Are the Players in an Interactive Belief Model Meta-certain of the Model Itself?*

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Abstract

In an interactive belief model, are the players "commonly meta-certain" of the model itself? This paper explicitly formalizes such implicit "common meta-certainty" assumption. To that end, the paper expands the objects of players' beliefs from events to functions defined on the underlying states. Then, the paper defines a player's belief-generating map: it associates, with each state, whether a player believes each event at that state. The paper formalizes what it means by: "a player is (meta-)certain of her own belief-generating map" or "the players are (meta-)certain of the profile of belief-generating maps (i.e., the model)." The paper shows: a player is (meta-)certain of her own belief-generating map if and only if her beliefs are introspective. The players are commonly (meta-)certain of the model if and only if, for any event which some player i believes, it is common belief that player i believes the event. This paper then asks whether the "common meta-certainty" assumption is needed for an epistemic characterization of game-theoretic solution concepts. The paper shows: if each player is logical and (meta-)certain of her own strategy and belief-generating map, then each player correctly believes her own rationality. Consequently, common belief in rationality alone leads to actions that survive iterated elimination of strictly dominated actions.

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The common knowledge assumption underlies all of game theory and much of economic theory. Whatever be the model under discussion, whatever complete or incomplete information, consistent or inconsistent, repeated or one-shot, cooperative or non-cooperative, the model itself must be assumed common knowledge; otherwise the model is insufficiently specified, and the analysis incoherent.

—Aumann (1987b)

1 Introduction

In an economic or game-theoretic model in which the players make their interactive reasoning about their strategies or rationality, the analysts implicitly ("from outside of the model") assume that the players understand the model itself in a meta-sense. The above quotation from Aumann (1987b) suggests that the analysts should assume "the model is commonly known by the players" since otherwise "the model is insufficiently specified, and the analysis incoherent."

This paper has two objectives. The first is to explicitly formalize the "common knowledge" assumption of a model within the model itself. An interactive belief/knowledge model formally represents players' beliefs/knowledge about its ingredients, that is, events. The model itself does not tell whether the players (commonly) believe/know the model itself, although the analysts assume that the players (commonly) believe/know the model in a meta-sense. I refer to the knowledge/belief of the model as the "meta-knowledge/meta-belief" of the model.

The second objective is to examine the role that "meta-knowledge" of a model plays in game-theoretic analyses such as epistemic characterizations of solution concepts, robustness of solution concepts, or robustness of behaviors with respect to players' beliefs/knowledge. For a given epistemic characterization of a game-theoretic solution concept such as iterated elimination of strictly dominated actions, do the outside analysts need to formally assume that the players "meta-know" an epistemic model of a game (that describes their interactive beliefs about their strategies and rationality)?

Are the players (commonly) meta-certain of a model itself?[1] This first question has been puzzling theorists since the pioneering work of Aumann (1976, 1987a,b, 1999) on interactive knowledge models.[2] To date, one informal solution is the use of

[1]Since different epistemic models may feature different notions of qualitative or probabilistic beliefs or knowledge, I use the word the "(meta-)certainty" of a model to refer generically to the meta-knowledge or meta-belief of the model. In a probabilistic-belief model, by (meta-)certainty, it means that the players meta-believe the model with probability one. In a model of qualitative belief or knowledge whose degree of beliefs are stronger than probability-one belief, by (meta-)certainty, it means that the players meta-believe the model in the absolute sense or they meta-know the model.

[2]For this question, see also Bacharach (1985, 1990), Binmore and Brandenburger (1990), Brandenburger and Dekel (1989, 1993), Brandenburger, Dekel, and Geanakoplos (1992), Brandenburger and Keisler (2006), Dekel and Gul (1997), Fagin et al. (1999), Gilboa (1988), Myerson (1991), Pires

a "universal" belief model in which each state encodes what the players believe at that state and in which the differences in the players' beliefs are all described within the underlying states themselves (see, for example, Brandenburger and Dekel (1993), Heifetz and Samet (1998), and Mertens and Zamir (1985)). Subsequently, the various strands of robustness literature relax the implicit "common meta-certainty" assumption of an environment among players by studying how equilibrium or non-equilibrium solutions (or allocations) would depend on specifications of players' interactive beliefs on a "universal" belief model.[3]

