Incomplete Information Robustness*

Stephen Morris† Takashi Ui‡

March 2020

Abstract

Consider an analyst who models a strategic situation in terms of an incomplete information game and makes a prediction about players' behavior. The analyst's model approximately describes each player's hierarchies of beliefs over payoff-relevant states, but the true incomplete information game may have correlated duplicated belief hierarchies, and the analyst has no information about the correlation. Under these circumstances, a natural candidate for the analyst's prediction is the set of belief-invariant Bayes correlated equilibria (BIBCE) of the analyst's incomplete information game. We introduce the concept of robustness for BIBCE: a subset of BIBCE is robust if every nearby incomplete information game has a BIBCE that is close to some BIBCE in this set. Our main result provides a sufficient condition for robustness by introducing a generalized potential function of an incomplete information game. A generalized potential function is a function on the Cartesian product of the set of states and a covering of the action space which incorporates some information about players' preferences. It is associated with a belief-invariant correlating device such that a signal sent to a player is a subset of the player's actions, which can be interpreted as a vague prescription to choose some action from this subset. We show that, for every belief-invariant correlating device that maximizes the expected value of a generalized potential function, there exists a BIBCE in which every player chooses an action from a subset of actions prescribed by the device, and that the set of such BIBCE is robust, which can differ from the set of potential maximizing BNE.

JEL classification: C72, D82.

Keywords: Bayes correlated equilibria, belief hierarchies, belief invariance, generalized potentials, incomplete information games, potential games.

---

*This work is supported by Grant-in-Aid for Scientific Research Grant Numbers 18H05217. Any update will be available at `http://www1.econ.hit-u.ac.jp/oui/iir.pdf`.

†Department of Economics, Massachusetts Institute of Technology.

‡Department of Economics, Hitotsubashi University.

1 Introduction

Consider an analyst who models a strategic situation in terms of an incomplete information game and makes a prediction about players' behavior. He believes that his model correctly describes the probability distribution over the players' Mertens-Zamir hierarchies of beliefs over payoff-relevant states (Mertens and Zamir, 1985). However, players may have observed signals generated by an individually uninformative correlating device (Liu, 2015), which allows the players to correlate their behavior. In other words, the true incomplete information game may have correlated duplicated belief hierarchies (Ely and Peski, 2006; Dekel et al., 2007). Then, a natural candidate for the analyst's prediction is the set of outcomes that can arise in some Bayes Nash equilibrium (BNE) of some incomplete information game with the same distribution over belief hierarchies. Liu (2015) shows that this set of outcomes can be characterized as the set of belief-invariant Bayes correlated equilibria (BIBCE) of the analyst's model. A BIBCE is a Bayes correlated equilibrium (BCE) in which a prescribed action does not reveal any additional information to the player about the opponents' types and the payoff-relevant state, thus preserving the player's belief hierarchy.[1]

Now imagine that the analyst believes that his model can be slightly different from the true distribution over belief hierarchies. That is, he believes that the true incomplete information game is in a "neighborhood" of the class of incomplete information games with the same belief hierarchies as those in his model. We call such a nearby incomplete information game an ε-elaboration, where $\varepsilon \geq 0$ is the distance between the ε-elaboration and the nearest incomplete information game measured by the maximum difference of probabilities of events. If a BIBCE of the analyst's model is not qualitatively different from some BIBCE of every ε-elaboration for sufficiently small ε, then the analyst will be justified in adopting this BIBCE as one of his prediction in the presence of small inaccuracies of the analyst's model. We apply this idea to a set of BIBCE and define the following set-valued concept: a set of BIBCE is robust if every ε-elaboration has a BIBCE which is close to some BIBCE in this set for sufficiently small ε.

[1] A BIBCE was introduced and analyzed by Forges (2006) and Lehrer et al. (2010) under the additional restriction that action recommendations could not depend upon the state. Bergemann and Morris (2013, 2016) introduced a BCE as an analyst's prediction when the analyst could not rule out players having additional payoff-relevant information. See Liu (2015) and Bergemann and Morris (2017) for more motivation for studying BIBCE and the relation to BCE and Bergemann and Morris (2019) for further discussion of the relation to other special cases of BCE.

To illustrate the idea, consider the following game of incomplete information with two players 1 and 2, two actions A and B for each player, and two payoff-relevant states S_1 and S_2 occurring with equal probability of $1/2$, which combines the information structure of an email game of Rubinstein (1989) and an example discussed in Ely and Peski (2006), Lehrer et al. (2010), and Liu (2015). Assume that each player's set of types is a singleton in the analyst's model; that is, it is common knowledge that each player has a uniform prior over $\{S_1, S_2\}$. The payoffs are summarized in the following table, where players would be better coordinated by choosing the same actions in state S_1 and the different actions in state S_2.

S_1	A	B
A	$1, 1$	$0, 0$
B	$0, 0$	$1, 1$

S_2	A	B
A	$0, 0$	$1, 1$
B	$1, 1$	$0, 0$

This game has an infinitely many number of BNE because each player has a single type and the expected payoff of each player is $1/2$ under every action profile. However, none of them is robust. We demonstrate it by constructing an ε-elaboration with a unique BNE which is not close to any BNE of the analyst's model. Let S_3 be another payoff-relevant state with the following payoffs, where player 1 has a dominant action A and player 2's payoffs are the same as those in state S_2.

S_3	A	B
A	$1, 0$	$1, 1$
B	$0, 1$	$0, 0$

Let p be a common prior over the product space of the set of payoff-relevant states $\{S_1, S_2, S_3\}$ and that of payoff-irrelevant states $\Omega = \{0, 1, 2, 3, \dots\}$ given by

$$p(X, \omega) = \begin{cases} \varepsilon & \text{if } \omega = 0 \text{ and } X = S_3, \\ \varepsilon(1-\varepsilon)^\omega & \text{if } \omega \geq 1 \text{ is odd and } X = S_1, \text{ or if } \omega \geq 2 \text{ is even and } X = S_2, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \varepsilon < 1$. Player 1 has an information partition $\{\{0\}, \{1, 2\}, \{3, 4\}, \{5, 6\}, \dots\}$ of Ω; player 2 has an information partition $\{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \dots\}$ of Ω. In the limit as ε goes to zero, it is common knowledge that each player has a uniform prior over $\{S_1, S_2\}$. When $\varepsilon > 0$, however, it is not the case, and we can iteratively eliminate dominated actions as follows: player 1 with

$\{0\}$ chooses a dominant action A; player 2 with $\{0, 1\}$ chooses B knowing that player 1 chooses A with probability $1/(2 - \varepsilon)$; player 1 with $\{1, 2\}$ chooses B knowing that player 2 chooses B with probability $1/(2-\varepsilon)$, and so on. That is, action profiles surviving iterated elimination of dominated actions are (A, B) at $\omega \in \{4k\}_{k=0}^{\infty}$, (B, B) at $\omega \in \{4k + 1\}_{k=0}^{\infty}$, (B, A) at $\omega \in \{4k + 2\}_{k=0}^{\infty}$, and (A, A) at $\omega \in \{4k + 3\}_{k=0}^{\infty}$. In the limit as ε goes to zero, players choose (A, A) and (B, B) with equal probability when S_1 realizes, and (A, B) and (B, A) with equal probability when S_2 realizes, as summarized in the following table of joint probabilities of actions and states.

S_1	A	B
A	$1/4$	0
B	0	$1/4$

S_2	A	B
A	0	$1/4$
B	$1/4$	0

The above joint probability distribution is a BIBCE of the analyst's model: players receive a recommendation to play one of their actions A or B according the table, and they know that it is optimal to accept the recommendation, while they do not learn anything about the state. Thus, the ε-elaboration has a unique BNE whose outcome is close to this BIBCE and qualitatively different from any BNE of the analyst's model. This implies that any BNE is not robust, and this BIBCE is the only candidate to be robust since it must arise under the ε-elaboration for every $\varepsilon > 0$. To establish robustness of this BIBCE, we must show that any other ε-elaboration also has a BIBCE whose outcome is close to it.

The purpose of this paper is to propose the concept of robust BIBCE and provide a sufficient condition for it. For this purpose, we introduce a generalized potential function of an incomplete information game, which is an extension of both a potential function of an incomplete information game (Monderer and Shapley, 1996; van Heumen et al., 1996) and a generalized potential function of a complete information game (Morris and Ui, 2005). A generalized potential function is a function of a payoff-relevant state and a covering of the action space, a collection of subsets of the action space such that the union of the subsets is the action space. It incorporates some information about players' preferences over the collection of subsets at each state. A special case is a potential function, where each covering is a singleton (i.e. an action profile). A generalized potential function is associated with a belief-invariant correlating device such that a signal sent to a player is a subset of the player's actions, which can be interpreted as a vague prescription to choose some action from this subset, and a signal does not reveal any additional information about the opponents' types and payoff-relevant states to the player. We can calculate the expected value of a generalized potential function with respect to the joint probability distribution of action subsets and payoff-relevant states induced by a correlating device together with a common prior of an incomplete information game. We focus on correlating devices that maximize the expected value of a generalized potential function and call them GP-maximizing correlating devices. By a property of a generalized potential function, every GP-maximizing correlating device is accompanied by a BIBCE where every player chooses an action from a prescribed subset of actions. Our main result states that the set of such BIBCE is robust. In particular, when an incomplete information game has a potential function, the set of potential maximizing BIBCE is robust. The incomplete information game in the preceding example has a potential function, which coincides with the identical payoff function of each player, and it is straightforward to show that the BIBCE discussed above uniquely maximizes the expected value of the potential function, so it is robust by our main result.

Robustness of BIBCE is essentially an extension of robustness of equilibria in complete information games proposed by Kajii and Morris (1997).[2] We briefly discuss the connection by focusing on an incomplete information game where payoff functions are common knowledge, i.e., a complete information game. Our definition of ε-elaborations applied to complete information games is equivalent to that of Kajii and Morris (1997). In Kajii and Morris (1997), a correlated equilibrium is said to be robust if every ε-elaboration has a BNE in a neighborhood of the correlated equilibrium. It is known that if a complete information game has a potential function, then the set of Nash equilibria that maximize the expected value of the potential function is robust (Ui, 2001; Morris and Ui, 2005).[3] In our paper, a correlated equilibrium is said to be robust if every ε-elaboration has a BIBCE in a neighborhood of the correlated equilibrium.[4] Thus, our definition of robustness applied to complete information games is weaker than that of Kajii and Morris (1997). However, as demonstrated in the above example, a set of potential maximizing BNE may not be robust, whereas a set of potential maximizing Nash equilibria is always robust, suggesting qualitative difference be-

[2]See Kajii and Morris (2020a,b) and Oyama and Takahashi (2020) for more details about recent developments in the study of robust equilibria of complete information games.

[3]The concept of robustness in this result is that of Kajii and Morris (2020a), which is slightly weaker than that of Kajii and Morris (1997).

[4]Pram (2019) also considers a similar variant of robustness of equilibria in complete information games. See Section 3.

tween the robustness exercise in incomplete information games and that in complete information games.

The organization of this paper is as follows. Section 2 introduces preliminary definitions and results. Section 3 formally defines robust BIBCE. Section 4 introduces a generalized potential function and provides the main result. Section 5 studies robust BIBCE of supermodular games.

2 Elaborations of incomplete information games

Fix a finite set of players I and a finite set of actions A_i for each player $i \in I$. An incomplete information game (T, Θ, π, u) consists of the following elements.

- $T = \prod_{i\in I} T_i$ is an at most countable set of type profiles, where T_i is a set of player i's types.
- $\Theta = \prod_{i\in I} \Theta_i$ is an at most countable set of payoff-relevant states, where Θ_i is a set of their components that determine player i's payoff function.
- $\pi \in \Delta(T \times \Theta)$ is a common prior.
- $u = (u_i)_{i\in I}$ is a payoff function profile, where $u_i : A \times \Theta \to \mathbb{R}$ is player i's payoff function such that $u_i(\cdot, \theta) = u_i(\cdot, \theta')$ if and only if $\theta_i = \theta'_i$.

Hereafter, we use $C = \prod_{i\in I} C_i$, $C_{-i} = \prod_{j\neq i} C_j$, $C_S = \prod_{i\in S} C_i$, and $C_{-S} = \prod_{i\notin S} C_i$ to denote the Cartesian products of $C_1, C_2, \ldots$ with generic elements $c \in C$, $c_{-i} \in C_{-i}$, $c_S \in C_S$, and $c_{-S} \in C_{-S}$, respectively.

Payoff functions are assumed to be bounded, i.e., $\sup_{i,a,\theta} |u_i(a, \theta)| < \infty$. For each $i \in I$, let $T_i^* \subseteq T_i$ and $\Theta_i^* \subseteq \Theta_i$ denote the sets of player i's types and payoff-relevant states on the support of π, respectively: $T_i^* = \{t_i \in T_i \mid \pi(t_i) > 0\}$ and $\Theta_i^* = \{\theta_i \in \Theta_i \mid \pi(\theta_i)\} > 0$, where $\pi(t_i) \equiv \sum_{t_{-i},\theta} \pi(t, \theta)$ and $\pi(\theta_i) \equiv \sum_{t,\theta_{-i}} \pi(t, \theta)$ are the marginal probabilities. Player i's belief is given by $\pi(t_{-i}, \theta|t_i) \equiv \pi(t, \theta)/\pi(t_i)$ when his type is $t_i \in T_i^*$.

Let $\pi^* \in \Delta(T^* \times \Theta^*)$ and $u_i^* : A \times \Theta^* \to \mathbb{R}$ denote the restriction of π to $T^* \times \Theta^*$ and that of u_i to $A \times \Theta^*$, respectively. Note that $(T^*, \Theta^*, \pi^*, u^*)$ is the minimum representation of (T, Θ, π, u) because every player with every type on the support of π in (T, Θ, π, u) has the same belief and payoffs as those in $(T^*, \Theta^*, \pi^*, u^*)$. We say that two incomplete information games (T, Θ, π, u) and

$(\bar{T}, \bar{\Theta}, \bar{\pi}, \bar{u})$ with the same set of players and the same set of actions are equivalent if they have the same minimum representation.

A decision rule is a mapping $\sigma : T \times \Theta \to \Delta(A)$, under which players choose an action profile $a \in A$ with a probability $\sigma(a|t, \theta)$ when $(t, \theta) \in T \times \Theta$ is realized. Let Σ denote the set of all decision rules. A decision rule σ together with a common prior π determines a joint probability distribution $\sigma \circ \pi \in \Delta(A \times T \times \Theta)$ given by $\sigma \circ \pi(a, t, \theta) \equiv \sigma(a|t, \theta)\pi(t, \theta)$, which is referred to as a distributional decision rule. The set of all distributional decision rules $\Sigma \circ \pi \equiv \{\sigma \circ \pi \in \Delta(A \times T \times \Theta) \mid \sigma \in \Sigma\}$ is readily shown to be a compact subset of a linear space $\{f \in \mathbb{R}^{A \times T \times \Theta} \mid \sum_{a,t,\theta} |f(a, t, \theta)| < \infty\}$ with the weak topology, which is metrizable.[5] Each distributional decision rule corresponds to an equivalence class of decision rules, where $\sigma, \sigma' \in \Sigma$ are equivalent if $\sigma \circ \pi(a, t, \theta) = \sigma' \circ \pi(a, t, \theta)$ for all $(a, t, \theta) \in A \times T^* \times \Theta^*$. When we discuss topology of Σ, we identify Σ with $\Sigma \circ \pi$ by regarding Σ as the set of the equivalence classes and considering the isomorphism from the set of the equivalence classes to $\Sigma \circ \pi$.

A decision rule σ is said to be *obedient* for player i of type t_i if

$$\sum_{a_{-i}, t_{-i}, \theta} \sigma(a|t, \theta)\pi(t, \theta)u_i(a, \theta) \geq \sum_{a_{-i}, t_{-i}, \theta} \sigma(a|t, \theta)\pi(t, \theta)u_i((a'_i, a_{-i}), \theta) \tag{1}$$

for all $a_i, a'_i \in A_i$, or equivalently,

$$\sum_{a_{-i}, t_{-i}, \theta} \sigma(a|t, \theta)\pi(t_{-i}, \theta|t_i)u_i(a, \theta) \geq \sum_{a_{-i}, t_{-i}, \theta} \sigma(a|t, \theta)\pi(t_{-i}, \theta|t_i)u_i((a'_i, a_{-i}), \theta)$$

if $T_i \in T^*_i$; that is, this player cannot increase the expected payoff by deviating from the action prescribed by the decision rule. In particular, if a decision rule σ is obedient for every player of every type, then σ is simply said to be obedient. An obedient decision rule is referred to as a Bayes correlated equilibrium (BCE).

A decision rule is *belief-invariant* if $\sigma(\{a_i\} \times A_{-i}| (t_i, t_{-i}), \theta)$ is independent of (t_{-i}, θ) for each $a_i \in A_i$, $t_i \in T_i$, and $i \in I$, or equivalently, there exists a mapping $\sigma_i : T_i \to \Delta(A_i)$ (i.e., player i's strategy) such that $\sigma_i(a_i|t_i) = \sigma(\{a_i\} \times A_{-i}| (t_i, t_{-i}), \theta)$. Belief-invariance implies that player i's action does not reveal any additional information to the player about the opponents' types and the state. In other words, from the viewpoint of player i who knows t_i and a_i, t_i is a sufficient statistic for (t_{-i}, θ). This property has played an important role in the literature on BCE (Forges, 1993,

[5] See Lemma 6 for more details.

2006; Lehrer et al., 2010; Liu, 2015). Let Σ^{BI} denote the set of all belief-invariant decision rules. It is readily shown that Σ^{BI} is a compact subset of Σ.

Note that a strategy profile $(\sigma_i)_{i\in I}$, where $\sigma_i : T_i \to \Delta(A_i)$ is player i's strategy, is regarded as a special case of a belief-invariant decision rule given by $\sigma(a|t,\theta) = \prod_{i\in I}\sigma_i(a_i|t_i)$. A strategy profile $(\sigma_i)_{i\in I}$ is said to be a Bayes Nash equilibrium (BNE) if it is obedient. Clearly, a BNE is a special case of a belief-invariant BCE (BIBCE). Because a BNE exists in our setting with at most a countable number of states, types, and actions (Milgrom and Weber, 1985), a BIBCE also exists.[6] It is easy to check that the set of all BIBCE has the following property.

Lemma 1. *The set of all BIBCE of (T,Θ,π,u) is a nonempty convex compact subset of Σ containing all BNE, where a convex combination $\alpha\sigma + (1-\alpha)\sigma'$ for $\sigma,\sigma' \in \Sigma^{BI}$ and $\alpha \in (0,1)$ is given by*

$$(\alpha\sigma + (1-\alpha)\sigma')(a|t,\theta) = \alpha\sigma(a|t,\theta) + (1-\alpha)\sigma'(a|t,\theta)$$

for all $a \in A$ and $(t,\theta) \in T\times\Theta$.

We introduce a correlating device. Let M_i be an at most countable set of signals (i.e. messages) player i receives. The probability distribution of a signal profile $m = (m_i)_{i\in I} \in M \equiv \prod_{i\in I} M_i$ is given by a mapping $\rho : T\times\Theta \to \Delta(M)$, under which $m \in M$ is drawn with a probability $\rho(m|t,\theta)$ when $(t,\theta) \in T\times\Theta$ is realized. This mapping ρ is referred to as a communication rule. Belief-invariance of a communication rule is defined similarly. That is, ρ is belief-invariant if $\rho\left(\{m_i\}\times M_{-i}|\left(t_i,t_{-i}\right),\theta\right)$ is independent of (t_{-i},θ) for each $m_i \in M_i$, $t_i \in T_i$, and $i \in I$, which implies that a signal received by player i does not reveal any additional information to the player about the opponents' types and the state.

Combining an incomplete information game (T,Θ,π,u) and a communication rule ρ, we can construct another incomplete information game $(\bar{T},\Theta,\bar{\pi},u)$ with the same sets of players and actions such that $\bar{T}_i = T_i \times M_i$ for each $i \in I$ and $\bar{\pi}(\bar{t},\theta) = \pi(t,\theta)\rho(m|t,\theta)$ for each $\bar{t} = ((t_i,m_i))_{i\in I} \in \bar{T}$ and $\theta \in \Theta$. This incomplete information game is referred to as the conjunction of (T,Θ,π,u) and ρ. Note that player i in $(\bar{T},\Theta,\bar{\pi},u)$ receives m_i as well as t_i, where t_i is drawn according to π and m_i is drawn according to ρ, and if ρ is belief-invariant, then this player's knowledge about (t_{-i},θ) is exactly the same as that in the original game (T,Θ,π,u). Using the conjunction when ρ is belief-invariant, we define an elaboration of (T,Θ,π,u).

[6] A BIBCE is also a solution to a linear programming problem with a countable number of variables and constraints.

Definition 1. An incomplete information game $(\bar{T}, \Theta, \bar{\pi}, u)$ is an elaboration of (T, Θ, π, u) if there exist a belief-invariant communication rule ρ and mappings $\tau_i : \bar{T}_i \to T_i$ and $\mu_i : \bar{T}_i \to M_i$ for each $i \in I$ such that the mapping $\bar{t}_i \mapsto (\tau_i(\bar{t}_i), \mu_i(\bar{t}_i))$ restricted to $\bar{T}^*$ is one-to-one and

$$\bar{\pi}(\bar{t}, \theta) = \pi(\tau(\bar{t}), \theta)\rho(\mu(\bar{t})|\tau(\bar{t}), \theta) \text{ for all } \bar{t} \in \bar{T}, \tag{2}$$

where $\tau(\bar{t}) = (\tau_i(\bar{t}_i))_{i \in I}$ and $\mu(\bar{t}) = (\mu_i(\bar{t}_i))_{i \in I}$.

Equation (2) implicitly requires that $\sum_{\bar{t} \in \tau^{-1}(t)} \rho(\mu(\bar{t})|t, \theta) = 1$ for all $(t, \theta) \in T^* \times \Theta^*$ and $\tau(\bar{T}^*) = T^*$ because $\bar{\pi}(\bar{T} \times \Theta) = 1$.

For example, if $(\bar{T}, \Theta, \bar{\pi}, u)$ is the conjunction of (T, Θ, π, u) and a belief-invariant communication rule ρ, then it is an elaboration of (T, Θ, π, u) with τ_i and μ_i given by $\tau_i(t_i, m_i) = t_i$ and $\mu_i(t_i, m_i) = m_i$ for all $\bar{t}_i = (t_i, m_i) \in \bar{T}_i$ and $i \in I$. Note that $(\bar{T}, \Theta, \bar{\pi}, u)$ is an elaboration of (T, Θ, π, u) if and only if it is isomorphic to a conjunction of (T, Θ, π, u) and some belief-invariant communication rule.

As the next lemma shows, we can provide a necessary and sufficient condition for an elaboration without using a communication rule.

Lemma 2. *An incomplete information game $(\bar{T}, \Theta, \bar{\pi}, u)$ is an elaboration of (T, Θ, π, u) if and only if, for each $i \in I$, there exists a mapping $\tau_i : \bar{T}_i \to T_i$ such that*

$$\bar{\pi}(\tau^{-1}(t), \theta) = \pi(t, \theta) \textit{ for all } (t, \theta) \in T^* \times \Theta^*, \tag{3}$$

$$\bar{\pi}(\tau_{-i}^{-1}(t_{-i}), \theta|\bar{t}_i) = \pi(t_{-i}, \theta|\tau_i(\bar{t}_i)) \textit{ for all } \bar{t}_i \in \bar{T}_i^* \textit{ and } (t_{-i}, \theta) \in T_{-i}^* \times \Theta^*, \tag{4}$$

where $\tau_i^{-1}(t_i) = \{\bar{t}_i \in \bar{T}_i \mid \tau_i(\bar{t}_i) = t_i\}$, $\tau^{-1}(t) = \prod_{i \in I} \tau_i^{-1}(t_i)$, and $\tau_{-i}^{-1}(t_{-i}) = \prod_{j \neq i} \tau_j^{-1}(t_j)$.

Proof. To show the "if" part, suppose that, for each $i \in I$, there exists a mapping $\tau_i : \bar{T}_i \to T_i$ satisfying (3) and (4). Let $M_i = \bar{T}_i$ and $\mu_i : \bar{T}_i \to M_i$ be such that $\mu_i(\bar{t}_i) = \bar{t}_i$ for all $\bar{t}_i \in \bar{T}_i$, by which the mapping $\bar{t}_i \mapsto (\tau_i(\bar{t}_i), \mu_i(\bar{t}_i))$ is one-to-one. Consider a communication rule $\rho : T \times \Theta \to \Delta(M)$ satisfying (2); that is, $\rho(\bar{t}|t, \theta)\pi(t, \theta) = \bar{\pi}(\bar{t}, \theta)$ if $\tau(\bar{t}) = t$ and $\rho(\bar{t}|t, \theta) = 0$ otherwise, which is well-defined by (3). Then, for $(t, \theta) \in T^* \times \Theta^*$ and $\bar{t}_i \in \bar{T}_i^*$ with $t_i = \tau_i(\bar{t}_i)$,

$$\rho(\{\bar{t}_i\} \times \bar{T}_{-i}|t, \theta) = \rho(\{\bar{t}_i\} \times \tau_{-i}^{-1}(t_{-i})|t, \theta) = \frac{\bar{\pi}(\{\bar{t}_i\} \times \tau_{-i}^{-1}(t_{-i}), \theta)}{\pi(t, \theta)} = \frac{\bar{\pi}(\tau_{-i}^{-1}(t_{-i}), \theta|\bar{t}_i) \times \bar{\pi}(\bar{t}_i)}{\pi(t_{-i}, \theta|t_i) \times \pi(t_i)} = \frac{\bar{\pi}(\bar{t}_i)}{\pi(t_i)}$$

by (4). Because $\rho(\{\bar{t}_i\} \times \bar{T}_{-i}|t,\theta) = \bar{\pi}(\bar{t}_i)/\pi(t_i)$ is independent of t_{-i} and θ, ρ is a belief-invariant communication rule, and thus $(\bar{T}, \Theta, \bar{\pi}, u)$ is an elaboration of (T, Θ, π, u).

