The Winner-Take-All Dilemma*

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Abstract

This paper considers collective decision-making when individuals are partitioned into groups (e.g., states or parties) endowed with voting weights. We study a game in which each group chooses an internal rule that specifies the allocation of its weight to the alternatives as a function of its members' preferences. We show that under quite general conditions, the game is a Prisoner's Dilemma: while the winner-take-all rule is a dominant strategy, the equilibrium is Pareto dominated. The Pareto set is fully characterized by the generalized proportional rules. We also show asymptotic Pareto dominance of the proportional rule. Our numerical computation for the US Electoral College verifies the sensibility of the asymptotic results.

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1 Introduction

Many social decisions are made by aggregating opinions of individuals who belong to distinct groups, such as states, constituencies, departments or parties. A fundamental question about collective decision making is thus how social decisions should reflect the opinions of individuals belonging to distinct groups. Typically, each group has a voting weight, in the form of a number of representatives or a weighted vote assigned to a unique representative. The groups allocate the weights to decision alternatives, and the one that receives the most weight becomes the social decision. In such cases, the quality of social decision-making depends not only on the apportionment of weights among the groups, but also on the rules that allocate the groups' weights to alternatives, based on the preferences of their individual members. The present paper is concerned with how the weight allocation rules affect individuals' welfare.

Existing institutions use different weight allocation rules. On the one hand, the *winner-take-all rule* devotes all the weight of a group to the alternative preferred by the majority of its members. Most states in the Untied States use this rule to allocate presidential electoral votes. A council of national ministers, each with a weighted vote (e.g., the Council of the European Union), is another example, provided the ministers represent their countries' majority interests. Party discipline frequently observed in legislative voting is also an example of the winner-take-all rule used by parties.

On the other hand, the *proportional rule* allocates a group's weight in proportion to the number of members who prefer the respective alternatives. In a wide range of parliamentary institutions at the regional, national or international level, each group (e.g., constituency, prefecture or state) elects a set of representatives whose composition proportionally reflects its residents' preferences. Alternatively, when the representatives are viewed as standing for parties rather than states or prefectures, the proportional rule corresponds to a party's rule that allows its representatives to vote for or against proposals based on their own preferences, provided the composition of the party's representatives proportionally reflects the opinions of all party members.

The weight allocation rules are often exogenously given to all groups, but there are also cases where each group chooses its own rule. For instance, in national parliaments, how the representatives are elected from the respective constituencies is stipulated by national law. By contrast, parties often have control over how their representatives vote, by punishing those who violate the party lines. As another example, the US Constitution stipulates that it is up to each state to decide the way in which the presidential electoral votes are allocated (Article II, Section 1, Clause 2).

If groups are allowed to choose their rules, it is possible that each group has an incentive to allocate the weight so as to increase the influence of its members' opinions on social decisions, at the cost of the other groups' influence. It is not clear whether such an incentive at the group level is compatible with desirable properties of the overall preference aggregation, such as Pareto efficiency. Many decision makings in the society which consists of distinct groups thus face a dilemma between the group's incentive and the social objective. To address this issue, we model the choice of rules as a non-cooperative game.

In this paper, we consider a model of social decision-making where individuals are partitioned into groups endowed with voting weights. The society makes a binary decision through two stages: first, all individual members vote; then each group allocates its weight to the alternatives, based on the number of votes they received from the group members. The winner is the alternative with the most weight. A *rule* for a group is a function that maps each possible vote result in the group to an allocation of its weight to the alternatives. Any Borel-measurable function is allowed, including the examples of the winner-take-all and proportional rules stated above. A *profile* is a specification of rules for all groups. We study the game in which the groups independently choose their rules, so as to maximize their members' expected welfare.

The main result of this paper is that the game is a n-player *Prisoner's*

Dilemma (Theorem 1). On the one hand, the winner-take-all rule is a *dominant strategy*, i.e., it is an optimal strategy for each group regardless of the rules chosen by the other groups. On the other hand, if each group has less than a half of the total weight, then the winner-take-all profile is *Pareto dominated*, i.e., some other profile makes *every* group better off. In brief, no group has an incentive to deviate from the winner-take-all rule, but every group will be better off if all groups jointly move to another profile. The dilemma structure exists for any number of groups (> 2) and with little restriction on the joint distribution of preferences (Assumption 1). Members' preferences are allowed to be biased and correlated within and across groups. For example, the model can be applied to the parties with distinct but overlapping political goals, or the states that tend to support specific alternatives, such as blue, red or swing states in the United States.

The observation that the winner-take-all rule is a dominant strategy is consistent with the fact that it has been dominantly employed by the states in the US Electoral College since 1830s in order to allocate presidential electoral votes,[1] and also with the widely observed party discipline in assemblies. Despite the various problems or limitations that have been pointed out concerning the winner-take-all rule,[2] it is still used prevalently.

Our conventional knowledge that direct majority voting by all individuals maximizes the *utilitarian* welfare of the society is not sufficient to see whether *every* group is better off under the proportional profile than the winner-take-all profile. We provide a counterexample later (Example 1): a small group may be strictly better off under the winner-take-all profile

[1]One of the recent attempts of reform by a state took place in 2004, when a ballot initiative for an amendment to the state constitution was raised in Colorado. The suggested procedure is the proportional rule, in which the state electoral votes are allocated proportionally to the state popular votes. The amendment did not pass, garnering only 34.1% approval.

[2]There are multiple arguments against the winner-take-all rule. First, the winner of the election may be inconsistent with that of the popular votes (May (1948), Feix et al. (2004)). Such a discrepancy has happened five times in the history of the US presidential elections, including recently in 2000 and 2016. Second, it may cause reduced dimensionality: (i) the parties have an incentive to concentrate campaign resources only in the battleground states, and (ii) voters' incentive to turn out or to invest in information may be small and/or uneven across states, since the probability of each voter to be pivotal is so small under the winner-take-all rule, and even smaller in the non-swing states. Although campaign resource allocation and voter turnout are important issues, they are beyond the scope of this paper.

than the proportional profile. Indeed, this is an oft-used argument by the advocates of the Electoral College in the US, on which their support for the winner-take-all rule is based. The welfare criterion used in Theorem 1 is Pareto dominance, which is obviously stronger than the utilitarian welfare evaluation: there exists a profile under which *every* group is better off than the winner-take-all profile. Example 1 shows that the dominant profile is not necessarily the proportional one. Then, what profile Pareto dominates the winner-take-all profile? A full characterization of the Pareto set is provided in Lemma 1.

To further study welfare properties, we turn to an asymptotic and normative analysis of the model. We consider situations where the number of groups is sufficiently large, and the preferences are independent across groups and distributed symmetrically with respect to the alternatives. In this case, we show that the proportional profile *Pareto dominates* every other symmetric profile (i.e., one in which all groups use the same rule), including the winner-take-all one. The assumptions on the preference distribution abstract from the fact that in reality, some groups tend to prefer specific alternatives. Such an abstraction would be reasonable on the grounds that normative judgment about rules should not favor particular groups because of their characteristic preference biases. To see how many groups are typically sufficient for the asymptotic result, we provide numerical computations in a model based on the US Electoral College, using the current apportionment of electoral votes. The numerical comparisons indicate that the proportional profile does Pareto dominate the winner-take-all profile in the model with fifty states and a federal district.

While the above result suggests that the proportional profile asymptotically performs well in terms of efficiency, it is silent about the equality of individuals' welfare. In fact, our model also provides some insight into how rules affect the distribution of welfare. We examine an asymmetric profile called the *congressional district profile*. This profile is inspired by the Congressional District Method currently used by Maine and Nebraska, in which two electoral votes are allocated by the winner-take-all rule and the remaining ones are awarded to the winner of each district-wide popular vote.[3] We show that the congressional district profile achieves a more equal

[3]The idea of allocating a part of the votes by the winner-take-all rule and allowing

distribution of welfare than any symmetric profile by making individuals in smaller groups better off.

A technical contribution of this paper is to develop an asymptotic method for analyzing players' expected welfare in weighted voting games. One of the major challenges in analyzing such games is their discreteness. By the nature of combinatorial problems, obtaining an analytical result often requires a large number of classifications by cases, which may include prohibitively tedious and complex tasks in order to obtain general insights. We overcome this difficulty by considering asymptotic properties of games in which there are a sufficiently large number of groups. This technique allows us to obtain an explicit formula that captures the asymptotic behavior of the probability of success for each individual, which holds for a wide class of distributions of weights among groups (the correlation lemma: Lemma 2).

1.1 Literature Review

The incentives for groups to use the winner-take-all rule have been studied by several papers. Hummel (2011) and Beisbart and Bovens (2008) analyze models of the US presidential elections. Gelman (2003) and Eguia (2011a,b) give theoretical explanations as to why voters in an assembly form parties or voting blocs to coordinate their votes. Their findings are coherent with our observation that the winner-take-all rule is a dominant strategy. In particular, Beisbart and Bovens (2008) and Gelman (2003) compare the winner-take-all and proportional profiles. Under the current apportionment of electoral votes in the US, Beisbart and Bovens (2008) numerically compares these profiles, in terms of inequality indices on citizens' voting power and the mean majority deficit, on the basis of *a priori* and *a posteriori* voting power measures. Gelman (2003) compares the case with coalitions of equal sizes in which voters coordinate their votes to the case without such coordination. Our analysis is based on Pareto dominance be-

the rest to be awarded to potentially distinct candidates can be seen as a compromise between the winner-take-all and the proportional rules. Symbolically, the two votes allocated by the winner-take-all rule is the same number as the Senators in each state, while the rest is equal to the number of the House representatives. The idea behind such a mixture is in line with the logic supporting bicameralism, which is supposed to provide checks and balances between the states' autonomy and the federal governance.

tween profiles, and provides results which hold under general distribution of groups' weights or sizes. In that sense, Beisbart and Bovens's positive analysis is complementary to our normative analysis of properties of the proportional profile.

De Mouzon et al. (2019) provides a welfare analysis of popular vote interstate compacts, and shows that, for the regional compact, welfare of the member states is single-peaked as a function of the number of the participating states, while it is monotonically decreasing for the non-member states. The second effect dominates in terms of the social welfare, unless a large majority (approximately more than $2/\pi \simeq 64\%$) of the states join the compact, implying that a small- or middle-sized regional compact is welfare detrimental. For the national compact, the total welfare is increasing, as it turns out that even the non-members would mostly benefit from the compact, implying that the social optimum is attained when a majority joins the compact, i.e. the winner is determined by the national popular vote. Their findings are coherent with ours: if the winner-take-all rule is applied only to a subset of the groups, then the member states enjoy the benefit at the expense of the welfare loss of the non-member states, and the total welfare decreases. The social optimum is attained when the entire nation uses the popular vote.

The winner-take-all rule has been a regular focus of the literature. The history, objectives, problems, and reforms of the US Electoral College are summarized, for example, in Edwards (2004) and Bugh (2010). One of the problems of the Electoral College most often scrutinized is its reduced dimensionality. The incentive of the candidates to concentrate their campaign resources in the swing and decisive states is modeled in Strömberg (2008), which is coherent with the findings of the seminal paper in probabilistic voting by Lindbeck and Weibull (1987). Strömberg (2008) also finds that uneven resource allocation and unfavorable treatment of minority states would be mitigated by implementing a national popular vote, which is coherent with the classical findings by Brams and Davis (1974). Voters' incentive to turn out is investigated by Kartal (2015), which finds that the winner-take-all rule discourages turnout when the voting cost is heterogeneous.

Constitutional design of weighted voting is studied extensively in the literature. Seminal contributions are found in the context of power measurement: Penrose (1946), Shapley and Shubik (1954), Banzhaf (1968) and Rae (1946). Excellent summaries of theory and applications of power measurement are given by, above all, Felsenthal and Machover (1998) and Laruelle and Valenciano (2008). The tools and insights obtained in the power measurement literature are often used in the apportionment problem: e.g., Barberà and Jackson (2006), Koriyama et al. (2013), and Kurz et al. (2017).

2 The Model

We consider social decision-making when individuals are partitioned into groups endowed with voting weights. We first describe the weighted voting mechanism (Section 2.1). We then construct a non-cooperative game in which each group chooses an internal rule that specifies the allocation of its weight to the alternatives as a function of its members' preferences (Section 2.2). Finally, we introduce social choice functions which include the weighted voting mechanism as a special case (Section 2.3).

2.1 Weighted Voting

Let us begin with the description of the social decision process. We consider a society partitioned into n disjoint groups: $i \in \{1, 2, \cdots, n\}$. Each group i is endowed with a voting weight $w_i > 0$.

The society makes a decision between two alternatives, denoted -1 and $+1$, through the following two voting stages: (i) each individual votes for his preferred alternative; (ii) each group allocates its weight between the alternatives, based on the group-wide voting result. The winner is the alternative that receives the majority of overall weight.