The main result regarding the first objective (Theorems 1A and 1B in the contexts of qualitative and probabilistic beliefs, respectively) in Section 4 characterizes the "implicit common meta-certainty" assumption as follows. According to the formal test to be discussed, the players are commonly (meta-)certain of a model if and only if, for any event which some player i believes at some state, it is common belief that player i believes the event at that state. A universal belief model, in which the differences in the players' beliefs are incorporated within the states, indeed passes the formal test provided that the players' beliefs are introspective. Indeed, a given belief model is mapped to the universal belief model under the belief-preserving map, and under certain conditions, the given belief model is isomorphic to the corresponding subset of the universal space. The players are commonly certain of such "belief subspace" of the universal model within itself. However, the test itself is not directly related to whether a given model is rich in describing the players' interactive beliefs. Also, the test tells for any belief model whether the players are commonly (meta-)certain of the model itself.

Moving on to the second question, Section 5 examines the role that the "common meta-certainty" assumption plays in the epistemic characterization of iterated elimination of strictly dominated actions (IESDA) in a strategic game. Informally, it states: if the players are "logical," if they are commonly meta-certain of a game, and if they commonly believe their rationality, then the resulting actions survive any process of IESDA. Formally, it states: if the players commonly believe their rationality and if their common belief in their rationality is correct, then the resulting actions survive any process of IESDA.[4] The main result regarding the second objective (Theorem 2) connects these two statements. If the players' beliefs are monotone (they believe any logical implication of their beliefs), consistent (i.e., they do not simultaneously believe an event and its negation), and finitely conjunctive (if they believe E and F then they believe its conjunction $E \cap F$), and if each player is certain of her own

(1994), Roy and Pacuit (2013), Samuelson (2004), Vassilakis and Zamir (1993), Werlang (1987), Werlang and Tan (1992), and Wilson (1987).

[3]For robust mechanism design, see, for instance, Bergemann and Morris (2005), Heifetz and Neeman (2006), and Neeman (2004). For robustness of solution concepts, see, for example, Weinstein and Yildiz (2007) in the context of rationalizability.

[4]The formal statement is taken from Fukuda (2020, Theorem 3), which holds irrespective of the nature of beliefs. For seminal papers on implications of common belief in rationality, see, for example, Brandenburger and Dekel (1987), Stalnaker (1994), and Tan and Werlang (1988).

strategy and the part of her own belief-generating process in the model (each player is not necessarily certain of how the opponents' beliefs are generated in the model), then each player correctly believes her own rationality, and hence they have correct common belief in their rationality. Thus, if the players are "logical" and each of them is meta-certain of the part of the model that governs her own beliefs, then common belief in rationality leads to actions that survive any process of IESDA.

Now, I formally introduce a (belief) model described in Section 2. The model consists of the following three ingredients. The first is a measurable space of states of the word $(\Omega, \mathcal{D})$. Each state $\omega \in \Omega$ is a list of possible specifications of what the world is like, and the collection $\mathcal{D}$ of events (i.e., subsets of Ω) are the objects of the players' beliefs. The second is the players' monotone belief operators $(B_i)_{i \in I}$. Player i's belief operator B_i associates, with each event E, the event that player i believes E. Iterative application of belief operators unpacks higher-order interactive beliefs. Monotonicity means that if player i believes E at a state and if E implies (i.e., is included in) F, then she believes F at that state. To focus on the (meta-)certainty of a model and to separate it from reasoning ability, this paper assumes that each player believes any logical implication of her own beliefs. The third is a common belief operator C, which associates, with each event E, the event that E is common belief among the players. Under certain assumptions on the players' beliefs, an event E is common belief if and only if everybody believes E, everybody believes that everybody believes E, and so on *ad infinitum*.

This framework nests various models of qualitative and probabilistic beliefs and knowledge. Broadly, the framework nests the following two standard models of belief or knowledge (or combinations thereof). First, the framework nests a possibility correspondence model of qualitative belief or knowledge when each player's belief or knowledge is induced by a possibility correspondence.[5] The possibility correspondence associates, with each state, the set of states that she considers possible. The player believes an event E at a state whenever the possibility set at ω implies (i.e., is included in) the event E. Second, the framework nests a Harsanyi (1967-1968) type space when each player's probabilistic beliefs are induced by her type mapping. The type mapping τ_i associates, with each state ω, her probability measure $\tau_i(\omega)$ on the underlying states at that state. The type mapping τ_i of player i induces her p-belief operator $B^p_{\tau_i}$ (Monderer and Samet, 1989): it associates, with each event E, the event that (i.e., the set of states at which) player i believes E with probability at least p (i.e., p-believes E). Certain properties of p-belief operators $(B^p_{\tau_i})_{p \in [0,1]}$ reproduce the underlying type mapping τ_i (Samet, 2000).