To show the "only if" part, suppose that $(\bar{T}, \Theta, \bar{\pi}, u)$ is an elaboration of (T, Θ, π, u); that is, there exists a belief-invariant communication rule ρ and mappings $\tau_i : \bar{T}_i \to T_i$ and $\mu_i : \bar{T}_i \to M_i$ for each $i \in I$ satisfying the condition in Definition 1. It is enough to show that τ and $\bar{\pi}$ satisfy (3) and (4). Let $(t, \theta) \in T^* \times \Theta^*$ and $\bar{t}_i \in \bar{T}_i$ be such that $\tau_i(\bar{t}_i) = t_i$. By (2),

$$\bar{\pi}(\tau^{-1}(t), \theta) = \pi(t,\theta) \sum_{\bar{t}:\tau(\bar{t})=t} \rho(\mu(\bar{t})|t,\theta) = \pi(t,\theta).$$

Thus, (3) holds. Moreover, by (2) again,

$$\begin{aligned}
\bar{\pi}(\tau_{-i}^{-1}(t_{-i}), \theta|\bar{t}_i) &= \frac{\bar{\pi}(\{\bar{t}_i\} \times \tau_{-i}^{-1}(t_{-i}), \theta)}{\sum_{t'_{-i},\theta'} \bar{\pi}(\{\bar{t}_i\} \times \tau_{-i}^{-1}(t'_{-i}), \theta')} \\
&= \frac{\pi(t,\theta)\left(\sum_{\bar{t}_{-i}:\tau_{-i}(\bar{t}_{-i})=t_{-i}} \rho(\mu(\bar{t})|t,\theta)\right)}{\sum_{t'_{-i},\theta'} \pi((t_i, t'_{-i}), \theta')\left(\sum_{\bar{t}_{-i}:\tau_{-i}(\bar{t}_{-i})=t'_{-i}} \rho(\mu(\bar{t})|(t_i, t'_{-i}), \theta')\right)} \\
&= \frac{\pi(t,\theta)}{\sum_{t'_{-i},\theta'} \pi((t_i, t'_{-i}), \theta')} = \pi(t,\theta|t_i),
\end{aligned}$$

where the third equality holds because

$$\sum_{\bar{t}_{-i}:\tau_{-i}(\bar{t}_{-i})=t_{-i}} \rho(\mu(\bar{t})|t,\theta) = \sum_{m_{-i}\in M_{-i}} \rho((\mu_i(\bar{t}_i), m_{-i})|t,\theta) = \rho(\{\mu_i(\bar{t}_i)\} \times M_{-i}|t,\theta)$$

is independent of t_{-i} and θ by the belief-invariance of ρ. Thus, (4) holds. □

The mapping τ in Lemma 2, which connects an elaboration to the original incomplete information game, is referred to as an elaboration mapping.

Consider the conjunction of (T, Θ, π, u) and a belief-invariant communication rule ρ. For a strategy profile $\bar{\sigma} = (\bar{\sigma}_i)_{i\in I}$ of the conjunction, where $\bar{\sigma}_i : T_i \times M_i \to \Delta(A_i)$, let σ be a decision rule of (T, Θ, π, u) given by

$$\sigma(a|t,\theta) = \sum_{m\in M} \prod_{i\in I} \bar{\sigma}_i(a_i|t_i, m_i)\rho(m|t,\theta),$$

which is referred to as a decision rule induced by $\bar{\sigma}$ and ρ. Then,

$$\sigma(\{a_i\} \times A_{-i}|t,\theta) = \sum_{m_i\in M_i} \bar{\sigma}_i(a_i|t_i, m_i)\rho(\{m_i\} \times M_{-i}|t,\theta)$$

is independent of (t_{-i}, θ) because ρ is belief-invariant; that is, the induced decision rule is belief-invariant. Moreover, it is straightforward to show that if $\bar{\sigma}$ is obedient, then σ is also obedient; that is, the induced decision rule is a BIBCE. The next proposition not only generalizes this observation but also establishes its converse; that is, the set of all BIBCE of an incomplete information game coincides with the set of all BNE of all elaborations of the incomplete information game (Liu, 2015).

Proposition 1. *If σ is a BIBCE of (T, Θ, π, u), then there exists an elaboration $(\bar{T}, \Theta, \bar{\pi}, u)$ with an elaboration mapping τ and its BNE $\bar{\sigma}$ satisfying*

$$\sigma(a|t,\theta)\pi(t,\theta) = \sum_{\bar{t}\in\tau^{-1}(t)} \prod_{i\in I} \bar{\sigma}_i(a_i|\bar{t}_i)\bar{\pi}(\bar{t},\theta) \tag{5}$$

for all $a \in A$ and $(t,\theta) \in T \times \Theta$. If $\bar{\sigma}$ is a BIBCE of an elaboration $(\bar{T}, \Theta, \bar{\pi}, u)$, then σ satisfying

$$\sigma(a|t,\theta)\pi(t,\theta) = \sum_{\bar{t}\in\tau^{-1}(t)} \bar{\sigma}(a|\bar{t},\theta)\bar{\pi}(\bar{t},\theta) \tag{6}$$

for all $a \in A$ and $(t,\theta) \in T \times \Theta$ is a BIBCE of (T, Θ, π, u). In particular, σ is a BIBCE of (T, Θ, π, u) if and only if there exists an elaboration $(\bar{T}, \Theta, \bar{\pi}, u)$ and its BNE $\bar{\sigma}$ satisfying (5).

Proof. Liu (2015) establishes the equivalence of BNE of an incomplete information game and BIBCE of the conjunctions of the incomplete information game and belief-invariant communication rules when T and M are finite sets. Proposition 1 is a straightforward extension of his result, where we add the formulae (5) and (6). We provide a proof for completeness.

To establish the first part, let σ be a BIBCE of (T, Θ, π, u). Construct a communication rule ρ as follows: $M_i = A_i$ for each $i \in I$ and $\rho(a|t,\theta) = \sigma(a|t,\theta)$ for each $a \in A$ and $(t,\theta) \in T \times \Theta$. Note that ρ is belief-invariant because σ is belief-invariant. Let $(\bar{T}, \Theta, \bar{\pi}, u)$ be the conjunction of (T, Θ, π, u) and ρ, where $\bar{\pi}(\bar{t},\theta) = \pi(t,\theta)\rho(a|t,\theta)$ if $\bar{t} = ((t_i, a_i))_{i\in I} \in \bar{T}$. Consider a strategy $\bar{\sigma}_i : \bar{T}_i \to \Delta(A_i)$ such that $\bar{\sigma}_i(a_i|\bar{t}_i) = 1$ if $\bar{t}_i = (t_i, a_i)$ and $\bar{\sigma}_i(a_i|\bar{t}_i) = 0$ otherwise. Then,

$$\prod_{i\in I} \bar{\sigma}_i(a_i|\bar{t}_i)\bar{\pi}(\bar{t},\theta) = \begin{cases} \sigma(a|t,\theta)\pi(t,\theta) & \text{if } \bar{t} = ((t_i,a_i))_{i\in I}, \\ 0 & \text{if } \bar{t} \neq ((t_i,a_i))_{i\in I}, \end{cases}$$

which implies (5). Thus, for each $i \in I$ and $\bar{t}_i = (t_i, b_i) \in \bar{T}_i$,

$$\sum_{a_{-i},\bar{t}_{-i},\theta} \prod_{j\in I} \bar{\sigma}_j(a_j|\bar{t}_j)\bar{\pi}(\bar{t},\theta)u_i((a'_i,a_{-i}),\theta) = \begin{cases} \displaystyle\sum_{a_{-i},t_{-i},\theta} \sigma(a|t,\theta)\pi(t,\theta)u_i((a'_i,a_{-i}),\theta) & \text{if } b_i = a_i, \\ 0 & \text{if } b_i \neq a_i, \end{cases}$$

which implies that $\bar{\sigma}$ is obedient because σ is obedient. Therefore, $\bar{\sigma}$ is a BNE of $(\bar{T}, \Theta, \bar{\pi}, u)$, which is also an elaboration of (T, Θ, π, u).

To establish the second part, let $\bar{\sigma}$ be a BIBCE of $(\bar{T}, \Theta, \bar{\pi}, u)$. Without loss of generality, we can assume that $(\bar{T}, \Theta, \bar{\pi}, u)$ is the conjunction of (T, Θ, π, u) and a belief-invariant communication rule $\rho : T \times \Theta \to M$. Then, for a decision rule σ of (T, Θ, π, u) satisfying (6), if $\pi(t, \theta) > 0$, then

$$\begin{aligned}
\sigma(\{a_i\} \times A_{-i}|t, \theta) &= \sum_{\bar{t} \in \tau^{-1}(t)} \bar{\sigma}(\{a_i\} \times A_{-i}|\bar{t}, \theta)\bar{\pi}(\bar{t}, \theta)/\pi(t, \theta) \\
&= \frac{\sum_{\bar{t} \in \tau^{-1}(t)} \bar{\sigma}_i(a_i|\bar{t}_i)\bar{\pi}(\bar{t}, \theta)}{\pi(t, \theta)} \\
&= \frac{\sum_{m \in M} \bar{\sigma}_i(a_i|t_i, m_i)\pi(t, \theta)\rho(m|t, \theta)}{\pi(t, \theta)} \\
&= \sum_{m_i \in M_i} \bar{\sigma}_i(a_i|t_i, m_i)\rho(\{m_i\} \times M_{-i}|t, \theta),
\end{aligned}$$

which is independent of (t_{-i}, θ) because ρ is belief-invariant. Thus, σ is belief-invariant. Moreover, by (6),

$$\begin{aligned}
\sum_{a_{-i}, t_{-i}, \theta} \sigma(a|t, \theta)\pi(t, \theta)u_i(a, \theta) &= \sum_{a_{-i}, t_{-i}, \theta} \sum_{\bar{t} \in \tau^{-1}(t)} \bar{\sigma}(a|\bar{t}, \theta)\bar{\pi}(\bar{t}, \theta)u_i(a, \theta) \\
&= \sum_{\bar{t}_i \in \tau_i^{-1}(t_i)} \sum_{a_{-i}, \bar{t}_{-i}, \theta} \bar{\sigma}(a|\bar{t}, \theta)\bar{\pi}(\bar{t}, \theta)u_i(a, \theta),
\end{aligned}$$

which implies that σ is obedient because $\bar{\sigma}$ is obedient. Consequently, σ is a BIBCE. $\square$

Before closing this section, we introduce another representation of decision and communication rules, which will be used when we discuss our main result. In the above discussion, we regard the conjunction of (T, Θ, π, u) and ρ as an incomplete incomplete information game, so a decision rule of the conjunction is defined as a mapping $\bar{\sigma} : \prod_{i \in I}(T_i \times M_i) \times \Theta \to \Delta(A)$. Instead of considering σ and ρ separately, we can also use a mapping $\gamma : T \times \Theta \to \Delta(A \times M)$ to represent a decision rule and a communication rule simultaneously, which is referred to as a decision-communication rule. A decision rule σ together with a communication rule ρ determines a decision-communication rule by $\gamma(a, m|t, \theta) = \sigma(a|(t_i, m_i)_{i \in I}, \theta)\rho(m|t, \theta)$ for all $(a, m) \in A \times M$ and $(t, \theta) \in T \times \Theta$. Conversely, a decision-communication rule γ determines a communication rule $\rho(m|t, \theta) = \gamma(A \times \{m\}|t, \theta)$ and a decision rule $\sigma(a|(t_i, m_i)_{i \in I}, \theta) = \gamma(a, m|t, \theta)/\rho(m|t, \theta)$ for all $(a, m) \in A \times M$ and $(t, \theta) \in T \times \Theta$ with $\rho(m|t, \theta) > 0$. We say that γ is obedient for player i with (t_i, m_i) if the induced decision rule σ is obedient for this player of this type in the conjunction of (T, Θ, π, u) and the induced communication rule ρ; that is,

$$\sum_{a_{-i}, m_{-i}, t_{-i}, \theta} \gamma(a, m|t, \theta)\pi(t, \theta)u_i(a, \theta) \geq \sum_{a_{-i}, m_{-i}, t_{-i}, \theta} \gamma(a, m|t, \theta)\pi(t, \theta)u_i((a_i', a_{-i}), \theta) \tag{7}$$

for all $a_i, a_i' \in A_i$. We simply say that γ is obedient if it is obedient for every player of every type. Belief-invariance of a decision-communication rule is defined similarly: γ is belief-invariant if $\gamma(\{a_i\} \times A_{-i} \times \{m_i\} \times M_{-i}|t, \theta)$ is independent of t_{-i} and θ for all $(a_i, m_i) \in A_i \times M_i$, $t_i \in T_i$, and $i \in I$; that is, there exists $\gamma_i : T_i \to \Delta(A_i \times M_i)$ such that $\gamma_i(a_i, m_i|t_i) = \gamma(\{a_i\} \times A_{-i} \times \{m_i\} \times M_{-i}|t, \theta)$. It is straightforward to show that if σ and ρ determined by γ are belief-invariant, then γ is also belief-invariant. An obedient belief-invariant decision-communication rule is referred to as a BIBCE with a communication rule (BIBCE-C). A BIBCE-C directly induces a BIBCE in the following sense.

Lemma 3. *If a belief-invariant decision-communication rule γ is a BIBCE-C, then $\sigma \in \Sigma^{BI}$ with $\sigma(a|t, \theta) = \gamma(\{a\} \times M|t, \theta)$ for all $(a, t, \theta) \in A \times T \times \Theta$ is a BIBCE.*

Proof. Let γ be a BIBCE-C. Because γ is obedient, (7) holds. By taking the summation of each side of (7) over $m_i \in M_i$, we obtain

$$\sum_{a_{-i}, t_{-i}, \theta} \gamma(\{a\} \times M|t, \theta)\pi(t, \theta)u_i(a, \theta) \geq \sum_{a_{-i}, t_{-i}, \theta} \gamma(\{a\} \times M|t, \theta)\pi(t, \theta)u_i((a_i', a_{-i}), \theta).$$

Thus, σ is obedient. In addition, because γ is belief-invariant, $\gamma(\{a_i\} \times A_{-i} \times \{m_i\} \times M_{-i}|t, \theta) = \gamma_i(a_i, m_i|t_i)$, which implies $\sigma(\{a_i\} \times A_{-i}|t, \theta) = \sum_{m_i \in M_i} \gamma_i(a_i, m_i|t_i)$. Thus, σ is belief-invariant as well; that is, σ is a BIBCE. □

3 ε-Elaborations and robustness

We define an ε-elaboration of (T, Θ, π, u), which is approximately an elaboration of (T, Θ, π, u).

Definition 2. For $\varepsilon \geq 0$, an incomplete information game $(\bar{T}, \bar{\Theta}, \bar{\pi}, \bar{u})$ is an ε-elaboration of (T, Θ, π, u) if the following condition is satisfied.

1. $\Theta^* \subseteq \bar{\Theta}$ and $\bar{u}_i(\cdot, \theta) = u_i(\cdot, \theta)$ for all $\theta \in \Theta^*$.

2. $\bar{\pi}(\bar{T}^\sharp) \geq 1 - \varepsilon$, where $\bar{T}^\sharp = \prod_{i \in I} \bar{T}_i^\sharp$ and $\bar{T}_i^\sharp = \{\bar{t}_i \in \bar{T}_i \mid \bar{\pi}(\Theta_i^* \times \bar{\Theta}_{-i}|\bar{t}_i) = 1\}$.

3. There exist a mapping $\tau_i : \bar{T}_i \to T_i$ and $\bar{T}_i^\flat \subseteq \bar{T}_i$ with $\bar{\pi}(\bar{T}_i^\flat) \geq 1 - \varepsilon$ such that

$$\sup_{E \subseteq T \times \Theta} \Big| \sum_{(t,\theta) \in E} \bar{\pi}(\tau^{-1}(t), \theta) - \sum_{(t,\theta)\in E} \pi(t,\theta) \Big| \leq \varepsilon, \tag{8}$$

$$\sup_{E_{-i} \subseteq T_{-i} \times \Theta} \Big| \sum_{(t_{-i},\theta) \in E_{-i}} \bar{\pi}(\tau_{-i}^{-1}(t_{-i}), \theta | \bar{t}_i) - \sum_{(t_{-i},\theta) \in E_{-i}} \pi(t_{-i}, \theta | \tau_i(\bar{t}_i)) \Big| \leq \varepsilon \text{ for all } \bar{t}_i \in \bar{T}_i^\flat. \tag{9}$$

The set of payoff-relevant states in an ε-elaboration includes that in the underlying incomplete information game by the first condition. All players in an ε-elaboration have the same payoff functions as those in the underlying incomplete information game with probability greater than $1 - \varepsilon$ by the second condition. The third condition means that if $\varepsilon = 0$ then an ε-elaboration is an elaboration with an elaboration mapping τ by Lemma 2. We also call τ in Definition 2 an elaboration mapping. It is straightforward to verify that the example of an ε-elaboration discussed in the introduction satisfies the condition in Definition 2.

As a special case, let $T = T^* = \{t\}$ and $\Theta = \Theta^* = \{\theta\}$, which are singletons. Then, (T, Θ, π, u) is a complete information game with a payoff function profile $(u_i(\cdot, \theta))_{i \in I}$. Kajii and Morris (1997) define an ε-elaboration of a complete information game: an incomplete information game $(\bar{T}, \bar{\Theta}, \bar{\pi}, \bar{u})$ is an ε-elaboration of the complete information game if $\bar{\pi}(\bar{T}^\sharp) = 1 - \varepsilon$; that is, with probability $1-\varepsilon$, every player knows that his payoff function is that of the complete information game. The following proposition establishes the equivalence of Definition 2 and the definition of Kajii and Morris (1997) when we focus on complete information games.

Proposition 2. *Suppose that $T = T^* = \{t\}$ and $\Theta = \Theta^* = \{\theta\}$. If $(\bar{T}, \bar{\Theta}, \bar{\pi}, \bar{u})$ is an ε-elaboration in the sense of Definition 2, then there exists $\varepsilon' > 0$ such that $(\bar{T}, \bar{\Theta}, \bar{\pi}, \bar{u})$ is an ε'-elaboration in the sense of Kajii and Morris (1997). Conversely if $(\bar{T}, \bar{\Theta}, \bar{\pi}, \bar{u})$ is an ε-elaboration in the sense of Kajii and Morris (1997), then there exists $\varepsilon' > 0$ such that $(\bar{T}, \bar{\Theta}, \bar{\pi}, \bar{u})$ is an ε'-elaboration in the sense of Definition 2.*

Proof. Suppose that $(\bar{T}, \bar{\Theta}, \bar{\pi}, \bar{u})$ is an ε-elaboration in the sense of Definition 2. Then, it is clear that $(\bar{T}, \bar{\Theta}, \bar{\pi}, \bar{u})$ is an ε'-elaboration in the sense of Kajii and Morris (1997) with $\varepsilon' \equiv 1 - \pi(\bar{T}^\sharp) \leq \varepsilon$.

Suppose that $(\bar{T}, \bar{\Theta}, \bar{\pi}, \bar{u})$ is an ε-elaboration in the sense of Kajii and Morris (1997). Then, $\bar{\pi}(\theta) \geq \bar{\pi}(\bar{T}^\sharp) = 1 - \varepsilon$, and thus $|\bar{\pi}(\tau^{-1}(t), \theta) - \pi(t, \theta)| = |\bar{\pi}(\theta) - 1| \leq \varepsilon \leq \sqrt{\varepsilon}$, where $\tau_i : \bar{T}_i \to \{t\}$. Thus, (8) holds with ε replaced by $\sqrt{\varepsilon}$. Let $\bar{T}_i^\flat = \{\bar{t}_i \in \bar{T}_i \mid \bar{\pi}(\theta | \bar{t}_i) \geq 1 - \sqrt{\varepsilon}\}$. Then, for all $\bar{t}_i \in \bar{T}_i^\flat$,

$$|\bar{\pi}(\tau_{-i}^{-1}(t_{-i}), \theta | \bar{t}_i) - \pi(t_{-i}, \theta | \tau_i(\bar{t}_i))| = |\bar{\pi}(\theta | \bar{t}_i) - \pi(\theta | \tau_i(\bar{t}_i))| = |\bar{\pi}(\theta | \bar{t}_i) - 1| \leq \sqrt{\varepsilon},$$

and thus (9) holds with ε replaced by $\sqrt{\varepsilon}$. Moreover, we must have $\bar{\pi}(\bar{T}_i^\flat) \geq 1 - \sqrt{\varepsilon}$ because

$$1 - \varepsilon \leq \bar{\pi}(\theta) = \sum_{\bar{t}_i \in \bar{T}_i^\flat} \bar{\pi}(\theta|\bar{t}_i)\bar{\pi}(\bar{t}_i) + \sum_{\bar{t}_i \notin \bar{T}_i^\flat} \bar{\pi}(\theta|\bar{t}_i)\bar{\pi}(\bar{t}_i) \leq \bar{\pi}(\bar{T}_i^\flat) + (1 - \sqrt{\varepsilon})(1 - \bar{\pi}(\bar{T}_i^\flat)) = \sqrt{\varepsilon}\bar{\pi}(\bar{T}_i^\flat) + 1 - \sqrt{\varepsilon}.$$

Therefore, $(\bar{T}, \bar{\Theta}, \bar{\pi}, \bar{u})$ is a $\sqrt{\varepsilon}$-elaboration in the sense of Definition 2. □

We say that a set of BIBCE of (T, Θ, π, u) is robust if every ε-elaboration has a BIBCE which is close to some BIBCE in this set.

Definition 3. A set of BIBCE of (T, Θ, π, u), $\mathcal{E} \subseteq \Sigma^{BI}$, is robust if, for every $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \leq \bar{\varepsilon}$, every ε-elaboration $(\bar{T}, \bar{\Theta}, \bar{\pi}, \bar{u})$ with an elaboration mapping τ has a BIBCE $\bar{\sigma}$ such that

$$\sup_{E \subseteq A \times T \times \Theta} \left| \sum_{(a,t,\theta) \in E} \bar{\sigma} \circ \bar{\pi}(a, \tau^{-1}(t), \theta) - \sum_{(a,t,\theta) \in E} \sigma \circ \pi(a, t, \theta) \right| \leq \delta \tag{10}$$

for some $\sigma \in \mathcal{E}$, where $\bar{\sigma} \circ \bar{\pi}(a, \bar{t}, \theta) = \bar{\sigma}(a|\bar{t}, \theta)\bar{\pi}(\bar{t}, \theta)$ and $\sigma \circ \pi(a, t, \theta) = \sigma(a|t, \theta)\pi(t, \theta)$. Equivalently, $\mathcal{E}$ is robust if every sequence of ε^k-elaborations $\{(\bar{T}^k, \bar{\Theta}^k, \bar{\pi}^k, \bar{u}^k)\}_{k=1}^\infty$ with $\lim_{k \to \infty} \varepsilon^k = 0$ has a sequence of BIBCE $\{\bar{\sigma}^k\}_{k=1}^\infty$ such that

$$\lim_{k \to \infty} \inf_{\sigma \in \mathcal{E}} \sup_{E \subseteq A \times T \times \Theta} \left| \sum_{(a,t,\theta) \in E} \bar{\sigma}^k \circ \bar{\pi}^k(a, (\tau^k)^{-1}(t), \theta) - \sum_{(a,t,\theta) \in E} \sigma \circ \pi(a, t, \theta) \right| = 0, \tag{11}$$

where τ^k is an elaboration mapping of $(\bar{T}^k, \bar{\Theta}^k, \bar{\pi}^k, \bar{u}^k)$ and $(\tau^k)^{-1}(t)$ is the inverse image of $t \in T$.

Robustness is defined in terms of a distributional decision rule $\sigma \circ \pi$. Thus, a set of BIBCE of (T, Θ, π, u), $\mathcal{E} \subseteq \Sigma^{BI}$, is robust if and only if the corresponding set of BIBCE of the minimum representation $(T^*, \Theta^*, \pi^*, u^*)$,

$$\mathcal{E}^* = \{\sigma^* : T^* \times \Theta^* \to \Delta(A) \mid \sigma^* \text{ is a restriction of } \sigma \in \mathcal{E} \text{ to } T^* \times \Theta^*\},$$

is robust. We use this observation to prove our main result in the next section.

Robustness of BIBCE is essentially an extension of robustness of correlated equilibria in complete information games due to Kajii and Morris (1997). However, there is a difference: we adopt BIBCE as a solution concept for ε-elaborations, whereas Kajii and Morris (1997) adopt BNE. To see the difference in more detail, let us compare Definition 3 and the definition of a set-valued version of robustness in complete information games due to Morris and Ui (2005). In Morris and

Ui (2005), T and Θ are singletons, and a set of correlated equilibria $\mathcal{E}$, which is also a set of BNE of elaborations, is said to be robust if every sequence of ε^k-elaborations $\{(\bar{T}^k, \bar{\Theta}^k, \bar{\pi}^k, \bar{u}^k)\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} \varepsilon^k = 0$ has a sequence of BNE satisfying (11). When a set of correlated equilibria is a singleton, a robust set is reduced to a robust equilibrium in Kajii and Morris (1997). Recently, Pram (2019) replaces BNE used in ε-elaborations in Kajii and Morris (1997) with an agent normal form BCE, where a prescribed action to each player of each type is independent of the opponents' types and the payoff-relevant state. Thus, when T and Θ are singletons, the requirement of robustness in Definition 3 is weaker than those in Kajii and Morris (1997), Morris and Ui (2005), and Pram (2019).

Let us explain why we adopt BIBCE by coming back to the example discussed in the introduction. We have demonstrated that there exists an ε-elaboration with a unique BNE. The BNE is close to the BIBCE of the analyst's model when ε is small, but it is not close to any BNE of the analyst's model. This implies that there is no robust BNE of the analyst's model, which explains why we are interested in robust BIBCE rather than robust BNE. Then, a solution concept for ε-elaborations must also be BIBCE because any BNE of an ε-elaboration may not be close to a BIBCE of the analyst model even if $\varepsilon = 0$. For example, every BNE of the analyst's model in the example, which is a 0-elaboration, is quite different form the BIBCE.

4 Main results

4.1 Generalized potentials

To provide a sufficient condition for robustness, we introduce a generalized potential function of an incomplete information game, which is an extension of a generalized potential function of a complete information game (Morris and Ui, 2005).

We first discuss a potential function in an incomplete information game (Monderer and Shapley, 1996; van Heumen et al., 1996; Ui, 2009), which is a special case of a generalized potential function. A function $v : A \times \Theta \to \mathbb{R}$ is a potential function of an incomplete information game (T, Θ, π, u) if, for each $\theta \in \Theta^*$, $v(\cdot, \theta)$ is a potential function of the ex-post game given by a payoff function profile $(u_i(\cdot,\theta))_{i\in I}$; that is,

$$u_i(a,\theta) - u_i((a_i', a_{-i}),\theta) = v(a,\theta) - v((a_i', a_{-i}),\theta) \tag{12}$$

for all $a_i, a_i' \in A_i$, $a_{-i} \in A_{-i}$, $i \in I$, and $\theta \in \Theta^*$. It is well known that v is a potential function if and only if there exists $q_i : A_{-i} \times \Theta \to \mathbb{R}$ such that $u_i(a,\theta) = v(a,\theta) + q_i(a_{-i},\theta)$ for all $i \in I$, $a \in A$, and $\theta \in \Theta^*$. Thus, (1) is equivalent to

$$\sum_{a_{-i},t_{-i},\theta} \sigma(a|t,\theta)\pi(t,\theta)v(a,\theta) \geq \sum_{a_{-i},t_{-i},\theta} \sigma(a|t,\theta)\pi(t,\theta)v((a_i', a_{-i}),\theta). \tag{13}$$

That is, σ is obedient in (T,Θ,π,u) if and only if σ is obedient in (T,Θ,π,u') with $u_i' = v$ for all $i \in I$, which is an incomplete information game with an identical payoff function v. This implies that if a belief-invariant decision rule $\sigma \in \Sigma^{BI}$ maximizes the expected value of the potential function, i.e., $\sigma \in \arg\max_{\sigma' \in \Sigma^{BI}} \sum_{a,t,\theta} \sigma \circ \pi(a,t,\theta)v(a,\theta)$, then σ is a BIBCE.

