# Specification Testing of DSGE Models Allowing for Indeterminacy and Weak Identification

Zhongjun Qu
BU

Denis Tkachenko
NUS

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# Abstract

We propose methods for testing the specification of linearized DSGE models allowing for multiple equilibria and weak identification. We build the framework from a frequency-domain perspective based on a new test statistic, enabling the separate evaluation of a model's static properties, dynamic properties, and properties over a chosen frequency band, such as the business cycle frequencies. The results are indicative of which variables and frequencies contribute to misspecification. We show that the test has a likelihood ratio interpretation and is consistent. In an application, the methods reject a typical small-scale DSGE model, revealing specification problems in the inflation dynamics and comovements between variables over business cycle frequencies.

# Highlights

- A family of specification tests for log-linearized DSGE models. ($H_0$: correct model specification; $H_1$: misspecification.)
- A frequency-domain perspective.
- Robust to weak and lack of identification.
- Allowing for indeterminacy (i.e., multiple equilibria).

# Outline

1. Two illustrative examples:
   interplay between parameter identification and specification analysis.
2. Test statistics.
3. A likelihood perspective.
4. Asymptotic properties.
5. A testing procedure robust to weak identification.
6. Simulation.
7. Applications to An and Schorfheide (2007): robust confidence intervals and specification tests.

# Related studies

- Frequency domain analysis of rational expectations models:
  Altug (1989)
  Hansen and Sargent (1993)
  Watson (1993)
  King and Watson (1996)
  Diebold, Ohanian and Berkowitz (1998)
  Christiano and Vigfusson (2003)
  Del Negro, Diebold and Schorfheide (2008)
  Qu and Tkachenko (2012, 2017)

- Integrated periodograms for misspecification testing:
  Grenander and Rosenblatt (1957)
  Bartlett (1955)

- Identification-robust inference for DSGE models:
  Guerron-Quintana, et al. (2013)
  Qu (2014)
  Andrews and Mikusheva (2015).

# Related studies (cont'd)

- Evaluating DSGE models: Del Negro et al. (2007), Del Negro and Schorfheide (2009), Inoue et al. (2020).
- Researchers often look at marginal likelihoods, recommending $M_1$ over $M_2$ if

$$p(M_1|Y) > p(M_2|Y).$$

  Unclear if the preferred model is compatible with data.
- No frequentist testing procedure for whether a DSGE model is compatible with data.
- We decompose the likelihood to provide formal tests of model specification.

# Motivating example I

$$y_t = \rho \mathbb{E}_t y_{t+1}.$$

- Regime 1: $|\rho| < 1$, determinacy,

$$y_t = \lim_{k \to \infty} \rho^k \mathbb{E}_t y_{t+k} = 0.$$

- Regime 2: $|\rho| > 1$, indeterminacy,

$$y_t = \frac{1}{\rho} y_{t-1} + \epsilon_t, \text{ where } \mathbb{E}_t \epsilon_{t+1} = 0.$$

- $\rho$ : structural parameter; $\epsilon_t$ : sunspot shock; $\sigma_\epsilon$ : sunspot parameter.
- Parameters are identified in Regime 2 but not in Regime 1.
- Different dynamic properties, therefore, different fits to data.

# Motivating example II

- An and Schorfheide (2007) with $(y_t, \pi_t, r_t)$ =(output, inflation, interest rate):

$$
\begin{aligned}
y_t &= \mathbb{E}_t y_{t+1} + g_t - \mathbb{E}_t g_{t+1} - \frac{1}{\tau}(r_t - \mathbb{E}_t \pi_{t+1} - \mathbb{E}_t z_{t+1}) \\
\pi_t &= \beta E_t \pi_{t+1} + \kappa(y_t - g_t) \\
r_t &= \rho_r r_{t-1} + (1-\rho_r)\psi_1 \pi_t + (1-\rho_r)\psi_2 (y_t - g_t) + \varepsilon_{rt} \\
g_t &= \rho_g g_{t-1} + \varepsilon_{gt} \\
z_t &= \rho_z z_{t-1} + \varepsilon_{zt},
\end{aligned}
$$