With the framework in mind, I formalize the (meta-)certainty of a model in two steps. In the first step, Section 3.1 expands the objects of the players' beliefs from events to functions defined on the underlying states. Examples of such functions are random variables, strategies, and type mappings. Any such function x has to be

[5]See, for instance, Aumann (1976, 1999), Brandenburger, Dekel, and Geanakoplos (1992), Dekel and Gul (1997), Geanakoplos (1989), and Morris (1996).

defined on the state space Ω, but the co-domain X can be any set such as the set $\mathbb{R}$ of real numbers (a random variable), a set A_i of player i's actions (her strategy), and the set $\Delta(\Omega)$ of probability measures on $(\Omega, \mathcal{D})$ (a type mapping). I call the function $x : \Omega \to X$ a signal if its co-domain X has "observational" contents $\mathcal{X}$ (where "observation" is broadly construed as being an object of reasoning): it is a collection of subsets of X such that each $F \in \mathcal{X}$ is deemed an event $x^{-1}(F)$. A *signal (mapping)* is a function $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ with the "measurability" condition $x^{-1}(\mathcal{X}) \subseteq \mathcal{D}$. Player i is *certain* of the value of the signal x at a state ω if, for any observational content F that holds at ω (i.e., $\omega \in x^{-1}(F)$), player i believes the event $x^{-1}(F)$ at ω (i.e., $\omega \in B_i(x^{-1}(F))$). Player i is *certain* of x if she is certain of the value of x at every state. For example, let x be the strategy of player i and let every singleton action be observable; then, player i is certain of her own strategy if, wherever she takes an action $x(\omega)$, she believes that she takes action $x(\omega)$. Having defined individual players' (meta-)certainty, the players are *commonly certain* of the value of the signal x at a state ω if, for any observational content F that holds at ω, the event $x^{-1}(F)$ is common belief at ω (i.e., $\omega \in C(x^{-1}(F))$). The players are *commonly certain* of the signal x if they are commonly certain of its value at every state.

In the second step, Section 3.2 formulates a players' "belief-generating map" as a signal that associates, with each state, her beliefs at that state. By the second step, I can apply the formalization of certainty and common certainty in the first step to the ingredients of a given model (i.e., players' belief-generating maps). To that end, take player i's belief operator B_i from the model. I define a qualitative-type mapping t_{B_i}: it associates, with each state, whether player i believes each event or not at that state (formally, a binary set function from the collection of events to the binary values $\{0, 1\}$ where 1 indicates the belief of an event). The qualitative-type mapping is a binary "type" mapping analogous to a type mapping τ_i that represents player i's probabilistic beliefs at each state in the context of probabilistic beliefs. That is, the type mapping τ_i assigns, to each state ω, her probabilistic beliefs $\tau_i(\omega) \in [0, 1]$ on $(\Omega, \mathcal{D})$ at ω. In a similar manner, the qualitative-type mapping t_{B_i} associates, with each state ω, her qualitative belief $t_{B_i}(\omega) \in \{0, 1\}$ (where $t_{B_i}(\omega)(E) = 1$ if and only if $\omega \in B_i(E)$) on $(\Omega, \mathcal{D})$ at ω. The qualitative-type mapping t_{B_i} is player i's belief-generating mapping. Since the belief operator B_i and the qualitative-type mapping t_{B_i} are equivalent means of representing player i's beliefs, a model means the profile of qualitative-type mappings. Thus, the formal test for whether the players are commonly certain of a given belief model is whether the players are certain of the profile of their qualitative-type mappings.

Before asking when a player is certain of all the players' qualitative-type mappings (i.e., the model), Section 3.3 characterizes when a player is certain of her own qualitative type-mapping in terms of her introspective properties of beliefs. Roughly, Propositions 1A and 1B show that each player is certain of her own qualitative-type mapping if and only if her belief is introspective. These results distinguish the fact that player i is certain of her own qualitative-type mapping and the fact that player i is (or the players are commonly) certain of the profile of the type mappings.