To introduce a generalized potential function, let $\mathcal{A}_i \subseteq 2^{A_i}\backslash\emptyset$ be a covering of A_i for each $i \in I$; that is, $\mathcal{A}_i$ is a collection of nonempty subsets of A_i such that $\bigcup_{X_i \in \mathcal{A}_i} X_i = A_i$. Each $X_i \in \mathcal{A}_i$ will be interpreted as a signal to player i which vaguely prescribes actions in X_i. We write $\mathcal{A} = \{X \mid X = \prod_{i\in I} X_i,\ X_i \in \mathcal{A}_i\}$ and $\mathcal{A}_{-i} = \{X_{-i} \mid X_{-i} = \prod_{j\neq i} X_j,\ X_j \in \mathcal{A}_j\}$.

Definition 4. A bounded function $F : \mathcal{A}\times\Theta \to \mathbb{R}$ is a generalized potential function of (T,Θ,π,u) if, for each $i \in I$ and $P_i \in \Delta(A_{-i} \times \mathcal{A}_{-i} \times \Theta)$ such that $P_i(A_{-i} \times \mathcal{A}_{-i} \times \Theta^*) = 1$ and $P_i(a_{-i}, X_{-i},\theta) = 0$ whenever $a_{-i} \notin X_{-i}$,

$$X_i \in \arg\max_{X_i' \in \mathcal{A}_i} \sum_{X_{-i},\theta} P_i(X_{-i},\theta)F((X_i', X_{-i}),\theta) \tag{14}$$

implies

$$X_i \cap \arg\max_{a_i' \in A_i} \sum_{a_{-i},\theta} P_i(a_{-i},\theta)u_i((a_i', a_{-i}),\theta) \neq \emptyset, \tag{15}$$

where $P_i(X_{-i},\theta) = \sum_{a_{-i}\in A_{-i}} P_i(a_{-i}, X_{-i},\theta)$ and $P_i(a_{-i},\theta) = \sum_{X_{-i}\in\mathcal{A}_{-i}} P_i(a_{-i}, X_{-i},\theta)$.

In the above definition, P_i is interpreted as player i's belief that $X_{-i} \in \mathcal{A}_{-i}$ and $\theta \in \Theta$ are drawn with probability $P(X_{-i},\theta)$, and then $a_{-i} \in X_{-i}$ is drawn with probability $P_i(a_{-i}|X_{-i},\theta) = P_i(a_{-i}, X_{-i},\theta)/P(X_{-i},\theta)$, where the opponents never choose $a_{-i} \notin X_{-i}$ because $P_i(a_{-i}, X_{-i},\theta) = 0$ if $a_{-i} \notin X_{-i}$. The definition implies that if X_i is optimal with respect to the expected value of a generalized potential function F then some action in X_i is optimal with respect to the expected value of player i's payoff function u_i.

At the extreme, consider $F : \mathcal{A} \times \Theta \to \mathbb{R}$ such that $\mathcal{A}_i = \{A_i\}$ for each $i \in I$. Note that $\mathcal{A} = \{\{A\}\}$. Clearly, every incomplete information game has a generalized potential function of this (trivial) type. At the other extreme, consider $F : \mathcal{A} \times \Theta \to \mathbb{R}$ such that $\mathcal{A}_i = \{\{a_i\} \mid a_i \in A_i\}$ for each $i \in I$. Note that $\mathcal{A} = \{\{a\} \mid a \in A\}$. For example, a potential function is a generalized potential function of this type, as the next lemma shows.

Lemma 4. *Suppose that $v : A \times \Theta \to \mathbb{R}$ is a potential function of (T, Θ, π, u). Then, $F : \mathcal{A} \times \Theta \to \mathbb{R}$ with $\mathcal{A}_i = \{\{a_i\} \mid a_i \in A_i\}$ for each $i \in I$ and $F(\{a\}, \theta) = v(a, \theta)$ for each $(a, \theta) \in A \times \Theta$ is a generalized potential function of (T, Θ, π, u).*

Proof. We show that F satisfies the condition in Definition 4. We can rewrite (14) and (15) as follows:

$$a_i \in \arg\max_{a_i' \in A_i} \sum_{a_{-i}, \theta} P_i(\{a_{-i}\}, \theta) v((a_i', a_{-i}), \theta), \tag{16}$$

$$a_i \in \arg\max_{a_i' \in A_i} \sum_{a_{-i}, \theta} P_i(a_{-i}, \theta) u_i((a_i', a_{-i}), \theta). \tag{17}$$

Note that $P_i(a_{-i}, \theta) = P_i(\{a_{-i}\}, \theta)$ since $P_i(a_{-i}, \{a'_{-i}\}, \theta) = 0$ if $a_{-i} \neq a'_{-i}$. Thus, (16) is equivalent to (17) by (12), which implies that F is a generalized potential function. $\square$

4.2 Robustness of GP-maximizing BIBCE

We regard $\mathcal{A}_i$ as a set of signals player i receives and consider a decision-communication rule $\gamma : T \times \Theta \to \Delta(A \times \mathcal{A})$. We say that γ is $\mathcal{A}$-consistent if $\gamma(a, X|t, \theta) = 0$ whenever $a \notin X$; that is, player i receiving X_i always chooses some $a_i \in X_i$. If $\mathcal{A} = \{\{a\} \mid a \in A\}$, then $\mathcal{A}$-consistency implies that $\gamma(a, \{a'\}|t, \theta) = 0$ whenever $a \neq a'$, so γ is equivalent to a decision rule σ with $\sigma(a|t, \theta) = \gamma(a, \{a\}|t, \theta)$. In the remainder of this paper, every decision-communication rule with a signal space $\mathcal{A}$ is assumed to be $\mathcal{A}$-consistent. Let Γ^{BI} denote the set of all ($\mathcal{A}$-consistent) belief-invariant decision-communication rules. For each $\gamma \in \Gamma^{BI}$, we use the following notation: $\gamma(a|t, \theta) = \sum_{X \in \mathcal{A}} \gamma(a, X|t, \theta)$, $\gamma(X|t, \theta) = \sum_{a \in A} \gamma(a, X|t, \theta)$, $\gamma_i(a_i, X_i|t_i) = \sum_{(a_{-i}, X_{-i}) \in A_{-i} \times \mathcal{A}_{-i}} \gamma(a, X|t, \theta)$, $\gamma_i(a_i|t_i) = \sum_{X_i \in \mathcal{A}_i} \gamma_i(a_i, X_i|t_i)$, and $\gamma_i(X_i|t_i) = \sum_{a_i \in A_i} \gamma_i(a_i, X_i|t_i)$. Note that γ_i does not depend upon (t_{-i}, θ) because γ is belief-invariant. Each $\gamma \in \Gamma^{BI}$ together with a prior π determines a joint probability distribution $\gamma \circ \pi \in \Delta(A \times \mathcal{A} \times T \times \Theta)$ given by $\gamma \circ \pi(a, X, t, \theta) = \gamma(a, X|t, \theta)\pi(t, \theta)$, which is referred to as a distributional decision-communication rule. We use $\Gamma^{BI} \circ \pi = \{\gamma \circ \pi \in$

$\Delta(A \times \mathcal{A} \times T \times \Theta) \mid \gamma \in \Gamma^{BI}\}$ to denote the set of all distributional ($\mathcal{A}$-consistent) belief-invariant decision-communication rules.

Let $\Gamma^F \subset \Gamma^{BI}$ denote the set of belief-invariant decision-communication rules that maximize the expected value of a generalized potential function F:

$$\Gamma^F \equiv \arg\max_{\gamma \in \Gamma^{BI}} \sum_{X,t,\theta} \gamma(X|t,\theta)\pi(t,\theta)F(X,\theta).$$

Then, Γ^F contains a BIBCE-C, as the next lemma shows.

Lemma 5. *There exists a BIBCE-C $\gamma \in \Gamma^F$.*

Proof. It can be readily shown that $\Gamma^{BI} \circ \pi$ is a compact set in the weak topology (see Lemma 6 in the next subsection). Thus,

$$\Gamma^F \circ \pi \equiv \arg\max_{\gamma\circ\pi \in \Gamma\circ\pi} \sum_{a,X,t,\theta} \gamma \circ \pi(a,X,t,\theta)F(X,\theta)$$

is nonempty, and Γ^F is also nonempty. Note that, for any $\gamma \in \Gamma^F$, $X_i \in \mathcal{A}_i$, and $t_i \in T_i$ with $\gamma_i(X_i|t_i) > 0$, it holds that

$$X_i \in \arg\max_{X_i' \in \mathcal{A}_i} \sum_{a,\,X_{-i},\,t_{-i},\,\theta} \gamma(a,X|t,\theta)\pi(t,\theta)F((X_i', X_{-i}),\theta), \tag{18}$$

which implies that

$$X_i \cap \arg\max_{a_i' \in A_i} \sum_{a,\,X_{-i},\,t_{-i},\,\theta} \gamma(a,X|t,\theta)\pi(t,\theta)u_i((a_i', a_{-i}),\theta) \neq \emptyset \tag{19}$$

by Definition 4, where we consider $P_i \in \Delta(A_{-i} \times \mathcal{A}_{-i} \times \Theta)$ given by

$$P_i(a_{-i}, X_{-i}, \theta) = \sum_{a_i \in A_i,\, t_{-i} \in T_{-i}} \gamma(a,X|t,\theta)\pi(t,\theta) / \sum_{a' \in A,\, X_{-i} \in \mathcal{A}_{-i},\, t_{-i} \in T_{-i},\, \theta' \in \Theta} \gamma(a',X|t,\theta)\pi(t,\theta').$$

Fix $\gamma \in \Gamma^F$ and let $\rho : T \times \Theta \to \Delta(\mathcal{A})$ be a communication rule with $\rho(X|t,\theta) = \gamma(X|t,\theta)$. Consider the conjunction of (T, Θ, π, u) and ρ and let $\Sigma_i^{\mathcal{A}}$ be the set of player i's strategies in the conjunction that always assign some $a_i \in X_i$ whenever player i receives $X_i \in \mathcal{A}_i$:

$$\Sigma_i^{\mathcal{A}} = \{\sigma_i : T_i \times \mathcal{A}_i \to \Delta(A_i) \mid \sigma_i(a_i|t_i, X_i) = 0 \text{ whenever } a_i \notin X_i\}.$$

Note that a decision-communication rule induced by ρ and a strategy profile $\sigma = (\sigma_i)_{i \in I} \in \Sigma^{\mathcal{A}}$, which is given by $\sigma \otimes \rho(a,X|t,\theta) \equiv \prod_{i \in I} \sigma_i(a_i|t_i, X_i)\rho(X|t,\theta)$, is an element of Γ^F because $\sigma \otimes$

$\rho(a, X|t, \theta) = 0$ if $a \notin X$ and $\sigma \otimes \rho(X|t, \theta) = \rho(X|t, \theta) = \gamma(X|t, \theta)$. Thus, for any $\sigma_{-i} \in \Sigma^{\mathcal{A}}_{-i}$, $X_i \in \mathcal{A}_i$, and $t_i \in T_i$ with $\gamma_i(X_i|t_i) > 0$, it holds that

$$X_i \cap \arg\max_{a_i' \in A_i} \sum_{a_{-i}, X_{-i}, t_{-i}, \theta} \prod_{j \neq i} \sigma_j(a_j|t_j, X_j)\rho(X|t, \theta)\pi(t, \theta)u_i(a_i', a_{-i}) \neq \emptyset$$

by (19). In other words, for any $\sigma_{-i} \in \Sigma^{\mathcal{A}}_{-i}$, there is $\sigma_i' \in \Sigma^{\mathcal{A}}_i$ that is a best response to σ_{-i}, and thus the best response correspondence $\Psi : \Sigma^{\mathcal{A}} \rightrightarrows \Sigma^{\mathcal{A}}$ given by

$$\Psi(\sigma) = \{\sigma' \in \Sigma^{\mathcal{A}} \mid \text{for each } i \in I \text{ and } (t_i, X_i) \in T_i \times \mathcal{A}_i, (\sigma_i', \sigma_{-i}) \text{ is obedient for player } i \text{ of type } (t_i, X_i)\}$$

is nonempty-valued. Moreover, it is readily shown that it is convex-valued and has a closed graph. Therefore, Ψ has a fixed point $\sigma^* \in \Sigma^{\mathcal{A}}$, which is a BNE of the conjunction of (T, Θ, π, u) and ρ. This implies that $\sigma^* \otimes \rho \in \Gamma^F$ is a BIBCE-C of (T, Θ, π, u). □

If $\gamma \in \Gamma^F$ is a BIBCE-C, then $\sigma \in \Sigma^{BI}$ with $\sigma(a|t, \theta) = \gamma(a|t, \theta)$ is a BIBCE by Lemma 3. Such a BIBCE is referred to as a GP-maximizing BIBCE. We denote the set of all GP-maximizing BIBCE by

$$\mathcal{E}^F \equiv \{\sigma \in \Sigma^{BI} \mid \gamma \in \Gamma^F \text{ is a BIBCE-C and } \sigma(a|t, \theta) = \gamma(a|t, \theta)\}.$$

In the following main result of this paper, we show that $\mathcal{E}^F$ is a robust set.

Theorem 1. *If (T, Θ, π, u) has a generalized potential function $F : \mathcal{A} \times \Theta \to \mathbb{R}$, then $\mathcal{E}^F$ is nonempty and robust.*

If the domain of a generalized potential function is $\mathcal{A} = \{\{A\}\}$, then $\mathcal{E}^F$ is the set of all BIBCE. In general, $\mathcal{E}^F$ is not always a minimal robust set, and Theorem 1 is useful only when we have a non-trivial generalized potential function.

For example, if an incomplete information game has a potential function $v : A \times \Theta \to \mathbb{R}$, it also has a generalized potential function $F : \mathcal{A} \times \Theta \to \mathbb{R}$ with $\mathcal{A} = \{\{a\} \mid a \in A\}$ and $F(\{a\}, \theta) = v(a, \theta)$ by Lemma 4, and it is straightforward to show that

$$\mathcal{E}^F = \Sigma^v \equiv \arg\max_{\sigma \in \Sigma^{BI}} \sum_{a,t,\theta} \sigma(a|t, \theta)\pi(t, \theta)v(a|t, \theta),$$

which is the set of potential maximizing BIBCE. Thus, we obtain the following corollary of Theorem 1.

Corollary 2. *If* (T, Θ, π, u) *has a potential function* $v : A \times \Theta \to \mathbb{R}$*, then* Σ^v *is nonempty and robust.*

As an application of this corollary, consider the example discussed in the introduction, which has an identical payoff function, i.e., a potential function. To calculate a potential maximizing BIBCE, we represent a decision rule in terms of the following table of a joint probability of an action profile and a state, where $p_1 + q_1 + r_1 + s_1 = p_2 + q_2 + r_2 + s_2 = 1/2$.

S_1	A	B
A	p_1	q_1
B	r_1	s_1

S_2	A	B
A	p_2	q_2
B	r_2	s_2

A decision rule is belief-invariant if and only if, for each player, the conditional probability of choosing A given S_1 is the same as that given S_2; that is, $p_1+q_1 = p_2+q_2$ and $p_1+r_1 = p_2+r_2$. Thus, in the case of a belief-invariant decision rule, we can rewrite the table using $x = p_1 + q_1 = p_2 + q_2$ and $y = p_1 + r_1 = p_2 + r_2$ as follows.

S_1	A	B
A	p_1	$x - p_1$
B	$y - p_1$	$1/2 + p_1 - x - y$

S_2	A	B
A	p_2	$x - p_2$
B	$y - p_2$	$1/2 + p_2 - x - y$

Then, the expected value of the potential function (i.e., the identical payoff function) is

$$
\begin{aligned}
\Pr[A, A, S_1] + &\Pr[B, B, S_1] + \Pr[A, B, S_2] + \Pr[B, A, S_2] \\
&= p_1 + (1/2 + p_1 - x - y) + (x - p_2) + (y - p_2) \\
&= 2(p_1 - p_2) + 1/2,
\end{aligned}
$$

which takes the maximum value if and only if $p_1 = 1/4$ and $p_2 = 0$. In this case, we must have $x = y = 1/4$ because $x - p_1 = x - 1/4 \geq 0$, $y - p_1 = y - 1/4 \geq 0$, and $1/2 + p_2 - x - y = 1/2 - x - y \geq 0$. Therefore, the potential maximizing BIBCE is given by the following table, which is exactly the same as the BIBCE discussed in the introduction, and it constitutes a unique minimal robust set of BIBCE by Corollary 2 together with the infection argument in the introduction.

S_1	A	B
A	$1/4$	0
B	0	$1/4$

S_2	A	B
A	0	$1/4$
B	$1/4$	0

4.3 Proof of Theorem 1

This subsection is devoted to the proof of Theorem 1. In the proof, for any countable set S, we regard the set of probability distributions $\Delta(S)$ as a subset of a linear space $\{f : S \to \mathbb{R} \mid \sum_{s\in S} |f(s)| < \infty\}$ endowed with the l_1-norm $\|f\|_1 = \sum_{s\in S} |f(s)|$. Because S is countable, it is straightforward to show that the topology of weak convergence in $\Delta(S)$ coincides with the topology induced by the l_1-norm in $\Delta(S)$. Thus, the following result holds by Prohorov's theorem,[7] which will be used several times in the proof.

Lemma 6. *Let $\Delta(S)$ be endowed with the topology induced by the l_1-norm. If $P \subset \Delta(S)$ is tight, i.e., for any $\varepsilon > 0$, there exists a finite set $K^\varepsilon \subset S$ such that $p(K^\varepsilon) > 1 - \varepsilon$ for all $p \in P$, then the closure of P is compact. Conversely, if the closure of $P \subset \Delta(S)$ is compact, then P is tight.*

The following lemma, which is a weaker version of Theorem 1, is the key to prove Theorem 1.

Lemma 7. *Let (T, Θ, π, u) be an incomplete information game with a generalized potential function $F : \mathcal{A}\times\Theta \to \mathbb{R}$. Then, every sequence of ε^k-elaborations $\{(T, \Theta, \pi^k, u)\}_{k=1}^\infty$ satisfying $\lim_{k\to\infty} \varepsilon^k = 0$ and sharing a common elaboration mapping τ has a sequence of BIBCE $\{\sigma^k\}_{k=1}^\infty$ such that*

$$\lim_{k\to\infty} \inf_{\sigma\in\mathcal{E}^F} \sum_{(a,t,\theta)\in A\times T\times\Theta} \left|\sigma^k \circ \pi^k(a, \tau^{-1}(t), \theta) - \sigma \circ \pi(a, t, \theta)\right| = 0. \tag{20}$$

In Lemma 7, an incomplete information game (T, Θ, π, u) and its ε-elaborations $\{(T, \Theta, \pi^k, u)\}_{k=1}^\infty$ have the same sets of types and states and the same payoff functions, where it is implicitly assumed that $T^* \subseteq \tau(T) \subsetneq T$ and $\Theta^* \subsetneq \Theta$. Note that the domain of F is essentially $\mathcal{A} \times \Theta^*$ by Definition 4, and $F(X, \theta)$ for $\theta \notin \Theta^*$ does not matter at all. Note also that (T, Θ, π^k, u) is an ε^k-elaboration of (T, Θ, π, u) if and only if it is an ε^k-elaboration of $(T^*, \Theta^*, \pi^*, u^*)$.

We can establish Theorem 1 using Lemma 7.

Proof of Theorem 1. Let $\{(\bar{T}^k, \bar{\Theta}^k, \bar{\pi}^k, \bar{u}^k)\}_{k=1}^\infty$ be a sequence of ε^k-elaborations of (T, Θ, π, u) with $\lim_{k\to\infty} \varepsilon^k = 0$. Without loss of generality, assume that $\bar{T}^k \cap T = \emptyset$, $\bar{T}^k \cap \bar{T}^l = \emptyset$, and $\bar{\Theta}^k \cap \bar{\Theta}^l = \Theta^*$ for all $k \neq l$. Define $\bar{T} = T \cup \left(\bigcup_{k=1}^\infty \bar{T}^k\right)$ and $\bar{\Theta} = \bigcup_{k=1}^\infty \bar{\Theta}^k$, which are countable.

We construct an equivalent sequence of ε^k-elaborations $\{(\bar{T}, \bar{\Theta}, \bar{\lambda}^k, \bar{u})\}_{k=1}^\infty$. Let $\bar{\lambda}^k \in \Delta(\bar{T} \times \bar{\Theta})$ be an extension of $\bar{\pi}^k$ to $\bar{T}$: $\bar{\lambda}^k(\bar{t}, \bar{\theta}) = \bar{\pi}^k(\bar{t}, \bar{\theta})$ if $(\bar{t}, \bar{\theta}) \in \bar{T}^k \times \bar{\Theta}^k$ and $\bar{\lambda}^k(\bar{t}, \bar{\theta}) = 0$ otherwise. Let

[7] See Billingsley (1999).

$\bar{u}_i : A \times \bar{\Theta} \to \mathbb{R}$ be such that $\bar{u}_i(\cdot, \theta) = u_i(\cdot, \theta)$ if $\theta \in \Theta^*$ and $\bar{u}_i(\cdot, \bar{\theta}) = \bar{u}^k_i(\cdot, \bar{\theta})$ if $\bar{\theta} \in \bar{\Theta}^k \setminus \Theta^*$ for each $i \in I$. Given an elaboration mapping τ^k of $(\bar{T}^k, \bar{\Theta}^k, \bar{\pi}^k, \bar{u}^k)$, an elaboration mapping τ of $(\bar{T}, \bar{\Theta}, \bar{\lambda}^k, \bar{u})$ is defined as $\tau_i(\bar{t}_i) = \tau^k_i(\bar{t}_i)$ if $\bar{t}_i \in \bar{T}^k_i$ for each $i \in I$. Clearly, $(\bar{T}, \bar{\Theta}, \bar{\lambda}^k, \bar{u})$ and $(\bar{T}^k, \bar{\Theta}^k, \bar{\pi}^k, \bar{u}^k)$ have the same minimum representation (every player of every type on the common support has the same belief and the same payoffs in both games), and $(\bar{T}, \bar{\Theta}, \bar{\lambda}^k, \bar{u})$ is also an ε^k-elaboration of (T, Θ, π, u).

We introduce another incomplete information game $(\bar{T}, \bar{\Theta}, \bar{\pi}, \bar{u})$, where $\bar{\pi} \in \Delta(\bar{T} \times \bar{\Theta})$ is an extension of π to $\bar{T} \times \bar{\Theta}$, i.e., $\bar{\pi}(t, \theta) = \pi(t, \theta)$ if $(t, \theta) \in T \times \Theta$ and $\bar{\pi}(t, \theta) = 0$ otherwise. Note that $(\bar{T}, \bar{\Theta}, \bar{\pi}, \bar{u})$ and (T, Θ, π, u) have the same minimum representation, and $(\bar{T}, \bar{\Theta}, \bar{\lambda}^k, \bar{u})$ is an ε^k-elaboration of $(\bar{T}, \bar{\Theta}, \bar{\pi}, \bar{u})$. Because an arbitrary extension of F to $\mathcal{A} \times \bar{\Theta}$ is a generalized potential function of $(\bar{T}, \bar{\Theta}, \bar{\pi}, \bar{u})$, $\bar{\sigma} : \bar{T} \times \bar{\Theta} \to \Delta(A)$ is a GP-maximizing BIBCE of $(\bar{T}, \bar{\Theta}, \bar{\pi}, \bar{u})$ if and only if the restriction of $\bar{\sigma}$ to $T \times \Theta$ is a GP-maximizing BIBCE of (T, Θ, π, u).

Now let $\overline{\mathcal{E}}^F$ be the set of all GP-maximizing BIBCE of $(\bar{T}, \bar{\Theta}, \bar{\pi}, \bar{u})$. Then, by Lemma 7, $\{(\bar{T}, \bar{\Theta}, \bar{\lambda}^k, \bar{u})\}_{k=1}^\infty$ has a sequence of BIBCE $\{\bar{\sigma}^k\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} \inf_{\bar{\sigma}\in\overline{\mathcal{E}}^F} \sum_{(a,t,\theta)\in A\times\bar{T}\times\bar{\Theta}} \left|\bar{\sigma}^k \circ \bar{\lambda}^k(a, \tau^{-1}(t), \theta) - \bar{\sigma} \circ \bar{\pi}(a, t, \theta)\right| = 0,$$

which is equivalent to

$$\lim_{k\to\infty} \inf_{\bar{\sigma}\in\overline{\mathcal{E}}^F} \sup_{E\subseteq A\times\bar{T}\times\bar{\Theta}} \left|\sum_{(a,t,\theta)\in E} \bar{\sigma}^k \circ \bar{\lambda}^k(a, \tau^{-1}(t), \theta) - \sum_{(a,t,\theta)\in E} \bar{\sigma} \circ \bar{\pi}(a, t, \theta)\right| = 0.$$

Let $\bar{\xi}^k : \bar{T}^k \times \bar{\Theta}^k \to \Delta(A)$ be the restriction of $\bar{\sigma}^k : \bar{T} \times \bar{\Theta} \to \Delta(A)$ to $\bar{T}^k \times \bar{\Theta}^k$. Then, $\bar{\xi}^k$ is a BIBCE of $(\bar{T}^k, \bar{\Theta}^k, \bar{\pi}^k, \bar{u}^k)$, and $\{\bar{\xi}^k\}_{k=1}^\infty$ satisfies

$$\lim_{k\to\infty} \inf_{\sigma\in\mathcal{E}^F} \sup_{E\subseteq A\times T\times\Theta} \left|\sum_{(a,t,\theta)\in E} \bar{\xi}^k \circ \bar{\pi}^k(a, (\tau^k)^{-1}(t), \theta) - \sum_{(a,t,\theta)\in E} \sigma \circ \pi(a, t, \theta)\right| = 0$$

because $\{\bar{\sigma} \circ \bar{\pi} \in \Delta(A \times \bar{T} \times \bar{\Theta}) \mid \bar{\sigma} \in \overline{\mathcal{E}}^F\}$ coincides with $\{\sigma \circ \pi \in \Delta(A \times T \times \Theta) \mid \sigma \in \mathcal{E}^F\}$ on their common support. Therefore, $\mathcal{E}^F$ is a robust set. □

The full proof of Lemma 7 is relegated to Appendix A. In the remainder of this subsection, we give its sketch for the special case when (T, Θ, π, u) has a potential function $v : A \times \Theta \to \mathbb{R}$. That is, we explain how we can prove Corollary 2.

The proof consists of two steps. In the first step, we construct a candidate for $\{\sigma^k\}_{k=1}^\infty$ satisfying (20). For an ε^k-elaboration (T, Θ, π^k, u), let $T^k_i \equiv \{t_i \in T_i \mid \pi^k(\Theta^*_i \times \Theta_{-i}|t_i) = 1\}$ be the set of player i's types who believe that their payoff functions are the same as those in $(T^*, \Theta^*, \pi^*, u^*)$ (the minimum representation of (T, Θ, π, u)). Note that $\pi^k(T^k) \geq 1 - \varepsilon^k$ by Definition 2. For each $t \in T$, we write $S^k(t) \equiv \{i \in I \mid t_i \in T_i^k\}$, which is the set of players whose types are in T_i^k when a type profile $t \in T$ is realized. We use the following notation.