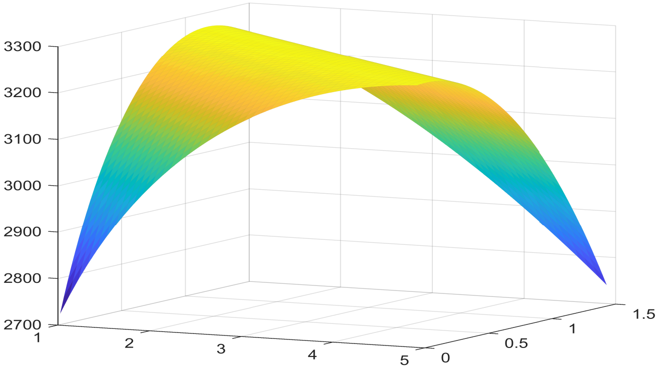
where

$$
\varepsilon_{rt} \overset{iid}{\sim} N(0, \sigma_r^2),\ \varepsilon_{gt} \overset{iid}{\sim} N(0, \sigma_g^2),\ \varepsilon_{zt} \overset{iid}{\sim} N(0, \sigma_z^2).
$$

- $\psi_1 > 1$: unique equilibrium; $(\rho_r, \psi_1, \psi_2, \epsilon_{rt})$ not separately identifiable.
- $0 < \psi_1 < 1$: indeterminacy; all parameters identified.
- Different dynamic properties, therefore, different fits to data.

# Likelihood surface of monetary policy parameters

$$r_t = \rho_r r_{t-1} + (1-\rho_r)\psi_1 \pi_t + (1-\rho_r)\psi_2 (y_t - g_t) + \varepsilon_{rt}$$




Similar identification issues arise in Lubik and Schorfheide (2004), Tan and Walker (2015) and Schmitt-Grohé and Uribe (2012).

# The spectrum of a DSGE model

- DSGE model (Sims, 2002):

$$\Gamma_0 S_t = \Gamma_1 S_{t-1} + \Psi \varepsilon_t + \Pi \eta_t. \tag{1}$$

In An and Schorfheide (2007):

$$\begin{aligned} S_t &= (r_t, y_t, \pi_t, g_t, z_t, \mathbb{E}_t(\pi_{t+1}), \mathbb{E}_t(y_{t+1}))', \\ \varepsilon_t &= (\varepsilon_{rt}, \varepsilon_{gt}, \varepsilon_{zt}, )', \\ \eta_t &= (\pi_t - \mathbb{E}_{t-1}(\pi_t), y_t - \mathbb{E}_{t-1}(y_t))'. \end{aligned}$$

- Model solution (Lubik and Schorfheide, 2003):

$$S_t = \Phi_1 S_{t-1} + \Phi_\varepsilon \varepsilon_t + \Phi_\epsilon \epsilon_t,$$

where $\varepsilon_t$ are structural shocks and $\epsilon_t$ sunspots. Thus,

$$S_t = (1 - \Phi_1 L)^{-1} [\Phi_\varepsilon, \quad \Phi_\epsilon] \begin{bmatrix} \varepsilon_t \\ \epsilon_t \end{bmatrix}.$$

# The spectrum of a DSGE model (cont'd)

- The observables satisfy

$$Y_t = A(L)S_t = \underbrace{A(L)(1-\Phi_1 L)^{-1}[\Phi_\varepsilon, \quad \Phi_\epsilon]}_{H(L;\theta)} \begin{bmatrix} \varepsilon_t \\ \epsilon_t \end{bmatrix}.$$

The spectral density matrix equals

$$f_\theta(\omega) = \frac{1}{2\pi} H(\exp(-i\omega);\theta)\Sigma(\theta)H(\exp(-i\omega);\theta)^*,$$

where $\theta$ is the entire parameter vector.

- Sample: $\{Y_1, Y_2, ..., Y_T\}$.
- Fourier frequencies: $\omega_j = 2\pi j/T$ $(j = 0, 1, ..., T-1)$.
- Fourier transform and periodogram:

$$\begin{aligned} w_T(\omega_j) &= (2\pi T)^{-1/2} \sum_{t=1}^{T} Y_t \exp(-i\omega_j t), \\ I_T(\omega_j) &= w_T(\omega_j) w_T(\omega_j)^*. \end{aligned}$$

# Proposed tests for model specification

- Consider a fixed parameter value $\theta_0$.
- A KS test for **dynamic specification**:

$$\mathcal{H}_{dT}^{W}(\theta_0) = \sup_{r\in[0,1]} \left\| \left(\frac{T}{2}\right)^{-1/2} \sum_{j=1}^{[Tr/2]} W(\omega_j)\,\mathrm{vec}\left\{ f_{\theta_0}^{-1/2}(\omega_j)\left[I_T(\omega_j) - f_{\theta_0}(\omega_j)\right] f_{\theta_0}^{-1/2}(\omega_j)\right\} \right\|_\infty ,$$

  $W(\omega_j)$ is a smooth or indicator function to select frequencies.