Let $\theta_i \in [-1, 1]$ denote the vote margin in group i at the first voting stage. That is, θ_i is the fraction of members of i preferring alternative $+1$ minus the fraction preferring -1.[4]

[4]For example, $\theta_i = -0.2$ means that 60% of members of i prefer $+1$ and 40% prefer -1.

At the second stage, each group's allocation of weight is determined as a function of the group-wide margin.

Definition 1. A *rule* for group i is defined as a Borel-measurable[5] function:

$$\phi_i : [-1, 1] \to [-1, 1].$$

The value $\phi_i(\theta_i)$ is the group-wide weight margin, i.e., the fraction of the weight w_i allocated to alternative $+1$ minus that allocated to -1, given that the vote margin is θ_i. That is, the rule allocates $w_i \phi_i(\theta_i)$ more weight to alternative $+1$ than alternative -1.[6]

Let

$$\Phi = \{\phi_i | \text{Borel-measurable}\}$$

be the set of all admissible rules.

Examples of rules. Among all admissible rules, the following ones deserve particular attention.

(i) *Winner-take-all rule*: $\phi_i^{\mathrm{WTA}}(\theta_i) = \operatorname{sgn} \theta_i$.

(ii) *Proportional rule*: $\phi_i^{\mathrm{PR}}(\theta_i) = \theta_i$.

(iii) *Mixed rules*: $\phi_i^a(\theta_i) = a\phi_i^{\mathrm{WTA}}(\theta_i) + (1-a)\phi_i^{\mathrm{PR}}(\theta_i)$, $0 \leq a \leq 1$.

The winner-take-all rule devotes all the weight of a group to the winning alternative in the group. The proportional rule allocates the weight in proportion to the vote shares of the respective alternatives in the group. The mixed rule ϕ^a allocates the fixed ratio a of the weight by the winner-take-all rule and the remaining $1-a$ part by the proportional rule.

Although we

The social decision is the alternative that receives the majority of overall weight. In the case of a tie, we assume that each alternative is chosen with

[5]Borel-measurability is needed to ensure that each $\phi_i(\Theta_i)$ is a well-defined random variable.

[6]For example, if $w_i = 50$ and $\phi_i(\theta_i) = -0.2$, it means that the rule allocates 20 (resp. 30) units of weight to the alternative $+1$ (resp. -1).

probability $\frac{1}{2}$. Thus, given the rules $\phi = (\phi_i)_{i=1}^n$ and the group-wide vote margins $\theta = (\theta_i)_{i=1}^n$, the social decision $d_\phi(\theta)$ is determined as follows:

$$d_\phi(\theta) = \begin{cases} \text{sgn } \sum_{i=1}^n w_i \phi_i(\theta_i) & \text{if } \sum_{i=1}^n w_i \phi_i(\theta_i) \neq 0 \\ \pm 1 \text{ equally likely} & \text{if } \sum_{i=1}^n w_i \phi_i(\theta_i) = 0. \end{cases} \tag{1}$$

2.2 The Game

We now define the non-cooperative game Γ in which the n groups choose their own rules.

The game is played under incomplete information about individuals' preferences, and hence about the group-wide vote margins. We assume that each group chooses a rule so as to maximize the expected welfare of its members. This assumption seems to be appropriate since in most actual collective decisions, the rules are fixed prior to the realization of preferences.

Let Θ_i be a random variable that takes values in $[-1, 1]$ and represents the vote margin in group i.[7] We impose little restriction on the joint distribution of the random vector $\Theta = (\Theta_i)_{i=1}^n$. The precise assumption on the distribution will be stated later in this section (Assumption 1).

The ex post payoff for group i is the average payoff for its members from the social decision. For simplicity, we assume that each individual obtains payoff 1 if he prefers the social decision and payoff -1 otherwise.[8] The average payoff of members of group i equals Θ_i or $-\Theta_i$ depending on whether the social decision is $+1$ or -1; more concisely, it is

$$\Theta_i d_\phi(\Theta).$$

The ex ante payoff for group i, denoted $\pi_i(\phi)$, is the expected value of the above expression:

$$\pi_i(\phi) = \mathbb{E}\left[\Theta_i d_\phi(\Theta)\right]. \tag{2}$$

[7]Throughout the paper, we use capital Θ_i for representation of a random variable, and small θ_i for the realization.

[8]Remark 2 shows that the model actually captures a more general situation in which members of each group may have different preference intensities.

Let $\pi_i(x_i, \phi_{-i}|\theta_i)$ denote the interim payoff for group i if it chooses the weight margin $x_i \in [-1, 1]$ given the realization of the vote margin θ_i. It is obtained by weighting the ex post payoffs θ_i and $-\theta_i$ from decisions $+1$ and -1 with the conditional probabilities:[9]

$$\begin{aligned} &\pi_i(x_i, \phi_{-i}|\theta_i) \\ &= \theta_i \mathbb{P}\left\{ w_i x_i + \textstyle\sum_{j \neq i} w_j \phi_j(\Theta_j) > 0 \middle| \Theta_i = \theta_i \right\} \\ &\quad - \theta_i \mathbb{P}\left\{ w_i x_i + \textstyle\sum_{j \neq i} w_j \phi_j(\Theta_j) < 0 \middle| \Theta_i = \theta_i \right\}. \end{aligned} \tag{3}$$

The ex ante and interim payoffs are related as follows:

$$\begin{aligned} &\phi_i \text{ maximizes } \pi_i(\phi_i, \phi_{-i}) \\ &\Leftrightarrow x_i = \phi_i(\theta_i) \text{ maximizes } \pi_i(x_i, \phi_{-i}|\theta_i) \text{ for almost every } \theta_i \in [-1, 1]. \end{aligned}$$

To summarize, the *game* Γ is the one in which: the players are the n groups; the strategy set for each group i is the set Φ of all rules; the payoff function for group i is π_i defined in (2).

The following is the assumption on the joint distribution of the group-wide margins.

Assumption 1. The joint distribution of group-wide margins $(\Theta_i)_{i=1}^n$ is absolutely continuous and has full support $[-1, 1]^n$.

Assumption 1 permits a wide variety of joint distributions of individuals' preferences, in which intra- and inter-group correlations and biases are possible. First, the assumption imposes no restriction on preference correlations within each group. Second, individuals' preferences may also be correlated across groups, since the group-wide margins $(\Theta_i)_{i=1}^n$ can be correlated. This allows us to capture situations where, for instance, residents of different states or members of different parties have common interest on some issues. Third, preferences may be biased toward a particular alternative, since Θ_i can be asymmetrically distributed. For instance, blue (resp. red) states in the US might be described as groups whose group-wide margins have a distribution biased to the left (resp. right). In contrast, swing

[9]The term corresponding to the event of a tie (i.e., $w_i x_i + \sum_{j \neq i} w_j \phi_j(\Theta_j) = 0$) does not appear in the formula below, since we assume that the tie is broken fairly.

states might be described as groups whose distributions are concentrated around zero.

Remark 1. *Success probability and voting power.* Our definition of group payoffs has the following interpretation based on the members' preferences. Let M_i be the set of individuals in group i, and $X_{im} \in \{-1, +1\}$ be the preferred alternative of member $m \in M_i$ in group i. Let us here redefine Θ_i as a *latent variable* that parametrizes the distribution of the random preferences in group i. Specifically, suppose X_{im} are independently and identically distributed conditional on the realization $(\theta_i)_{i=1}^n$ with the following probabilities for all $i = 1, \cdots, n$ and $m \in M_i$:

$$\begin{cases} \mathbb{P}\left\{X_{im} = +1 | \Theta_1 = \theta_1, \cdots, \Theta_n = \theta_n\right\} = \left(1 + \theta_i\right)/2, \\ \mathbb{P}\left\{X_{im} = -1 | \Theta_1 = \theta_1, \cdots, \Theta_n = \theta_n\right\} = \left(1 - \theta_i\right)/2. \end{cases} \tag{4}$$

Then, as the group size becomes large ($|M_i| \to \infty$), the Law of Large Numbers implies that the group-wide margin $\frac{1}{M_i}\sum_{m \in M_i} X_{im}$ indeed converges to Θ_i almost surely, which is consistent with our original definition of Θ_i as the group-wide margin. Moreover,

$$\begin{aligned}
&\mathbb{P}\left\{X_{im} = d_\phi(\Theta)\right\} \\
&= \mathbb{E}\left[\mathbb{P}\left\{X_{im} = d_\phi(\Theta) | \Theta\right\}\right] \\
&= \mathbb{E}\left[\mathbb{P}\left\{X_{im} = 1, d_\phi(\Theta) = 1 | \Theta\right\} + \mathbb{P}\left\{X_{im} = -1, d_\phi(\Theta) = -1 | \Theta\right\}\right] \\
&= \mathbb{E}\left[\mathbb{P}\left\{d_\phi(\Theta) = 1 | \Theta\right\} \frac{1 + \Theta_i}{2} + \mathbb{P}\left\{d_\phi(\Theta) = -1 | \Theta\right\} \frac{1 - \Theta_i}{2}\right] \\
&= \frac{1}{2}\left(1 + \mathbb{E}\left[\mathbb{P}\left\{d_\phi(\Theta) = 1 | \Theta\right\} \Theta_i + \mathbb{P}\left\{d_\phi(\Theta) = -1 | \Theta\right\}(-\Theta_i)\right]\right) \\
&= \frac{1}{2}\left(1 + \mathbb{E}\left[\Theta_i d_\phi(\Theta)\right]\right).
\end{aligned}$$

Therefore, $\pi_i(\phi) = \mathbb{E}\left[\Theta_i d_\phi(\Theta)\right]$ is an affine transformation of the probability that the preferred alternative of a member m in group i coincides with the social decision ($X_{im} = d_\phi(\Theta)$), which is called *success* in the literature of voting power measurement (Laruelle and Valenciano (2008)). The objective of the group, formulated as the maximization of π_i, is thus equivalent to maximization of the probability of success.

Under the winner-take-all profile ϕ^{WTA}, π_i is closely related to the classical voting power indices studied in the literature. If $(\Theta_i)_{i=1}^n$ are independently, identically and symmetrically distributed (thus each group's preferred alternative is independently and equally distributed over $\{-1, +1\}$, called *Impartial Culture*), then π_i corresponds to the Banzhaf-Penrose index (Banzhaf (1965), Penrose (1946)) and $\mathbb{P}\{X_{im} = d_\phi(\Theta)\}$ to the Rae index (Rae (1946)), up to a multiplication by the constant $\mathbb{E}[|\Theta_i|]$. If $(\Theta_i)_{i=1}^n$ are perfectly correlated and symmetrically distributed (called *Impartial Anonymous Culture*. See, for example, Le Breton et al. (2016)), then π_i corresponds to the Shapley-Shubik index (Shapley and Shubik (1954)). □

Remark 2. *Intra-group heterogeneity.* We have assumed that all individuals have the same preference intensities (i.e., each individual receives a unit payoff whenever he prefers the social decision), and each group's objective is to maximize the ex ante average payoff of its members. We stick with this assumption to avoid unnecessary complications. However, our formal definition (2) is consistent with more general situations. In fact, it only requires that each group maximizes the expected value of a "group-wide payoff" from the social decision. It requires neither that the group-wide payoff be the average of members' payoffs, nor that the members' preference intensities be identical.

To be more precise, suppose each group i receives a random payoff U_i^+ or U_i^- depending on whether the social decision is $+1$ or -1, where U_i^+, U_i^- are assumed to take values in $[0, 1]$. Redefine the variable Θ_i as the payoff difference: $\Theta_i := U_i^+ - U_i^-$. Then the group's ex ante payoff from the social decision under profile ϕ is

$$\begin{aligned} u_i(\phi) &= \mathbb{E}\left[U_i^+ \frac{1 + d_\phi(\Theta)}{2} + U_i^- \frac{1 - d_\phi(\Theta)}{2}\right] \\ &= \frac{1}{2}\mathbb{E}\left[\Theta_i d_\phi(\Theta)\right] + \frac{1}{2}\mathbb{E}\left[U_i^+ + U_i^-\right] \\ &= \frac{1}{2}\pi_i(\phi) + \text{constant}. \end{aligned}$$

Since this is a positive affine transformation of $\pi_i(\phi)$, our model captures the general case where each group maximizes the expected group-wide payoff u_i. In particular, the group-wide payoffs U_i^+, U_i^- can be any function of members' payoffs, which may or may not respect different preference intensities among the members.

The above argument also reminds us that our model can be translated into the more abstract context of collective decisions by n agents, whether they stand for individuals or groups. In Section 3.2, we will elaborate on the implications of our analysis for mechanism design problems in which agents' preference intensities are private information. □

2.3 Social Choice Functions

Although the social decision in game Γ is determined by the sum of the groups' weights allocated to each alternative, we can consider more general social choice functions.