- For each $\sigma \in \Sigma^{BI}$, we write $\sigma_S(a_S|t,\theta) \equiv \sigma(\{a_S\} \times A_{-S}|t,\theta)$. In particular, when $S = \{i\}$, we write $\sigma_i(a_i|t_i) \equiv \sigma_{\{i\}}(a_i|t,\theta)$ because σ is belief-invariant. Let

  $$\Sigma^k \equiv \{\sigma \in \Sigma^{BI} \mid \sigma(a|t,\theta) = \sigma_{S^k(t)}(a_{S^k(t)}|t,\theta) \prod_{i \notin S^k(t)} \sigma_i(a_i|t_i)\}$$

  denote the collection of belief-invariant decision rules in which types not in T_i^k choose independent actions. Note that Σ^k is convex with respect to the following convex combination: for $\sigma, \sigma' \in \Sigma^k$, $\sigma'' = \lambda\sigma + (1-\lambda)\sigma' \in \Sigma^k$ is given by

  $$\sigma''_{S^k(t)}(\cdot|t,\theta) = \lambda\sigma_{S^k(t)}(\cdot|t,\theta) + (1-\lambda)\sigma'_{S^k(t)}(\cdot|t,\theta),$$

  $$\prod_{i \notin S^k(t)} \sigma''_i(\cdot|t_i) = \prod_{i \notin S^k(t)} \Big(\lambda\sigma_i(\cdot|t_i) + (1-\lambda)\sigma'_i(\cdot|t_i)\Big).$$

- For each $\sigma \in \Sigma^k$, let

  $$\Sigma^k[\sigma] = \{\sigma' \in \Sigma^k \mid \sigma'_i(a_i|t_i) = \sigma_i(a_i|t_i) \text{ for all } a_i \in A_i,\ t_i \notin T_i^k, \text{ and } i \in I\}.$$

  This is the collection of decision rules in Σ^k under which player $i \in I$ of type $t_i \notin T_i^k$ chooses an action according to the same probability distribution as that under σ. Note that $\Sigma^k[\sigma]$ is a convex subset of Σ^k.

- Fix $\theta^* \in \Theta^*$. For each $\theta \in \Theta$, let $\phi(\theta) = (\phi_i(\theta))_{i \in I} \in \Theta^* = \prod_{i \in I} \Theta_i^*$ be such that $\phi_i(\theta) = \theta_i$ if $\theta_i \in \Theta_i^*$ and $\phi_i(\theta) = \theta_i^*$ if $\theta_i \notin \Theta_i^*$. For $\sigma \in \Sigma^k$, let

  $$\begin{aligned}\Sigma^{k,F}[\sigma] &\equiv \arg\max_{\sigma' \in \Sigma^k[\sigma]} \sum_{a,t,\theta} \sigma'(a|t,\theta)\pi^k(t,\theta)v(a,\phi(\theta)) \\ &= \arg\max_{\sigma' \in \Sigma^k[\sigma]} \sum_{a,t,\theta} \sigma' \circ \pi^k(a,t,\theta)v(a,\phi(\theta)),\end{aligned}$$

  which is nonempty because $\{\sigma' \circ \pi^k \in \Delta(A \times T \times \Theta) \mid \sigma' \in \Sigma^k[\sigma]\}$ is tight and closed, and thus it is compact by Lemma 6.

Then, we can show that (T, Θ, π^k, u) has is a BIBCE $\sigma^k \in \Sigma^k$ with $\sigma^k \in \Sigma^{k,F}[\sigma^k]$ by observing that σ^k is a fixed point of a correspondence $\Psi : \Sigma^k \rightrightarrows \Sigma^k$ given by the following condition: $\sigma' \in \Psi(\sigma)$ if and only if

$$\sigma'(a|t,\theta) = \sigma^2_{S^k(t)}(a_{S^k(t)}|t,\theta) \prod_{i \notin S^k(t)} \sigma^1_i(a_i|t_i)$$

for all (a, t, θ), where

- $\sigma^1 \in \Sigma^k$ and, for each i and $t_i \notin T^k_i$, $\sigma^1_i(a_i|t_i) > 0$ implies

$$a_i \in \arg\max_{a'_i \in A_i} \sum_{a_{-i}, t_{-i}, \theta} \sigma_{N \setminus \{i\}}(a_{-i}|t,\theta) \pi^k(t,\theta) u_i((a'_i, a_{-i}), \theta),$$

- $\sigma^2 \in \Sigma^{k,F}[\sigma]$ and, for each $i \in I$ and $t_i \in T^k_i$, σ^2 is obedient for player i of type t_i.

Under $\sigma' \in \Psi(\sigma)$, each $t_i \notin T^k_i$ follows $\sigma^1 \in \Sigma^k$ and chooses a best response to σ, whereas each $t_i \in T^k_i$ follows $\sigma^2 \in \Sigma^{k,F}(\sigma)$ and simultaneously chooses a best response to each other under the assumption that each $t_i \notin T^k_i$ adopts the same mixed action as that given by σ. Thus, if $\sigma^k \in \Psi(\sigma^k)$, then $\sigma^k \in \Sigma^{k,F}[\sigma^k]$ and it is obedient; that is, $\sigma^k \in \Sigma^{k,F}[\sigma^k]$ is a BIBCE. It can be readily shown that Ψ is nonempty-valued, convex-valued, and has a closed graph. Therefore, Ψ has a fixed point by the Kakutani-Fan-Glicksberg fixed point theorem.

In the second step, we show that $\{\sigma^k\}_{k=1}^\infty$ satisfies (20). To this end, let $\eta^k \in \Delta(A \times T \times \Theta)$ be given by $\eta^k(a,t,\theta) \equiv \sigma^k \circ \pi^k(a, \tau^{-1}(t), \theta) = \sum_{t' \in \tau^{-1}(t)} \sigma^k(a|t',\theta)\pi^k(t',\theta)$. Note that $\eta^k(t,\theta) \equiv \sum_{a \in A} \eta^k(a,t,\theta) = \pi^k(\tau^{-1}(t),\theta)$. We write $\eta^k(a|t,\theta) \equiv \eta^k(a,t,\theta)/\eta^k(t,\theta)$ for $(t,\theta) \in T \times \Theta$ with $\eta^k(t,\theta) > 0$. It can be readily shown that $\{\eta^k\}_{k=1}^\infty$ is tight, so it has a convergent subsequence by Lemma 6, which is denoted by $\{\eta^{k_l}\}_{l=1}^\infty$ with $\lim_{l\to\infty} \eta^{k_l} = \eta^*$. Note that, for each $(t,\theta) \in T^* \times \Theta^*$, $\eta^*(t,\theta) = \lim_{l\to\infty} \eta^{k_l}(t,\theta) = \lim_{l\to\infty} \pi^{k_l}(\tau^{-1}(t),\theta) = \pi(t,\theta)$ by Definition 2. Thus, we have $\eta^*(a|t,\theta)\pi(t,\theta) = \eta^*(a,t,\theta)$. Then, (20) is written as

$$\lim_{k\to\infty} \inf_{\sigma \in \mathcal{E}^F} \sum_{a,t,\theta} \left|\eta^k(a,t,\theta) - \sigma \circ \pi(a,t,\theta)\right| = \lim_{k\to\infty} \inf_{\sigma \in \Sigma^v} \sum_{a,t,\theta} \left|\eta^k(a,t,\theta) - \sigma \circ \pi(a,t,\theta)\right| = 0. \tag{21}$$

To prove (21), it is enough to show that, for every convergent subsequence $\{\eta^{k_l}\}_{l=1}^\infty$, there exists $\sigma \in \Sigma^v$ such that $\eta^*(a,t,\theta) = \sigma(a|t,\theta)\pi(t,\theta)$; that is, $\eta^*(\cdot|t,\theta) \in \Sigma^v$ (with some abuse of notation). Recall that $\Sigma^v = \arg\max_{\sigma \in \Sigma^{BI}} \sum_{a,t,\theta} \sigma(a|t,\theta)\pi(t,\theta)v(a|t,\theta)$.

We now denote a convergent subsequence by $\{\eta^k\}_{k=1}^{\infty}$ with $\eta^* = \lim_{k\to\infty} \eta^k$ rather than $\{\eta^{k_l}\}_{l=1}^{\infty}$ to simplify notation. We can prove $\eta^*(\cdot|t,\theta) \in \Sigma^v$ by showing that

$$\sum_{a,t,\theta} \eta^*(a|t,\theta)\pi(t,\theta)v(a,\theta) \geq \sum_{a,t,\theta} \hat{\sigma}(a|t,\theta)\pi(t,\theta)v(a,\theta) \tag{22}$$

for arbitrary $\hat{\sigma} \in \Sigma^v$, and that $\eta^*(\cdot|t,\theta)$ is belief-invariant and obedient in $(T^*, \Theta^*, \pi^*, u^*)$. Here we give a proof for (22) (see Appendix A for belief-invariance and obedience). Fix $\hat{\sigma} \in \Sigma^v$ and let $\hat{\sigma}^k \in \Sigma^k[\sigma^k]$ be such that

$$\hat{\sigma}^k_{S^k(t)}(a_{S^k(t)}|t,\theta) = \hat{\sigma}_{S^k(t)}(a_{S^k(t)}|\tau(t),\theta)$$

for all (t,θ), which is well-defined because $\hat{\sigma}$ is belief-invariant. Note that $\hat{\sigma}^k(a|t,\theta) = \hat{\sigma}(a|\tau(t),\theta)$ if $t \in T^k$. Then, $\sigma^k \in \Sigma^{k,F}[\sigma^k]$ implies that

$$\sum_{a,t,\theta} \sigma^k(a|t,\theta)\pi^k(t,\theta)v(a,\phi(\theta)) \geq \sum_{a,t,\theta} \hat{\sigma}^k(a|t,\theta)\pi^k(t,\theta)v(a,\phi(\theta)). \tag{23}$$

We first evaluate the limit of the left-hand side of (23) as $k \to \infty$. Because

$$\sum_{a,t,\theta} \sigma^k(a|t,\theta)\pi^k(t,\theta)v(a,\phi(\theta)) = \sum_{a,t,\theta} \sum_{t' \in \tau^{-1}(t)} \sigma^k(a|t',\theta)\pi^k(t',\theta)v(a,\phi(\theta)) = \sum_{a,t,\theta} \eta^k(a,t,\theta)v(a,\phi(\theta)),$$

we have

$$\lim_{k\to\infty} \sum_{a,t,\theta} \sigma^k(a|t,\theta)\pi^k(t,\theta)v(a,\phi(\theta)) = \sum_{a,t,\theta} \eta^*(a,t,\theta)v(a,\phi(\theta)) = \sum_{a,t,\theta} \eta^*(a|t,\theta)\pi(t,\theta)v(a,\theta). \tag{24}$$

We next evaluate the limit of the right-hand side of (23) as $k \to \infty$. Note that

$$\sum_{a,t,\theta} \hat{\sigma}^k(a|t,\theta)\pi^k(t,\theta)v(a,\phi(\theta)) \geq \sum_{a,\theta} \sum_{t \in T^k} \hat{\sigma}(a|\tau(t),\theta)\pi^k(t,\theta)v(a,\phi(\theta)) + \pi^k(T \setminus T^k) \inf_{a,\theta} v(a,\theta),$$

and the first term in the right-hand side is evaluated as

$$\begin{aligned}
\sum_{a,\theta} \sum_{t \in T^k} \hat{\sigma}(a|\tau(t),\theta)\pi^k(t,\theta)v(a,\phi(\theta)) &= \sum_{a,t,\theta} \hat{\sigma}(a|t,\theta)\pi^k(\tau^{-1}(t) \cap T^k,\theta)v(a,\phi(\theta)) \\
&\geq \sum_{a,t,\theta} \hat{\sigma}(a|t,\theta)\pi^k(\tau^{-1}(t),\theta)v(a,\phi(\theta)) - \pi^k(T \setminus T^k) \sup_{a,\theta} v(a,\theta) \\
&= \sum_{a,t,\theta} \hat{\sigma}(a|t,\theta)\eta^k(t,\theta)v(a,\phi(\theta)) - \pi^k(T \setminus T^k) \sup_{a,\theta} v(a,\theta).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\lim_{k\to\infty}\sum_{a,t,\theta}\hat{\sigma}^k(a|t,\theta)\pi^k(t,\theta)v(a,\phi(\theta)) \\
&\quad\geq \lim_{k\to\infty}\sum_{a,t,\theta}\hat{\sigma}(a|t,\theta)\eta^k(t,\theta)v(a,\phi(\theta)) + \pi^k(T\setminus T^k)(\inf_{a,\theta} v(X,\theta) - \sup_{a,\theta} v(a,\theta)) \\
&\quad= \sum_{a,t,\theta}\hat{\sigma}(a|t,\theta)\eta^*(t,\theta)v(a,\phi(\theta)) = \sum_{a,t,\theta}\hat{\sigma}(a|t,\theta)\pi(t,\theta)v(a,\theta). \qquad (25)
\end{aligned}$$

Combining (23), (24), and (25), we obtain (22).

5 Robust BIBCE of supermodular games

This section focuses on supermodular games and studies their robust BIBCE.

5.1 Supermodular potential games

Recall that a robust BIBCE of the example in the introduction is not a BNE. This is why we have to adopt BIBCE as a solution concept rather than BNE. Then, a natural question arises: in what class of incomplete information games can a BNE be robust? This subsection demonstrates that a robust set of BIBCE given by Theorem 1 contains BNE if the game is a supermodular potential game.

Let A_i be linearly ordered with $\geq_i$. We write $\geq^P$ for the product order: $a \geq^P b$ if $a_i \geq_i b_i$ for all $i \in I$. We say that (T,Θ,π,u) is a supermodular incomplete information game if, for each $\theta\in\Theta$, the ex-post game with a payoff function profile $(u_i(\cdot,\theta))_{i\in I}$ satisfies strategic complementarities; that is, for each $i\in I$ and $a,b\in A$ with $a\geq^P b$,

$$u_i(a_i,a_{-i}) - u_i(b_i,a_{-i}) \geq u_i(a_i,b_{-i}) - u_i(b_i,b_{-i}).$$

Proposition 3. *Assume that a supermodular incomplete information game (T,Θ,π,u) has a potential function $v : A\times\Theta\to\mathbb{R}$. Then, the following holds.*

- *For a potential maximizing BIBCE $\sigma\in\Sigma^v$, let $\overline{\sigma} = (\overline{\sigma}_i)_{i\in I}$ and $\underline{\sigma} = (\underline{\sigma}_i)_{i\in I}$ be strategy profiles such that, for each $t_i\in T_i$ and $i\in I$, $\overline{\sigma}_i(\overline{a}_i|t_i) = 1$ if $\overline{a}_i = \max\{a_i \mid \sigma_i(a_i|t_i) > 0\}$ and $\underline{\sigma}_i(\underline{a}_i|t_i) = 1$ if $\underline{a}_i = \min\{a_i \mid \sigma_i(a_i|t_i) > 0\}$, respectively, where $\sigma_i(a_i|t_i) = \sigma(\{a_i\}\times A_{-i}|t,\theta)$. Then, $\overline{\sigma},\underline{\sigma}\in\Sigma^v$.*

- *Let $\overline{\sigma}^* = (\overline{\sigma}_i^*)_{i\in I}$ and $\underline{\sigma}^* = (\underline{\sigma}_i^*)_{i\in I}$ be strategy profiles such that, for each $t_i \in T_i$ and $i \in I$, $\overline{\sigma}_i^*(\overline{a}_i|t_i) = 1$ if $\overline{a}_i = \max\{a_i \,|\, \sigma_i(a_i|t_i) > 0,\ \sigma \in \Sigma^v\}$ and $\underline{\sigma}_i^*(\underline{a}_i|t_i) = 1$ if $\underline{a}_i = \min\{a_i \,|\, \sigma_i(a_i|t_i) > 0,\ \sigma \in \Sigma^v\}$, respectively. Then, $\overline{\sigma}^*, \underline{\sigma}^* \in \Sigma^v$.*

- *If a potential maximizing BIBCE is unique, then it is a potential maximizing BNE. Conversely, if a potential maximizing BNE is unique, then it is a potential maximizing BIBCE.*

Proof. See Appendix B. □

By Proposition 3, for each potential maximizing BIBCE, the strategy profile that is the supremum (or infimum) of the BIBCE is a potential maximizing BIBCE. Moreover, the strategy profile that is the supremum (or infimum) of the set of potential maximizing BIBCE is also a potential maximizing BIBCE. This implies that the robust set of potential maximizing BIBCE contains BNE, and in particular, if the set is a singleton, then the unique potential maximizing BIBCE must be a BNE.

As an application of Proposition 3, we consider an incomplete information game with two players 1 and 2 and two actions 0 and 1. The set of states is $\Theta = \{0, 1, 2, 3, \ldots\}$. We assume that state $k \in \Theta$ occurs with probability $\varepsilon(1-\varepsilon)^k > 0$, where $\varepsilon \in (0, 1)$. Player 1 has an information partition $\{\{0\}, \{1, 2\}, \{3, 4\}, \ldots\}$, and player 2 has an information partition $\Pi_2 = \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \ldots\}$. The type of a player with a partition $\{k-1, k\}$ (or $\{k\}$) is referred to as type k. Thus, the sets of player 1's types and player 2's types are $T_1 = \{0, 2, 4, \ldots\}$ and $T_2 = \{1, 3, 5, \ldots\}$, respectively. Note that the prior $\pi \in \Delta(T \times \Theta)$ is given by

$$\pi(t,\theta) = \begin{cases} \varepsilon(1-\varepsilon)^k & \text{if } t_1 = k,\, t_2 = k+1, \text{ and } \theta = k \text{ is even,} \\ \varepsilon(1-\varepsilon)^k & \text{if } t_1 = k+1,\, t_2 = k, \text{ and } \theta = k \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases}$$

which implies that

$$\pi(t_2,\theta|t_1) = \begin{cases} 1 & \text{if } t_1 = 0,\, t_2 = 1, \text{ and } \theta = 0, \\ (1-\varepsilon)/(2-\varepsilon) & \text{if } t_1 = k,\, t_2 = k+1, \text{ and } \theta = k \neq 0 \text{ is even,} \\ 1/(2-\varepsilon) & \text{if } t_1 = k+1,\, t_2 = k, \text{ and } \theta = k \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\pi(t_1, \theta | t_2) = \begin{cases} (1-\varepsilon)/(2-\varepsilon) & \text{if } t_1 = k+1,\ t_2 = k, \text{ and } \theta = k \text{ is odd,} \\ 1/(2-\varepsilon) & \text{if } t_1 = k,\ t_2 = k+1, \text{ and } \theta = k \neq 0 \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

We assume the following payoff table for each state $k \in \Theta$, where $\delta \in (0, 1)$.

	1	0
1	δ^k, δ^k	$\delta^k - 1, 0$
0	$0, \delta^k - 1$	$0, 0$

This game is a supermodular potential game with the following potential function.

	1	0
1	δ^k	0
0	0	$1 - \delta^k$

A pure-strategy profile is represented by a sequence of actions $x = (x_k)_{k=0}^{\infty} \in \{0, 1\}^{T_1 \cup T_2}$, where $x_k \in \{0, 1\}$ is an action of player 1 with type k if k is even and an action of player 2 with type k if k is odd. We denote by X the set of all such sequences. Let $z^n \in X$ be the decreasing sequence such that $z^n_k = 1$ if $k \leq n - 1$ and $z^n_k = 0$ if $k \geq n$. Then, the minimum BNE is represented by z^0, and the maximum BNE is represented by z^∞. Note that z^n with $n \geq 1$ (i.e., type 0 chooses 1) is a BNE if and only if $n \geq \underline{n}$, where $\underline{n}$ is the minimum integer satisfying $\delta^{\underline{n}} \leq (1 - \varepsilon)/(2 - \varepsilon)$, which can be verified by the standard argument of iterated elimination of dominated actions.

We characterize a potential maximizing BNE. Note that the expected value of the potential function under $x \in X$ is

$$f(x) = \sum_{k=0}^{\infty} \varepsilon(1-\varepsilon)^k \Big(\delta^k x_k x_{k+1} + (1 - \delta^k)(1 - x_k)(1 - x_{k+1})\Big).$$

Note that

$$f(z^n) = \begin{cases} \sum_{k=n}^{\infty} \varepsilon(1-\varepsilon)^k (1-\delta^k) & \text{if } n \leq 1, \\ \sum_{k=0}^{n-2} \varepsilon(1-\varepsilon)^k \delta^k + \sum_{k=n}^{\infty} \varepsilon(1-\varepsilon)^k (1-\delta^k) & \text{if } n \geq 2. \end{cases}$$

Thus, for $n \geq 1$,

$$f(z^{n+1}) - f(z^n) = \varepsilon(1-\varepsilon)^{n-1}\delta^{n-1} - \varepsilon(1-\varepsilon)^n(1-\delta^n) = \varepsilon(1-\varepsilon)^{n-1}(\delta^{n-1}(1 + (1-\varepsilon)\delta) - (1-\varepsilon)),$$

and

$$f(z^{n+1}) \gtreqqless f(z^n) \;\Leftrightarrow\; \delta^{n-1} \gtreqqless (1-\epsilon)/(1+(1-\varepsilon)\delta) \;\Leftrightarrow\; n \lesseqqgtr k^*,$$

where $k^* \in \mathbb{R}$ is the unique solution to

$$\delta^{k^*} = (1-\epsilon)\delta/(1+(1-\varepsilon)\delta). \tag{26}$$

Therefore, $f(z^n) \geq f(z^{n'})$ for all n' if $n = n^* \equiv \lceil k^* \rceil$.

Suppose that k^* is not an integer. We show that the strategy profile represented by z^{n^*} is a robust BIBCE. By Corollary 2 and Proposition 3, it is enough to establish $f(z^{n^*}) > f(x)$ for all $x \neq z^{n^*}$. Consider the following two types of non-optimal strategy profiles.

- Let $x \in X$ be such that $x_0 = 0$. Then, for $x' \in X$ given by $x'_0 = x'_1 = 1$ and $x'_k = x_k$ for all $k \geq 2$,

  $$f(x') - f(x) \geq \varepsilon - \varepsilon(1-\varepsilon)(1-\delta) > 0.$$

- Let $x \in X$ be such that $x_0 = 1$ and $n \equiv \min\{k \mid x_k = 0\} \leq n^* - 1$. Then, for $x' \in X$ given by $x'_n = 1$ and $x'_k = x_k$ for all $k \neq n$,

  $$f(x') - f(x) \geq \varepsilon(1-\varepsilon)^{n-1}\delta^{n-1} - \varepsilon(1-\varepsilon)^n(1-\delta^n) = \varepsilon(1-\varepsilon)^{n-1}(\delta^{n-1}(1+(1-\varepsilon)\delta) - (1-\varepsilon)) > 0$$

  because $n \leq n^* - 1 < k^*$.

The above discussion implies that if $x^* \in \arg\max_{x\in X} f(x)$, then $x^*_k \geq z^{n^*}_k$ for all k. We then consider the following type of non-optimal strategy profiles.

- Let $x \in X$ be such that there exists $n \geq n^*$ satisfying $x_n = 1$ and $x_{n+1} = 0$. Then, for $x' \in X$ given by $x'_n = 0$ and $x'_k = x_k$ for all $k \neq n$,

  $$f(x') - f(x) \geq \varepsilon(1-\varepsilon)^n(1-\delta^n) - \varepsilon(1-\varepsilon)^{n-1}\delta^{n-1} = \varepsilon(1-\varepsilon)^{n-1}(1-\varepsilon-\delta^{n-1}(1+(1-\varepsilon)\delta) > 0$$

  because $n \geq n^* > k^*$.

The above discussion implies that if $x^* \in \arg\max_{x\in X} f(x)$, then $x^* = z^{n^*}$ or $x^* = z^\infty$, but we know that $f(z^{n^*}) > f(z^\infty)$. Therefore, $x^* = z^{n^*}$.

5.2 Binary-action supermodular games with monotone potential functions

Let (T, Θ, π, u) be a binary-action incomplete information game with $A_i = \{0, 1\}$, where we write $a_i \geq b_i$ and $a \geq b$ rather than $a_i \geq_i b_i$ and $a \geq^P b$, respectively, for the sake of notational simplicity. For $S \subseteq I$, we denote by $\mathbf{1}_S$ the action profile such that all players in S play action 1, and the others play action 0. We write $\mathbf{1} = \mathbf{1}_I$ by convention. We provide a sufficient condition for robust BIBCE of binary-action games, in particular, binary-action supermodular games.

Suppose that there exists a function $v : A \times \Theta \to \mathbb{R}$ and a constant $\lambda_i > 0$ for each $i \in I$ such that

$$\lambda_i(u_i((1, a_{-i}), \theta) - u_i((0, a_{-i}), \theta)) \geq v((1, a_{-i}), \theta) - v((0, a_{-i}), \theta) \tag{27}$$

for all $a_{-i} \in A_{-i}$ and $\theta \in \Theta$. This function v is referred to as a monotone potential function (Morris and Ui, 2005). If $\lambda_i = 1$ for all $i \in I$ and the equality holds in (27), then a monotone potential function is a potential function. We say that a monotone potential function v is supermodular if, for each $i \in I$ and $a, b \in A$ with $a \geq b$, it holds that $v(a_i, a_{-i}) - v(b_i, a_{-i}) \geq v(a_i, b_{-i}) - v(b_i, b_{-i})$.

The next lemma shows that a monotone potential function is a special case of a generalized potential function if the game or the monotone potential function is supermodular, which generalizes the corresponding result in the case of complete information games (Morris and Ui, 2005).