- A KS test for **static** specification:

$$\mathcal{H}_{sT}(\theta_0) = \sup_{r\in[0,1]} \left\| \frac{1}{\sqrt{2\pi T}} f_{\theta_0}^{-1/2}(0) \sum_{j=1}^{[Tr]} (Y_t - \mu(\theta_0)) \right\|_\infty .$$

- A joint test for **both**:

$$\mathcal{H}_T(\theta_0) = \max\left(\mathcal{H}_{sT}(\theta_0), \mathcal{H}_{dT}^{W}(\theta_0)\right).$$

# A likelihood interpretation

- The frequency-domain log-likelihood (dynamic properties only):

$$L_T(\theta) = -\frac{1}{2T^{1/2}} \sum_{j=1}^{T-1} \left[\log\det\left(f_\theta(\omega_j)\right) + \mathrm{tr}\left\{f_\theta^{-1}(\omega_j) I_T(\omega_j)\right\}\right].$$

- Its expectation at the true DGP:

$$-\frac{1}{2T^{1/2}} \sum_{j=1}^{T-1} \left\{\log\det(f_0(\omega_j)) + n_Y\right\} + o(1).$$

- The difference (omitting components independent of data):

$$-\frac{1}{2T^{-1/2}} \sum_{j=1}^{T-1} \mathrm{vec}\left\{f_\theta(\omega_j)^{-1/2}\left(I_T(\omega_j) - f_\theta(\omega_j)\right) f_\theta(\omega_j)^{-1/2}\right\}.$$

- The test statistic has a KL interpretation because it gauges the distance between the likelihood and its expected value under the null hypothesis.

# Extension: testing a model segment

- E.g., testing inflation-output dynamics only.
- Let $A$ be a variable selection matrix.
- Construct the test after replacing $Y_t$ and $f_{\theta_0}(\omega)$ with $AY_t$ and $Af_{\theta_0}(\omega)A'$.

# Extension: Factor augmented VAR

$$
\begin{aligned}
Y_t &= \lambda(L) f_t + D(L) Y_{t-1} + v_t, \\
f_t &= \Gamma(L) f_{t-1} + \zeta_t.
\end{aligned}
$$

- Under stationarity,

$$Y_t = H_1(L;\theta)\zeta_t + H_2(L;\theta)v_t$$

with

$$
\begin{aligned}
H_1(L;\theta) &= [I - D(L)L]^{-1} \lambda(L) [I - \Gamma(L)L]^{-1}, \\
H_2(L;\theta) &= [I - D(L)L]^{-1}.
\end{aligned}
$$

- The spectral density of $Y_t$ is

$$
\begin{aligned}
f_\theta(\omega) &= \frac{1}{2\pi} H_1(\exp(-i\omega);\theta) H_1(\exp(-i\omega);\theta)^* \\
&\quad + \frac{1}{2\pi} H_2(\exp(-i\omega);\theta) \Sigma H_2(\exp(-i\omega);\theta)^*.
\end{aligned}
$$

- Construct the test using $Y_t$ and $f_\theta(\omega)$.

# Asymptotic properties – assumptions:

- **Assumption 1.** $\theta_0 \in \Theta \subset \mathbb{R}^q$ with a compact $\Theta$.

- **Assumption 2.** The model solution is

$$Y_t^d(\theta) = H(L;\theta)u_t(\theta) \quad \text{with} \quad H(L;\theta) = \sum_{j=0}^{\infty} h_j(\theta)L^j,$$

  where $u_t(\theta)$ are serially uncorrelated with a nonsingular covariance $\Sigma(\theta)$.