Let $\mathcal{R}$ be the set of all random variables taking values in $\{-1, +1\}$. A *social choice function (SCF)* is a Borel-measurable function[10]

$$d : [-1, 1]^n \to \mathcal{R}.$$

The SCF assigns to each profile of realized vote margins $\theta = (\theta_i)_{i=1}^n \in [-1, 1]^n$ a social decision $d(\theta)$ which may randomize between alternatives -1 and $+1$. The decision function d_ϕ in game Γ is an example of an SCF.[11]

With slight abuse of notation, we denote by $\pi_i(d)$ the ex ante payoff for group i under SCF d. By extending formula (2), we have the following expression:

$$\pi_i(d) = \mathbb{E}\left[\Theta_i d(\Theta)\right].$$

Our main analysis in Section 3.1 is based on game Γ, but the results have implications to mechanism design problems with general SCFs, which we summarize in Section 3.2.

[10]More precisely, an SCF is a function $d(\theta, \omega)$ of two variables, $\theta \in [-1, 1]$ and $\omega \in \Omega$ for some sample space Ω, such that: for each θ, $d(\theta, \cdot) : \Omega \to \{-1, +1\}$ is a random variable; for each ω, $d(\cdot, \omega) : [-1, 1]^n \to \{-1, +1\}$ is a Borel-measurable function.

[11]Randomness of $d_\phi(\theta)$ occurs when the weighted vote is tied.

3 The Dilemma

3.1 The Main Result

In game Γ, a rule (or strategy) ϕ_i for group i *weakly dominates* another rule ψ_i if $\pi_i(\phi_i, \phi_{-i}) \geq \pi_i(\psi_i, \phi_{-i})$ for any ϕ_{-i}, with strict inequality for at least one ϕ_{-i}. A rule ϕ_i is a *weakly dominant strategy* for group i if it weakly dominates every rule not equivalent to ϕ_i, where we call two rules ϕ_i and ψ_i *equivalent* if $\phi_i(\theta_i) = \psi_i(\theta_i)$ for almost every θ_i (with respect to Lebesgue measure on $[-1, 1]$).

A profile ϕ *Pareto dominates* another profile ψ if $\pi_i(\phi) \geq \pi_i(\psi)$ for all i, with strict inequality for at least one i. If ϕ is not Pareto dominated by any profile, it is called *Pareto efficient.* Pareto dominance of SCFs d is defined in the same way, based on the payoff functions $\pi_i(d)$ (see Section 2.3).

We first consider the case in which there is no 'dictator' group that can determine the winner by putting all its weight to an alternative (Theorem 1). Later we consider the case with such a group (Proposition **??**).

Assumption 2. Each group has less than half the total weight: $w_i < \frac{1}{2}\sum_{j=1}^{n} w_j$ for all $i = 1, \cdots, n$.

Theorem 1. *Under Assumptions 1 and 2, game Γ is a Prisoner's Dilemma:*

(i) the winner-take-all rule ϕ_i^{WTA} is the weakly dominant strategy[12] *for each group i;*

(ii) the winner-take-all profile ϕ^{WTA} is Pareto dominated.

We use the following lemma to prove the theorem. An SCF d is called a *cardinal weighted majority rule* if there exists a vector $(\lambda_i)_{i=1}^n \in \mathbb{R}_+^n \setminus \{0\}$ such that:

$$d(\theta) = \operatorname{sgn} \sum_{i=1}^{n} \lambda_i \theta_i \text{ for almost every } \theta \in [-1, 1]^n.$$

[12] By the definition of weak dominance, ϕ_i^{WTA} is the unique weakly dominant strategy up to equivalence of rules.

In game Γ, a profile ϕ is called a *generalized proportional profile* if there exists a vector $(\lambda_i)_{i=1}^n \in [0,1]^n \setminus \{0\}$ such that for each i,

$$\phi_i(\theta_i) = \lambda_i \theta_i \text{ for almost every } \theta_i \in [-1,1].$$

Two profiles ϕ and ψ are called *equivalent* if $d_\phi(\theta) = d_\psi(\theta)$ for almost every $\theta \in [-1,1]^n$.

Lemma 1. *(Characterization of the Pareto set) Under Assumption 1, the following statements hold:*

(i) An SCF d is Pareto efficient in the set of all SCFs if and only if it is a cardinal weighted majority rule.

(ii) In game Γ, a profile $\phi = (\phi_i)_{i=1}^n$ is Pareto efficient in the set of all profiles if and only if it is equivalent to a generalized proportional profile.

The proof of Lemma 1 is relegated to Appendix.

Proof of Theorem 1. Part (i). We first check that

$$\pi_i(\phi_i^{\mathrm{WTA}}, \phi_{-i}) \geq \pi_i(\phi_i, \phi_{-i}) \tag{5}$$

for any (ϕ_i, ϕ_{-i}). By (3), if $\theta_i > 0$ (resp. $\theta_i < 0$), then the interim payoff $\pi_i(x_i, \phi_{-i}|\theta_i)$ is non-decreasing (resp. non-increasing) in $x_i \in [-1,1]$. We thus have $\pi_i(\phi_i^{\mathrm{WTA}}(\theta_i), \phi_{-i}|\theta_i) \geq \pi_i(\phi_i(\theta_i), \phi_{-i}|\theta_i)$ for any (ϕ_i, ϕ_{-i}) and $\theta_i \neq 0$. Since $\Theta_i = 0$ occurs with probability 0, This implies (5).

Now we show that for any profile ϕ_{-i} in which each $\phi_j(\Theta_i)$ $(j \neq i)$ has full support $[-1,1]$ (e.g., ϕ_j^{PR}), the strict inequality

$$\pi_i(\phi_i^{\mathrm{WTA}}, \phi_{-i}) > \pi_i(\phi_i, \phi_{-i}) \tag{6}$$

holds for any rule ϕ_i that differs from ϕ_i^{WTA} on a set $A \subset [-1,1]$ of positive measure. To see this, note that for such ϕ_{-i} and any θ_i, the conditional distribution of $\sum_{j \neq i} w_j \phi_j(\Theta_j)$ given $\Theta_i = \theta_i$ has support

$$I = \left[-\sum_{j \neq i} w_j, \sum_{j \neq i} w_j \right].$$

Since $w_i < \sum_{j \neq i} w_j$ by Assumption 2, as x_i moves in $[-1, 1]$, $w_i x_i$ moves in interval I. Formula (3) thus implies that if $\theta_i > 0$ (resp. $\theta_i < 0$), then $\pi_i(x_i, \phi_{-i}|\theta_i)$ is strictly increasing (resp. decreasing) in $x_i \in [-1, 1]$. Hence $\pi_i(\phi_i^{\text{WTA}}(\theta_i), \phi_{-i}|\theta_i) > \pi_i(\phi_i(\theta_i), \phi_{-i}|\theta_i)$ at any $\theta_i \in A$. Since Θ_i has full support, result (6) follows.

Part (ii). By the characterization of the Pareto set (Lemma 1(ii)), it suffices to check that ϕ^{WTA} is not equivalent to any generalized proportional profile. Suppose, on the contrary, that ϕ^{WTA} is equivalent to a generalized proportional profile with coefficients $\lambda \in [0, 1]^n \setminus \{0\}$. Then, since $(\Theta_i)_{i=1}^n$ has full support,

$$d_{\phi^{\text{WTA}}}(\theta) = \operatorname{sgn} \sum_{i=1}^{n} w_i \lambda_i \theta_i \text{ at almost every } \theta \in [-1, 1]^n. \tag{7}$$

Since no group dictates the social decision, the coefficients λ_i are positive for at least two groups. Without loss of generality, assume $\lambda_1 > 0$ and $\lambda_2 > 0$. Now, fix θ_i for $i \neq 1, 2$ so that they are sufficiently small in absolute value. Then, according to (7), for (almost any) sufficiently small $\varepsilon > 0$, $d_{\phi^{\text{WTA}}}(\theta) = +1$ if $\theta_1 = 1 - \varepsilon$ and $\theta_2 = -\varepsilon$, while $d_{\phi^{\text{WTA}}}(\theta) = -1$ if $\theta_1 = \varepsilon$ and $\theta_2 = -1 + \varepsilon$. This contradicts the fact that $d_{\phi^{\text{WTA}}}(\theta)$ depends only on the signs of $(\theta_i)_{i=1}^n$. □

Theorem 1 shows that, while the dominant strategy for each group is the winner-take-all rule, the dominant-strategy equilibrium is Pareto dominated by a generalized proportional profile. This typical Social Dilemma (or, n-player Prisoner's Dilemma) situation suggests that a Pareto efficient outcome is not expected to be achieved under decentralized decision making, and a coordination device is necessary in order to attain a Pareto improvement.

The observation that groups have an incentive to use the winner-take-all rule is not new. Beisbart and Bovens (2008) consider Colorado's deviation from the winner-take-all rule to the proportional rule, following the state's attempt in 2004 to amend the state constitution, and show that the citizens in Colorado are worse off under both *a priori* and *a posteriori* measures. Hummel (2011) shows that a majority of the voters in a state is worse off by unilaterally switching to the proportional rule from the winner-take-all profile.

Our results are also consistent with the findings in the literature of the coalition formation games in which individuals may have incentive to raise their voices by forming a coalition and aligning their votes. Gelman (2003) illustrates that individuals are better off by forming a coalition and assign all their weights to one alternative. Eguia (2011a) considers a game in which the members in an assembly decide whether to accept the party discipline to align their votes, and shows that the voting blocs form in equilibrium if preferences are sufficiently polarized. Eguia (2011b) considers a dynamic model and show the conditions under which voters form two polarized voting blocs in a stationary equilibrium.

A novelty of Theorem 1 lies in its generality. Earlier studies have introduced a specific structure either on the distribution of the preferences and/or of the weights, or on the set of the rules that groups can use.[13] In contrast, we only impose fairly mild conditions on the preference distribution (in particular, Assumption 1 imposes no restriction on across-group correlation), on the weight distribution (Assumption 2 imposes no specific weight structure such as one big group and small ones, or equally sized ones), and on the set of the available rules (Definition 1 admits all Borel-measurable rules, not just the winner-take-all and the proportional rules).

Most importantly, the generality of our model allows for a welfare analysis which does not require introduction of a specific structure on the weight and/or the preference distribution, or the set of available rules. Since our model incorporates *all* Borel-measurable profiles, the Pareto set obtained in Lemma 1 leads us to an explicit characterization of the set of *first-best* outcomes which can be attained.

The key welfare implication of our result is that the dominant-strategy equilibrium is Pareto dominated by generalized proportional profiles. This provides us two important insights in welfare analysis of groupwise preference aggregation problems. First, the game is in Prisoner's dilemma so

[13]Beisbart and Bovens (2008) consider Colorado's strategic choice between the winner-take-all and the proportional rules in the US Electoral College. Hummel (2011) either introduces a correlation structure in the preference distribution or assumes weights to be constant in other states. Gelman (2003) shows interesting computations but all claims are based on observations from examples. Eguia (2011a) introduces a three-group preference structure, left, right and independent, and Eguia (2011b)'s main results focus on a nine-voter example, and the internal rules are assumed to be (super) majority rules.

that a coordination device is necessary for a Pareto improvement. Second, once such a device is available, our characterization lemma tells us that the first-best rules that the society can use are characterized by the generalized proportional profiles.

It is worth emphasizing that the result does not imply merely utilitarian inefficiency of the equilibrium profile. The profile is Pareto dominated, implying that it is in *every* group's interest to move from the winner-take-all equilibrium to another profile. From the utilitarian perspective, it is straightforward to see that the social optimum is obtained by the *popular vote*, i.e., direct majority voting by all individuals. However, this observation is not sufficient to establish that the winner-take-all profile is Pareto dominated.[14] After all, the utilitarian optimum is merely one point in the Pareto set.

The following example illustrates that the winner-take-all profile is not always Pareto dominated by either popular vote or proportional profile.

Example 1. Consider a society which consists of two large groups with an equal weight and one small group. For an illustrative purpose, let us consider three states in the US, Florida, New York and Wyoming. The population and weights are summarized in Table 1.

Table 1: Comparison of the expected payoffs in an example of the society which consists of three states: Florida, New York and Wyoming. Weights are the electoral votes assigned in the Electoral College in 2020. Population is an estimation of the voting-age population in 2018 (thousands). Source: US Census Bureau.