Lemma 8. *Let $\mathcal{A}_i = \{\{0, 1\}, \{1\}\}$, and define a one-to-one mapping $\Lambda_i : A_i \to \mathcal{A}_i$ by $\Lambda_i(1) = \{1\}$ and $\Lambda_i(0) = \{0, 1\}$ for each $i \in I$. Let $v : A \times \Theta \to \mathbb{R}$ be a monotone potential function of (T, Θ, π, u). If (T, Θ, π, u) or v is supermodular, then $F : \mathcal{A} \times \Theta \to \mathbb{R}$ given by $F(\Lambda(a), \theta) = v(a, \theta)$ is a generalized potential function, where $\Lambda(a) = \prod_{i \in I} \Lambda_i(a_i) = \{a' \in A \mid a' \geq a\}$.*

Proof. Let $P_i \in \Delta(A_{-i} \times \mathcal{A}_{-i} \times \Theta)$ satisfy $P_i(a_{-i}, X_{-i}, \theta) = 0$ whenever $a_{-i} \notin X_{-i}$. Recall that we write $P_i(a_{-i}, \theta) = \sum_{X_{-i}} P_i(a_{-i}, X_{-i}, \theta)$ and $P_i(X_{-i}, \theta) = \sum_{a_{-i}} P_i(a_{-i}, X_{-i}, \theta)$. Then, for each increasing subset $B_{-i} \subseteq A_{-i}$ (i.e. $a_{-i} \in B$ and $a'_{-i} \geq a_{-i}$ imply $a'_{-i} \in B_{-i}$),

$$\sum_{a_{-i} \in B_{-i}} P_i(a_{-i}, \theta) = \sum_{a_{-i} \in B_{-i}} \sum_{X_{-i} \subseteq A_{-i}} P_i(a_{-i}, X_{-i}, \theta),$$

$$\sum_{a_{-i} \in B_{-i}} P_i(\Lambda_{-i}(a_{-i}), \theta) = \sum_{X_{-i} \subseteq B_{-i}} \sum_{a_{-i} \in B_{-i}} P_i(a_{-i}, X_{-i}, \theta).$$

Thus,

$$\sum_{a_{-i} \in B_{-i}} P_i(a_{-i}, \theta) \geq \sum_{a_{-i} \in B_{-i}} P_i(\Lambda_{-i}(a_{-i}), \theta).$$

That is, $P_i(a_{-i}|\theta) \in \Delta(A_{-i})$ first-order stochastically dominates $P_i(\Lambda_{-i}(a_{-i})|\theta) \in \Delta(A_{-i})$. This implies that, if the game is supermodular, then

$$\sum_{a_{-i},\theta} P_i(a_{-i},\theta)(u_i((1,a_{-i}),\theta) - u_i((0,a_{-i}),\theta)) \geq \sum_{a_{-i},\theta} P_i(\Lambda_{-i}(a_{-i}),\theta)(u_i((1,a_{-i}),\theta) - u_i((0,a_{-i}),\theta)), \quad (28)$$

and if the monotone potential function is supermodular, then

$$\sum_{a_{-i},\theta} P_i(a_{-i},\theta)(v((1,a_{-i}),\theta) - v((0,a_{-i}),\theta)) \geq \sum_{a_{-i},\theta} P_i(\Lambda_{-i}(a_{-i}),\theta)(v((1,a_{-i}),\theta) - v((0,a_{-i}),\theta)). \quad (29)$$

Using either (28) or (29), we show that F is a generalized potential function.

If

$$\{0,1\} \in \arg\max_{X'_i \in \mathcal{A}_i} \sum_{X_{-i},\theta} P_i(X_{-i},\theta) F((X'_i, X_{-i}),\theta),$$

then it is obvious that

$$\{0,1\} \cap \arg\max_{a'_i \in A_i} \sum_{a_{-i},\theta} P_i(a_{-i},\theta) u_i((a'_i, a_{-i}),\theta) \neq \emptyset.$$

If

$$\{1\} \in \arg\max_{X'_i \in \mathcal{A}_i} \sum_{X_{-i},\theta} P_i(X_{-i},\theta) F((X'_i, X_{-i}),\theta),$$

then

$$\sum_{a_{-i},\theta} P_i(\Lambda_{-i}(a_{-i}),\theta)(v((1,a_{-i}),\theta) - v((0,a_{-i}),\theta) \geq 0. \quad (30)$$

Thus, if the game is supermodular, then

$$\begin{aligned}\sum_{a_{-i},\theta} P_i(a_{-i},\theta)(u_i((1,a_{-i}),\theta) - u_i((0,a_{-i}),\theta)) &\geq \sum_{a_{-i},\theta} P_i(\Lambda_{-i}(a_{-i}),\theta)(u_i((1,a_{-i}),\theta) - u_i((0,a_{-i}),\theta)) \\ &\geq \lambda_i^{-1} \sum_{a_{-i},\theta} P_i(\Lambda_{-i}(a_{-i}),\theta)(v((1,a_{-i}),\theta) - v((0,a_{-i}),\theta)) \geq 0\end{aligned}$$

by (27), (28), and (30), and if the monotone potential function is supermodular, then

$$\begin{aligned}\sum_{a_{-i},\theta} P_i(a_{-i},\theta)(u_i((1,a_{-i}),\theta) - u_i((0,a_{-i}),\theta)) &\geq \lambda_i^{-1} \sum_{a_{-i},\theta} P_i(a_{-i},\theta)(v((1,a_{-i}),\theta) - v((0,a_{-i}),\theta)) \\ &\geq \lambda_i^{-1} \sum_{a_{-i},\theta} P_i(\Lambda_{-i}(a_{-i}),\theta)(v((1,a_{-i}),\theta) - v((0,a_{-i}),\theta)) \geq 0\end{aligned}$$

by (27), (29), and (30). Each case implies that

$$1 \in \arg\max_{a'_i \in A_i} \sum_{a_{-i},\theta} P_i(a_{-i},\theta) u_i((a'_i, a_{-i}),\theta).$$

Therefore, F is a generalized potential function. $\square$

Let

$$\Sigma^v \equiv \arg\max_{\sigma \in \Sigma^{BI}} \sum_{a,t,\theta} \sigma(a|t,\theta)\pi(t,\theta)v(a,\theta)$$

denote the set of all belief-invariant decision rules that maximize the expected value of a monotone potential function v. Using Σ^v, we provide a sufficient conditions for BIBCE in the next proposition.

Proposition 4. *Let (T, Θ, π, u) be a binary-action incomplete information game with a monotone potential function $v : T \times \Theta \to \mathbb{R}$. Assume that the game or the monotone potential function is supermodular. Then,*

$$\mathcal{E}^v \equiv \{\sigma \in \Sigma^{BI} \mid \sigma \text{ is a BIBCE such that, for each } (t,\theta) \in T \times \Theta,\ \sigma(\cdot|t,\theta) \in \Delta(A) \text{ first-order stochastically dominates } \sigma'(\cdot|t,\theta) \in \Delta(A) \text{ for some } \sigma' \in \Sigma^v\}$$

is a robust set of BIBCE. In particular, if $\Sigma^v = \{\sigma^*\}$ with $\sigma^*(\mathbf{1}|t,\theta) = 1$ for all $(t,\theta) \in T \times \Theta$, then σ^* is a robust BIBCE. If the monotone potential function is supermodular, then $\mathcal{E}^v$ is reduced to

$$\mathcal{E}^v = \{\sigma \in \Sigma^{BI} \mid \sigma \text{ is a BIBCE such that, for each } i \in I \text{ and } t_i \in T_i,\ \sigma_i(1|t_i) = 1 \text{ if } \sigma'_i(1|t_i) = 1 \text{ for all } \sigma' \in \Sigma^v\}.$$

To prove Proposition 4, we use the following lemma, which holds not only for monotone potential functions of binary-action supermodular games but also for arbitrary generalized potential functions of arbitrary incomplete information games.

Lemma 9. *For any $\mathcal{A}$-consistent decision-communication rule γ, it holds that*

$$\sum_{a \in B} \gamma(a|t,\theta) \geq \sum_{X \subseteq B, X \in \mathcal{A}} \gamma(X|t,\theta) \tag{31}$$

for all $B \in 2^A$ and $(t,\theta) \in T \times \Theta$. Conversely, for any $\sigma : T \times \Theta \to \Delta(A)$ and $\rho : T \times \Theta \to \Delta(\mathcal{A})$ such that

$$\sum_{a \in B} \sigma(a|t,\theta) \geq \sum_{X \subseteq B, X \in \mathcal{A}} \rho(X|t,\theta) \tag{32}$$

for all $B \in 2^A$ and $(t,\theta) \in T \times \Theta$, there exists an $\mathcal{A}$-consistent decision-communication rule γ such that $\gamma(a|t,\theta) = \sigma(a|t,\theta)$ and $\gamma(X|t,\theta) = \rho(X|t,\theta)$ for all $a \in A$, $X \in \mathcal{A}$, and $(t,\theta) \in T \times \Theta$.

Proof. The first part holds because

$$\sum_{a\in B}\gamma(a|t,\theta)=\sum_{a\in B}\sum_{X\in\mathcal{A}}\gamma(a,X|t,\theta)\geq\sum_{a\in B}\sum_{X\subseteq B,X\in\mathcal{A}}\gamma(a,X|t,\theta)=\sum_{X\subseteq B,X\in\mathcal{A}}\gamma(X|t,\theta).$$

To prove the second part, fix (t,θ) and we write $\sigma(a)$ and $\rho(X)$ instead of $\sigma(a|t,\theta)$ and $\rho(X|t,\theta)$, respectively, ignoring (t,θ) to simplify notation. Let $\nu : 2^A \to [0,1]$ be such that

$$\nu(B)=\sum_{X\subseteq B,X\in\mathcal{A}}\rho(X)$$

for each $B \in 2^A$. Then, ν can be interpreted as a totally monotone transferable utility game (with a fictitious set of players A), where the dividend of a coalition $X \in 2^A$ is $\rho(X)$ if $X \in \mathcal{A}$ and zero otherwise. Moreover, (32) is rewritten as

$$\sum_{a\in B}\sigma(a)\geq\nu(B),$$

which implies that σ is in the core of ν. It is well known that σ is in the core if and only if, for each $X \in 2^A$ with a positive dividend (i.e. $X \in \mathcal{A}$), there exists $\mu_X \in \Delta(X)$ such that

$$\sigma(a)=\sum_{X\in\mathcal{A}}\mu_X(a)\rho(X).$$

Because the above discussion is valid for each fixed $(t,\theta)\in T\times\Theta$, there exists $\mu_X(\cdot|t,\theta)\in\Delta(X)$ for each $X \in \mathcal{A}$ and $(t,\theta)\in T\times\Theta$ such that

$$\sigma(a|t,\theta)=\sum_{X\in\mathcal{A}}\mu_X(a|t,\theta)\rho(X|t,\theta).$$

Then, a decision-communication rule γ given by $\gamma(a,X|t,\theta)=\mu_X(a|t,\theta)\rho(X|t,\theta)$ satisfies the condition in the second part of the proposition. □

We are ready to prove Proposition 4.

Proof of Proposition 4. It is enough to show that $\mathcal{E}^\nu$ is the set of all GP-maximizing BIBCE, where the generalized potential function F is given by Lemma 8. Note that

$$\mathcal{E}^\nu=\{\sigma\in\Sigma^{BI}\mid\sigma\text{ is a BIBCE such that, for each increasing subset }B\subseteq A\text{ and }(t,\theta)\in T\times\Theta,$$
$$\sum_{a\in B}\sigma(a|t,\theta)\geq\sum_{a\in B}\sigma'(a|t,\theta)\text{ for some }\sigma'\in\Sigma^\nu\}.$$

Let Γ^v be the set of belief-invariant decision-communication rules that maximize the expected value of F:

$$\Gamma^v \equiv \arg\max_{\gamma\in\Gamma^{BI}} \sum_{X,t,\theta} \gamma(X|t,\theta)\pi(t,\theta)F(X,\theta) = \arg\max_{\gamma\in\Gamma^{BI}} \sum_{a,t,\theta} \gamma(\Lambda(a)|t,\theta)\pi(t,\theta)v(a,\theta), \tag{33}$$

where the last equality holds by Lemma 8.

Suppose that $\sigma \in \Sigma^{BI}$ is a GP-maximizing BIBCE. Then, there exists a BIBCE-C $\gamma \in \Gamma^v$ such that $\sigma(a|t,\theta) = \gamma(a|t,\theta)$ for all $(a,t,\theta) \in A\times T\times\Theta$. Let $\sigma' \in \Sigma^{BI}$ be such that $\sigma'(a|t,\theta) = \gamma(\Lambda(a)|t,\theta)$ for all $(a,t,\theta) \in A\times T\times\Theta$. Then, $\sigma' \in \Sigma^v$ by (33), and

$$\sum_{a\in B} \sigma(a|t,\theta) = \sum_{a\in B} \gamma(a|t,\theta) \geq \sum_{X\subseteq B} \gamma(X|t,\theta) = \sum_{a\in B} \gamma(\Lambda(a)|t,\theta) = \sum_{a\in B} \sigma'(a|t,\theta) \tag{34}$$

for all increasing subset $B \subseteq A$ by (31). Therefore, $\sigma \in \mathcal{E}^v$ holds.

Suppose that $\sigma \in \mathcal{E}^v$. Then, there exists $\sigma' \in \Sigma^v$ satisfying (34) for all increasing subset $B \subseteq A$. Let $\rho : T\times\Theta \to \Delta(\mathcal{A})$ be such that $\rho(\Lambda(a)|t,\theta) = \sigma'(a|t,\theta)$ for all $(a,t,\theta) \in A\times T\times\Theta$. Then, for each increasing subset $B \subseteq A$,

$$\sum_{a\in B} \sigma(a|t,\theta) \geq \sum_{a\in B} \sigma'(a|t,\theta) = \sum_{a\in B} \rho(\Lambda(a)|t,\theta) = \sum_{X\subseteq B} \rho(X|t,\theta). \tag{35}$$

In addition, (35) holds for all $B \subseteq A$ because $B' \equiv \bigcup_{X\subset B, X\in\mathcal{A}} X \subseteq B$ is an increasing subset and

$$\sum_{a\in B} \sigma(a|t,\theta) \geq \sum_{a\in B'} \sigma(a|t,\theta) \geq \sum_{X\subseteq B'} \rho(X|t,\theta) = \sum_{X\subseteq B} \rho(X|t,\theta). \tag{36}$$

Thus, by Lemma 9, there exists an $\mathcal{A}$-consistent decision-communication rule γ such that $\gamma(a|t,\theta) = \sigma(a|t,\theta)$ and $\gamma(X|t,\theta) = \rho(X|t,\theta)$ for all $a \in$A, $X \in \mathcal{A}$, and $(t,\theta) \in T\times\Theta$. Note that $\gamma \in \Gamma^v$ because $\sigma' \in \Sigma^v$, and that γ is a BIBCE-C because σ is a BIBCE. Therefore, $\sigma \in \mathcal{E}^v$ is a GP-maximizing BIBCE.

By the above discussion, we conclude that $\mathcal{E}^v$ coincides with the set of all GP-maximizing BIBCE, which is a robust set of BIBCE by Theorem 1.

Finally, assume that v is supermodular. Let $\sigma^\circ \in \Sigma^v$ be such that, for each $t_i \in T_i$ and $i \in I$, $\sigma_i^\circ(0|t_i) = \sigma^\circ(\{0\}\times A_{-i}|t,\theta) > 0$ if $\sigma_i'(0|t_i) > 0$ for some $\sigma' \in \Sigma^v$. Note that σ^0 exists because Σ^v is a convex set and we can construct σ° in terms of a convex combination of elements in $\{\sigma' \in \Sigma^v \mid \sigma_i'(0|t_i) > 0 \text{ for some } t_i \in T_i \text{ and } i \in I\}$. Then, the strategy profile $\underline{\sigma}^\circ = (\underline{\sigma}_i^\circ)_{i\in I}$ given by $\underline{\sigma}_i^\circ(\min\{a_i \mid \sigma_i^\circ(a_i|t_i) > 0\}|t_i) = 1$ for each $t_i \in T_i$ and $i \in I$ is an element of Σ^v by Proposition 3. Moreover, it can be readily shown that, for each $\sigma \in \Sigma^v$ and $(t,\theta) \in T \times \Theta$, $\sigma(\cdot|t,\theta) \in \Delta(A)$ first-order stochastically dominates $\underline{\sigma}^\circ(\cdot|t,\theta) \in \Delta(A)$. Thus,

$$\begin{aligned}\mathcal{E}^v = \{\sigma \in \Sigma^{BI} \mid \sigma &\text{ is a BIBCE such that, for each } (t,\theta) \in T \times \Theta,\ \sigma(\cdot|t,\theta) \in \Delta(A) \text{ first-order} \\ &\text{stochastically dominates } \underline{\sigma}^\circ(\cdot|t,\theta) \in \Delta(A)\},\end{aligned}$$

which establishes the last claim of the proposition. □

5.3 The generalized critical path theorem

Using the result in the previous subsection on binary-action supermodular games, we generalize the critical path theorem of Kajii and Morris (1997) and Oyama and Takahashi (2020), which evaluates the probability of a common belief event. Kajii and Morris (1997) consider a common $\mathbf{p}$-belief event, where $\mathbf{p} = (p_i)_{i\in I}$ is a vector of constant probabilities. On the other hand, Oyama and Takahashi (2020) adopt a generalized belief operator (Morris and Shin, 2007; Morris et al., 2016) and consider a common $\mathbf{f}$-belief event, where $\mathbf{f} = (f_i)_{i\in I}$ is a vector of real-valued functions defined on the set of all coalitions consisting of the opponents. In contrast, we allow each f_i to depend upon a state as well; that is, we introduce a more general notion of a common belief event.

Let (T,Θ,π,u) be a binary-action supermodular incomplete information game with a monotone potential function $v : A \times \Theta \to \mathbb{R}$ such that $\Sigma^v = \{\sigma^*\}$ with $\sigma^*(\mathbf{1}|t,\theta) = 1$ for all $(t,\theta) \in T \times \Theta$. Note that σ^* is a robust BIBCE by Proposition 4. As demonstrated below, this result implies the generalized critical path theorem.

We write $\mathcal{I} = 2^I$, $\mathcal{I}_{-i} = 2^{I\setminus\{i\}}$, $\mathcal{T}_i = 2^{T_i}$, $\mathcal{T} = \{E = \prod_{i\in I} E_i \mid E_i \in \mathcal{T}_i\} \subset 2^T$, and $\mathcal{T}_{-i} = \{E_{-i} = \prod_{j\neq i} E_j \mid E_j \in \mathcal{T}_j\} \subset 2^{T_{-i}}$. Each $E = \prod_{i\in I} E_i \in \mathcal{T}$ is associated with the strategy profile $\sigma = (\sigma_i)_{i\in I}$ given by $\sigma_i(1|t_i) = 1$ if $t_i \in E_i$ and $\sigma_i(0|t_i) = 1$ if $t_i \notin E_i$.

For each $i \in I$, we define the "payoff increment" function $f_i : \mathcal{I}_{-i} \times \Theta \to \mathbb{R}$ by

$$f_i(S,\theta) = u_i(\mathbf{1}_{S\cup\{i\}},\theta) - u_i(\mathbf{1}_S,\theta)$$

for $(S,\theta) \in \mathcal{I}_{-i} \times \Theta$. That is, $f_i(S,\theta)$ is the payoff increment for player i by switching his action from 0 to 1 when the set of the opponents playing action 1 is S and the state is θ. Because (T,Θ,π,u) is a supermodular incomplete information game, $f_i(S,\theta) \le f_i(S',\theta)$ whenever $S \subseteq S'$.

We introduce f_i-belief. When Θ is a singleton, it is the same as f_i-belief introduced by Morris and Shin (2007) and Morris et al. (2016).

For $i \in I$ and $E_{-i} = \prod_{j \neq i} E_j \in \mathcal{T}_{-i}$, define the function $S_{E_{-i}} : T_{-i} \to \mathcal{I}_{-i}$ by

$$S_{E_{-i}}(t_{-i}) = \{j \neq i \mid t_j \in E_j\}.$$

The conditional expected value of $f_i(S_{E_{-i}}(t_{-i}), \theta)$ given t_i,

$$\mathrm{E}[f_i(S_{E_{-i}}(t_{-i}), \theta)|t_i] = \sum_{t_{-i},\theta} \pi(t_{-i}, \theta|t_i) f_i(S_{E_{-i}}(t_{-i}), \theta),$$

is the expected payoff increment for player i of type t_i when each $j \neq i$ chooses action 1 if $t_j \in E_j$ and action 0 if $t_j \notin E_j$.

Definition 5. For $i \in I$, $t_i \in T_i$, and $E = \prod_j E_j \in \mathcal{T}$, type t_i is said to have f_i-belief about E if $t_i \in E_i$ and $\mathrm{E}[f_i(S_{E_{-i}}(t_{-i}), \theta)|t_i] \geq 0$. Player i's f_i-belief operator $B_i^{f_i} : \mathcal{T} \to \mathcal{T}_i$ is defined by

$$B_i^{f_i}(E) = \{t_i \in E_i \mid \mathrm{E}[f_i(S_{E_{-i}}(t_{-i}), \theta)|t_i] \geq 0\}$$

for each $E = \prod_j E_j \in \mathcal{T}$; that is, $B_i^{f_i}(E)$ is the set of player i's types that have f_i-belief about E.

For example, let

$$f_i(S, \theta) = \begin{cases} 1 - p_i(\theta) & \text{if } S = I \setminus \{i\}, \\ -p_i(\theta) & \text{otherwise} \end{cases}$$

for $(S, \theta) \in \mathcal{I}_{-i} \times \Theta$, where $p_i(\theta) \in [0, 1]$ for each $\theta \in \Theta$. Then,

$$\mathrm{E}[f_i(S_{E_{-i}}(t_{-i}), \theta)|t_i] = \sum_{t_{-i} \in E_{-i}, \theta \in \Theta} \pi(t_{-i}, \theta|t_i)(1 - p_i(\theta)) - \sum_{t_{-i} \notin E_{-i}, \theta \in \Theta} \pi(t_{-i}, \theta|t_i) p_i(\theta) = \pi(E_{-i}|t_i) - \mathrm{E}[p_i(\theta)|t_i].$$

Thus,

$$B_i^{f_i}(E) = \{t_i \in E_i \mid \pi(E_{-i}|t_i) \geq \mathrm{E}[p_i(\theta)|t_i]\}.$$

If $p_i(\theta) = p_i$ is a constant independent of θ, then $B_i^{f_i}$ is reduced to player i's p_i-belief operator.

If $F = \prod_j F_j \in \mathcal{T}$ satisfies $F_i \subset B_i^{f_i}(F)$ for each $i \in I$, we say that F is $\mathbf{f}$-evident, where $\mathbf{f} = (f_i)_{i \in I}$. For an $\mathbf{f}$-evident event F, let $\sigma = (\sigma_i)_{i \in I}$ be the associated strategy profile; that is, $\sigma_i(1|t_i) = 1$ if $t_i \in F_i$ and $\sigma_i(0|t_i) = 1$ if $t_i \notin F_i$ for each $i \in I$. Then, it is clear that, for each $t_i \in F_i$ and $i \in I$, action 1 is a best response to σ_{-i}.

A typical **f**-evident event is given by a common **f**-belief operator, which is defined as follows. For a payoff increment function profile $\mathbf{f} = (f_i)_{i\in I}$ and $E = \prod_i E_i \in \mathcal{T}$, let

$$B_i^{\mathbf{f},0}(E) = E_i,$$
$$B_i^{\mathbf{f},n+1}(E) = B_i^{f_i}\left(\prod_j B_j^{\mathbf{f},n}(E)\right) \text{ for } n \geq 0,$$
$$CB_i^{\mathbf{f}}(E) = \bigcap_{n=0}^{\infty} B_i^{\mathbf{f},n}(E).$$

We say that $t_i \in T_i$ has common **f**-belief about $E \in \mathcal{T}$ if $t_i \in CB_i^{\mathbf{f}}(E)$. We write $CB^{\mathbf{f}}(E) = \prod_j CB_j^{\mathbf{f}}(E)$ and call $CB^{\mathbf{f}} : \mathcal{T} \to \mathcal{T}$ a common **f**-belief operator. The following result is a straightforward generalization of the corresponding result for common **p**-belief and common **f**-belief when Θ is a singleton.

Proposition 5. *For each $E \in \mathcal{T}$, $CB^{\mathbf{f}}(E)$ is the largest* **f***-evident event contained in E.*

We are ready to state the generalized critical path theorem, which is implied by the robustness of GP-maximizing BIBCE in Proposition 4.

Proposition 6. *Let (T, Θ, π, u) be a binary-action supermodular game with a monotone potential function $v : A \times \Theta \to \mathbb{R}$ and a payoff increment function profile $\mathbf{f} = (f_i)_{i\in I}$. Assume that*

$$\Sigma^v = \{\sigma^*\}, \tag{37}$$

where $\sigma^*(\mathbf{1}|t, \theta) = 1$ for all $(t, \theta) \in T \times \Theta$. Then, for any $\delta > 0$, there exists $\varepsilon > 0$ such that, for every $E = \prod_i E_i \in \mathcal{T}$ with $\pi(E) \geq 1 - \varepsilon$, it holds that

$$\pi(CB^{\mathbf{f}}(E)) \geq 1 - \delta. \tag{38}$$

Proof. See Appendix C. □

In the following special case, we can obtain ε in a closed-form expression.

Proposition 7. *Let (T, Θ, π, u) be a binary-action supermodular game with a monotone potential function $v : A\times\Theta \to \mathbb{R}$ and a payoff increment function profile $\mathbf{f} = (f_i)_{i\in I}$. For each $E = \prod_i E_i \in \mathcal{T}$, let*

$$M(v, \pi, E) \equiv \inf_{t\in E\cap T^*, S\neq I} \sum_\theta \pi(\theta|t)(v(\mathbf{1}, \theta) - v(\mathbf{1}_S, \theta)),$$

$$M^*(v,\pi) \equiv M(v,\pi,T^*) = \inf_{t\in T^*, S\neq I} \sum_{\theta} \pi(\theta|t)(v(\mathbf{1},\theta) - v(\mathbf{1}_S,\theta)),$$

$$M^{**}(v) \equiv \inf_{S\neq I, \theta\in\Theta} v(\mathbf{1},\theta) - v(\mathbf{1}_S,\theta).$$

Then, the following holds.

1. *If* $M^*(v,\pi) > 0$, *then* (37) *holds. Moreover, for any* $\delta > 0$ *and* $E = \prod_i E_i \in \mathcal{T}$ *with* $\pi(E) \geq 1 - \delta/\kappa^*(v,\pi)$, (38) *holds, where*

$$\kappa^*(v,\pi) \equiv 1 + \frac{\sup_{S\subseteq S'\neq I, \theta\in\Theta} v(\mathbf{1}_S,\theta) - v(\mathbf{1}_{S'},\theta)}{M^*(v,\pi)} > 0.$$

   That is,

$$\pi(CB^{\mathbf{f}}(E)) \geq 1 - \kappa^*(v,\pi)(1-\pi(E)). \tag{39}$$

   In particular, if $M^{**}(v) > 0$, *then*

$$\pi(CB^{\mathbf{f}}(E)) \geq 1 - \kappa^{**}(v)(1-\pi(E)), \tag{40}$$

   where

$$\kappa^{**}(v) \equiv 1 + \frac{\sup_{S\subseteq S'\neq I, \theta\in\Theta} v(\mathbf{1}_S,\theta) - v(\mathbf{1}_{S'},\theta)}{M^{**}(v)} > 0.$$

2. *For any* $E \in \mathcal{T}$ *with* $M(v,\pi,E) > 0$,

$$\pi(CB^{\mathbf{f}}(E)) \geq 1 - \kappa(v,\pi,E)(1-\pi(E)), \tag{41}$$

   where

$$\kappa(v,\pi,E) \equiv 1 + \frac{\sup_{S\subseteq S'\neq I, \theta\in\Theta} v(\mathbf{1}_S,\theta) - v(\mathbf{1}_{S'},\theta)}{M(v,\pi,E)} > 0.$$

Proof. See Appendix C. □

The second part of Proposition 7, which is the critical path theorem for an event E satisfying $M(v,\pi,E) > 0$, is valid even if (37) does not hold. The first part of Proposition 7 is reduced to the critical path theorem of Oyama and Takahashi (2020) when Θ is a singleton, and it is reduced to the critical path theorem of Kajii and Morris (1997) when f_i-belief is p_i-belief.