- **Assumption 3.** There are $C_L$ and $C_U$ such that for $\omega \in [-\pi, \pi]$ and $\theta \in \Theta$:
  (i) $C_L \leq \mathrm{eig}(f_\theta(\omega)) \leq C_U$;
  (ii) $f_\theta(\omega)$ belongs to Lip($\beta$) with $\beta > 1/2$;
  (iii) $||\partial \operatorname{vec} f_\theta(\omega)/\partial\theta'|| \leq C_U$;
  (iv) $\partial \operatorname{vec} f_\theta(\omega)/\partial\theta$ belongs to Lip($\beta$) with $\beta > 1/2$.
  (v) $||\partial\mu(\theta)/\partial\theta'|| \leq C_U$.

- **Assumption 4.** $\{Y_t\}_{t=1}^T$ are multivariate normal random vectors.

# Distributions under the null

**Theorem 1**. Suppose $\{Y_t\}_{t=1}^T$ satisfy Assumptions 1-4. Then,

1.
$$\mathcal{H}_{dT}^W(\theta_0) \Rightarrow G_d^W(s) \equiv \sup_{r\in[0,1]} \left\| \int_0^r W(s) \mathrm{d} G_d(s) \right\|_\infty .$$

   The first $n_Y$ elements of $G_d(r)$ are independent Wiener processes. The remaining $n_Y(n_Y-1)/2$ elements are independent copies of $\tilde{B}(r) = (B_1(r) + iB_2(r))/\sqrt{2}$ with $B_1(r)$ and $B_2(r)$ being independent Wiener processes.

2.
$$\mathcal{H}_{sT}(\theta_0) \Rightarrow \sup_{r\in[0,1]} \|G_s(r)\|_\infty ,$$

   where $G_s(r)$ is an $n_Y$-vector of independent Wiener processes.

3.
$$\mathcal{H}_T(\theta_0) \Rightarrow \max\left( \sup_{r\in[0,1]} \left\| G_d^W(r) \right\|_\infty , \sup_{r\in[0,1]} \|G_s(r)\|_\infty \right),$$

   where the elements of $G_d^W(r)$ and $G_s(r)$ are mutually independent.

# Consistency

**Theorem 2**. Suppose $\{Y_t\}$ is weakly stationary with $\mathbb{E}Y_t = \mu_0$ and its spectral density $f_0(\omega)$ satisfies Assumptions 1-4. Suppose $\mu(\theta_0)$ and $f_{\theta_0}(\omega)$ are the mean and spectral density implied by the DSGE model, satisfying Assumptions 1-4. Let $\delta > 0$ be an arbitrary constant independent of $T$. Then:

1. $\mathcal{H}_{dT}^{W}(\theta_0) \to \infty$ if $\|f_0(\omega) - f_{\theta_0}(\omega)\| > \delta$ for some $\omega$ with $W(\omega) = 1$.
2. $\mathcal{H}_{sT}(\theta_0) \to \infty$ if $\|\mu_0 - \mu(\theta_0)\| > \delta$.
3. $\mathcal{H}_T(\theta_0) \to \infty$ if $\|\mu_0 - \mu(\theta_0)\| > \delta$ or $\|f_0(\omega) - f_{\theta_0}(\omega)\| > \delta$ for some $\omega$ with $W(\omega) = 1$.

# A robust specification testing procedure

**Two steps**:

1. Obtain a robust confidence set by inverting the score test of Qu (2014).
2. Compute the KS tests using all values in this set.

**Remarks**:

- Reject the null if all points are rejected in Step 2.
- Use a Bonferroni adjustment to control the overall size.
- This procedure first provides a set of plausible parameter values and then checks whether any of them are compatible with the data.

# The score test of Qu (2014) for the confidence set

$$S_T(\theta) = D_T(\theta)' M_T^{+}(\theta) D_T(\theta)$$

with

$$D_T(\theta) = 2\pi T^{-1/2} \sum_{j=1}^{T-1} W(\omega_j) \left( \frac{\partial \operatorname{vec} f_\theta(\omega_j)}{\partial \theta'} \right)^* (f_\theta^{-1}(\omega_j)' \otimes f_\theta^{-1}(\omega_j)) \operatorname{vec} \left( I_T(\omega_j) - f_\theta(\omega_j) \right)$$

and

$$M_T(\theta) = 8\pi^2 T^{-1} \sum_{j=1}^{T-1} W(\omega_j) \left( \frac{\partial \operatorname{vec} f_\theta(\omega_j)}{\partial \theta'} \right)^* \left( f_\theta^{-1}(\omega_j)' \otimes f_\theta^{-1}(\omega_j) \right) \left( \frac{\partial \operatorname{vec} f_\theta(\omega_j)}{\partial \theta'} \right),$$

- Under Assumptions 1-4,

$$\lim_{T\to\infty} \Pr\left(S_T(\theta_0) \le c\right) \to \Pr\left(\chi^2_{q-q_3} \le c\right),$$

  where $q = \dim(\theta_0)$ and $q_3$ is the number of unidentified parameters.
- The pivotal property enables fast computation over many replications.