State	Weight	Population	$\pi_i\left(\phi^{\text{WTA}}\right)$	$\pi_i\left(\phi^{\text{PR}}\right)$	$\pi_i\left(\phi^{\text{POP}}\right)$	$\pi_i\left(\hat{\phi}\right)$
Florida	29	15,047	0.250	0.332	0.343	0.271
New York	29	13,684	0.250	0.332	0.323	0.271
Wyoming	3	422	0.250	0.034	0.008	0.271
per capita average			0.250	0.328	0.329	0.271

As defined above, ϕ^{WTA} and ϕ^{PR} are the winner-take-all and proportional profiles. The vote margins $(\Theta_i)_{i=1,2,3}$ are drawn from the uniform distribution on $[-1, 1]$ independently across the states. The payoff of the popular vote $\pi_i\left(\phi^{\text{POP}}\right)$ is defined as the ex ante expected payoff of a repre-

[14] Obviously, utilitarian optimality does not imply Pareto dominance.

sentative voter in each state, which is obtained by letting the social decision d as the popular vote winner in (2).

Since there is no dictator state (*i.e.* Assumpion 2 is satisfied) in this example, any pair of two states is a minimal winning coalition under the winner-take-all profile, implying that the expected payoffs are exactly the same across states under ϕ^{WTA}.

Two large states are better off under the proportional profile ϕ^{PR}, while the small state is worse off. This is because the social decision is more likely to coincide with the alternative preferred by the majority of the large states under ϕ^{PR}. As a consequence, the differences in the weights are reflected more directly on the differences in the expected payoffs.

Even though the small state is better off under ϕ^{WTA} in this particular example, it is worth underlining that whether the winner-take-all profile favors small states comparing to the proportional profile depends on the weight distribution. For example, if one state is a dictator (*i.e.* violating Assumption 2), the payoffs of the two other state are zero under ϕ^{WTA}, while they are (probably small but) strictly positive under ϕ^{PR}.

Under the popular vote ϕ^{POP}, the expected payoff of the small state is even smaller than under ϕ^{PR}. This comes from the fact that the weight assigned to the small state is larger than the large states in the *per capita* measure. In this example, Wyoming has more weights than it would if assigned proportionally to the population.[15] Under the popular vote, the citizens in the small state lose such an advantage assigned through the weights. We can also observe that the utilitarian welfare is maximized under the popular vote ϕ^{POP} by comparing the per capita average of the expected payoffs.

Finally, let $\hat{\phi}$ be the generalized proportional profile with coefficients $\lambda_i = 1/w_i$. We observe that it Pareto dominates ϕ^{WTA}. Remember that our characterization lemma tells us that a profile is Pareto efficient if and only if it is equivalent to a generalized proportional profile. We can show that among the profiles which Pareto dominate the equilibrium profile ϕ^{WTA}, one is obtained by letting $\lambda_i = 1/w_i$, because the expected payoffs are equal

[15] The digressive proportionality is a consequence of the rule specified in the US Constitution. The electoral votes of each state is the sum of the numbers of Senate members (constant) and of House (proportional to population in principle). Under such a rule, per capita weight is decreasing in population.

across the states in this example and we can obtain the particular point in the Pareto set with the equal Pareto coefficients by setting $\lambda_i = 1/w_i$.

This example illustrates that the winner-take-all, proportional profiles and the popular vote may be all Pareto imcomparable. Even though Theorem 1 shows that the winner-take-all profile is Pareto dominated, it may be neither by the proportional profile nor the popular vote. This may happen when the number of groups is small. For the cases in which there are sufficiently many groups, we provide clear-cut insights in Section 4 by using an asymptotic model and numerical simulations. □

To summarize, we have the following propositions.

Proposition 1. *Under Assumption 1, the proportional profile* ϕ^{PR} *and the popular vote* ϕ^{POP} *are both Pareto efficient.*

Proof. Trivially, the proportional profile is a generalized proportional profile by letting $\lambda_i = 1$ for all i. The outcome of the popular vote coincides with that of the generalized proportional profile with $\lambda_i = n_i/w_i$ for all i. By Lemma 1, we obtain the result. □

Proposition 2. *Under Assumption 1, the winner-take-all profile* ϕ^{WTA} *is Pareto dominated if and only if Assumption 2 is satisfied.*

Proof. The "if" part is already proven in Theorem 1 (ii). To show the "only if" part, suppose that Assumption 2 is violated. Then, there exists a dictator state i^* that can determine the winner by putting all its weight to the alternative preferred by the majority of the state. Hence, ϕ^{WTA} is equivalent to the generalized proportional profile with coefficients $\lambda_{i^*} > 0$ and $\lambda_i = 0$ for all $i \neq i^*$. By Lemma 1, we obtain the result. □

3.2 Implications to Mechanism Design Problems

The analysis in Section 3.1 provides insight into more general mechanism design problems. Our model of indirect voting is formally analogous to a Bayesian collective decision model in which agents' preference intensities are private information. This analogy allows us to apply the basic idea of Theorem 1 to establish the impossibility of implementing ex ante efficient social choice. In this section, we present the impossibility result, and then relate it to earlier work in the literature of mechanism design.

Consider a society of n agents ($i = 1, \cdots, n$). The society must decide between two alternatives -1 and $+1$. Agent i's (von-Neumann-Morgenstern) utilities from alternatives -1 and $+1$ are denoted U_i^- and U_i^+. We assume that the utilities are random variables taking values in $[0, 1]$. Here we use the notation Θ_i to represent the utility difference $\Theta_i := U_i^+ - U_i^-$. Then Θ_i is a random variable taking values in $[-1, 1]$. We call Θ_i agent i's *type*. The types are private information: each agent can observe only his own type.

We only impose Assumption 1: the joint distribution of types $\Theta = (\Theta_i)_{i=1}^n$ has a density with full support $[-1, 1]^n$. This allows for correlations of types, and ex ante asymmetries with respect to the agents and the alternatives.

A social choice function (SCF) d is defined as in Section **??**. In the present context, it assigns to each profile of realized types $\theta = (\theta_i)_{i=1}^n$ a decision $d(\theta)$ which may randomize between alternatives -1 and $+1$. The SCF is *dictatorial* if there exists an agent i such that $d(\theta) = \operatorname{sgn} \theta_i$ for almost every $\theta \in [-1, 1]^n$. Note that a cardinal weighted majority rule with $\lambda_i > 0$ for some i and $\lambda_j = 0$ for all $j \neq i$ is dictatorial.

The *direct mechanism* associated with SCF d is the game in which the n agents simultaneously report (possibly false) types, based on which an alternative is chosen according to d. A strategy for agent i is a Borel-measurable function $\sigma_i : [-1, 1] \to [-1, 1]$ that assigns to each realization of type $\theta_i \in [-1, 1]$ a reported type $\sigma_i(\theta_i) \in [-1, 1]$. A strategy σ_i is called *truthful* if $\sigma_i(\theta_i) = \theta_i$ for almost every θ_i. Given a strategy profile $\sigma = (\sigma_i)_{i=1}^n$, the ex ante payoff for agent i is

$$\pi_i(\sigma; d) = \mathbb{E}\left[\Theta_i d(\sigma(\Theta))\right],$$

where $\sigma(\Theta) = (\sigma_j(\Theta_j))_{j=1}^n$ is the profile of reported types.

The game Γ defined in Section 2.2 can be seen as the direct mechanism for the cardinal weighted majority rule d with coefficients $\lambda_i = w_i$ ($i = 1, \cdots, n$). Call group i in game Γ agent i, and its group-wide vote margin Θ_i the agent's type. The strategy set Φ for group i in that game is the same as the strategy set for agent i in the mechanism. The definition (1) of the social decision $d_\phi(\theta)$ in game Γ is exactly the same as the decision $d(\phi(\theta))$ in the direct mechanism when the agents play the strategy profile $\sigma = \phi$. Therefore, the ex ante payoff functions in the two models also coincide.

We consider two concepts of incentive compatibility corresponding to alternative solution concepts. An SCF is *dominant-strategy incentive compatible* if for each agent, the truthful strategy is a weakly dominant strategy in the direct mechanism.[16] An SCF is *Bayesian incentive compatible* if the profile of truthful strategies is a Bayesian Nash equilibrium of the direct mechanism.[17] By the revelation principle, it is without loss of generality to consider only direct mechanisms.

Proposition 3. *Under Assumption 1, the following statements hold:*

(i) An SCF is Pareto efficient and dominant-strategy incentive compatible if and only if it is dictatorial.

(ii) An SCF is Pareto efficient and Bayesian incentive compatible if and only if it is dictatorial.

Proof. (i) It is obvious that any dictatorial SCF is Pareto efficient and dominant-strategy incentive compatible. By Lemma 1(i), it only remains to check that any non-dictatorial cardinal weighted majority rule d (i.e., one with $\lambda_i > 0$ for at least two i) is not dominant-strategy incentive compatible. By the analogy of the direct mechanism for the rule d and game Γ which we noted above, Theorem 1(i) implies that in the direct mechanism, the unique (up to equivalence) weakly dominant strategy for each agent i is $\sigma_i(\theta_i) = \operatorname{sgn} \theta_i$, i.e., the analogue of the winner-take-all rule. Under the full-support assumption, σ_i is not a truthful strategy.

(ii) As in the proof of (i), it suffices to check that any non-dictatorial cardinal weighted majority rule d is not Bayesian incentive compatible. In

[16]A strategy σ_i for agent i weakly dominates another strategy σ_i' if $\pi_i(\sigma_i, \sigma_{-i}; d) \geq \pi_i(\sigma_i, \sigma_{-i}; f)$ for all σ_{-i}, with strict inequality for some σ_{-i}. Strategy σ_i is *weakly dominant* if it weakly dominates every other strategy (which differs from σ_i on a set of positive measure). Note that in the literature of mechanism design, dominant-strategy incentive compatibility is often used in the weaker sense that truth-telling is always a best response for each agent whatever the other agents' strategies are.

[17]The profile σ^* of truthful strategies is a *Bayesian Nash equilibrium* if for each agent i, $\pi_i(\sigma^*; d) = \max_{\sigma_i} \pi_i(\sigma_i, \sigma^*_{-i}; d)$.

the proof of Theorem 1(i), we have shown that in game Γ, if ϕ_{-i} is such that each $\phi_j(\Theta_j)$ ($j \neq i$) has full support $[-1, 1]$, the unique (up to equivalence) best response for group i is the winner-take-all rule. Thus, in the direct mechanism for d, the unique (up to equivalence) best response for each agent i against the profile in which all other agents play a truthful strategy is $\sigma_i(\theta_i) = \operatorname{sgn} \theta_i$, which is again not a truthful strategy. Thus the profile of truthful strategies is not a Bayesian Nash equilibrium. □

Proposition 3 summarizes the implications of our discussion about indirect voting for mechanism design problems. It shows the impossibility of implementing an efficient social choice rule, except in the case of dictatorship. This is parallel to the fact stated in Theorem 1 that the strategic behavior of groups to maximize their members' welfare is incompatible with efficient social decision making.

There is large literature on Bayesian mechanism design without monetary transfers as we considered above. The work most relevant to Proposition 3 is Börger and Postl (2009). They consider the case with three alternatives and two agents having opposite preference rankings. The agents' vNM utilities from the middle-ranked alternative (the compromise) are private information. Assuming that the two agents' utilities are independent, they show an impossibility theorem of the same form as Proposition 3(ii). While their results are not directly comparable to ours because we focus on the two-alternative case, our Proposition 3 holds for any number of agents and without the independence assumption.

Azrieli and Kim (2014) consider the case with n agents and two alternatives, which is the same as our setting except for assumptions on the distribution of types. Assuming that the agents' types are independent, they study second-best SCFs, i.e., ones that are Pareto efficient in the set of Bayesian incentive compatible SCFs.[18] They show that ex ante second-best SCFs constitute a subset of *ordinal* qualified weighted majority rules. They also provide a similar characterization of interim second-best SCFs. Our Proposition 3 complements their results by showing that first-best

[18]Schmitz and Tröger (2012) is another paper that considers an environment with n agents and two alternatives. Assuming that the agents and the alternatives are symmetric ex ante, but allowing for correlated preferences, they study the utilitarian optimal social choice rule.

Pareto efficiency is indeed unattainable, except in the case of dictatorship.

A key observation behind Azrieli and Kim's characterization theorems is that the incentive-compatibility constraints rule out effective use of cardinal information of agents' preferences. This basic observation is strengthened by Ehlers, Majumdar and Sen (2020), who show that in a more general setup with any number of alternatives, every incentive-compatible SCF that satisfies a certain continuity condition is ordinal. These results provide a general perspective to understand the incentive of each group in our model to adopt the winner-take-all rule, which only uses the group's ordinal preference (i.e., which alternative the majority of its members prefers).

4 Asymptotic and Computational Results

4.1 Asymptotic Analysis

We saw above that the game is a Prisoner's Dilemma. In this section, we provide further insights on the welfare properties, by focusing on the following situations in which: (i) the number of groups is sufficiently large, and (ii) the preferences of the members are distributed symmetrically. These properties allows us to provide an asymptotic and normative analysis.