Section 3.4 provides an alternative characterization of the fact that player i is certain of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ in terms of her reasoning of the signal. If she is certain of the signal x, then she would be able to rank the underlying states based on the collection of observational contents that hold at each state. Call a state ω *at least as informative as* a state ω' (according to the signal x) if, for any observational content F that holds at ω', it holds at ω. Section 3.4 then characterizes properties of beliefs and the certainty of a mapping in terms of the notion of informativeness. In the literature on type spaces such as Mertens and Zamir (1985), a player is "certain" of her own type if, at each state ω, she believes, with probability one, the set of states indistinguishable from (i.e., equally informative to) ω. I show that such "Harsanyi" property holds if and only if the player is (meta-)certain of her own type mapping in the strongest sense. Hence, I characterize the original idea behind Harsanyi (1967-1968) that each player "is certain of" her own type mapping.

To the best of my knowledge, this is the first paper which systematically formalizes the statement that the players are (commonly) (meta-)certain of any given belief model within the model itself. The main result on this question, nevertheless, is related to Gilboa (1988). He constructs a particular syntactic model in which the statement that the model is common knowledge is incorporated within itself. He formulates the sense in which the model is commonly known from Positive Introspection of common knowledge: if a statement is common knowledge then it is commonly known that the statement is common knowledge. In Theorems 1A and 1B, in contrast, the players are commonly certain of a given model if and only if, at each state and for any event which some player believes at that state, it is common belief that the player believes the event at that state. Thus, in this paper, the key criteria is the positive introspective property of common belief with respect to each player's beliefs. Whenever some individual player believes some event, it is common belief that she believes it. Bacharach (1985, 1990), in the context of partitional possibility correspondence models, formalizes the event that a player has an information partition by regarding it as a function.

The paper is organized as follows. Section 2 defines the basic framework of the paper, i.e., a belief model. Section 3 characterizes the sense in which each player is certain of how her belief is generated in a model. Section 4 examines the sense in which the players are commonly certain of a model itself (i.e., how the players' beliefs are generated in the model). Section 5 studies how the assumption that the players are commonly certain of a model itself can make game-theoretic analyses coherent. Section 6 provides concluding remarks. The proofs are relegated to Appendix A.

2 Framework

Throughout the paper, let I denote a non-empty finite set of *players*. The framework represents players' interactive beliefs by belief operators on a state space so that it can capture various forms of qualitative and probabilistic beliefs and knowledge. Section 2.1 first defines a belief model. Section 2.2 then defines properties of beliefs.

2.1 A Belief Model

A *belief model (of I)* is a tuple $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}), (B_i)_{i \in I}, C\rangle$, where: (i) $(\Omega, \mathcal{D})$ is a non-empty measurable space of states of the world (call Ω the *state space*); (ii) $B_i : \mathcal{D} \to \mathcal{D}$ is player i's (monotone) *belief operator*; and (iii) $C : \mathcal{D} \to \mathcal{D}$ is a (monotone) *common belief operator* to be defined in Expression (1) below.

While Ω constitutes a non-empty set of *states* of the world, each element E of $\mathcal{D}$ is an *event* about which the players reason. The assumption that $(\Omega, \mathcal{D})$ forms a measurable space accommodates players' qualitative and probabilistic beliefs in the same framework. Conceptually, the assumption means that: (i) any form of tautology Ω is an object of beliefs; (ii) if E is an object of beliefs, then so is its complement E^c (denote it also by $\neg E$); and that (iii) if $(E_n)_{n \in \mathbb{N}}$ are objects of beliefs, then so are its union $\bigcup_{n \in \mathbb{N}} E_n$ and its intersection $\bigcap_{n \in \mathbb{N}} E_n$.[6]

For each event E, the set $B_i(E)$ denotes the event that (i.e., the set of states at which) a player i believes E. Thus, the player $i \in I$ believes an event $E \in \mathcal{D}$ at a state $\omega \in \Omega$ if $\omega \in B_i(E)$. I assume that each player's belief operator satisfies *Monotonicity*: $E \subseteq F$ implies $B_i(E) \subseteq B_i(F)$. It means that if player i believes some event then she believes any of its logical consequences.