# Simulations

- DGP: An and Schorfheide (2007):

$$
\begin{aligned}
y_t &= \mathbb{E}_t y_{t+1} + g_t - \mathbb{E}_t g_{t+1} - \frac{1}{\tau}(r_t - \mathbb{E}_t \pi_{t+1} - \mathbb{E}_t z_{t+1}) \\
\pi_t &= \beta E_t \pi_{t+1} + \kappa(y_t - g_t) \\
r_t &= \rho_r r_{t-1} + (1-\rho_r)\psi_1 \pi_t + (1-\rho_r)\psi_2 (y_t - g_t) + \varepsilon_{rt} \\
g_t &= \rho_g g_{t-1} + \varepsilon_{gt} \\
z_t &= \rho_z z_{t-1} + \varepsilon_{zt},
\end{aligned}
$$

with $\varepsilon_{rt} \overset{iid}{\sim} N(0, \sigma_r^2)$, $\varepsilon_{gt} \overset{iid}{\sim} N(0, \sigma_g^2)$, $\varepsilon_{zt} \overset{iid}{\sim} N(0, \sigma_z^2)$.

- Parameter values (empirical estimates):

$$
\begin{aligned}
\theta_0 &= (\tau, \kappa, \psi_1, \psi_2, \rho_r, \rho_g, \rho_z, \sigma_r, \sigma_g, \sigma_z, r^*, \pi^*, y^*) \\
&= (2, 0.15, 1.5, 1.0, 0.6, 0.95, 0.65, 0.2, 0.8, 0.45, 0.40, 4.00, 0.50).
\end{aligned}
$$

- Rejection frequencies of $\mathcal{H}_{dT}^{W}(\theta_0), \mathcal{H}_{dT}(\theta_0), \mathcal{H}_T(\bar{\theta}_0)$.

# Size

Table 1. Rejection frequencies under the null hypothesis

| Level | $T$ | BC frequencies $\mathcal{H}^{W}_{dT}(\theta_0)$ | Full spectrum $\mathcal{H}_{dT}(\theta_0)$ | Mean and spectrum $\mathcal{H}_{T}(\bar{\theta}_0)$ |
|---|---|---|---|---|
| 10% | 80 | 0.096 | 0.092 | 0.078 |
| | 160 | 0.102 | 0.093 | 0.076 |
| | 240 | 0.099 | 0.103 | 0.079 |
| | 320 | 0.090 | 0.094 | 0.078 |
| 5% | 80 | 0.067 | 0.055 | 0.051 |
| | 160 | 0.062 | 0.049 | 0.038 |
| | 240 | 0.056 | 0.050 | 0.036 |
| | 320 | 0.051 | 0.052 | 0.037 |

# Power

| Table 2. Rejection frequencies under the alternative hypothesis | | | | |
|---|---|---|---|---|
| | | BC frequencies | Full spectrum | Mean and spectrum |
| Level | $T$ | $\mathcal{H}_{dT}^{W}(\theta_0)$ | $\mathcal{H}_{dT}(\theta_0)$ | $\mathcal{H}_{T}(\bar{\theta}_0)$ |
| Randomly perturb an element of $\theta_0$ by 20% | | | | |
| 10% | 80 | 0.231 | 0.381 | 0.353 |
| | 160 | 0.363 | 0.540 | 0.516 |
| | 240 | 0.399 | 0.629 | 0.632 |
| | 320 | 0.436 | 0.676 | 0.694 |
| Randomly perturb an elements of $\theta_0$ by 40% | | | | |
| 10% | 80 | 0.458 | 0.634 | 0.657 |
| | 160 | 0.594 | 0.790 | 0.792 |
| | 240 | 0.641 | 0.850 | 0.865 |
| | 320 | 0.689 | 0.871 | 0.900 |

# Application

- An and Schorfheide (2007), a popular small-scale model.
- Data: quarterly observations on output, inflation, and interest rate for 1965:I-2021:IV.
- 95% confidence intervals.
- 10% specification tests based on the full spectrum and business cycle frequencies.