Often the difficulty of analysis arises from the discrete nature of the problem. Since the social decision D_ϕ is determined as a function of the sum of the weights allocated to the alternatives across the groups, computing the expected payoffs may require classification of a large number of success configurations which increases exponentially as the number of groups increases, rendering the analysis prohibitively costly. We overcome this difficulty by studying asymptotic properties. In order to check the sensibility of our analysis, we provide Monte Carlo simulation results later in the section, using the example of the US Electoral College.

In order to study asymptotic properties, let us consider a sequence of weights $(w_i)_{i=1}^{\infty}$, exogenously given as a fixed parameter.

Assumption 3. The sequence of weights $(w_i)_{i=1}^{\infty}$ satisfies the following properties.

(i) $w_1, w_2, \cdots$ are in a finite interval $[\underline{w}, \bar{w}]$ for some $0 \leq \underline{w} < \bar{w}$.

(ii) As $n \to \infty$, the statistical distribution G_n induced by $(w_i)_{i=1}^n$ weakly converges to a distribution G with support $[\underline{w}, \bar{w}]$.[19]

Assumption 3 guarantees that for large n, the statistical distribution of weights G_n is sufficiently close to some well-behaved distribution G, on which our asymptotic analysis is based.

Additionally, we impose an impartiality assumption for our normative analysis:

Assumption 4. The variables $(\Theta_i)_{i=1}^{\infty}$ are drawn independently from a common symmetric distribution F.

As in Felsenthal and Machover (1998), a normative analysis requires impartiality, and a study of fundamental rules in the society, such as a constitution, should be free from any dependence on the ex post realization of the group characteristics. Assumption 4 allows our normative analysis to abstract away the distributional details. Of course, a normative analysis is best complemented by a positive analysis which takes into account the actual characteristics of the distributions (as in Beisbart and Bovens (2008)).

Following the symmetry of the preferences, our analysis also focuses on *symmetric* profiles, in which all groups use the same rule: $\phi_i = \phi$ for all i. With a slight abuse of notation, we write ϕ both for a *single rule* ϕ and for the *symmetric profile* $(\phi, \phi, \cdots)$, which should not create any confusion as long as we refer to symmetric profiles. As for the alternatives, it is natural to consider that the label should not matter when the group-wide vote margin is translated into the weight allocation, given the symmetry of the preferences.

Assumption 5. We assume that the rule is monotone and neutral, that is, ϕ is a non-decreasing, odd function: $\phi(\theta_i) = -\phi(-\theta_i)$.

Let $\pi_i(\phi; n)$ denote the expected payoff for group $i (\leq n)$ under profile ϕ when the set of groups is $\{1, \cdots, n\}$ and each group j's weight is w_j, the

[19] The statistical distribution function G_n induced by $(w_i)_{i=1}^n$ is defined by $G_n(x) = \#\{i \leq n : w_i \leq x\}/n$ for each x. G_n weakly converges to G if $G_n(x) \to G(x)$ at every point x of continuity of G.

jth component of the sequence of weights. The definition of $\pi_i(\phi; n)$ is the same as $\pi_i(\phi)$ in the preceding sections; the new notation just clarifies its dependence on the number of groups n.

The main welfare criterion employed in this section is the asymptotic Pareto dominance.

Definition 2. For two symmetric profiles ϕ and ψ, we say that ϕ *asymptotically Pareto dominates* ψ if there exists N such that for all $n > N$ and all $i = 1, \cdots, n$,

$$\pi_i(\phi; n) > \pi_i(\psi; n).$$

4.2 Pareto Dominance

The following is the main result in our asymptotic analysis.

Theorem 2. *Under Assumptions 1-5, the proportional profile asymptotically Pareto dominates all other symmetric profiles. In particular, it asymptotically Pareto dominates the dominant-strategy equilibrium of the game, i.e., the symmetric winner-take-all profile.*

We use the following lemma to prove Theorem 2. The proof of Lemma 2 is relegated to the Appendix. The proof of part (ii) uses a more general result, Lemma 3, stated in the next subsection, whose proof also appears in the Appendix.

Lemma 2. *Under Assumptions 1-5, the following statements hold.*

(i) For any symmetric profile ϕ,

$$\pi_i(\phi; n) = 2\int_0^1 \theta_i \mathbb{P}\left\{ -w_i\phi(\theta_i) < \sum_{j\le n,\, j\ne i} w_j \phi(\Theta_j) \le w_i\phi(\theta_i)\right\} dF(\theta_i).$$

(ii) For any symmetric profile ϕ, as $n \to \infty$,

$$\sqrt{2\pi n}\,\pi_i(\phi; n) \to 2w_i \sqrt{\frac{\mathbb{E}[\Theta^2]}{\int_{\underline{w}}^{\bar{w}} w^2 dG(w)}}\,\mathrm{Corr}\,[\Theta, \phi(\Theta)],$$ [20]

[20]Since Θ and $\phi(\Theta)$ are symmetrically distributed, the correlation is given by

uniformly in $w_i \in [\underline{w}, \bar{w}]$, where Θ is a random variable having the same distribution F as Θ_i. The limit depends on the profile ϕ only through the factor $\mathrm{Corr}[\Theta, \phi(\Theta)]$.

Proof of Theorem 2.
The heart of the proof is in the correlation result shown in part (ii) of Lemma 2. It follows that if $\phi(\Theta)$ is more correlated with Θ than $\psi(\Theta)$ is, then for each group i, there exists N_i such that if the number of groups (n) is greater than N_i, group i ($\leq n$) will be better off under ϕ than ψ.

Note that the convergence in part (ii) of Lemma 2 is uniform in $w_i \in [\underline{w}, \bar{w}]$. This implies that the convergence is uniform in $i = 1, 2, \cdots$.[21] Thus there is N with the above property, without subscript i, which applies to all groups $i = 1, 2, \cdots$. Therefore, if $\phi(\Theta)$ is more correlated with Θ than $\psi(\Theta)$ is, then ϕ asymptotically Pareto dominates ψ.

Since the perfect correlation $\mathrm{Corr}[\Theta, \phi^{\mathrm{PR}}(\Theta)] = 1$ is attained by the proportional rule, Theorem 2 follows. □

The above results show that the winner-take-all rule is characterized by its strategic dominance, while the proportional rule is characterized by its asymptotic Pareto dominance. The following proposition provides a complete Pareto order among all the linear combinations of the two rules.

Remember that we defined the mixed rules in Section 2 above. For $0 \leq a \leq 1$, a fraction a of the weight is assigned to the winner of the group-wide vote, while the rest, $1 - a$, is distributed proportionally to each alternative:

$$\phi^a(\theta_i) = a\phi^{\mathrm{WTA}}(\theta_i) + (1 - a)\phi^{\mathrm{PR}}(\theta_i).$$

Proposition 4. *Under Assumptions 1-4, mixed profile ϕ^a asymptotically Pareto dominates mixed profile $\phi^{a'}$ for any $0 \leq a < a' \leq 1$. In particular,*

$\mathrm{Corr}\,[\Theta, \phi(\Theta)] = \mathbb{E}[\Theta\phi(\Theta)]/\sqrt{\mathbb{E}[\Theta^2]\mathbb{E}[\phi(\Theta)^2]}$ unless $\phi(\Theta)$ is almost surely zero. If $\phi(\Theta)$ is almost surely zero, then the correlation is zero.

[21] A more detailed explanation of this step is the following. By Lemma 2 (i), $\sqrt{2\pi n}\pi_i(\phi; n))$ asymptotically behaves as $2\sqrt{2\pi n}\int_0^1 \theta\mathbb{P}\{-w_i\phi(\theta) < \sum_{j\leq n} w_j\phi(\Theta_j) \leq w_i\phi(\theta)\}dF(\theta)$, where whether the sum $\sum_{j\leq n} w_j\phi(\Theta_j)$ includes the ith term or not is immaterial in the limit. The estimate of $\sqrt{2\pi n}\pi_i(\phi; n)$ therefore has the form $f_n(w_i)$, where $f_n(x) := 2\sqrt{2\pi n}\int_0^1 \theta\mathbb{P}\{-x\phi(\theta) < \sum_{j\leq n} w_j\phi(\Theta_j) \leq x\phi(\theta)\}dF(\theta)$. Lemma 2 (ii) implies that $f_n(x)$ converges uniformly in $x \in [\underline{w}, \bar{w}]$, which in turn implies that the convergence of $\sqrt{2\pi n}\pi_i(\phi; n) \approx f_n(w_i)$ is uniform in $i = 1, 2, \cdots$.

the proportional profile asymptotically Pareto dominates any mixed profile ϕ^a for $0 < a < 1$, which in turn asymptotically Pareto dominates the winner-take-all profile. In other words, all mixed profiles can be ordered by asymptotic Pareto dominance, from the proportional profile as the best, to the winner-take-all profile as the worst.

Proof. In Appendix. □

The winner-take-all rule is not only asymptotically Pareto inefficient, but the worst among the symmetric mixed profiles. Is it worse than *any* other symmetric profile? We provide an answer in Remark 3 below.

Remark 3. *What is the worst profile?* Theorem 2 leaves the natural question of whether the winner-take-all profile is the worst among all symmetric profiles, in terms of asymptotic Pareto dominance. The answer is negative. To see this, note first that, for the winner-take-all profile, the correlation in Lemma 2 is strictly positive: $\mathrm{Corr}[\Theta, \phi^{\mathrm{WTA}}(\Theta)] = \mathbb{E}(|\Theta|)/\sqrt{\mathbb{E}(\Theta^2)} > 0$. On the other hand, for the symmetric profile ϕ^0 in which the rule is defined by $\phi^0(\theta) = 0$ for almost all θ, the correlation is obviously zero. This rule assigns exactly half of the weight to each alternative, regardless of the group-wide vote. Thus the profile ϕ^0 is the worst among all symmetric profiles, as the social decision is made by a coin toss almost surely, yielding expected payoff 0 to all groups. In the following, we exclude such a trivial profile from our consideration.

4.3 Congressional District Method

The analysis in the preceding subsection suggests that the proportional profile is optimal in terms of Pareto efficiency. However, our model also implies that this profile produces an unequal distribution of welfare; in fact, this unequal nature pertains to all symmetric profiles. The Correlation Lemma 2 (ii) shows that for these profiles, the expected payoff for a group is asymptotically proportional to its weight, providing high expected payoffs to the members in a group with a large weight.

In this subsection, we examine whether such inequality can be alleviated without impairing efficiency by using an asymmetric profile, based on the Congressional District Method, currently used in Maine and Nebraska. This profile allocates a fixed amount c of each group's weight by the winner-take-all rule and the rest by the proportional rule:

$$w_i \phi^{\mathrm{CD}}(\theta_i, w_i) = c\phi^{\mathrm{WTA}}(\theta_i) + (w_i - c)\phi^{\mathrm{PR}}(\theta_i).$$

We consider the *congressional district profile* ϕ^{CD} in which the rule is used by all groups. Note that this profile is not symmetric as it depends on w_i, but the way ϕ^{CD} depends on w_i is the same for all groups. To ensure that the profile is well-defined, we impose that its parameter c is below the lower bound of weights: $c \in [0, \underline{w}]$.

Theorem 3. *Under Assumptions 1-5, let us consider the congressional district profile with parameter $c \leq \underline{w}$. For any symmetric profile ϕ, there exists $w^* \in [\underline{w}, \bar{w}]$ with the following property: for any $\varepsilon > 0$, there is N such that for all $n > N$ and $i = 1, \cdots, n$,*

$$\begin{aligned} w_i < w^* - \varepsilon &\Rightarrow \pi_i(\phi^{\mathrm{CD}}; n) > \pi_i(\phi; n), \\ w_i > w^* + \varepsilon &\Rightarrow \pi_i(\phi^{\mathrm{CD}}; n) < \pi_i(\phi; n). \end{aligned}$$

The proof of Theorem 3 uses the following lemma, which shows that the correlation lemma holds for a class of profiles such that the weight allocation rules have the following specific form of separability. Its proof and the Local Limit Theorem used in the proof are relegated to the Appendix.

Assumption 6. Let $\phi = (\phi_i)_{i=1}^{\infty}$ be a profile. There exist functions h_1, h_2, h_3 such that

$$w_i \phi_i(\theta_i, w_i) = h_1(w_i) h_2(\theta_i) + h_3(w_i) \operatorname{sgn} \theta_i, \text{ for all } i$$

where (i) h_1 is bounded, (ii) h_2 is an odd function such that the support of the distribution of $h_2(\Theta_i)$ contains 0, and (iii) h_3 is continuous but not constant.[22]

[22]Under this form, $\phi_i(\cdot, \cdot)$ is the same for all i so that we can omit subscript i whenever there is no confusion.