Appendix

A Proof of Lemma 7

We give the proof of Lemma 7. Let $T_i^k \equiv \{t_i \in T_i \mid \pi^k(\Theta_i^* \times \Theta_{-i}|t_i) = 1\}$ and $S^k(t) \equiv \{i \in I \mid t_i \in T_i^k\}$, which are the same as those in the sketch of the proof in Section 4.3. We also use the following notation.

- For each $\gamma \in \Gamma^{BI}$, we write $\gamma_S(a_s, X_S|t,\theta) \equiv \gamma(\{a_S\} \times A_{-S} \times \{X_S\} \times \mathcal{A}_{-S}|t,\theta)$ for the conditional distribution of $(a_S, X_S) \in A_S \times \mathcal{A}_S$ given $(t,\theta) \in T \times \Theta$. In particular, when $S = \{i\}$, we simply write $\gamma_i(a_i, X_i|t_i) \equiv \gamma_{\{i\}}(a_i, X_i|t,\theta)$ because γ is belief-invariant. We also write $\gamma_S(a_S|t,\theta) = \gamma_S(\{a_S\} \times \mathcal{A}_S|t,\theta)$ and $\gamma_S(X_S|t,\theta) = \gamma_S(A_S \times \{X_S\}|t,\theta)$ for the marginal distributions. Let

$$\Gamma^k \equiv \{\gamma \in \Gamma^{BI} \mid \gamma(a, X|t,\theta) = \gamma_{S^k(t)}(a_{S^k(t)}, X_{S^k(t)}|t,\theta) \prod_{i \notin S^k(t)} \gamma_i(a_i, X_i|t_i)\}$$

  denote the collection of belief-invariant decision-communication rules in which types not in T_i^k receive independent signals and choose independent actions. Note that Γ^k is convex with respect to the following convex combination: for $\gamma, \gamma' \in \Gamma^k$, $\gamma'' = \lambda\gamma + (1-\lambda)\gamma' \in \Gamma^k$ is given by

$$\gamma''_{S^k(t)}(\cdot|t,\theta) = \lambda\gamma_{S^k(t)}(\cdot|t,\theta) + (1-\lambda)\gamma'_{S^k(t)}(\cdot|t,\theta),$$
$$\prod_{i \notin S^k(t)} \gamma''_i(\cdot|t_i) = \prod_{i \notin S^k(t)} \Big(\lambda\gamma_i(\cdot|t_i) + (1-\lambda)\gamma'_i(\cdot|t_i)\Big).$$

- For each $\gamma \in \Gamma^k$, let

$$\Gamma^k[\gamma] = \{\gamma' \in \Gamma^k \mid \gamma'_i(a_i|t_i) = \gamma_i(a_i|t_i) \text{ for all } a_i \in A_i,\ t_i \notin T_i^k, \text{ and } i \in I\}.$$

  This is the collection of decision-communication rules in Γ^k under which player $i \in I$ of type $t_i \notin T_i^k$ chooses an action according to the same probability distribution as that under γ. Note that $\Gamma^k[\gamma]$ is a convex subset of Γ^k.

- Fix $\theta^* \in \Theta^*$. For each $\theta \in \Theta$, let $\phi(\theta) = (\phi_i(\theta))_{i\in I} \in \Theta^* = \prod_{i\in I} \Theta_i^*$ be such that $\phi_i(\theta) = \theta_i$ if $\theta_i \in \Theta_i^*$ and $\phi_i(\theta) = \theta_i^*$ if $\theta_i \notin \Theta_i^*$.

In the next lemma, we construct a candidate for $\{\sigma^k\}_{k=1}^\infty$ satisfying (20).

Lemma A. *An ε^k-elaboration (T, Θ, π^k, u) has is a BIBCE-C $\gamma^k \in \Gamma^k$ satisfying*

$$\gamma^k \in \arg\max_{\gamma \in \Gamma^k[\gamma^k]} \sum_{(X,t,\theta) \in \mathcal{A} \times T \times \Theta} \gamma(X|t,\theta)\pi^k(t,\theta)F(X,\phi(\theta)). \tag{A1}$$

Thus, there exists a BIBCE $\sigma^k \in \Sigma^{BI}$ given by $\sigma^k(a|t,\theta) = \gamma^k(a|t,\theta)$.

Proof. We first characterize γ^k as a fixed point of a correspondence on Γ^k, which is composed of two correspondences. Then, we establish the existence of a fixed point. The proof consists of the following four steps.

Step 1: We define the first correspondence $\Psi^1 : \Gamma^k \rightrightarrows \Gamma^k$:

$$\begin{aligned}\Psi^1(\gamma) = \{\gamma' \in \Gamma^k \mid & \text{ for each } t_i \notin T_i^k,\ \gamma_i'(a_i|t_i) > 0 \text{ implies} \\ & a_i \in \arg\max_{a_i' \in A_i} \sum_{a_{-i}, t_{-i}, \theta} \gamma_{N\setminus\{i\}}(a_{-i}|t,\theta)\pi^k(t,\theta)u_i((a_i', a_{-i}),\theta), \\ & \gamma'(a, X|t,\theta) = \gamma(a_{S^k(t)}, X_{S^k(t)}|t,\theta) \prod_{i \notin S^k(t)} \gamma_i'(a_i, X_i|t_i) \text{ for all } (a,t,\theta)\}.\end{aligned}$$

Under $\gamma' \in \Psi^1(\gamma)$, each type $t_i \notin T_i^k$ chooses a best response to γ, whereas each type $t_i \in T_i^k$ follows γ. Note that $\Psi^1(\gamma)$ is a nonempty convex subset of Γ^k.

Step 2: To define the second correspondence, we use the following collection of decision-communication rules: for $\gamma \in \Gamma^k$,

$$\begin{aligned}\Gamma^{k,F}[\gamma] &\equiv \arg\max_{\gamma' \in \Gamma^k[\gamma]} \sum_{X,t,\theta} \gamma'(X|t,\theta)\pi^k(t,\theta)F(X,\phi(\theta)) \\ &= \arg\max_{\gamma' \in \Gamma^k[\gamma]} \sum_{X,t,\theta} \gamma' \circ \pi^k(X,t,\theta)F(X,\phi(\theta)),\end{aligned}$$

which is nonempty because $\{\gamma' \circ \pi^k \in \Delta(A \times \mathcal{A} \times T \times \Theta) \mid \gamma' \in \Gamma^k[\gamma]\}$ is tight and closed, and thus it is compact by Lemma 6. Note that (A1) is written as $\gamma^k \in \Gamma^{k,F}[\gamma^k]$. The second correspondence $\Psi^2 : \Gamma^k \rightrightarrows \Gamma^k$ is given by

$$\Psi^2(\gamma) = \{\gamma' \in \Gamma^{k,F}[\gamma] \ \mid \text{for each } i \in I \text{ and } (t_i, X_i) \in T_i^k \times \mathcal{A}_i,\ \gamma' \text{ is obedient for } i \text{ with } (t_i, X_i)\}.$$

Under $\gamma' \in \Psi^2(\gamma)$, each type $t_i \notin T_i^k$ adopts the same mixed action as that given by γ, whereas each type $t_i \in T_i^k$ simultaneously chooses a best response to each other; that is, given the mixed actions of types not in T_i^k, types in T_i^k behave as if they follow a BIBCE-C.

Note that $\Psi^2(\gamma)$ is a convex subset of Γ^k. We show that $\Psi^2(\gamma)$ is nonempty. Fix $\gamma^0 \in \Gamma^{k,F}[\gamma]$ and let $\Gamma^0 = \{\gamma' \in \Gamma^{k,F}[\gamma] \mid \gamma'(X|t,\theta) = \gamma^0(X|t,\theta)$ for all $(X,t,\theta) \in \mathcal{A} \times T \times \Theta\}$. Fix $i \in I$, $t_i \in T_i^k$, and $X_i \in \mathcal{A}_i$ with $\sum_{X_{-i},t_{-i},\theta} \gamma^0(X|t,\theta)\pi^k(t,\theta) > 0$. Then, for arbitrary $\gamma' \in \Gamma^0$, it holds that

$$
\begin{aligned}
X_i &\in \arg\max_{X_i' \in \mathcal{A}_i} \sum_{X_{-i},t_{-i},\theta} \gamma^0(X|t,\theta)\pi^k(t,\theta)F((X_i', X_{-i}), \phi(\theta)) \\
&= \arg\max_{X_i' \in \mathcal{A}_i} \sum_{X_{-i},t_{-i},\theta} \gamma'(X|t,\theta)\pi^k(t,\theta)F((X_i', X_{-i}), \phi(\theta)) \qquad \text{(A2)}
\end{aligned}
$$

because $t_i \in T_i^k$ and $\gamma^0 \in \Gamma^{k,F}[\gamma]$. Define $P_i \in \Delta(A_{-i} \times \mathcal{A}_{-i} \times \Theta^*)$ by

$$
P_i(a_{-i}, X_{-i}, \theta) = \sum_{a_i \in A_i,\, t_{-i} \in T_{-i},\, \theta' \in \phi^{-1}(\theta)} \gamma'(a, X|t, \theta')\pi^k(t, \theta')/Z,
$$

where $Z = \sum_{a, X_{-i}, t_{-i}, \theta' \in \phi^{-1}(\theta)} \gamma'(a, X|t, \theta')\pi^k(t, \theta')$. Note that $P_i(a_{-i}, X_{-i}, \theta) = 0$ if $a_{-i} \notin X_{-i}$. Then,

$$
\begin{aligned}
\sum_{X_{-i},t_{-i},\theta} \gamma'(X|t,\theta)\pi^k(t,\theta)F((X_i', X_{-i}), \phi(\theta)) &= \sum_{X_{-i},t_{-i},\theta} \left(\sum_{\theta' \in \phi^{-1}(\theta)} \gamma'(X|t,\theta')\pi^k(t,\theta') \right) F((X_i', X_{-i}), \theta) \\
&= Z \sum_{(X_{-i},\theta) \in \mathcal{A}_{-i} \times \Theta^*} P_i(X_{-i}, \theta) F((X_i', X_{-i}), \theta), \qquad \text{(A3)}
\end{aligned}
$$

where $P_i(X_{-i}, \theta) = \sum_{a_{-i} \in A_{-i}} P_i(a_{-i}, X_{-i}, \theta)$, and thus (A2) is rewritten as

$$
X_i \in \arg\max_{X_i' \in \mathcal{A}_i} \sum_{(X_{-i},\theta) \in \mathcal{A}_{-i} \times \Theta^*} P_i(X_{-i}, \theta) F((X_i', X_{-i}), \theta),
$$

which implies that

$$
X_i \cap \arg\max_{a_i' \in A_i} \sum_{(a_{-i},\theta) \in A_{-i} \times \Theta^*} P_i(a_{-i}, \theta) u_i((a_i', a_{-i}), \theta) \neq \emptyset \qquad \text{(A4)}
$$

by Definition 4, where $P_i(a_{-i}, \theta) = \sum_{X_{-i} \in \mathcal{A}_{-i}} P_i(a_{-i}, X_{-i}, \theta)$. Because $t_i \in T_i^k$ and $u_i(a,\theta) = u_i(a,\theta')$ whenever $\theta_i = \theta_i' \in \Theta_i^*$, if $\pi^k(t,\theta) > 0$, then $\theta_i \in \Theta_i^*$ and $u_i(a,\theta) = u_i(a,\phi(\theta))$. This implies that

$$
\begin{aligned}
&Z \sum_{(a_{-i},\theta) \in A_{-i} \times \Theta^*} P_i(a_{-i}, \theta) u_i((a_i', a_{-i}), \theta) \\
&= \sum_{(a_{-i},\theta) \in A_{-i} \times \Theta^*} \left(\sum_{a_i \in A_i,\, X_{-i} \in \mathcal{A}_{-i},\, t_{-i} \in T_{-i},\, \theta' \in \phi^{-1}(\theta)} \gamma'(a, X|t, \theta')\pi^k(t, \theta') \right) u_i((a_i', a_{-i}), \theta) \\
&= \sum_{(a_{-i},\theta) \in A_{-i} \times \Theta^*} \left(\sum_{a_i \in A_i,\, X_{-i} \in \mathcal{A}_{-i},\, t_{-i} \in T_{-i},\, \theta' \in \phi^{-1}(\theta)} \gamma'(a, X|t, \theta')\pi^k(t, \theta') u_i((a_i', a_{-i}), \theta') \right) \\
&= \sum_{a \in A,\, X_{-i} \in \mathcal{A}_{-i}\, t_{-i} \in T_{-i},\, \theta \in \Theta} \gamma'(a, X|t, \theta)\pi^k(t, \theta') u_i((a_i', a_{-i}), \theta).
\end{aligned}
$$

Thus, by (A4),

$$X_i \cap \arg\max_{a_i' \in A_i} \sum_{a, X_{-i}, t_{-i}, \theta} \gamma'(a, X|t, \theta)\pi^k(t, \theta)u_i((a_i', a_{-i}), \theta) \neq \emptyset. \tag{A5}$$

To summarize, we have shown that if $t_i \in T_i^k$ and $X_i \in \mathcal{A}_i$ with $\sum_{X_{-i}, t_{-i}, \theta} \gamma^0(X|t, \theta)\pi^k(t, \theta) > 0$, then (A5) holds for arbitrary $\gamma' \in \Gamma^0$.

Now let $\rho : T \times \Theta \to \Delta(\mathcal{A})$ be a communication rule with $\rho(X|t, \theta) = \gamma^0(X|t, \theta)$. Consider the conjunction of (T, Θ, π, u) and ρ and let $\Sigma_i^{\mathcal{A}}[\gamma^0]$ be the set of player i's strategies in the conjunction that always assign some $a_i \in X_i$ whenever player i receives a signal $X_i \in \mathcal{A}_i$ and player i of type $t_i \notin T_i^k$ follows γ^0:

$$\Sigma_i^{\mathcal{A}}[\gamma^0] = \{\sigma_i : T_i \times \mathcal{A}_i \to \Delta(A_i) \mid \sigma_i(a_i|t_i, X_i) = 0 \text{ whenever } a_i \notin X_i,$$
$$\sigma_i(a_i|t_i, X_i)\gamma_i^0(X_i|t_i) = \gamma_i^0(a_i, X_i|t_i) \text{ for all } t_i \notin T_i^k\}.$$

Note that any decision-communication rule induced by ρ and $\sigma \in \Sigma^{\mathcal{A}}[\gamma^0]$ (which is given by $\gamma(a, X|t, \theta) = \prod_i \sigma_i(a_i|t_i, X_i)\rho(X|t, \theta)$) is an element of Γ^0. Thus, the correspondence $\Phi : \Sigma^{\mathcal{A}}[\gamma^0] \rightrightarrows \Sigma^{\mathcal{A}}[\gamma^0]$ given by

$$\Phi(\sigma) = \{\sigma' \in \Sigma^{\mathcal{A}}[\gamma^0] \mid \text{for each } i \in I \text{ and } (t_i, X_i) \in T_i^k \times \mathcal{A}_i, (\sigma_i', \sigma_{-i}) \text{ is obedient for } (t_i, X_i)\}$$

is nonempty-valued by (A5) because, for any $\sigma \in \Sigma^{\mathcal{A}}[\gamma^0]$, $i \in I$, $t_i \in T_i^k$, and $X_i \in \mathcal{A}_i$ with $\sum_{X_{-i}, t_{-i}, \theta} \rho(X|t, \theta)\pi^k(t, \theta) > 0$, it holds that

$$X_i \cap \arg\max_{a_i' \in A_i} \sum_{a_{-i}, X_{-i}, t_{-i}, \theta} \left(\prod_{j \neq i} \sigma_j(a_j|t_j, X_j) \right) \rho(X|t, \theta)\pi^k(t, \theta)u_i((a_i', a_{-i}), \theta) \neq \emptyset.$$

Moreover, it is readily shown that Φ is convex-valued and has a closed graph. Therefore, Φ has a fixed point $\sigma^* \in \Sigma^{\mathcal{A}}[\gamma^0]$ by the Kakutani-Fan-Glicksberg fixed point theorem. Then, the decision-communication rule induced by σ^* and ρ is obedient for each $i \in I$ and $t_i \in T_i^k$, so it is an element of $\Psi^2(\gamma)$.

Step 3: In this step, we construct another correspondence Ψ using Ψ^1 and Ψ^2 and show that its fixed point is a BIBCE-C satisfying (A1). Let a correspondence $\Psi : \Gamma^k \rightrightarrows \Gamma^k$ be given by

$$\Psi(\gamma) = \{\gamma' \in \Gamma^k \mid \gamma'(a, X|t, \theta) = \gamma^2_{S^k(t)}(a_{S^k(t)}, X_{S^k(t)}|t, \theta) \prod_{i \notin S^k(t)} \gamma_i^1(a_i, X_i|t_i) \text{ for all } (a, X, t, \theta),$$
$$\text{where } \gamma^1 \in \Psi^1(\gamma) \text{ and } \gamma^2 \in \Psi^2(\gamma)\}.$$

If γ is a fixed point, i.e., $\gamma \in \Psi(\gamma)$, then $\gamma \in \Psi^1(\gamma)$ and $\gamma \in \Psi^2(\gamma)$. Thus, γ satisfies (A1) by $\gamma \in \Psi^2(\gamma)$. Moreover, γ is obedient for player i of type $t_i \notin T_i^k$ because $\gamma \in \Psi^1(\gamma)$, and it is obedient for player i of type $t_i \in T_i^k$ because $\gamma \in \Psi^2(\gamma)$; that is, γ is a BIBCE-C. Thus, to prove Lemma A, it is enough to show that Ψ has a fixed point.

Step 4: We show that Ψ has a fixed point. Consider

$$\Gamma^k \circ \pi^k \equiv \{\gamma \circ \pi^k \in \Delta(A \times \mathcal{A} \times T \times \Theta) \mid \gamma \circ \pi^k(a, X, t, \theta) = \gamma(a, X|t, \theta)\pi^k(t, \theta),\ \gamma \in \Gamma^k\}$$

and identify Γ^k with $\Gamma^k \circ \pi^k$, i.e., we regard Γ^k as the set of equivalence classes induced by each $\gamma \circ \pi^k \in \Gamma^k \circ \pi^k$, where γ and γ' are equivalent if $\gamma \circ \pi^k = \gamma' \circ \pi^k$. Note that $\Gamma^k \circ \pi^k$ is a tight closed subset of $\Delta(A \times \mathcal{A} \times T \times \Theta)$. Thus, by Lemma 6, it is a compact convex subset of a topological vector space. This implies that we can regard Ψ as a correspondence defined on such a set, which possesses the following properties.

- $\Psi(\gamma)$ is nonempty because $\Psi^1(\gamma)$ and $\Psi^2(\gamma)$ are nonempty.
- $\Psi(\gamma)$ is convex because $\Psi^1(\gamma)$ and $\Psi^2(\gamma)$ are convex.
- Because expected payoffs are continuous with respect to decision-communication rules, $\Psi^1(\gamma)$ and $\Psi^2(\gamma)$ are easily shown to be closed subsets of a compact set Γ^k. Thus, $\Psi(\gamma)$ is compact.
- We can establish the closed graph property of Ψ because each of Ψ^1 and Ψ^2 has a closed graph.
  - Because Ψ^1 is a best response correspondence for types not in T^k, we can readily show that Ψ^1 has a closed graph by the maximum theorem.
  - We show that Ψ^2 has a closed graph. Let $\{\gamma^l\}_{l=1}^{\infty}, \{\xi^l\}_{l=1}^{\infty} \subset \Gamma^k$ be convergent sequences such that $\xi^l \in \Psi^2(\gamma^l)$ for each l, $\lim_{l\to\infty} \gamma^l = \gamma^*$, and $\lim_{l\to\infty} \xi^l = \xi^*$. Note that $\gamma^*, \xi^* \in \Gamma^k$ since Γ^k is compact. Because expected payoffs are continuous with respect to decision-communication rules and the correspondence $\gamma \mapsto \Gamma^k[\gamma]$ is continuous, we have $\xi^* \in \Gamma^{k,F}[\gamma^*]$ by the maximum theorem. Moreover, because ξ^l is obedient for payer i of type $t_i \in T_i^k$ for each l, ξ^* must be also obedient for player i of type $t_i \in T_i^k$. Therefore, we can conclude that $\xi^* \in \Psi^2(\gamma^*)$, which implies that Ψ^2 has a closed graph.

Then, by the Kakutani-Fan-Glicksberg fixed point theorem and the above argument, Ψ has a fixed point. □

We now prove Lemma 7 using Lemma A.

Proof of Lemma 7. Let γ^k and σ^k be the BIBCE-C and the BIBCE of (T, Θ, π^k, u) given in Lemma A, respectively. Let $\eta^k \in \Delta(A \times \mathcal{A} \times T \times \Theta)$ be given by $\eta^k(a, X, t, \theta) \equiv \gamma^k \circ \pi^k(a, X, \tau^{-1}(t), \theta) = \sum_{t' \in \tau^{-1}(t)} \gamma^k(a, X|t', \theta)\pi^k(t', \theta)$. Note that $\eta^k(t, \theta) \equiv \sum_{(a,X)\in A\times\mathcal{A}} \eta^k(a, X, t, \theta) = \pi^k(\tau^{-1}(t), \theta)$. We write

$$\eta^k(a, X|t, \theta) \equiv \eta^k(a, X, t, \theta)/\eta^k(t, \theta),\ \eta^k(a|t, \theta) \equiv \sum_{X\in\mathcal{A}} \eta^k(a, X|t, \theta),\ \eta^k(X|t, \theta) \equiv \sum_{a\in A} \eta^k(a, X|t, \theta)$$

for $(t, \theta) \in T \times \Theta$ with $\eta^k(t, \theta) > 0$. It can be readily shown that $\{\eta^k\}_{k=1}^{\infty}$ is tight, so it has a convergent subsequence by Lemma 6, which is denoted by $\{\eta^{k_l}\}_{l=1}^{\infty}$ with $\lim_{l\to\infty} \eta^{k_l} = \eta^*$. Note that, for each $(t, \theta) \in T^* \times \Theta^*$, $\eta^*(t, \theta) = \lim_{l\to\infty} \eta^{k_l}(t, \theta) = \lim_{l\to\infty} \pi^{k_l}(\tau^{-1}(t), \theta) = \pi(t, \theta)$. Thus, we have $\eta^*(a, X|t, \theta)\pi(t, \theta) = \eta^*(a, X, t, \theta)$.

To establish Lemma 7, it is enough to show that $\{\sigma^k\}_{k=1}^{\infty}$ satisfies (20), which is written as

$$\lim_{k\to\infty} \inf_{\sigma\in\mathcal{E}^F} \sum_{(a,t,\theta)\in A\times T\times\Theta} \left|\eta^k(a, t, \theta) - \sigma \circ \pi(a, t, \theta)\right| = 0, \tag{A6}$$

where $\eta^k(a, t, \theta) = \sum_{X\in\mathcal{A}} \eta^k(a, X, t, \theta)$. To prove (A6), it is enough to show that, for every convergent subsequence $\{\eta^{k_l}\}_{l=1}^{\infty}$, there exists $\sigma \in \mathcal{E}^F$ such that $\eta^*(a, t, \theta) = \sigma(a|t, \theta)\pi(t, \theta)$. In fact, if (A6) does not hold, then there exists a convergent subsequence $\{\eta^{k_l}\}_{l=1}$ such that

$$\begin{aligned}&\lim_{l\to\infty} \inf_{\sigma\in\mathcal{E}^F} \sum_{(a,t,\theta)\in A\times T\times\Theta} \left|\eta^{k_l}(a, t, \theta) - \sigma \circ \pi(a, t, \theta)\right| \\ &\qquad = \inf_{\sigma\in\mathcal{E}^F} \sum_{(a,t,\theta)\in A\times T\times\Theta} |\eta^*(a, t, \theta) - \sigma(a|t, \theta)\pi(t, \theta)| > 0,\end{aligned}$$

which implies that there is no $\sigma \in \mathcal{E}^F$ with $\eta^*(a, t, \theta) = \sigma(a|t, \theta)\pi(t, \theta)$. In the remainder of the proof, we denote a convergent subsequence by $\{\eta^k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} \eta^k = \eta^*$ rather than $\{\eta^{k_l}\}_{l=1}^{\infty}$.