# Confidence intervals

Table 3. 95% confidence intervals based on the mean and the spectrum

| Parameter | $\theta_0$ | Bounds | CI |
|---|---|---|---|
| $\tau$ | risk aversion | [1E-2,5] | [1.63,5] |
| $\kappa$ | Phillips curve slope | [1E-2, 3] | [0.16,2.97] |
| $\psi_1$ | inflation target | [1.01,5] | [1.01,1.30] |
| $\psi_2$ | output target | [1E-2,2] | [1E-2,2] |
| $\rho_r$ | interest rate smoothing | [1E-2, 0.95] | [0.70, 0.89] |
| $\rho_g$ | exogenous spending AR | [1E-2, 0.99] | [0.93, 0.98] |
| $\rho_z$ | technology shock AR | [1E-2, 0.99] | [0.95, 0.98] |
| $100\sigma_r$ | monetary policy shock SD | [1E-2, 3] | [0.22, 0.34] |
| $100\sigma_g$ | exogenous spending SD | [1E-2, 3] | [1.02, 1.36] |
| $100\sigma_z$ | technology shock SD | [1E-2, 3] | [0.03, 0.14] |
| $r^*$ | real interest rate SS | [1E-2, 15] | [1E-2, 1.84] |

# Tests

- Full spectrum: rejected (with $W(\omega) = 1 - \omega$) with

$$\begin{aligned} \mathcal{H}^W_{dT}(\theta) &> 1.26 \\ \text{critical value} &: 1.13. \end{aligned}$$

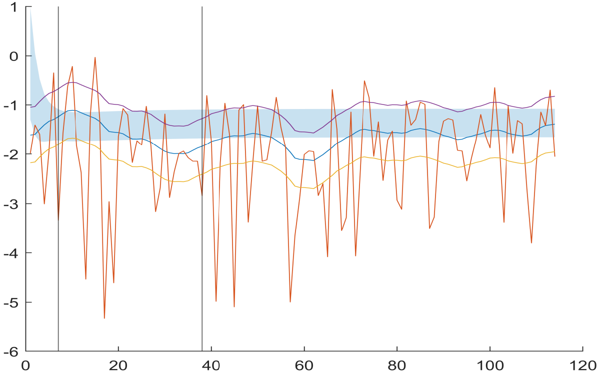
- BC frequencies: rejected with

$$\begin{aligned} \mathcal{H}^W_{dT}(\theta_0) &> 1.53 \\ \text{critical value} &: 1.38. \end{aligned}$$

# A closer look (BC frequencies)

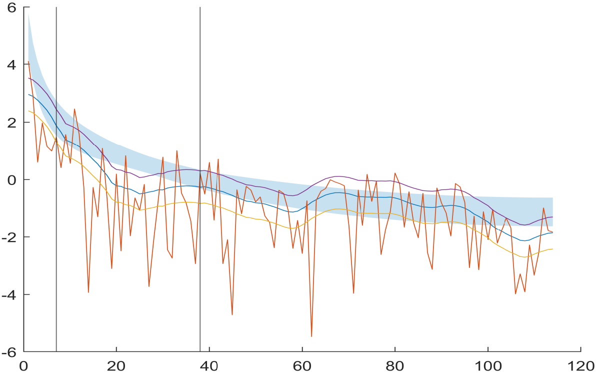
- Testing output growth only: 0 percent draws rejected.
- Testing inflation only: 99 percent draws rejected.
- Testing interest rate only: 0.3 percent draws rejected.

# Log spectrum of GDP growth



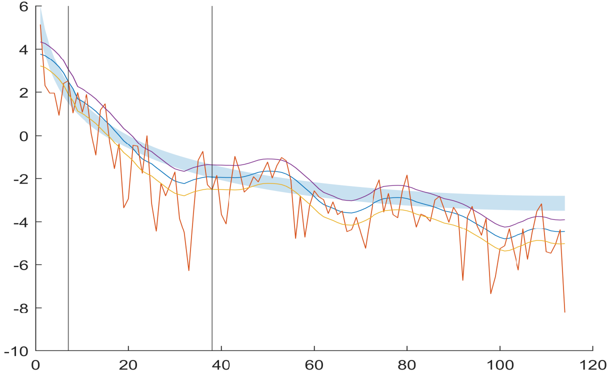

Shaded area: model-implied spectrum. Band: nonparametric estimates and 95% CIs. Note business cycle frequencies.