It is straightforward to show that Assumption 6 is satisfied for any symmetric profile as well as the congressional district profile. For a symmetric profile ϕ, let $h_1(w_i) = w_i$, $h_2(\theta_i) = \phi(\theta_i) - r \operatorname{sgn} \theta_i$, and $h_3(w_i) = w_i r$ where $r > 0$ is any positive number in the support of the distribution of $\phi(\Theta)$.[23] For the congressional district profile ϕ^{CD}, let $h_1(w_i) = w_i - c$, $h_2(\theta_i) = \theta_i - \operatorname{sgn} \theta_i$, and $h_3(w_i) = w_i$.

Lemma 3. *Under Assumptions 1-5, let ϕ be a profile which satisfies Assumption 6. Then, as $n \to \infty$,*

$$\sqrt{2\pi n}\pi_i(\phi; n) \to \frac{2w_i \mathbb{E}[\Theta\phi(\Theta, w_i)]}{\sqrt{\int_{\underline{w}}^{\bar{w}} w^2 \mathbb{E}[\phi(\Theta, w)^2] dG(w)}},$$

uniformly in $w_i \in [\underline{w}, \bar{w}]$, where Θ is a random variable having the same distribution F as Θ_i.

Proof of Theorem 3. By Lemma 3, the expected payoff for group i under a symmetric profile ϕ tends to a linear function of w_i. Let A^ϕ be the coefficient:

$$\begin{aligned} \lim_{n\to\infty} \sqrt{2\pi n}\pi_i(\phi; n) &= \frac{2w_i \mathbb{E}[\Theta\phi(\Theta)]}{\sqrt{\mathbb{E}[\phi(\Theta)^2] \int_{\underline{w}}^{\bar{w}} w^2 dG(w)}} \\ &=: A^\phi w_i. \end{aligned} \tag{8}$$

For the congressional district profile, remember the definition:

$$\begin{aligned} w_j \phi^{\mathrm{CD}}(\theta_j, w_j) &= c\phi^{\mathrm{WTA}}(\theta_j) + (w_j - c)\phi^{\mathrm{PR}}(\theta_j) \\ &= c \operatorname{sgn}(\theta_j) + (w_j - c)\theta_j. \end{aligned}$$

We claim that the limit function is affine in w_i:

$$\lim_{n\to\infty} \sqrt{2\pi n}\pi_i(\phi^{\mathrm{CD}}; n) = B w_i + C. \tag{9}$$

[23]This is possible since $\phi(\Theta)$ is symmetrically distributed, and since we exclude the trivial case in which $\phi(\Theta) = 0$ almost surely.

To see that, let us apply Lemma 3 again:

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{2\pi n}\pi_i(\phi^{\mathrm{CD}}; n) &= 2 \cdot \frac{w_i \mathbb{E}\left[\Theta \phi^{\mathrm{CD}}(\Theta, w_i)\right]}{\sqrt{\int_{\underline{w}}^{\bar{w}} w^2 \mathbb{E}\left[\phi^{\mathrm{CD}}(\Theta, w)^2\right] dG(w)}} \\
&= 2 \cdot \frac{c\mathbb{E}\left[|\Theta|\right] + (w_i - c)\,\mathbb{E}\left[\Theta^2\right]}{\sqrt{\int_{\underline{w}}^{\bar{w}} w^2 \mathbb{E}\left[\phi^{\mathrm{CD}}(\Theta, w)^2\right] dG(w)}}.
\end{aligned}$$

Since $|\theta| \geq \theta^2$ with a strict inequality for $0 < |\theta| < 1$, the full support condition for Θ implies $\mathbb{E}[|\Theta|] > \mathbb{E}[\Theta^2]$, which induces that the intercept C is positive. The coefficient of w_i is:

$$B = \frac{2\mathbb{E}\left[\Theta^2\right]}{\sqrt{\int_{\underline{w}}^{\bar{w}} w^2 \mathbb{E}\left[\phi^{\mathrm{CD}}(\Theta, w)^2\right] dG(w)}}.$$

If $A^\phi < B$, combined with $C > 0$, the right-hand side of (9) is above that of (8). Then, set $w^* = \bar{w}$. If $A^\phi > B$, again combined with $C > 0$, the two limit functions (8) and (9) intersect only once at a positive value $\hat{w}$. Let $w^* = \max\{\underline{w}, \min\{\hat{w}, \bar{w}\}\}$.

Since the convergences (8) and (9) are uniform in w_i, for any $\varepsilon > 0$ there is N with the property stated in Theorem 3. □

Theorem 3 implies that the congressional district profile makes the members of groups with small weights better off, compared with *any* symmetric profile. If the weight is an increasing function of the group size, it means that the congressional district profile is favorable for the members of small groups.

The intuitive reason why the congressional district profile is advantageous for small groups is as follows. Under this profile, the ratio of weights cast by the winner-take-all rule (i.e. c/w_i) is higher for small groups than large groups. The congressional district profile therefore resembles the situation where the rules used by the smaller groups are relatively close to the winner-take-all rule, whereas those by the larger groups are close to the proportional rule. The strategic dominance of the winner-take-all rule suggests that this deviation is profitable for the small groups. We provide a numerical result in the following subsection using an example of the US Electoral College.

In addition to Theorem 3, we can also show that the congressional district profile distributes payoffs more equally than any symmetric profile does, in the sense of Lorenz dominance. A profile of per capita payoffs for the groups, $\pi = (\pi_1, \cdots, \pi_n)$, is said to *Lorenz dominate* another profile $\pi' = (\pi'_1, \cdots, \pi'_n)$ if the share of payoffs acquired by any bottom fraction of groups is larger in the former profile than in the latter.[24] Lorenz dominance, whenever it occurs, agrees with equality comparisons by various inequality indices including coefficient of variation, Gini coefficient, Atkinson index, and Theil index (see Fields and Fei (1978) and Atkinson (1970)). To see why the congressional district profile is more equal than any symmetric profile, recall equations (8) and (9) in the proof of Theorem 3, which assert that when the number of groups is large, the per capita payoff for group i is approximately $A^{\phi} w_i$ for the symmetric profile, and it is approximately $Bw_i + C$ for the congressional district profile. The constant term $C > 0$ for the congressional district profile means equal additions to all groups' payoffs, which result in a more equal distribution than when there is no such term. More precisely, we can prove the following statement. The proof is relegated to the Appendix.

Theorem 4. *Under Assumptions 1-5, let us consider the payoff profile under the congressional district profile:* $\pi\left(\phi^{\mathrm{CD}}; n\right) = \left(\pi_i\left(\phi^{\mathrm{CD}}; n\right)\right)_{i=1}^{n}$. *Let* ϕ *be any symmetric profile and* $\pi(\phi; n) = (\pi_i(\phi; n))_{i=1}^{n}$ *the payoff profile under* ϕ. *For sufficiently large* n, $\pi\left(\phi^{\mathrm{CD}}; n\right)$ *Lorenz dominates* $\pi(\phi; n)$.

4.4 Computational Results

The results in the previous subsection concern cases with a large number of groups. The question remains as to whether the conclusions obtained there are also valid for a finite number of groups. In this section, we study this question by numerically analyzing a model of the US presidential election.

There are 50 states and one federal district. The weight w_i for state i is

[24]Formally, for each $x \in [-1, 1]$, let $H_\pi(x)$ be the total population share of those groups whose per capita welfare is not greater than x under the payoff profile π. Then H_π is a distribution function. The *Lorenz curve* of H_π is the graph of the function $\int_0^{H_\pi^{-1}(p)} x dH_\pi(x) / \int_0^1 x dH_\pi(x)$, $0 \le p \le 1$, where we define $H_\pi^{-1}(p) = \sup\{x : H_\pi(x) \le p\}$. A payoff profile π *Lorenz dominates* another profile π' if the Lorenz curve of H_π lies above that of $H_{\pi'}$.

the number of electoral votes currently assigned to it. This number equals the state's total number of seats in the Senate and House of Representatives. Thus, w_i is two plus a number that is roughly proportional to the state's population. The first and second columns of Table 2 describe the distribution of weights among the states.

We assume IAC* (Impartial Anonymous Culture*): the statewide popular vote margins Θ_i are independent and uniformly distributed on $[-1, 1]$, first introduced by May (1948) and studied thoroughly by, for example, De Mouzon et al. (2019). For any profile ϕ, we can compute the per capita payoff for state i via the formula:

$$\pi_i(\phi) = 0.5^{50} \int_{-1}^{1} \cdots \int_{-1}^{1} \theta_i 1_A(\theta_1, \cdots, \theta_{51}) d\theta_1 \cdots d\theta_{51} \tag{10}$$

where $A = \left\{(\theta_1, \cdots, \theta_{51}) \middle| \sum_{j=1}^{51} w_j \phi_j(\theta_j) > 0\right\}$.[25]

We consider four distinct profiles: ϕ^{WTA}, ϕ^{PR}, ϕ^a with $a = 102/538$, and ϕ^{CD} with coefficient $c = 2$. As before, these are respectively the winner-take-all profile, the proportional profile, a mixed profile, and a congressional district profile. The parameter $c = 2$ of the congressional district profile is the number currently used in Maine and Nebraska, namely, it corresponds to two seats assigned to each state in the Senate. The parameter $a = 102/538$ of the mixed profile is chosen so that the proportion of electoral votes allocated on the winner-take-all basis is the same for all states, and the total number of electoral votes allocated in this way is the same as in the congressional district profile.

We compute (10) under these four profiles by a Monte Carlo simulation with 10^{10} iterations. The results are summarized in Tables 2 and 3. Table 2 shows the per capita payoff ($\pi_i(\phi)$) under the respective profiles. Table 3 shows the ratios of per capita payoff between different profiles ($\pi_i(\phi)/\pi_i(\psi)$). If the ratio is below 1, state i prefers ψ to ϕ.

It follows from Lemma 2 (ii) that as the number n of states increases, the ratios $\pi_i\left(\phi^{\mathrm{WTA}}\right)/\pi_i\left(\phi^{\mathrm{PR}}\right)$ and $\pi_i\left(\phi^a\right)/\pi_i\left(\phi^{\mathrm{PR}}\right)$ converge to the respective correlations $\mathrm{Corr}[\Theta, \phi^{\mathrm{WTA}}(\Theta)] \approx 0.866$ and $\mathrm{Corr}[\Theta, \phi^a(\Theta)] \approx 0.989$, where the values are computed for Θ uniformly distributed on $[-1, 1]$. Table

[25] It is easy to check that under the uniform distribution assumption, (10) is equivalent to the expression in Lemma 2 (i).

3 indicates that for the present example with 50 states plus DC, these ratios are indeed close to the respective correlations, which suggests that convergence of the π-ratios is fairly quick. In particular, as expected by Theorem 2, the proportional profile Pareto dominates the winner-take-all profile in the present case. As suggested by Proposition 4, all states prefer the mixed profile ϕ^a to the winner-take-all profile, and the proportional profile to ϕ^a.

The ratios $\pi_i(\phi^{\mathrm{CD}})/\pi_i(\phi^{\mathrm{PR}})$ in Table 3 are consistent with the result in Theorem 3. Small states prefer the congressional district profile to the proportional one.

In addition, the values of $\pi_i(\phi^{\mathrm{CD}})/\pi_i(\phi^{\mathrm{WTA}})$ in the table show that the winner-take-all profile is Pareto dominated by the congressional district profile, and the welfare improvement by switching to the congressional district profile is greater for small states than for large states in terms of the ratio.