First, we show that $\eta^*(\cdot|t, \theta)$ is belief-invariant as a decision-communication rule of $(T^*, \Theta^*, \pi^*, u^*)$. Fix $i \in I$ and $(t, \theta) \in T^* \times \Theta^*$, and we focus on sufficiently large k satisfying $\varepsilon^k < \pi^k(\tau_i^{-1}(t_i))$ (recall that $\lim_{k\to\infty} \pi^k(\tau_i^{-1}(t_i))) = \pi(t) > 0$). Define

$$\zeta_i^k(a_i, X_i|t_i) \equiv \sum_{t_i'\in\tau_i^{-1}(t_i)} \gamma_i^k(a_i, X_i|t_i')\pi^k(t_i')/\pi^k(\tau_i^{-1}(t_i)),$$

where $\gamma_i^k(a_i, X_i|t_i') \equiv \sum_{a_{-i},X_{-i}} \gamma^k(a, X|t', \theta)$, which is well-defined because γ^k is belief-invariant. Recall that

$$\eta^k(a, X|t, \theta) = \sum_{t' \in \tau^{-1}(t)} \gamma^k(a, X|t', \theta)\pi^k(t', \theta)/\pi^k(\tau^{-1}(t), \theta),$$

and thus

$$\eta^k(a_i, X_i|t, \theta) = \sum_{t_i' \in \tau_i^{-1}(t_i)} \gamma_i^k(a_i, X_i|t_i')\pi^k(\{t_i'\} \times \tau_{-i}^{-1}(t_{-i}), \theta)/\pi^k(\tau^{-1}(t), \theta).$$

To establish the belief-invariance of $\eta^*(\cdot|t, \theta)$, it is enough to show that

$$\begin{aligned}
&\lim_{k\to\infty}|\eta^k(a_i, X_i|t, \theta) - \zeta_i^k(a_i, X_i|t_i)| \\
&\quad = \lim_{k\to\infty}\left|\sum_{t_i' \in \tau_i^{-1}(t_i)} \gamma_i^k(a_i, X_i|t_i')\left(\frac{\pi^k(\{t_i'\} \times \tau_{-i}^{-1}(t_{-i}), \theta)}{\pi^k(\tau^{-1}(t), \theta)} - \frac{\pi^k(t_i')}{\pi^k(\tau_i^{-1}(t_i))}\right)\right| = 0 \qquad \text{(A7)}
\end{aligned}$$

because this implies that

$$\begin{aligned}
&|\eta^*(a_i, X_i|t, \theta) - \eta^*(a_i, X_i|(t_i, t_{-i}'), \theta')| \\
&= \lim_{k\to\infty} |\eta^k(a_i, X_i|t, \theta) - \eta^k(a_i, X_i|(t_i, t_{-i}'), \theta')| \\
&\leq \lim_{k\to\infty} |\eta^k(a_i, X_i|t, \theta) - \zeta_i^k(a_i, X_i|t_i)| + \lim_{k\to\infty} |\eta^k(a_i, X_i|(t_i, t_{-i}'), \theta') - \zeta_i^k(a_i, X_i|t_i)| = 0.
\end{aligned}$$

To show (A7), observe that

$$\begin{aligned}
\left|\frac{\pi^k(\{t_i'\} \times \tau_{-i}^{-1}(t_{-i}), \theta)}{\pi^k(\tau^{-1}(t), \theta)} - \frac{\pi^k(t_i')}{\pi^k(\tau_i^{-1}(t_i))}\right| &\leq \left|\frac{\pi^k(\{t_i'\} \times \tau_{-i}^{-1}(t_{-i}), \theta)}{\pi^k(\tau^{-1}(t), \theta)} - \frac{\pi^k(t_i')}{\pi(t_i)}\frac{\pi(t, \theta)}{\pi^k(\tau^{-1}(t), \theta)}\right| \\
&\quad + \left|\frac{\pi^k(t_i')}{\pi(t_i)}\frac{\pi(t, \theta)}{\pi^k(\tau^{-1}(t), \theta)} - \frac{\pi^k(t_i')}{\pi(t_i)}\right| + \left|\frac{\pi^k(t_i')}{\pi(t_i)} - \frac{\pi^k(t_i')}{\pi^k(\tau_i^{-1}(t_i))}\right|.
\end{aligned}$$

Using the definition of an ε-elaboration, we can evaluate each term in the right-hand side as follows: there exists $T_i^{\flat,k} \subset T_i$ with $\pi^k(T_i^{\flat,k}) \geq 1 - \varepsilon^k$ such that, for all $t_i' \in \tau_i^{-1}(t) \cap T_i^{\flat,k}$ (which is nonempty because $\pi^k(\tau_i^{-1}(t_i)) > \varepsilon^k$),

$$\begin{aligned}
\left|\frac{\pi^k(\{t_i'\} \times \tau_{-i}^{-1}(t_{-i}), \theta)}{\pi^k(\tau^{-1}(t), \theta)} - \frac{\pi^k(t_i')}{\pi(t_i)}\frac{\pi(t, \theta)}{\pi^k(\tau^{-1}(t), \theta)}\right| &= \frac{\pi^k(t_i')}{\pi^k(\tau^{-1}(t), \theta)}\left|\pi^k(\tau_{-i}^{-1}(t_{-i}), \theta|t_i') - \pi(t_{-i}, \theta|t_i)\right| \\
&\leq \frac{\pi^k(t_i')}{\pi^k(\tau^{-1}(t), \theta)}\varepsilon^k,
\end{aligned}$$

$$\left|\frac{\pi^k(t_i')}{\pi(t_i)}\frac{\pi(t, \theta)}{\pi^k(\tau^{-1}(t), \theta)} - \frac{\pi^k(t_i')}{\pi(t_i)}\right| = \frac{\pi^k(t_i')}{\pi(t_i)\pi^k(\tau^{-1}(t), \theta)}\left|\pi(t, \theta) - \pi^k(\tau^{-1}(t), \theta)\right| \leq \frac{\pi^k(t_i')}{\pi(t_i)\pi^k(\tau^{-1}(t), \theta)}\varepsilon^k,$$

$$\left|\frac{\pi^k(t_i')}{\pi(t_i)} - \frac{\pi^k(t_i')}{\pi^k(\tau_i^{-1}(t_i))}\right| = \frac{\pi^k(t_i')}{\pi(t_i)\pi^k(\tau_i^{-1}(t_i))}\left|\pi^k(\tau_i^{-1}(t_i)) - \pi(t_i)\right| \leq \frac{\pi^k(t_i')}{\pi(t_i)\pi^k(\tau_i^{-1}(t_i))}\varepsilon^k.$$

Thus,

$$\begin{aligned}
&\left|\frac{\pi^k(\{t_i'\}\times\tau_{-i}^{-1}(t_{-i}),\theta)}{\pi^k(\tau^{-1}(t),\theta)} - \frac{\pi^k(t_i')}{\pi^k(\tau_i^{-1}(t_i))}\right| \\
&\quad\leq \varepsilon^k\pi^k(t_i')\left(\frac{1}{\pi^k(\tau^{-1}(t),\theta)} + \frac{1}{\pi(t_i)\pi^k(\tau^{-1}(t),\theta)} + \frac{1}{\pi(t_i)\pi^k(\tau_i^{-1}(t_i))}\right).
\end{aligned}$$

Then,

$$\begin{aligned}
&|\eta^k(a_i, X_i|t,\theta) - \zeta_i^k(a_i, X_i|t_i)| \\
&= \Bigg|\sum_{t_i'\in\tau_i^{-1}(t_i)\cap T_i^{\flat,k}} \gamma_i^k(a_i, X_i|t_i')\left(\frac{\pi^k(\{t_i'\}\times\tau_{-i}^{-1}(t_{-i}),\theta)}{\pi^k(\tau^{-1}(t),\theta)} - \frac{\pi^k(t_i')}{\pi^k(\tau_i^{-1}(t_i))}\right) \\
&\qquad + \sum_{t_i'\in\tau_i^{-1}(t_i)\setminus T_i^{\flat,k}} \gamma_i^k(a_i, X_i|t_i')\left(\frac{\pi^k(\{t_i'\}\times\tau_{-i}^{-1}(t_{-i}),\theta)}{\pi^k(\tau^{-1}(t),\theta)} - \frac{\pi^k(t_i')}{\pi^k(\tau_i^{-1}(t_i))}\right)\Bigg| \\
&\leq \varepsilon^k\pi^k(\tau_i^{-1}(t_i)\cap T_i^{\flat,k})\left(\frac{1}{\pi^k(\tau^{-1}(t),\theta)} + \frac{1}{\pi(t_i)\pi^k(\tau^{-1}(t),\theta)} + \frac{1}{\pi(t_i)\pi^k(\tau_i^{-1}(t_i))}\right) + \pi^k(\tau_i^{-1}(t_i)\setminus T_i^{\flat,k}) \\
&\leq \varepsilon^k\left(\frac{1}{\pi^k(\tau^{-1}(t),\theta)} + \frac{1}{\pi(t_i)\pi^k(\tau^{-1}(t),\theta)} + \frac{1}{\pi(t_i)\pi^k(\tau_i^{-1}(t_i))} + 1\right) \to 0
\end{aligned}$$

as $k\to\infty$, which implies (A7).

Next, we show that $\eta^*(\cdot|t,\theta)$ is obedient as a decision-communication rule of $(T^*, \Theta^*, \pi^*, u^*)$. Because a decision-communication rule γ^k is obedient in (T, Θ, π^k, u), it holds that

$$\sum_{a_{-i}, X_{-i}, t_{-i}, \theta} \gamma^k(a, X|t,\theta)\pi^k(t,\theta)u_i(a,\theta) \geq \sum_{a_{-i}, X_{-i}, t_{-i}, \theta} \gamma^k(a, X|t,\theta)\pi^k(t,\theta)u_i((a_i', a_{-i}),\theta)$$

for all $t_i \in T_i$, $X_i \in \mathcal{A}_i$, $a_i \in X_i$, $a_i' \in A_i$, and $i \in I$. This is written as

$$\sum_{a_{-i},X_{-i},t_{-i},\theta}\left(\sum_{t_{-i}'\in\tau_{-i}^{-1}(t_{-i})} \gamma^k(a, X|t',\theta)\pi^k(t',\theta)\right)u_i(a,\theta) \geq \sum_{a_{-i},X_{-i},t_{-i},\theta}\left(\sum_{t_{-i}'\in\tau_{-i}^{-1}(t_{-i})} \gamma^k(a, X|t',\theta)\pi^k(t',\theta)\right)u_i((a_i', a_{-i}),\theta),$$

where we replace t_i with t_i'. Then, by summing each side of the above inequality over $t_i' \in \tau_i^{-1}(t_i)$ for $t_i \in T_i^*$, we obtain

$$\sum_{a_{-i},X_{-i},t_{-i},\theta}\left(\sum_{t'\in\tau^{-1}(t)} \gamma^k(a, X|t',\theta)\pi^k(t',\theta)\right)u_i(a,\theta) \geq \sum_{a_{-i},X_{-i},t_{-i},\theta}\left(\sum_{t'\in\tau^{-1}(t)} \gamma^k(a, X|t',\theta)\pi^k(t',\theta)\right)u_i((a_i', a_{-i}),\theta),$$

which is written as

$$\sum_{a_{-i},X_{-i},t_{-i},\theta} \eta^k(a,X,t,\theta)u_i(a,\theta) \geq \sum_{a_{-i},X_{-i},t_{-i},\theta} \eta^k(a,X,t,\theta)u_i((a_i',a_{-i}),\theta).$$

In the limit as k goes to infinity, we have

$$\sum_{a_{-i},X_{-i},t_{-i},\theta} \eta^*(a,X,t,\theta)u_i(a,\theta) \geq \sum_{a_{-i},X_{-i},t_{-i},\theta} \eta^*(a,X,t,\theta)u_i((a_i',a_{-i}),\theta).$$

Because $\eta^*(t,\theta) = \pi(t,\theta)$, the above is reduced to

$$\sum_{a_{-i},X_{-i},t_{-i},\theta} \eta^*(a,X|t,\theta)\pi(t,\theta)u_i(a,\theta) \geq \sum_{a_{-i},X_{-i},t_{-i},\theta} \eta^*(a,X|t,\theta)\pi(t,\theta)u_i((a_i',a_{-i}),\theta).$$

Therefore, $\eta^*(\cdot|t,\theta)$ is obedient.

Finally, we show that $\eta^*(\cdot|t,\theta)$ is an element of Γ^F; that is,

$$\sum_{X,t,\theta} \eta^*(X|t,\theta)\pi(t,\theta)F(X,\theta) \geq \sum_{X,t,\theta} \hat{\gamma}(X|t,\theta)\pi(t,\theta)F(X,\theta) \tag{A8}$$

for arbitrary $\hat{\gamma} \in \Gamma^F$. Fix $\hat{\gamma} \in \Gamma^F$ and let $\hat{\gamma}^k \in \Gamma^k[\gamma^k]$ be such that

$$\hat{\gamma}^k_{S^k(t)}(a_{S^k(t)}, X_{S^k(t)}|t,\theta) = \hat{\gamma}_{S^k(t)}(a_{S^k(t)}, X_{S^k(t)}|\tau(t),\theta),$$

which is well-defined because $\hat{\gamma}$ is belief-invariant. Note that $\hat{\gamma}^k(a,X|t,\theta) = \hat{\gamma}(a,X|\tau(t),\theta)$ if $t \in T^k$. Then, $\gamma^k \in \Gamma^{k,F}[\gamma^k]$ implies that

$$\sum_{X,t,\theta} \gamma^k(X|t,\theta)\pi^k(t,\theta)F(X,\phi(\theta)) \geq \sum_{X,t,\theta} \hat{\gamma}^k(X|t,\theta)\pi^k(t,\theta)F(X,\phi(\theta)). \tag{A9}$$

We first evaluate the limit of the left-hand side of (A9) as $k \to \infty$. Because

$$\begin{aligned}\sum_{X,t,\theta} \gamma^k(X|t,\theta)\pi^k(t,\theta)F(X,\phi(\theta)) &= \sum_{X,t,\theta} \sum_{t'\in\tau^{-1}(t)} \gamma^k(X|t',\theta)\pi^k(t',\theta)F(X,\phi(\theta)) \\ &= \sum_{X,t,\theta} \eta^k(X,t,\theta)F(X,\phi(\theta)),\end{aligned}$$

we have

$$\begin{aligned}\lim_{k\to\infty} \sum_{X,t,\theta} \gamma^k(X|t,\theta)\pi^k(t,\theta)F(X,\phi(\theta)) &= \sum_{X,t,\theta} \eta^*(X,t,\theta)F(X,\phi(\theta)) \\ &= \sum_{X,t,\theta} \eta^*(X,t|t,\theta)\pi(t,\theta)F(X,\theta). \qquad \text{(A10)}\end{aligned}$$

We next evaluate the limit of the right-hand side of (A9) as $k \to \infty$. Note that

$$\sum_{X,t,\theta} \hat{\gamma}^k(X|t,\theta)\pi^k(t,\theta)F(X,\phi(\theta)) \geq \sum_{X,\theta}\sum_{t\in T^k} \hat{\gamma}(X|\tau(t),\theta)\pi^k(t,\theta)F(X,\phi(\theta)) + \pi^k(T \setminus T^k)\inf_{X,\theta} F(X,\theta),$$

and the first term in the right-hand side is reduced to

$$\begin{aligned}
\sum_{X,\theta}\sum_{t\in T^k} \hat{\gamma}(X|\tau(t),\theta)\pi^k(t,\theta)F(X,\phi(\theta)) &= \sum_{X,t,\theta} \hat{\gamma}(X|t,\theta)\pi^k(\tau^{-1}(t)\cap T^k,\theta)F(X,\phi(\theta)) \\
&\geq \sum_{X,t,\theta} \hat{\gamma}(X|t,\theta)\pi^k(\tau^{-1}(t),\theta)F(X,\phi(\theta)) - \pi^k(T\setminus T^k)\sup_{X,\theta} F(X,\theta) \\
&= \sum_{X,t,\theta} \hat{\gamma}(X|t,\theta)\eta^k(t,\theta)F(X,\phi(\theta)) - \pi^k(T\setminus T^k)\sup_{X,\theta} F(X,\theta).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\lim_{k\to\infty} \sum_{X,t,\theta} \hat{\gamma}^k(X|t,\theta)\pi^k(t,\theta)F(X,\phi(\theta)) \\
&\quad \geq \lim_{k\to\infty} \sum_{X,t,\theta} \hat{\gamma}(X|t,\theta)\eta^k(t,\theta)F(X,\phi(\theta)) + \pi^k(T\setminus T^k)(\inf_{X,\theta} F(X,\theta) - \sup_{X,\theta} F(X,\theta)) \\
&\quad = \sum_{X,t,\theta} \hat{\gamma}(X|t,\theta)\eta^*(t,\theta)F(X,\phi(\theta)) \\
&\quad = \sum_{X,t,\theta} \hat{\gamma}(X|t,\theta)\pi(t,\theta)F(X,\theta). \qquad \text{(A11)}
\end{aligned}$$

Combining (A9), (A10), and (A11), we obtain (A8).

Because a decision-communication rule $\eta^*(a,X|t,\theta)$ is an element of Γ^F and it is a BIBCE-C, a decision rule $\eta^*(a|t,\theta)$ is an element of $\mathcal{E}^F$, which completes the proof. □

B Proof of Proposition 3

We give the proof of Proposition 3. A function $f : A \to \mathbb{R}$ is a supermodular function if, for any $a, b \in A$,

$$f(a \vee b) + f(a \wedge b) \geq f(a) + f(b),$$

where $a \vee b = (\max\{a_i, b_i\})_{i\in I}$ is the join of a and b, and $a \wedge b = (\min\{a_i, b_i\})_{i\in I}$ is the meet of a and b. We use the following lemmas.

Lemma B. *Let $v : A \times \Theta \to \mathbb{R}$ be a potential function of a supermodular incomplete information game. Then, $v(\cdot,\theta) : A \to \mathbb{R}$ is a supermodular function for each $\theta \in \Theta$.*

Proof. It is well known that, if a function $f : A \to \mathbb{R}$ satisfies increasing differences, i.e., $f(a_i, a_{-i}) - f(b_i, a_{-i}) \geq f(a_i, b_{-i}) - f(b_i, b_{-i})$ for each $i \in I$ and $a, b \in A$ with $a \geq^P b$, and A_i is a lattice for each $i \in I$, then f is a supermodular function. Strategic complementarities and increasing differences are equivalent, and a linearly ordered finite set A_i is a lattice. □

Lemma C. *Let $f : A \to \mathbb{R}$ be a supermodular function. For arbitrary $\mu \in \Delta(A)$, there exists $\mu^* \in \Delta(A)$ satisfying the following conditions.*

- *The marginal probability distributions of μ and μ^* on A_i are the same for each $i \in I$.*
- $\sum_a \mu^*(a) f(a) \geq \sum_a \mu(a) f(a)$.
- *The support of μ^* denoted by $\mathrm{supp}(\mu^*)$ is linearly ordered with respect to $\geq^P$; that is, for any $a, b \in \mathrm{supp}(\mu^*)$, either $a \geq^P b$ or $b \geq^P a$.*

Proof. We use the following notation.

- Let $\geq^L$ denote the lexicographic order on A; that is, $a >^L b$ if there exists $i \in I$ such that $a_i > b_i$ and $a_j = b_j$ for all $j < i$.
- For $A' = \{a^1, \ldots, a^K\} \subseteq A$, which is lexicographically ordered with $a^1 >^L a^2 >^L \cdots >^L a^K$, let $\kappa_1(A') \in \{1, \ldots, K\}$ be the smallest number such that there exists $k \geq \kappa_1(A') + 1$ satisfying $a^{\kappa_1(A')} \ngeq^P a^k$. Let $\kappa_2(A') \geq \kappa_1(A') + 1$ be the smallest number satisfying $a^{\kappa_1(A')} \ngeq^P a^{\kappa_2(A')}$. Note that, for each $k < \kappa_1(A')$ and $l > k$, $a^k >^P a^l$ as well as $a^k >^L a^l$. We write $\alpha(A') = a^{\kappa_1(A')}$ and $\beta(A') = a^{\kappa_2(A')}$.
- For each $\mu \in \Delta(A)$ and $a, b \in A$, let $\phi[\mu|a, b] \in \Delta(A)$ be such that

$$\phi[\mu|a,b](x) = \begin{cases} \mu(x) - \min_{x' \in \{a,b\}} \mu(x') & \text{if } x \in \{a, b\}, \\ \mu(x) + \min_{x' \in \{a,b\}} \mu(x') & \text{if } x \in \{a \vee b, a \wedge b\}, \\ \mu(x) & \text{if } x \notin \{a, b, a \vee b, a \wedge b\}. \end{cases}$$

  Then,

$$\sum_x \phi[\mu|a,b](x) f(x) - \sum_x \phi[\mu|a,b](x) f(x) = \min_{x' \in \{a,b\}} \mu(x')(f(a \vee b) + f(a \wedge b) - f(a) - f(b)) \geq 0$$

  because f is a supermodular function. The marginal probability distributions of $\phi[\mu|a, b]$ and μ on A_i are the same for each $i \in I$ because $\{a_i, b_i\} = \{(a \vee b)_i, (a \wedge b)_i\}$.

Fix $\mu \in \Delta(A)$. Let $\mu^0 = \mu$ and $A^0 = \text{supp}(\mu^0)$. For each $n \geq 1$, if $\kappa_1(A^{n-1})$ exists, define $\mu^n \in \Delta(A)$ and $A^n = \text{supp}(\mu^n)$ recursively by the rule

$$\mu^n = \phi[\mu^{n-1}|\alpha(A^{n-1}), \beta(A^{n-1})].$$

If $\kappa_1(A^{n-1})$ does not exist, set $\mu^* = \mu^{n-1}$. Then, $\text{supp}(\mu^*)$ is linearly ordered with respect to $\geq^P$, and μ and μ^* have the same marginal distributions because, for each k, μ^k and μ^{k+1} have the same marginal distributions. Moreover, $\sum_x \mu^*(a)f(a) \geq \sum_x \mu(a)f(a)$.

We show that the above procedure terminates in finite steps, which implies that μ^* exists. To this end, it is enough to show that $\kappa_1(A^n) \geq \kappa_1(A^{n-1})$ and $\kappa_2(A^n) \geq \kappa_2(A^{n-1})$, and that at least one of the inequalities is strict. In fact, if this is true, then, for sufficiently large n, $\kappa_1(A^n)$ does not exist because $\kappa_1(A^n) \leq \#|A| - 1$ and $\kappa_2(A^n) \leq \#|A|$, so we can obtain μ^* in finite steps.

Observe that

$$A^n = (A^{n-1} \cup \{\alpha(A^{n-1}) \vee \beta(A^{n-1}), \alpha(A^{n-1}) \wedge \beta(A^{n-1})\}) \setminus \arg\min_{x \in \{a,b\}} f(x).$$

Thus, the rank of $\alpha(A^{n-1}) \vee \beta(A^{n-1})$ in A^n is $\kappa_1(A^{n-1})$, and that of $\alpha(A^{n-1}) \wedge \beta(A^{n-1})$ is greater than or equal to $\kappa_2(A^{n-1})$. By construction, we must have $\alpha(A^{n-1}) \vee \beta(A^{n-1}) \geq^L \alpha(A^n)$ and $\alpha(A^{n-1}) \wedge \beta(A^{n-1}) \geq^L \beta(A^n)$; that is, $\kappa_1(A^n) \geq \kappa_1(A^{n-1})$ and $\kappa_2(A^n) \geq \kappa_2(A^{n-1})$. Now suppose that $\kappa_1(A^n) = \kappa_1(A^{n-1})$, i.e., $\alpha(A^n) = \alpha(A^{n-1}) \vee \beta(A^{n-1})$. Because $\alpha(A^{n-1}) \vee \beta(A^{n-1}) >^P \alpha(A^{n-1}) \wedge \beta(A^{n-1})$, it follows that $\alpha(A^{n-1}) \wedge \beta(A^{n-1}) >^L \beta(A^n)$, which implies that $\kappa_2(A^n) > \kappa_2(A^{n-1})$. Consequently, we have either $\kappa_1(A^n) = \kappa_1(A^{n-1})$ and $\kappa_2(A^n) > \kappa_2(A^{n-1})$, or $\kappa_1(A^n) > \kappa_1(A^{n-1})$ and $\kappa_2(A^n) \geq \kappa_2(A^{n-1})$. □

We are ready to prove Proposition 3.

Proof of Proposition 3. We first prove the first claim. By Lemma B, $v(\cdot, \theta) : A \to \mathbb{R}$ is a supermodular function. Thus, by Lemma C, there exists $\sigma^* \in \Sigma^{BI}$ such that, for each $(t, \theta) \in T \times \Theta$, $\sigma^*(\cdot|t, \theta) \in \Delta(A)$ satisfies the following conditions.

- The marginal probability distributions of $\sigma(\cdot|t, \theta) \in \Delta(A)$ and $\sigma^*(\cdot|t, \theta) \in \Delta(A)$ on A_i are the same for each $i \in I$. That is, $\sigma^*(\{a_i\} \times A_{-i}|t, \theta) = \sigma(\{a_i\} \times A_{-i}|t, \theta) = \sigma_i(a_i|t_i)$ is independent of t_{-i} and θ.

- $\sum_a \sigma^*(a|t, \theta)v(a, \theta) \geq \sum_a \sigma(a|t, \theta)v(a, \theta)$.

- The support of $\sigma^*(\cdot|t,\theta)$ is linearly ordered with respect to $\geq^P$.

By the second condition, it holds that

$$\sum_{a,t,\theta} \sigma^*(a|t,\theta)\pi(t,\theta)v(a,\theta) \geq \sum_{a,t,\theta} \sigma(a|t,\theta)\pi(t,\theta)v(a,\theta),$$

which implies that σ^* is also a potential maximizing BIBCE.

Given σ^*, we construct a function $\alpha : [0,1] \times T \times \Theta \to A$ satisfying the following conditions.

- For each $(z,t,\theta) \in [0,1] \times T \times \Theta$, $\alpha(z,t,\theta) \in A$ is in the support of $\sigma^*(\cdot|t,\theta) \in \Delta(A)$, and $\alpha(z,t,\theta)$ is right continuous and increasing in z; that is, $\lim_{z'\to z+} \alpha(z',t,\theta) = \alpha(z,t,\theta)$ and $\alpha(z,t,\theta) \geq^P \alpha(z',t,\theta)$ if $z \geq z'$.

- For each $(a,t,\theta) \in A \times T \times \Theta$,

$$\sigma^*(a|t,\theta) = \int_0^1 \delta_a(\alpha(z,t,\theta))dz,$$

  where $\delta_a : A \to \{0,1\}$ is given by $\delta_a(x) = 1$ if $x = a$ and $\delta_a(x) = 0$ if $x \neq a$.

Let $\alpha_i(z,t,\theta) \in A_i$ denote player i's action in the action profile $\alpha(z,t,\theta) \in A$, i.e., $\alpha(z,t,\theta) = (\alpha_i(z,t,\theta))_{i\in I}$. Note that

$$\sum_{a_{-i}\in A_{-i}} \delta_a(\alpha(z,t,\theta)) = \delta_{a_i}(\alpha_i(z,t,\theta)),$$

where $\delta_{a_i} : A_i \to \{0,1\}$ is given by $\delta_{a_i}(x_i) = 1$ if $x_i = a_i$ and $\delta_{a_i}(x_i) = 0$ if $x_i \neq a_i$. Thus,

$$\sigma_i^*(a_i|t_i) = \sigma^*(\{a_i\} \times A_{-i}|t,\theta) = \sum_{a_{-i}\in A_{-i}} \int_0^1 \delta_a(\alpha(z,t,\theta))dz = \int_0^1 \delta_{a_i}(\alpha_i(z,t,\theta))dz$$

for each $a_i \in A_i$, which implies that $\alpha_i(z,t,\theta)$ is independent of t_{-i} and θ because $\alpha_i(z,t,\theta)$ is increasing in z. This observation allows us to write $\alpha_i(z,t_i) = \alpha_i(z,t,\theta)$ and $\alpha(z,t) = (\alpha_i(z,t_i))_{i\in I} = \alpha(z,t,\theta)$. That is, for each $a \in A$, we can write

$$\sigma^*(a|t,\theta) = \int_0^1 \delta_a(\alpha(z,t))dz.$$

Now, for each $\rho \in (0,1)$, define decision rules $\underline{\sigma}^\rho, \overline{\sigma}^\rho \in \Sigma$ by

$$\underline{\sigma}^\rho(a|t,\theta) = \int_0^1 \delta_a(\alpha(\rho z,t))dz,$$
$$\overline{\sigma}^\rho(a|t,\theta) = \int_0^1 \delta_a(\alpha(\rho + (1-\rho)z,t))dz.$$

Note that $\underline{\sigma}^\rho$ and $\overline{\sigma}^\rho$ are belief-invariant because

$$\underline{\sigma}^\rho(\{a_i\}\times A_{-i}|t,\theta) = \int_0^1 \delta_{a_i}(\alpha_i(\rho z, t_i))dz,$$
$$\overline{\sigma}^\rho(\{a_i\}\times A_{-i}|t,\theta) = \int_0^1 \delta_{a_i}(\alpha_i(\rho + (1-\rho)z, t_i))dz,$$

which are independent of t_{-i} and θ. Note also that, for each $(a,t,\theta)\in A\times T\times\Theta$,

$$\sigma^*(a|t,\theta) = \rho\underline{\sigma}^\rho(a|t,\theta) + (1-\rho)\overline{\sigma}^\rho(a|t,\theta).$$

Thus,

$$\sum_{a,t,\theta}\sigma^*(a|t,\theta)\pi(t,\theta)v(a,\theta) = \rho\sum_{a,t,\theta}\underline{\sigma}^\rho(a|t,\theta)\pi(t,\theta)v(a,\theta) + (1-\rho)\sum_{a,t,\theta}\overline{\sigma}^\rho(a|t,\theta)\pi(t,\theta)v(a,\theta),$$

which implies that

$$\sum_{a,t,\theta}\sigma^*(a|t,\theta)\pi(t,\theta)v(a,\theta) = \sum_{a,t,\theta}\underline{\sigma}^\rho(a|t,\theta)\pi(t,\theta)v(a,\theta) = \sum_{a,t,\theta}\overline{\sigma}^\rho(a|t,\theta)\pi(t,\theta)v(a,\theta)$$

because σ^* is a potential maximizing BIBCE and $\underline{\sigma}^\rho$ and $\overline{\sigma}^\rho$ are belief-invariant decision rules. Clearly, $\underline{\sigma}(a|t) = \lim_{\rho\to 0}\underline{\sigma}^\rho(a|t,\theta)$ and $\overline{\sigma}(a|t) = \lim_{\rho\to 1}\overline{\sigma}^\rho(a|t,\theta)$ for each $(a,t,\theta)\in A\times T\times\Theta$. Moreover, it can be readily shown that $\lim_{\rho\to 0}\underline{\sigma}^\rho = \underline{\sigma}$ and $\lim_{\rho\to 0}\overline{\sigma}^\rho = \overline{\sigma}$ in the topology of the set of distributional decision rules discussed in Section 2. Therefore,

$$\sum_{a,t,\theta}\sigma^*(a|t,\theta)\pi(t,\theta)v(a,\theta) = \sum_{a,t,\theta}\underline{\sigma}(a|t,\theta)\pi(t,\theta)v(a,\theta) = \sum_{a,t,\theta}\overline{\sigma}(a|t,\theta)\pi(t,\theta)v(a,\theta),$$

and thus $\underline{\sigma}$ and $\overline{\sigma}$ are potential maximizing BIBCE, which completes the proof of the first claim.