Since the players' beliefs are monotone, I introduce the common belief operator $C : \mathcal{D} \to \mathcal{D}$ following Monderer and Samet (1989). Call an event E *publicly evident* (e.g., Milgrom (1981)) if $E \subseteq B_I(E) := \bigcap_{i \in I} B_i(E)$. That is, everybody believes E whenever E is true. Denote by $\mathcal{J}_{B_I}$ the collection of publicly-evident events. An event E is *common belief* at a state ω if there is a publicly-evident event that is true at ω and that implies the mutual belief in E: that is, $\omega \in F \subseteq B_I(E)$ for some $F \in \mathcal{J}_{B_I}$. Now, C is assumed to satisfy that the set of states at which E is common belief is an event for each $E \in \mathcal{D}$:

$$C(E) := \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J}_{B_I} \text{ with } \omega \in F \subseteq B_I(E)\}. \tag{1}$$

[6]Technically, one can assume that $(\Omega, \mathcal{D})$ forms a κ-complete algebra and that I is a non-empty set with $|I| < \kappa$, where κ is a (regular) infinite cardinal or a symbol ∞ that plays a role of $\lambda < \infty$ for any cardinal λ. The assumption on I ensures that mutual beliefs are well-defined. The pair $(\Omega, \mathcal{D})$ is a *κ-complete algebra* if $\Omega \in \mathcal{D}$ and if $\mathcal{D}$ is closed under complementation and under any intersection of a collection of events with cardinality strictly less than κ. It is an *(∞-)complete algebra* if $\Omega \in \mathcal{D}$ and if $\mathcal{D}$ is closed under complementation and under arbitrary intersection. In this formulation, a measurable space is an $\aleph_1$-complete algebra, where $\aleph_1$ is the least uncountable cardinal.

Since players' beliefs are monotone and since $\mathcal{D}$ is closed under countable intersection, if E is common belief, then everybody believes E, everybody believes that everybody believes E, and so forth *ad infinitum*: $C(E) \subseteq \bigcap_{n \in \mathbb{N}} B_I^n(E)$. The converse (set inclusion) holds, for example, when the mutual belief operator B_I satisfies *Countable Conjunction*: $\bigcap_{n \in \mathbb{N}} B_I(E_n) \subseteq B_I(\bigcap_{n \in \mathbb{N}} E_n)$, meaning that everybody believes the countable conjunction of events whenever everybody believes each of them (also, B_I satisfies Countable Conjunction if every player's belief operator B_i satisfies it). Hence, if, for example, the mutual belief operator satisfies Countable Conjunction, then C is always a well-defined operator without incorporating it into the assumption.

While a possibility correspondence model often allows any subset of Ω to be an event, I represent the players' beliefs on a measurable space $(\Omega, \mathcal{D})$ instead of the power set algebra $(\Omega, \mathcal{P}(\Omega))$ so that I can analyze players' qualitative and probabilistic beliefs (such as the possibility correspondence and type space models) under the same framework. I will analyze the players' (countably-additive) probabilistic beliefs on a measurable space $(\Omega, \mathcal{D})$ by p-belief operators (Monderer and Samet, 1989).[7] For each $p \in [0, 1]$, player i's p-belief operator $B_i^p : \mathcal{D} \to \mathcal{D}$ associates, with each event E, the event that player i believes E with probability at least p (she *p-believes* E). I will also introduce the common p-belief operator C^p. Samet (2000) specifies conditions on p-belief operators under which a player's beliefs are equivalently represented by a type mapping $\tau_i : \Omega \to \Delta(\Omega)$ that associates, with each state of the world, the player's probabilistic beliefs at that state, where $\Delta(\Omega)$ denotes the set of countably-additive probability measures on $(\Omega, \mathcal{D})$. I will also analyze both qualitative and probabilistic beliefs at the same time: for example, in an extensive-form game with perfect information, each player has knowledge about players' past moves while she has beliefs about the future moves of the opponents.[8]

2.2 Properties of Beliefs

Next, I introduce additional eight properties of beliefs. Various possibility correspondence models of qualitative beliefs and knowledge are represented as belief models that satisfy certain properties specified below. Fix a player i. I first introduce the following five logical properties of beliefs.

1. *Necessitation*: $B_i(\Omega) = \Omega$.

2. *Countable Conjunction*: $\bigcap_{n \in \mathbb{N}} B_i(E_n) \subseteq B_i(\bigcap_{n \in \mathbb{N}} E_n)$ (for any events $(E_n)_{n \in \mathbb{N}}$).

3. *Finite Conjunction*: $B_i(E) \cap B_i(F) \subseteq B_i(E \cap F)$.

[7] While one can analyze finitely-additive or non-additive beliefs, for ease of exposition I focus on countably-additive probabilistic beliefs when it comes to quantitative beliefs.