Table 2: Estimated payoffs in the US presidential election, based on the apportionment in 2016, via Monte Carlo simulation with 10^{10} iterations. The estimated standard errors are in the range between 3.9 and 4.1×10^{-6}.

electoral votes	number of states	$\pi(\phi^{\mathrm{WTA}})$	$\pi(\phi^{\mathrm{PR}})$	$\pi(\phi^{a})$	$\pi(\phi^{\mathrm{CD}})$
3	8	0.0113	0.0133	0.0130	0.0167
4	5	0.0151	0.0177	0.0174	0.0209
5	3	0.0189	0.0221	0.0217	0.0251
6	6	0.0226	0.0266	0.0261	0.0293
7	3	0.0264	0.0310	0.0305	0.0335
8	2	0.0302	0.0354	0.0348	0.0377
9	3	0.0340	0.0399	0.0392	0.0419
10	4	0.0378	0.0443	0.0436	0.0461
11	4	0.0416	0.0488	0.0479	0.0503
12	1	0.0454	0.0532	0.0523	0.0545
13	1	0.0492	0.0577	0.0567	0.0587
14	1	0.0531	0.0622	0.0611	0.0630
15	1	0.0569	0.0666	0.0655	0.0672
16	2	0.0607	0.0711	0.0699	0.0715
18	1	0.0684	0.0801	0.0788	0.0800
20	2	0.0762	0.0891	0.0877	0.0885
29	2	0.1120	0.1303	0.1284	0.1275
38	1	0.1494	0.1729	0.1706	0.1677
55	1	0.2356	0.2614	0.2615	0.2507

Table 3: Ratios between payoffs.

electoral votes	number of states	$\frac{\pi(\phi^{\mathrm{WTA}})}{\pi(\phi^{\mathrm{PR}})}$	$\frac{\pi(\phi^{a})}{\pi(\phi^{\mathrm{PR}})}$	$\frac{\pi(\phi^{\mathrm{CD}})}{\pi(\phi^{\mathrm{PR}})}$	$\frac{\pi(\phi^{\mathrm{CD}})}{\pi(\phi^{\mathrm{WTA}})}$
3	8	0.852	0.982	1.260	1.479
4	5	0.852	0.982	1.182	1.387
5	3	0.852	0.982	1.134	1.331
6	6	0.852	0.982	1.103	1.294
7	3	0.852	0.982	1.080	1.268
8	2	0.852	0.982	1.064	1.248
9	3	0.852	0.982	1.050	1.232
10	4	0.853	0.983	1.040	1.220
11	4	0.853	0.983	1.031	1.210
12	1	0.853	0.983	1.024	1.201
13	1	0.853	0.983	1.018	1.194
14	1	0.853	0.983	1.013	1.187
15	1	0.854	0.983	1.009	1.181
16	2	0.854	0.983	1.005	1.177
18	1	0.854	0.983	0.998	1.168
20	2	0.855	0.983	0.993	1.161
29	2	0.859	0.985	0.978	1.138
38	1	0.864	0.987	0.970	1.122
55	1	0.901	1.000	0.959	1.064

5 Concluding Remarks

This paper shows that the decentralized choice of the weight allocation rule in representative voting constitutes a Prisoner's Dilemma: the winner-take-all rule is a dominant strategy for each group, whereas the Nash equilibrium is Pareto dominated. Each group has an incentive to put its entire weight on the alternative supported by the majority of its members in order to reflect their preferences in the social decision, although it fails to efficiently aggregate the preferences of all members in the society. The society which consists of distinct groups thus faces a dilemma between the group's incentive and the social objective.

We also provide an asymptotic model and show that the proportional rule Pareto dominates every other symmetric profile when the number of the groups is sufficiently large.

Our model may provide explanations for the phenomena that we observe in existing collective decision making. In the United States Electoral College, the rule used by the states varied in early elections until it converged by 1832 to the winner-take-all rule, which remains dominantly employed by nearly all states since then. In many parliamentary voting situations, we often observe parties and/or factions forcing their members to align their votes in order to maximally reflect their preferences in the social decision, although some members may disagree with the party's alignment. The voting outcome obtained by the winner-take-all rule may fail to efficiently aggregate preferences, as observed in the discrepancy between the electoral result and the national popular vote winner in the US presidential elections in 2000 and 2016. Party discipline or factional voting may also cause welfare loss when each group pushes their votes maximally toward their ideological goals, failing to reflect all members' preferences in the social decision.

The Winner-Take-All Dilemma tells us that the society should call for some device different from each group's unilateral effort, in order to obtain a socially preferable outcome. As we see in the failure of various attempts to modify or abolish the winner-take-all rule, such as the ballot initiative for an amendment to the State Constitution in Colorado in 2004, each state has no incentive to unilaterally deviate from the equilibrium. The National Popular Vote Interstate Compact is a well-suited example of a coordination device (Koza et al. (2013)). As it comes into effect only when the number of electoral votes attains the majority, each state does not suffer from the payoff loss by a unilateral (or coalitional) deviation until sufficient coordination is attained. The emergence of such an attempt is coherent with the insights obtained in this paper that the game is a Prisoner's Dilemma, and a coordination device is necessary for a Pareto improvement.

Our analysis is abstract in that we do not impose assumptions on the preferences distribution based on the observed characteristics in the real representative voting problems. Additionally, we impose an impartiality assumption in our asymptotic analysis. Obviously, our normative analysis would be best complemented by a positive analysis, which we leave for future research.

We have assumed that social decisions are binary. There are situations where this assumption may not fit. In the US presidential elections, third-party or independent candidates can, and do, have a non-negligible impact on the election outcome. It is not clear how the presence of such candidates alters the comparison of rules to allocate electoral votes. When the model is applied to legislative voting, the assumption of binary decision might be justified on the grounds that choices are ultimately made between the status quo and a proposal. However, such an argument abstracts away the process that gives rise to the particular pair of alternatives (e.g., what becomes the status quo, how much proposal power each party has, and so on). Cases with more than two alternatives require further investigation.

Appendix

A1 Proof of Lemma 1

(i) Let D be the set of all SCFs, and $\pi(d) = \{(\pi_i(d))_{i=1}^n | d \in D\}$ the set of (ex ante) payoff vectors generated by SCFs. Then $\pi(d)$ is convex.[26] Let $\mathrm{Pa}\,(\pi(D))$ be the Pareto frontier of $\pi(D)$, i.e., the set of payoff profiles $u \in \pi(D)$ for which there is no $d' \in \pi(D)$ such that $u'_i \geq u_i$ for all i and $u'_i > u_i$ for some i.

We divide the proof of (i) into two steps.

Step 1. *Let $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$. Then the unique solution to the following maximization problem (11) is the payoff vector $u^\lambda := \left(\pi_i(d^\lambda)\right)_{i=1}^n$ under a cardinal λ-weighted majority rule d^λ.*[27]

$$\max_{u \in \pi(D)} \sum_{i=1}^n \lambda_i u_i. \tag{11}$$

Moreover, an SCF d satisfies $(\pi_i(d))_{i=1}^n = u^\lambda$ if and only if d is a cardinal λ-weighted majority rule.

[26] This is because for any two SCFs d and d', any convex combination of the payoff vectors corresponding to d and d' can be realized as a compound SCF that randomizes between d and d'.

[27] Recall that a cardinal weighted majority rule with a given weight vector is unique only up to differences on a set of measure zero. But all versions of the rule induce the same payoff vector.

Let $d \in D$ be any SCF. Then

$$\sum_{i=1}^{n} \lambda_i \pi_i(d) = \sum_{i=1}^{n} \lambda_i \mathbb{E}\left[\Theta_i d(\Theta)\right] = \mathbb{E}\left[d(\Theta) \sum_{i=1}^{n} \lambda_i \Theta_i\right]. \tag{12}$$

Since Θ is absolutely continuous, and so $\sum_{i=1}^{n} \lambda_i \Theta_i \neq 0$ almost surely, d maximizes (12) if and only if $d(\Theta) = \operatorname{sgn} \sum_{i=1}^{n} \lambda_i \Theta_i$ almost surely. That is,

$$d \text{ maximizes (12)} \Leftrightarrow d \text{ is a cardinal } \lambda\text{-weighted majority rule.} \tag{13}$$

This implies the first sentence of Step 1. Result (13) also implies that if d is not a cardinal λ-weighted majority rule, then $\pi_i(d) \neq \pi_i(d^\lambda)$ for some i, which proves the "only if" part of the second sentence of Step 1. The "if" part is trivial.

Step 2. *A payoff vector $u \in \pi(D)$ is in the Pareto frontier $\mathrm{Pa}\,(\pi(D))$ if and only if there exists $\lambda \in \mathbb{R}_+^n \setminus \{0\}$ such that $u = (\pi_i(d^\lambda))_{i=1}^n =: u^\lambda$, where d^λ is a cardinal λ-weighted majority rule.*

Since $\pi(D)$ is convex, we can apply Mas-Colell et al. (1995, Proposition 16.E.2) to show the "only if" part of Step 2.

To show the "if" part, suppose on the contrary that $u^\lambda \notin \mathrm{Pa}\,(\pi(D))$ for some $\lambda \in \mathbb{R}_+^n \setminus \{0\}$. Then there exists $u \in \pi(D)$ such that $u \neq u^\lambda$ and $u_i \geq u_i^\lambda$ for all i. Then $\sum_{i=1}^{n} \lambda_i u_i \geq \sum_{i=1}^{n} \lambda_i u_i^\lambda$. This contradicts the fact that u^λ is the unique solution to problem (11).

(ii) This follows from the trivial fact that the set of SCFs d_ϕ induced by profiles ϕ that are equivalent to a generalized proportional profile coincides with the set of all cardinal weighted majority rules.[28] $\square$

[28] Indeed, if ϕ is equivalent to a generalized proportional profile with the vector of coefficients $\lambda \in [0,1]^n \setminus \{0\}$, the induced SCF d_ϕ is a cardinal μ-weighted majority rule, where the weights are defined by $\mu_i := w_i \lambda_i$; conversely, if d is a cardinal μ-weighted majority rule, then $d = d_\phi$ for some profile ϕ that is equivalent to the generalized proportional profile with coefficients $\lambda_i := \frac{\mu_i}{w_i}$.

A2 Proof of Part (i) of Lemma 2

We prove the statement for group 1. Let $\pi_1(\phi; n|\theta_1)$ be the conditional expected payoff for group 1 given that the group-wide margin is $\Theta_1 = \theta_1$, which by (3) is:

$$\pi_1(\phi; n|\theta_1) = \theta_1(\mathbb{P}\{w_1\phi(\theta_1) + S_{\phi_{-1}} > 0\} - \mathbb{P}\{w_1\phi(\theta_1) + S_{\phi_{-1}} < 0\}).$$

Since $S_{\phi_{-1}}$ is symmetrically distributed, the second probability can be written as $\mathbb{P}\{-w_1\phi(\theta_1) + S_{\phi_{-1}} > 0\}$. Thus, for $\theta_1 \in [0,1]$, the above expression equals

$$\pi_1(\phi; n|\theta_1) = \theta_1\mathbb{P}\{-w_1\phi(\theta_1) < S_{\phi_{-1}} \leq w_1\phi(\theta_1)\}.$$

By symmetry, twice the integral of this expression over $\theta_1 \in [0,1]$ (instead of $[-1,1]$) equals the unconditional expected payoff $\pi_1(\phi; n)$, which proves part (i) of Lemma 2. □

A3 Local Limit Theorem

We quote a version of the Local Limit Theorem shown in Mineka and Silverman (1970). We will use it in the proof of part (ii) of Lemma 2.

LLT. *(Mineka and Silverman (1970, Theorem 1)) Let (X_i) be a sequence of independent random variables with mean 0 and variances $0 < \sigma_i^2 < \infty$. Write F_i for the distribution of X_i. Write also $S_n = \sum_{i=1}^n X_i$ and $s_n^2 = \sum_{i=1}^n \sigma_i^2$. Suppose the sequence (X_i) satisfies the following conditions:*

(α) *There exists $\bar{x} > 0$ and $c > 0$ such that for all i,*

$$\frac{1}{\sigma_i^2}\int_{|x|<\bar{x}} x^2 dF_i(x) > c.$$

(β) *Define the set*

$$A(t,\varepsilon) = \{x : |x| < \bar{x} \text{ and } |xt - \pi m| > \varepsilon \text{ for all integer } m \text{ with } |m| < \bar{x}\}.$$

Then, for some bounded sequence (a_i) such that $\inf_i \mathbb{P}\{|X_i - a_i| <$

$\delta\} > 0$ for all $\delta > 0$, and for any $t \neq 0$, there exists $\varepsilon > 0$ such that

$$\frac{1}{\log s_n} \sum_{i=1}^{n} \mathbb{P}\{X_i - a_i \in A(t, \varepsilon)\} \to \infty.$$

(γ) (Lindeberg's condition.) For any $\varepsilon > 0$,

$$\frac{1}{s_n^2} \sum_{i=1}^{n} \int_{|x|/s_n > \varepsilon} x^2 dF_i(x) \to 0.$$

Under conditions (α)-(γ), if $s_n^2 \to \infty$, we have

$$\sqrt{2\pi s_n^2}\mathbb{P}\{S_n \in (a, b]\} \to b - a.\text{[29]} \tag{14}$$

A4 Proof of Lemma 3

Preliminaries. We prove the lemma for group 1. In the proof, we use the notation of LLT. Let

$$X_i := w_i \phi(\Theta_i, w_i),\ i = 1, 2, \cdots,$$

and $S_n := \sum_{i=1}^n X_i$. Then X_i has mean 0 and variance $\sigma_i^2 := w_i^2 \mathbb{E}[\phi(\Theta, w_i)^2]$, and so the partial sum of variances is $s_n^2 := \sum_{i=1}^n w_i^2 \mathbb{E}[\phi(\Theta, w_i)^2]$, where Θ represents a random variable that has the same distribution F as Θ_i.

Define the event

$$\Omega_n(\theta_1, w_1) = \left\{ -w_1\phi(\theta_1, w_1) < \sum_{i=2}^{n} X_i \leq w_1\phi(\theta_1, w_1) \right\}.$$

We divide the proof into several claims. Claims 5.1-5.3 show that the sequence (X_i) defined above satisfies the conditions of the Local Limit

[29] The original conclusion of Theorem 1 in Mineka and Silverman (1970) is stated in terms of the open interval (a, b). Applying the theorem to $(a, b + c)$ and $(b, b + c)$ and then taking the difference gives the result for $(a, b]$. In addition, the original statement allows for cases where s_n^2 does not go to infinity, and also mentions uniform convergence. These considerations are not necessary for our purpose, so we omit them.