To prove the second claim, note that Σ^v is convex; that is, any convex combination of potential maximizing BIBCE is a potential maximizing BIBCE. Thus, there exists $\sigma'\in\Sigma^v$ such that, for each $t_i\in T_i$ and $i\in I$, $\sigma'_i(\overline{a}_i|t_i)>0$ and $\sigma'_i(\underline{a}_i|t_i)>0$ for $\overline{a}_i = \max\{a_i\,|\,\sigma_i(a_i|t_i)>0,\ \sigma\in\Sigma^v\}$ and $\underline{a}_i = \min\{a_i\,|\,\sigma_i(a_i|t_i)>0,\ \sigma\in\Sigma^v\}$. Then, we can obtain the second claim by applying the first claim to $\overline{\sigma}'$ and $\underline{\sigma}'$.

The last claim is implied by the first claim. □

C Proofs of Propositions 6 and 7

Proof of Proposition 6. Fix $\delta>0$. Because σ^* is a robust BIBCE of (T,Θ,π,u), there exists $\bar{\varepsilon}>0$ such that, for all $\varepsilon\le\bar{\varepsilon}$, every ε-elaboration $(\bar{T},\bar{\Theta},\bar{\pi},\bar{u})$ with an elaboration mapping τ has a BIBCE

$\bar{\sigma}$ such that

$$\sup_{D \subseteq A\times T\times\Theta} \left| \sum_{(a,t,\theta)\in D} \bar{\sigma} \circ \bar{\pi}(a, \tau^{-1}(t), \theta) - \sum_{(a,t,\theta)\in D} \sigma^* \circ \pi(a, t, \theta) \right| \leq \delta,$$

which implies that

$$\sum_{(\bar{t},\theta)\in \bar{T}\times\Theta} \bar{\sigma}(\mathbf{1}|\bar{t}, \theta)\bar{\pi}(\bar{t}, \theta) = \sum_{(t,\theta)\in T\times\Theta} \bar{\sigma} \circ \bar{\pi}(\mathbf{1}, \tau^{-1}(t), \theta) \geq 1 - \delta. \tag{C1}$$

Using (C1), we prove the proposition, and to this end, we construct an appropriate ε-elaboration.

Fix $\varepsilon \leq \bar{\varepsilon}^2$, where $\bar{\varepsilon}$ is given above. For an arbitrary $E = \prod_{i\in I} E_i \in \mathcal{T}$ with $\pi(E) = 1 - \varepsilon$, we construct a $\sqrt{\varepsilon}$-elaboration $(T, \bar{\Theta}, \bar{\pi}, \bar{u})$ as follows.

- An elaboration mapping $\tau : T \to T$ is the identity mapping, i.e., $\tau(t) = (\tau_i(t_i))_{i\in I} = t$ for each $t \in T$.
- $\bar{\Theta}_i = \Theta_i \cup \{0\}$ and $\bar{\Theta} = \prod_{i\in I} \bar{\Theta}_i$ (recall that $\Theta = \prod_{i\in I} \Theta_i$).
- For each $i \in I$,

$$\bar{u}_i(a, \theta) = \begin{cases} u_i(a, (\theta_i, \theta'_{-i})) & \text{if } \theta_i \in \Theta_i, \text{ where } \theta'_{-i} \in \Theta_{-i} \text{ (which can be arbitrary)}, \\ 1 & \text{if } a_i = 0 \text{ and } \theta_i = 0, \\ 0 & \text{if } a_i = 1 \text{ and } \theta_i = 0. \end{cases}$$

  That is, action 0 is a dominant action when player i's payoff type is 0.

- For $t \in T$ with $J = \{i \in I \mid t_i \notin E_i\}$,

$$\bar{\pi}(t, \theta) = \begin{cases} \pi(t, \theta) & \text{if } J = \emptyset \text{ and } \theta \in \Theta, \\ \sum_{\theta'_J \in \Theta_J} \pi(t, (\theta_{I\setminus J}, \theta'_J)) & \text{if } J \neq \emptyset,\ \theta_i \in \Theta_i \text{ for } i \in I \setminus J, \text{ and } \theta_i = 0 \text{ for } i \in J, \\ 0 & \text{otherwise.} \end{cases}$$

To show that $(T, \bar{\Theta}, \bar{\pi}, \bar{u})$ is a $\sqrt{\varepsilon}$-elaboration, we check the three conditions in Definition 2 (the definition of an ε-elaboration). Clearly, the first condition is satisfied. To check the second condition, let $T_i^\sharp = E_i$ for each $i \in I$. Then, $\bar{\pi}(T^\sharp) = \pi(E) = 1 - \varepsilon \geq 1 - \sqrt{\varepsilon}$. In addition, if $t_i \in T_i^\sharp$, then $\bar{\pi}(\Theta_i \times \bar{\Theta}_{-i}|t_i) = 1$, and if $t_i \notin T_i^\sharp$, then $\bar{\pi}(\Theta_i \times \bar{\Theta}_{-i}|t_i) = 0$. Thus, $T_i^\sharp = \{t_i \in T_i \mid \bar{\pi}(\Theta_i \times \bar{\Theta}_{-i}|t_i) = 1\}$, which implies the second condition. We check the third condition. For each $D \subseteq T \times \Theta$,

$$\begin{aligned}
&\Big|\sum_{(t,\theta)\in D} \bar{\pi}(\tau^{-1}(t),\theta) - \sum_{(t,\theta)\in D} \pi(t,\theta)\Big| \\
&= \Big|\sum_{(t,\theta)\in D} (\bar{\pi}(t,\theta) - \pi(t,\theta))\Big| \\
&= \Big|\sum_{(t,\theta)\in D, t\in T^\sharp} (\bar{\pi}(t,\theta) - \pi(t,\theta)) + \sum_{(t,\theta)\in D, t\notin T^\sharp} (\bar{\pi}(t,\theta) - \pi(t,\theta))\Big| \\
&= \Big|\sum_{(t,\theta)\in D, t\notin T^\sharp} (\bar{\pi}(t,\theta) - \pi(t,\theta))\Big| = \sum_{(t,\theta)\in D, t\notin T^\sharp} \pi(t,\theta) \leq \pi(T \setminus T^\sharp) = \varepsilon \leq \sqrt{\varepsilon}.
\end{aligned}$$

Let $T_i^\flat = \{t_i \in T_i^\sharp \mid \bar{\pi}(T_{-i}^\sharp|t_i) \geq 1 - \sqrt{\varepsilon}\} = \{t_i \in T_i^\sharp \mid \pi(T_{-i}^\sharp|t_i) \geq 1 - \sqrt{\varepsilon}\}$. Then, for each $t_i \in T_i^\flat$ and $D_{-i} \subseteq T_{-i} \times \Theta$,

$$\begin{aligned}
&\Big|\sum_{(t_{-i},\theta)\in D_{-i}} \bar{\pi}(\tau_{-i}^{-1}(t_{-i}),\theta|t_i) - \sum_{(t_{-i},\theta)\in D_{-i}} \pi(t_{-i},\theta|\tau_i(t_i))\Big| \\
&= \Big|\sum_{(t_{-i},\theta)\in D_{-i}} (\bar{\pi}(t_{-i},\theta|t_i) - \pi(t_{-i},\theta|t_i))\Big| \\
&= \Big|\sum_{(t_{-i},\theta)\in D_{-i}, t_{-i}\in T_{-i}^\sharp} (\bar{\pi}(t_{-i},\theta|t_i) - \pi(t_{-i},\theta|t_i)) + \sum_{(t_{-i},\theta)\in D_{-i}, t_{-i}\notin T_{-i}^\sharp} (\bar{\pi}(t_{-i},\theta|t_i) - \pi(t_{-i},\theta|t_i))\Big| \\
&= \Big|\sum_{(t_{-i},\theta)\in D_{-i}, t_{-i}\notin T_{-i}^\sharp} (\bar{\pi}(t_{-i},\theta|t_i) - \pi(t_{-i},\theta|t_i))\Big| = \sum_{(t_{-i},\theta)\in D_{-i}, t_{-i}\notin T_{-i}^\sharp} \pi(t_{-i},\theta|t_i) \leq \pi(T_{-i} \setminus T_{-i}^\sharp|t_i) \leq \sqrt{\varepsilon}.
\end{aligned}$$

Moreover,

$$1 - \varepsilon = \pi(T^\sharp) = \sum_{t_i\in T_i^\flat} \pi(T_{-i}^\sharp|t_i)\pi(t_i) + \sum_{t_i\in T_i^\sharp\setminus T_i^\flat} \pi(T_{-i}^\sharp|t_i)\pi(t_i) \leq \pi(T_i^\flat) + (1 - \sqrt{\varepsilon})(1 - \pi(T_i^\flat)),$$

and thus $\bar{\pi}(T_i^\flat) = \pi(T_i^\flat) \geq 1 - \sqrt{\varepsilon}$. Therefore, the third condition of Definition 2 is satisfied.

Let $\bar{\sigma} \in \Sigma^{BI}$ be a BIBCE of the $\sqrt{\varepsilon}$-elaboration $(T, \bar{\Theta}, \bar{\pi}, \bar{u})$ satisfying (C1); that is,

$$\sum_{(t,\theta)\in T\times\Theta} \bar{\sigma}(\mathbf{1}|t,\theta)\bar{\pi}(t,\theta) \geq 1 - \delta.$$

Let $E_i^* \subseteq E_i$ be the set of player i's types who choose action 1 with a positive probability under $\bar{\sigma}$:

$$E_i^* = \{t_i \in E_i \mid \bar{\sigma}_i(1|t_i) > 0\}.$$

Let $\sigma^* = (\sigma_i^*)_{i\in I}$ be the strategy profile associated with $E^* = \prod_{i\in I} E_i^*$; that is, $\sigma_i^*(1|t_i) = 1$ if $t_i \in E_i^*$ and $\sigma_i^*(0|t_i) = 1$ if $t_i \notin E_i^*$. Then, it is straightforward to show that, for each $(t,\theta) \in T \times \Theta$,

$\sigma^*(\cdot|t) \equiv \prod_i \sigma_i^*(\cdot|t_i) \in \Delta(A)$ first-order stochastically dominates $\bar{\sigma}(\cdot|t,\theta) \in \Delta(A)$. In particular, we have $\sigma^*(\mathbf{1}|t) \geq \bar{\sigma}(\mathbf{1}|t,\theta)$, and thus,

$$\bar{\pi}(E^*) = \sum_{(t,\theta)\in T\times\Theta} \sigma^*(\mathbf{1}|t)\bar{\pi}(t,\theta) \geq \sum_{(t,\theta)\in T\times\Theta} \bar{\sigma}(\mathbf{1}|t,\theta)\bar{\pi}(t,\theta) \geq 1-\delta.$$

For $(t,\theta) \in T \times \bar{\Theta}$ and $i \in I$ with $t_i \in E_i^*$, let $\bar{\sigma}_{-i}(a_{-i}|t,\theta) = \bar{\sigma}((1,a_{-i})|t,\theta)/\bar{\sigma}_i(1|t_i)$ and $\sigma_{-i}^*(a_{-i}|t_{-i}) = \prod_{j\neq i}\sigma_j^*(a_j|t_j)$. We show that $\sigma_{-i}^*(\cdot|t_{-i}) \in \Delta(A_{-i})$ first-order stochastically dominates $\bar{\sigma}_{-i}(\cdot|t,\theta) \in \Delta(A_{-i})$. To this end, fix $(t,\theta) \in T \times \bar{\Theta}$ and $i \in I$ with $t_i \in E_i^*$, and let $a^* \in A$ be such that $a_j^* = 1$ if $t_j \in E_j^*$ and $a_j^* = 0$ if $t_j \notin E_j^*$, which implies that $\sigma_{-i}^*(a_{-i}^*|t_{-i}) = 1$. Then, for any deceasing set $B_{-i} \subseteq A_{-i}$ (i.e., $a_{-i} \in B_{-i}$ implies $a'_{-i} \in B_{-i}$ for each $a'_{-i} \leq a_{-i}$), if $a_{-i}^* \notin B_{-i}$, then $\sum_{a_{-i}\in B_{-i}} \sigma_{-i}^*(a_{-i}|t_{-i}) = 0 \leq \sum_{a_{-i}\in B_{-i}} \bar{\sigma}_{-i}(a_{-i}|t,\theta)$, and if $a_{-i}^* \in B_{-i}$, then $\sum_{a_{-i}\in B_{-i}} \sigma_{-i}^*(a_{-i}|t_{-i}) = 1 = \sum_{a_{-i}\in B_{-i}} \bar{\sigma}_{-i}(a_{-i}|t,\theta)$, where the last equality holds because, if

$$1 > \sum_{a_{-i}\in B_{-i}} \bar{\sigma}_{-i}(a_{-i}|t,\theta) \geq \sum_{a_{-i}\leq a_{-i}^*} \bar{\sigma}_{-i}(a_{-i}|t,\theta),$$

then there exists $a'_{-i} \nleq a_{-i}^*$ satisfying $\bar{\sigma}_{-i}(a'_{-i}|t,\theta) > 0$; that is, there exists $j \in I$ with $1 > a_j^* = 0$ and $\bar{\sigma}_j(1|t_j) > 0$, which contradicts the choice of a_j^*. Therefore, we conclude that $\sigma_{-i}^*(\cdot|t_{-i})$ first-order stochastically dominates $\bar{\sigma}_{-i}(\cdot|t,\theta)$.

By obedience of the BIBCE $\bar{\sigma}$ of $(T,\bar{\Theta},\bar{\pi},\bar{u})$, for each $i \in I$ with $t_i \in E_i^*$, it holds that

$$\sum_{a_{-i},t_{-i},\theta} \bar{\sigma}((1,a_{-i})|t,\theta)\bar{\pi}(t,\theta)(\bar{u}_i((1,a_{-i}),\theta) - \bar{u}_i((0,a_{-i}),\theta)) \geq 0,$$

or equivalently,

$$\sum_{a_{-i},t_{-i},\theta} \bar{\sigma}_{-i}(a_{-i}|t,\theta)\bar{\pi}(t,\theta)(\bar{u}_i((1,a_{-i}),\theta) - \bar{u}_i((0,a_{-i}),\theta)) \geq 0. \tag{C2}$$

Because $\bar{u}_i((1,a_{-i}),\theta) - \bar{u}_i((0,a_{-i}),\theta)$ is increasing in a_{-i} and $\sigma_{-i}^*(\cdot|t_{-i})$ first-order stochastically dominates $\bar{\sigma}_{-i}(\cdot|t,\theta)$, we have

$$\sum_{a_{-i}} \sigma_{-i}^*(a_{-i}|t_{-i})(\bar{u}_i((1,a_{-i}),\theta) - \bar{u}_i((0,a_{-i}),\theta)) \geq \sum_{a_{-i}} \bar{\sigma}_{-i}(a_{-i}|t,\theta)(\bar{u}_i((1,a_{-i}),\theta) - \bar{u}_i((0,a_{-i}),\theta))$$

for each t_{-i} and θ with $\bar{\pi}(t,\theta) > 0$. Thus, by (C2),

$$\sum_{a_{-i},t_{-i},\theta} \sigma_{-i}^*(a_{-i}|t_{-i})\bar{\pi}(t,\theta)(\bar{u}_i((1,a_{-i}),\theta) - \bar{u}_i((0,a_{-i}),\theta)) \geq 0,$$

or equivalently,

$$\sum_{a_{-i},t_{-i},\theta} \sigma^*_{-i}(a_{-i}|t_{-i})\bar{\pi}(t_{-i},\theta|t_i)(\bar{u}_i((1,a_{-i}),\theta) - \bar{u}_i((0,a_{-i}),\theta)) \geq 0. \tag{C3}$$

For each $i \in I$, let $\bar{f}_i : \mathcal{I}_{-i} \times \bar{\Theta} \to \mathbb{R}$ be the payoff increment function of $\bar{u}_i$: $\bar{f}_i(S,\theta) = \bar{u}_i(\mathbf{1}_{S\cup\{i\}},\theta) - \bar{u}_i(\mathbf{1}_S,\theta)$. Then, noting that $\sigma^*_{-i}(a_{-i}|t_{-i}) = 1$ if $\{j \neq i \mid a_j = 1\} = \{j \neq i \mid t_j \in E^*_j\}$ and $\sigma^*_{-i}(a_{-i}|t_{-i}) = 0$ otherwise, we can rewrite (C3) as

$$\sum_{t_{-i},\theta} \bar{\pi}(t_{-i},\theta|t_i)\bar{f}_i(S_{E^*_{-i}}(t_{-i}),\theta) \geq 0, \tag{C4}$$

which holds for each $t_i \in E^*_i$.

We now consider (T,Θ,π,u). For each $i \in I$, let $f_i : \mathcal{I}_{-i} \times \Theta \to \mathbb{R}$ be the payoff increment function of u_i. Note that $f_i(S,\theta) = \bar{f}_i(S,\theta)$ if $\theta_i \in \Theta_i$. Thus, by (C4) and the construction of $\bar{\pi}$,

$$\mathrm{E}[f_i(S_{E^*_{-i}}(t_{-i}),\theta)|t_i] = \sum_{t_{-i},\theta} \pi(t_{-i},\theta|t_i)f_i(S_{E^*_{-i}}(t_{-i}),\theta) \geq 0$$

for each $t_i \in E^*_i$, which implies that

$$E^*_i \subseteq B^{f_i}_i(E^*) = \{t_i \mid \mathrm{E}[f_i(S_{E^*_{-i}}(t_{-i}),\theta)|t_i] \geq 0\}.$$

That is, E^* is an $\mathbf{f}$-evident event contained in E. Therefore, $E^* \subseteq CB^{\mathbf{f}}(E)$ and $\pi(CB^{\mathbf{f}}(E)) \geq \pi(E^*) = \bar{\pi}(E^*) \geq 1-\delta$, thus establishing the proposition. □

Proof of Proposition 7. This proof generalizes the proofs of Ui (2001), Morris and Ui (2005), and the alternative proof of the critical path theorem in Oyama and Takahashi (2019) (the working paper version of Oyama and Takahashi (2020)).

We first prove the second claim of the proposition. Fix $E = \prod_i E_i \in \mathcal{T}$ satisfying $M(v,\pi,E) > 0$. Let $E^* = \prod_i E^*_i \in \mathcal{T}$ be such that $E^* \subseteq E$ and

$$\sum_{(t,\theta)\in T\times\Theta} \pi(t,\theta)v(\mathbf{1}_{S_{E^*}(t)},\theta) \geq \sum_{(t,\theta)\in T\times\Theta} \pi(t,\theta)v(\mathbf{1}_{S_{E'}(t)},\theta) \tag{C5}$$

for all $E' = \prod_i E'_i \in \mathcal{T}$ with $E' \subseteq E$, where $S_{E'}(t) = \{i \in I \mid t_i \in E'_i\}$. Then, for each $t_i \in E^*_i$, it holds that

$$\begin{aligned} E[f_i(S_{E^*_{-i}}(t_{-i},\theta)|t_i] &= \sum_{t_{-i},\theta} \pi(t_{-i},\theta|t_i)(u_i(\mathbf{1}_{S_{E^*_{-i}}(t_{-i})\cup\{i\}},\theta) - u_i(\mathbf{1}_{S_{E^*_{-i}}(t_{-i})},\theta)) \\ &\geq \frac{1}{\lambda_i} \sum_{t_{-i},\theta} \pi(t_{-i},\theta|t_i)(v(\mathbf{1}_{S_{E^*_{-i}}(t_{-i})\cup\{i\}},\theta) - v(\mathbf{1}_{S_{E^*_{-i}}(t_{-i})},\theta)) \geq 0 \end{aligned}$$

by (C5), which implies that E^* is an $\mathbf{f}$-evident event; that is, $E^* \subseteq CB^{\mathbf{f}}(E)$. Observe that

$$
\begin{aligned}
0 \leq & \sum_{t,\theta} \pi(t,\theta) v(\mathbf{1}_{S_{E^*(t)}}, \theta) - \sum_{t,\theta} \pi(t,\theta) v(\mathbf{1}_{S_{E(t)}}, \theta) \\
= & \sum_{t \in E \setminus E^*, \theta \in \Theta} \pi(t,\theta)(v(\mathbf{1}_{S_{E^*(t)}}, \theta) - v(\mathbf{1}, \theta)) + \sum_{t \in T \setminus E, \theta \in \Theta} \pi(t,\theta)(v(\mathbf{1}_{S_{E^*(t)}}, \theta) - v(\mathbf{1}_{S_{E(t)}}, \theta)) \\
\leq & (\pi(E) - \pi(E^*)) c_1 + (1 - \pi(E)) c_2,
\end{aligned}
$$

where

$$
c_1 = \sup_{t \in E \cap T^*, S \neq I} \sum_{\theta} \pi(\theta|t)(v(\mathbf{1}_S, \theta) - v(\mathbf{1}, \theta)) = -M(v, \pi, E) < 0,
$$

$$
c_2 = \sup_{S \subseteq S' \neq I, \theta \in \Theta} v(\mathbf{1}_S, \theta) - v(\mathbf{1}_{S'}, \theta) \geq 0.
$$

Thus,

$$
\pi(CB^{\mathbf{f}}(E)) \geq \pi(E^*) \geq 1 - (1 - c_2/c_1)(1 - \pi(E)) = 1 - \kappa(v, \pi, E)(1 - \pi(E)).
$$

If $M^*(v, \pi) > 0$, then $M(v, \pi, E) \geq M^*(v, \pi) > 0$ for all $E \in \mathcal{T}$, so

$$
\pi(CB^{\mathbf{f}}(E)) \geq 1 - \kappa(v, \pi, E)(1 - \pi(E)) \geq 1 - \kappa^*(v, \pi)(1 - \pi(E)).
$$

If $M^{**}(v) > 0$, then $M^*(v, \pi) \geq M^{**}(v) > 0$, so

$$
\pi(CB^{\mathbf{f}}(E)) \geq 1 - \kappa^*(v, \pi)(1 - \pi(E)) \geq 1 - \kappa^{**}(v)(1 - \pi(E)),
$$

which completes the proof. □

References

Bergemann, D., Morris, S., 2013. Robust predictions in games with incomplete information. Econometrica 81, 1251–1308.

Bergemann, D., Morris, S., 2016. Bayes correlated equilibrium and the comparison of information structures in games. Theor. Econ. 11, 487–522.

Bergemann, D., Morris, S., 2017. Belief-free rationalizability and informational robustness. Games Econ. Behav. 104, 744–759.

Bergemann, D., Morris, S., 2019. Information design: A unified perspective. J. Econ. Lit. 57, 44–95.

Billingsley, P., 1999. Convergence of Probability Measures, 2nd Edition. John Wiley & Sons.

Dekel, E., Fudenberg, D., Morris, S., 2007. Interim correlated rationalizability. Theor. Econ. 2, 15–40.

Ely, J., Peski, M., 2006. Hierarchies of belief and interim rationalizability. Theor. Econ. 1, 19–65.

Forges, F., 1993. Five legitimate definitions of correlated equilibrium in games with incomplete information. Theory Dec. 35, 277–310.

Forges, F., 2006. Correlated equilibrium in games with incomplete information revisited. Theory Dec. 61, 329–344.

van Heumen, R., Peleg, B., Tijs, S., Borm, P., 1996. Axiomatic characterizations of solutions for Bayesian games. Theory Dec. 40, 103–129.

Kajii, A., Morris, S. 1997. The Robustness of equilibria to incomplete information. Econometrica 65, 1283–1309.

Kajii, A., Morris, S. 2020a. Refinements and higher order beliefs: A unified survey. Jpn. Econ. Rev. 71, 7–34.

Kajii, A., Morris, S. 2020b. Notes on “refinements and higher order beliefs.” Jpn. Econ. Rev. 71, 35–41.

Lehrer, E., Rosenberg, D., Shmaya, E., 2010. Signaling and mediation in games with common interest. Games Econ. Behav. 68, 670–682.

Liu, Q., 2015. Correlation and common priors in games with incomplete information. J. Econ. Theory 157, 49–75.

Mertens, J.-F., Zamir, S., 1985. Formulation of Bayesian analysis for games with incomplete information. Int. J. Game Theory 14, 1–29.

Milgrom, P. R., Weber, R. J., 1985. Distributional strategies for games with incomplete information. Math. Oper. Res. 10, 619–632.

Morris, S., Shin, H. S., 2007. Common belief foundations of global games. Working paper.

Morris, S., Shin, H. S., Yildiz, M., 2016. Common belief foundations of global games. J. Econ. Theory 163, 826–848.

Morris, S., Ui, T., 2005. Generalized potentials and robust sets of equilibria. J. Econ. Theory 124, 45–78.

Monderer,D., Shapley, L. S., 1996. Potential games. Games Econ. Behav. 14, 124–143.

Oyama, D., Takahashi, S., 2019. Generalized belief operator and the impact of small probability events on higher order beliefs. Working paper.

Oyama, D., Takahashi, S., 2020. Generalized belief operator and robustness in binary-action supermodular games. Econometrica 88, 693–726.

Pram, K., 2019. On the Equivalence of robustness to canonical and general elaborations. J. Econ. Theory 180, 1–10.

Rubinstein, A., 1989. The electronic mail game: Strategic behavior under "Almost Common Knowledge." Amer. Econ. Rev. 385–391.

Ui, T., 2001. Robust equilibria of potential games. Econometrica 69, 1373–1380.

Ui, T., 2009. Bayesian potentials and information structures: Team decision problems revisited. Int. J. Econ. Theory 5, 271–291.