall j \neq i.$$

First, there is no other information structure $\{G_i'\}_{i\in\mathcal{I}}$ such that $p_i' > p_i$, since the good will be allocated to agent i only if $s_i \geq R + c$. Thus suppose there is a Pareto-improvement information structure $\{G_i'\}_{i\in\mathcal{I}}$. There must be at least one $j \neq i$ such that $p_j > 0$, therefore for agent j,

$$Prob\{s_j \geq R + c\} > 0.$$

In this case, if we choose the equal tie-breaking rule, the probability of getting the good for agent i is

$$\begin{aligned}p_i' =& \frac{1}{2}(1 - F_i(s_i^\dagger))Prob\{s_j = R + c\} + (1 - F_i(s_i^\dagger))\left(1 - Prob\{s_j \geq R + c\}\right) \\ =& (1 - F_i(s_i^\dagger))\left(\frac{1}{2}Prob\{s_j = R + c\} + 1 - Prob\{s_j \geq R + c\}\right) \\ \leq& (1 - F_i(s_i^\dagger))\left(+1 - \frac{1}{2}Prob\{s_j \geq R + c\}\right) < (1 - F_i(s_i^\dagger)) = p_i.\end{aligned}$$

Therefore, if we strictly benefit some agents, we must strictly hurt some other agents. Hence a Pareto improvement is impossible given $\{G_i\}_{i\in\mathcal{I}}$.

Appendix C Agents' information acquisition game

A principal is to decide whether or not to allocate an indivisible good to multiple agents. We study the context in which agents can strategically engage in information acquisition. That is, each agent i does not know the realization of t_i *á priori* but can choose to acquire information about t_i. We assume that agents cannot communicate and can only choose their own signal structure. Formally, each agent i can independently choose a signal distribution $G_i \in \mathcal{G}_i$. After seeing the agents' information acquisition behavior, the principal will implement the optimal favored-agent mechanism with respect to the action profile $\mathbf{G}$ (Proposition 6).[19] An agent's utility is simply the probability of receiving the good after the optimal mechanism is executed.

In what follows, we provide an asymptotically symmetric Nash equilibrium of the game at the information acquisition stage, which describes the strategic behaviors of the agents when acquiring information.

Proposition 10. *Given any F, the strategy profile where every agent chooses $G_i =$*

[19] Yang (2019, 2021) study such a similar information acquisition game among agents in the auction model.

$F \in \mathcal{G}_F$, i.e., fully revelation, forms a symmetric Nash equilibrium in the information acquisition game as $n \to \infty$.

Proof. We argue that any deviation $G_i(s_i)$ for agent i would not be profitable. This argument contains two parts:

1. First, if agent i's realization $s_i < \bar{t}$, then agent i has zero probability to win this good. Thus, in this case, agent i will gain zero profit. To see this, since $\bar{t}$ is on the support of F, there exists a positive $\varepsilon > 0$, such that for any other agent j,

$$\text{Prob}\{s_j | s_i < s_j \leq \bar{t}\} > \varepsilon.$$

   Therefore, the probability of the largest realization of other agents $\max\{x_{-i}\}$ which is strictly greater than s_i is

$$\text{Prob}\{s | s = \max\{s_{-i}\} > s_i\} > 1 - (1-\varepsilon)^n,$$

   which converges to 1 as $n \to \infty$.

2. Second, we consider the case that agent i has the chance to win with positive probability, that is $s_i = \bar{t}$. However, in this case, we claim that the probability that $s_i = \bar{t}$ under F is no less than that under any other distributions $G_i \in \mathcal{G}_F$. Hence, in this case, any deviation is still not profitable.

   This directly follows from the definition of the mean-preserving contraction. To see this, since $G_i \in \mathcal{G}_F$, for any $s_i \in [\underline{t}, \bar{t}]$,

$$\int_{s_i}^{\bar{t}} F(s)\mathrm{d}s \geq \int_{s_i}^{\bar{t}} G_i(s)\mathrm{d}s$$

   Now suppose that $\text{Prob}_F\{s_i = \bar{t}\} < \text{Prob}_{G_i}\{s_i = \bar{t}\}$. Then as $s_i \to \bar{t}$,

$$\int_{s_i}^{\bar{t}} F(s)\mathrm{d}s < \int_{s_i}^{\bar{t}} G_i(s)\mathrm{d}s,$$

   which contradicts the definition of the mean-preserving contraction.

□

While for finite agents, there may not exist a pure-strategy Nash equilibrium, which is illustrated by the following Example 3.

Example 3 (Nonexistence of pure-strategy Nash equilibrium)**.** Let F be an uniform distribution on $[0,1]$, $R = 1/4$ and $c = 1/8$ with two agents. The maximal s^*_{max} is $\mathbb{E}(t)+c = 1/2 + 1/8 = 5/8$, which can be induced by any $G \in \mathcal{G}_F$ such that supp $G \subseteq [0, s^*_{max}]$. Without loss of generality, let G be

$$G(s) = \begin{cases} s, & \text{if } s \in [0, 1/4]; \\ 1/4, & \text{if } s \in [1/4, 5/8); \\ 1, & \text{if } s \in [5/8, 1]. \end{cases}$$

Suppose (G, G) is a pure-strategy Nash equilibrium. Any deviation should be not profitable. Now define $\hat{G}$ to be

$$\hat{G}(s) = \begin{cases} s, & \text{if } s \in [0, 1/4 + 2\varepsilon]; \\ 1/4 + 2\varepsilon, & \text{if } s \in [1/4 + 2\varepsilon, 5/8 + \varepsilon); \\ 1, & \text{if } s \in [5/8 + \varepsilon, 1], \end{cases}$$

for a small enough $\varepsilon > 0$. This $\hat{G}$ has two features: first, the correspondent $\hat{s}^*$ is strictly smaller than s^*_{max}; second, it puts as much as possible mass above s^*_{max}. Each agent has the incentive to deviate from G to $\hat{G}$ since

$$1 - (1/4 + 2\varepsilon) > 1/2.$$

Then we show the following claim:

Claim 3. *Any strategy profile (G, G) with $s^* < s^*_{max}$ can not be an Nash equilibrium.*

Proof. Since $s^* < s^*_{max}$, there must be s^* is strictly within the supp G. Now define $\tilde{G}$ to be

$$\tilde{G}(s) = \begin{cases} G(s) & \text{if } s \in [0, s^* - \varepsilon_1]; \\ \\ G(s^* - \varepsilon_1), & \text{if } s \in [s^* - \varepsilon_1, s^*); \\ \\ G(s^* + \varepsilon_2) & \text{if } s \in [s^*, s^* + \varepsilon_2); \\ \\ G(s), & \text{if } s \in [s^* + \varepsilon_2, 1], \end{cases}$$

where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are chosen such that $\tilde{G} \in \mathcal{G}_F$ and $G(s^*) + \varepsilon > G(s^* + \varepsilon_2) > G(s^*)$ for a small enough $\varepsilon > 0$. The last condition ensures that $\tilde{s}^* > s^*$. In this case, each agent has incentives to deviate from G to $\tilde{G}$ since the benefit is much more than the cost.

$$G^2(s^* - \varepsilon_1) > \varepsilon.$$

□

Remark. Even under the case $c > 1/4$ that the deviation from G to $\hat{G}$ is nonprofitable, we are still hard to say that (G, G) is a pure-strategy Nash equilibrium, since the deviation within the class with s^*_{max} may still be profitable. We leave the characterization of the symmetric pure-strategy (or even mixed-strategy) Nash equilibrium for future research.

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