Theorem (LLT) in Section A4. Claim 5.4 applies LLT to complete the proof of Lemma 3.

Claim 5.1. $\frac{s_n^2}{n} \to \int_{\underline{w}}^{\bar{w}} w^2 \mathbb{E}[\phi(\Theta, w)^2] dG(w)$.

Proof of Claim 5.1. This holds since sequence (σ_i^2) is bounded and the statistical distribution G_n induced by $(w_i)_{i=1}^n$ converges weakly to G. □

Claim 5.2. *Conditions (α) and (γ) in LLT hold.*

Proof of Claim 5.2. This immediately follows from the fact that sequence (X_i) is bounded and $s_n^2 \to \infty$. In particular, it is enough to define $\bar{x}$ to be any finite number greater than $\bar{w}$. □

Claim 5.3. *Condition (β) in LLT holds.*

Proof of Claim 5.3. Recall that ϕ has the form

$$w_i \phi(\theta_i, w_i) = h_1(w_i) h_2(\theta_i) + h_3(w_i) \operatorname{sgn} \theta_i.$$

Let $a_i = h_3(w_i)$. We first check that the sequence (a_i) satisfies the requirements in condition (β). First, (a_i) is bounded since h_3 is bounded. Now, for any i and any $\delta > 0$,

$$\begin{aligned} \mathbb{P}\{|X_i - a_i| < \delta\} &\geq \mathbb{P}\{|X_i - a_i| < \delta \text{ and } \Theta_i > 0\} \\ &\geq \mathbb{P}\left\{|w_i \phi(\Theta_i, w_i) - h_3(w_i) \operatorname{sgn} \Theta_i| < \delta \text{ and } \Theta_i > 0\right\} \\ &= \mathbb{P}\{|h_1(w_i) h_2(\Theta_i)| < \delta \text{ and } \Theta_i > 0\}. \end{aligned}$$

Letting $\bar{h}_1 > 0$ be an upper bound of $|h_1|$ and Θ a random variable distributed as Θ_i, the last expression has the following lower bound independent of i:

$$\mathbb{P}\{|h_2(\Theta)| < \delta / \bar{h}_1 \text{ and } \Theta > 0\} > 0,$$

which is positive by the assumptions on h_2 and on the distribution of Θ.

Next we check the limit condition in (β). Recall that $A(t, \varepsilon)$ is the union of intervals

$$\left(\frac{\pi m + \varepsilon}{|t|}, \frac{\pi(m+1) - \varepsilon}{|t|} \right), \ m = 0, \pm 1, \pm 2, \cdots,$$

restricted to $(-\bar{x}, \bar{x})$, where we can choose $\bar{x}$ to be any number greater than $\bar{w}$. To prove the limit condition in (β), it therefore suffices to verify that one such interval contains $X_i - a_i$ with probability bounded away from zero, for all groups i in some sufficiently large subset of groups. To do this, note that if $\Theta_i < 0$, then $X_i - a_i = h_1(w_i)h_2(\Theta_i) - 2h_3(w_i)$. The assumptions on h_2 and on the distribution of Θ imply that for any $\eta > 0$, there exists a set $O_\eta \subset [-1, 0]$ with $\mathbb{P}\{\Theta \in O_\eta\} > 0$ such that if $\Theta \in O_\eta$ then $|h_2(\Theta)| \leq \eta$. Therefore,

$$\Theta_i \in O_\eta \Rightarrow X_i - a_i \in T_{w_i,\eta},$$

where

$$T_{w_i,\eta} := [-2h_3(w_i) - \eta h_1(w_i),\ -2h_3(w_i) + \eta h_1(w_i)].$$

Since h_1 is bounded, we can make $T_{w_i,\eta}$ an arbitrarily small interval around $-2h_3(w_i)$ by letting $\eta > 0$ sufficiently small. Moreover, since h_3 is continuous and not a constant, we can find a sufficiently small interval $[\underline{v}, \bar{v}] \subset [\underline{w}, \bar{w}]$ with $\underline{v} < \bar{v}$ such that if $w_i \in [\underline{v}, \bar{v}]$, then $-2h_3(w_i)$ is between, and bounded away from, $\frac{\pi m}{|t|}$ and $\frac{\pi(m+1)}{|t|}$ for some integer m. Fix such an interval $[\underline{v}, \bar{v}]$ and define

$$I := \{i : w_i \in [\underline{v}, \bar{v}]\}.$$

Then, for sufficiently small $\eta > 0$ and $\varepsilon > 0$, we have $T_{w_i,\eta} \subset A(t, \varepsilon)$ for all $i \in I$. Fixing such $\eta > 0$ and $\varepsilon > 0$, it follows that

$$\Theta_i \in O_\eta \text{ and } i \in I \Rightarrow X_i - a_i \in A(t, \varepsilon).$$

This implies that

$$\mathbb{P}\{X_i - a_i \in A(t, \varepsilon)\} \geq \mathbb{P}\{\Theta \in O_\eta\} =: p > 0 \text{ for all } i \in I,$$

and hence

$$\frac{1}{\log s_n} \sum_{i=1}^{n} \mathbb{P}\{X_i - a_i \in A(t, \varepsilon)\} \geq \frac{n}{\log s_n} \cdot \frac{\sharp\{i \in I : i \leq n\}}{n} \cdot p.$$

As $n \to \infty$, the first factor on the right-hand side tends to ∞ since s_n has an asymptotic order of $\sqrt{n}$. The second factor tends to $G(\bar{v}) - G(\underline{v}) > 0$, which is positive since G has full support on $[\underline{w}, \bar{w}]$. Therefore the left-hand side tends to ∞. $\square$

Claim 5.4. *As $n \to \infty$, uniformly in $w_1 \in [\underline{w}, \bar{w}]$,*

$$2\int_0^1 \theta_1 \sqrt{2\pi n}\mathbb{P}\{\Omega_n(\theta_1, w_1)\}dF(\theta_1) \to \frac{2w_1\mathbb{E}[\Theta\phi(\Theta, w_1)]}{\sqrt{\int_{\underline{w}}^{\bar{w}} w^2\mathbb{E}[\phi(\Theta, w)^2]dG(w)}}. \quad (15)$$

By part (i) of Lemma 2,[30] *the left-hand side of (15) is $\sqrt{2\pi n}\pi_i(\phi; n)$, and therefore Lemma 3 holds.*

Proof of Claim 5.4. By Claims 5.2 and 5.3, we may apply LLT to obtain

$$\sqrt{2\pi s_n^2}\mathbb{P}\{\Omega_n(\theta_1, w_1)\} \to 2w_1\phi(\theta_1, w_1).$$

By Claim 5.1, this means that

$$\sqrt{2\pi n}\theta_1\mathbb{P}\{\Omega_n(\theta_1, w_1)\} \to \frac{2w_1\theta_1\phi(\theta_1, w_1)}{\sqrt{\int_{\underline{w}}^{\bar{w}} w^2\mathbb{E}[\phi(\Theta, w)^2]dG(w)}}. \quad (16)$$

Letting $\theta_1 = 1$ maximizes the left-hand side of (16) with the maximum value $\sqrt{2\pi n}\mathbb{P}\{\Omega_n(1, w_1)\}$. This maximum value itself converges to a finite limit. Hence the expression $\sqrt{2\pi n}\theta_1\mathbb{P}\{\Omega_n(\theta_1, w_1)\}$ is uniformly bounded for all n and $\theta_1 \in [0, 1]$. By the Bounded Convergence Theorem,

$$2\int_0^1 \theta_1 \sqrt{2\pi n}\mathbb{P}\{\Omega_n(\theta_1, w_1)\}dF(\theta_1) \to 2 \cdot \frac{2w_1 \int_0^1 \theta_1\phi(\theta_1, w_1)dF(\theta_1)}{\sqrt{\int_{\underline{w}}^{\bar{w}} w^2\mathbb{E}[\phi(\Theta, w)^2]dG(w)}}.$$

Since F is symmetric and ϕ is odd, this limit is exactly the one in (15).

To check the uniform convergence, note that for each n, the integral on the left-hand side of (15) is non-decreasing in w_1, since event $\Omega_n(\theta_1, w_1)$ weakly expands as w_1 increases.[31] We have shown that this integral converges pointwise to a limit that is proportional to the factor $w_1\mathbb{E}[\Theta\phi(\Theta, w_1)]$, which is continuous in w_1.[32] Therefore, the convergence

[30] It is easy to check that part (i) of Lemma 2 holds for rules $\phi(\cdot, w_i)$ that depend on weight w_i as well.

[31] Let $\theta_1 \in [0, 1]$. If ϕ is a symmetric profile, i.e. if $\phi(\theta_1, w_1) = \phi(\theta_1)$, then $w_1\phi(\theta_1)$ is non-decreasing in w_1. If $\phi = \phi^{\text{CD}}$, then $w_1\phi^{\text{CD}}(\theta_1, w_1) = c\,\text{sgn}(\theta_1) + (w_1 - c)\theta_1$, which is non-decreasing in w_1 again. Thus event $\Omega_n(\theta_1, w_1)$ weakly expands as w_1 increases.

[32] If ϕ is a symmetric profile, this factor is linear in w_i. If $\phi = \phi^{\text{CD}}$, the factor equals

in (15) is uniform in $w_1 \in [\underline{w}, \bar{w}]$.[33] □

A5 Proof of Part (ii) of Lemma 2

This follows immediately from Lemma 3, by noting that if ϕ is a symmetric profile, each group's rule can be written as $\phi(\theta_j, w_j) = \phi(\theta_j)$. □

A6 Proof of Proposition 2

By part (ii) of Lemma 2, we must show that $\text{Corr}\,[\Theta, \phi^a(\Theta)]$ is decreasing in $a \in [0,1]$. By simple calculation,

$$\mathbb{E}(\Theta^2) \cdot \text{Corr}\,[\Theta, \phi^a(\Theta)]^2 = \frac{a\mathbb{E}(|\Theta|) + (1-a)\mathbb{E}(\Theta^2)}{a^2 + 2a(1-a)\mathbb{E}(|\Theta|) + (1-a)^2\mathbb{E}(\Theta^2)}.$$

The derivative of this expression with respect to a has the same sign as

$$\begin{aligned}
&\left\{\tfrac{d}{da}(a\mathbb{E}(|\Theta|) + (1-a)\mathbb{E}(\Theta^2))^2\right\}\Big(a^2 + 2a(1-a)\mathbb{E}(|\Theta|) + (1-a)^2\mathbb{E}(\Theta^2)\Big) \\
&- \Big(a\mathbb{E}(|\Theta|) + (1-a)\mathbb{E}(\Theta^2)\Big)^2\left\{\tfrac{d}{da}(a^2 + 2a(1-a)\mathbb{E}(|\Theta|) + (1-a)^2\mathbb{E}(\Theta^2))\right\} \\
&= a(a\mathbb{E}(|\Theta|) + (1-a)\mathbb{E}(\Theta^2))(\mathbb{E}(|\Theta|)^2 - \mathbb{E}(\Theta^2)).
\end{aligned}$$

This is negative for any $a \in (0,1]$, since $\mathbb{E}(|\Theta|)^2 \leq \mathbb{E}(\Theta^2)$ in general, and the full-support assumption implies that this holds with strict inequality. □

A7 Proof of Theorem 4

Clearly, Lorenz dominance is invariant to linear transformations of payoffs. Thus, it suffices to prove that for large enough n, the payoff profile $\sqrt{2\pi n}\pi(\phi^{\text{CD}}; n)$ Lorenz dominates the payoff profile $\sqrt{2\pi n}\pi(\phi; n)$. By equations (8) and (9) in the proof of Theorem 3, as $n \to \infty$ these amounts converge to $Bw_i + C$ and $A^\phi w_i$, respectively. A result by Moyes (1994, Proposition 2.3) implies that if f and g are continuous, nondecreasing,

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$c\mathbb{E}(|\Theta|) + (w_i - c)\mathbb{E}(\Theta^2)$, which is affine in w_i.

[33] It is known that if (f_n) is a sequence of non-decreasing functions on a fixed finite interval and f_n converges pointwise to a continuous function, then the convergence is uniform. See Buchanan and Hildebrandt (1908).

and positive-valued functions such that $f(w_i)/g(w_i)$ is decreasing in w_i, then the distribution of $f(w_i)$ Lorenz dominates that of $g(w_i)$. The ratio $(Bw_i+C)/(A^\phi w_i)$ is decreasing in w_i, and so the claimed Lorenz dominance holds in the limit as $n \to \infty$. Recalling that the convergences are uniform, the dominance holds for sufficiently large n. □

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