# Log spectrum of inflation




Shaded area: model-implied spectrum. Band: nonparametric estimates and 95% CIs.

# Log spectrum of interest rate

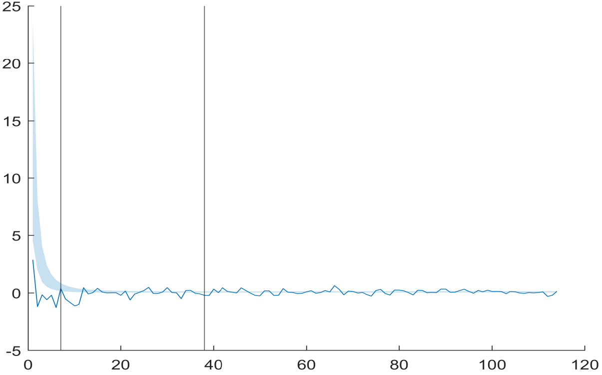



Shaded area: model-implied spectrum. Band: nonparametric estimates and 95% CIs.

# A closer look at comovement (BC frequencies)

- Testing output-inflation: 91 percent draws rejected.
- Testing inflation-interest rate: 83 percent draws rejected.
- Testing output-interest rate: 72 percent draws rejected.

# Comovement of output growth and inflation

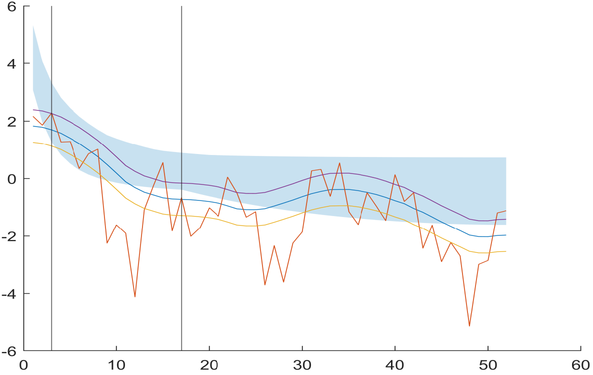



Shaded area: model-implied cross spectrum (real part). Curve: the actual cross periodogram (real part).

# Volcker-Greenspan period only

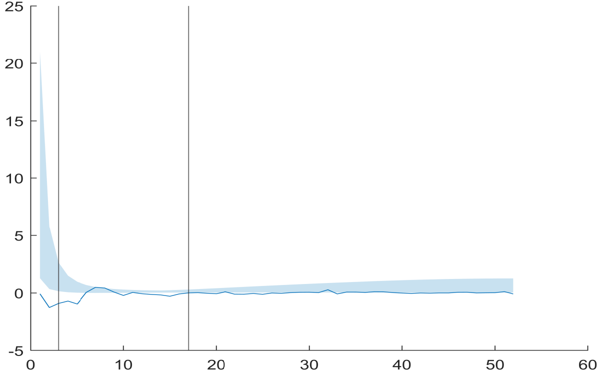
- 1979:III-2005:IV – 104 obs.
- Full Spectrum: the model is not rejected, but only 7 percent draws survive the testing. BC frequencies: 33 percent draws survive.
- Inflation only: 30 percent draws survive based on the full spectrum.
- Comovement: 4 percent draws survive pair-wise testing based on the full spectrum.
- Figures are similar. Next page.

# Log spectrum of inflation




Shaded area: model-implied spectrum. Band: nonparametric estimates and 95% CIs.

# Comovement of output growth and inflation




Shaded area: model-implied cross spectrum (real part). Curve: the actual cross periodogram (real part).

# Summary

- We propose specification tests for DSGE models, allowing for weak identification and multiple equilibria.
- The tests can evaluate a model along multiple dimensions.
- We show a typical small-scale DSGE model is rejected based on the full spectrum and over business cycle frequencies.
- We trace misspecification to inflation dynamics and comovements between variables.
- Applications to medium-scale models are in progress.