[8] Moreover, the framework could naturally be extended to generalized state space models of unawareness (e.g., Heifetz, Meier, and Schipper (2006, 2013)). A state space Ω consists of multiple sub-spaces, and $\mathcal{D}$ is the collection of events (in their contexts). Players' belief and common belief operators are defined on the collection of events.

4. The *Kripke property*: $B_i(E) = \{\omega \in \Omega \mid b_{B_i}(\omega) \subseteq E\}$, where $b_{B_i}(\omega) := \bigcap\{E \in \mathcal{D} \mid \omega \in B_i(E)\}$ is the set of states player i considers *possible* at ω.

5. *Consistency*: $B_i(E) \subseteq (\neg B_i)(E^c)$.

First, Necessitation means that the player believes a tautology such as $E \cup E^c$. Second, as discussed, Countable Conjunction means that if the player beliefs each of a countable collection of events, then she believes its conjunction. In the probabilistic environment, if the player believes E_n with probability one for each $n \in \mathbb{N}$, she believes the intersection $\bigcap_{n\in\mathbb{N}} E_n$ with probability one. Third, Finite Conjunction is weaker than Countable Conjunction: if the player believes E and F then she believes its conjunction $E \cap F$. Fourth, to discuss the Kripke property, the player considers ω' possible at ω if, for any event E which she believes at ω, E is true at ω'. The Kripke property provides the condition under which i's belief is induced by her *possibility correspondence* $b_{B_i} : \Omega \to \mathcal{P}(\Omega)$: she believes E at ω if and only if (hereafter, iff) her possibility set $b_{B_i}(\omega)$ at ω implies E. In fact, B_i satisfies the Kripke property iff B_i is induced by some possibility correspondence $b_i : \Omega \to \mathcal{P}(\Omega)$: $B_i(E) = B_{b_i}(E) := \{\omega \in \Omega \mid b_i(\omega) \subseteq E\}$ (Fukuda, 2019a). Under the Kripke property, $B_i = B_{b_{B_i}}$ and $b_i = b_{B_{b_i}}$, i.e., the belief operator B_i and the possibility correspondence are equivalent representations of beliefs. The Kripke property implies the previous three properties as well as Monotonicity. Fifth, Consistency means that the player cannot simultaneously believe an event E and its negation E^c.

Next, I move on to truth and introspective properties.

6. *Truth Axiom*: $B_i(E) \subseteq E$ (for all $E \in \mathcal{D}$).

7. *Positive Introspection*: $B_i(\cdot) \subseteq B_iB_i(\cdot)$ (i.e., $B_i(E) \subseteq B_iB_i(E)$ for all $E \in \mathcal{D}$).

8. *Negative Introspection*: $(\neg B_i)(\cdot) \subseteq B_i(\neg B_i)(\cdot)$.

Sixth, Truth Axiom says that the player can only "know" what is true. Truth Axiom turns belief into knowledge in that knowledge has to be true while belief can be false. Truth Axiom implies Consistency. While knowledge satisfies Truth Axiom, qualitative and probabilistic beliefs are often assumed to satisfy Consistency. Seventh, Positive Introspection states that if the player believes some event then she believes that she believes it. Eighth, Negative Introspection states that if the player does not believe some event then she believes that she does not believe it. Truth Axiom and Negative Introspection yield Positive Introspection (e.g., Aumann (1999)).

Three remarks are in order. First, the introspective properties will play important roles in whether a player is (meta-)certain of a belief model. Intuitively, Positive Introspection provides the sense in which the player believes her own belief (at least at face value) while Negative Introspection yields the sense in which the player believes the lack of her own belief. To see these points formally, an event E is *self-evident* to i if $E \subseteq B_i(E)$. That is, i believes E whenever E is true. Positive Introspection means that i's belief in E is self-evident to i, and Negative Introspection means that i's lack of belief in E is self-evident to i. Denote by $\mathcal{J}_{B_i}$ the collection of self-evident events to i.

Second, the last four properties can be restated in terms of b_{B_i} under the Kripke property: B_i satisfies Consistency iff b_i is serial (i.e., $b_{B_i}(\cdot) \neq \emptyset$); B_i satisfies Truth Axiom iff b_{B_i} is reflexive (i.e., $\omega \in b_{B_i}(\omega)$ for all $\omega \in \Omega$); B_i satisfies Positive Introspection iff b_{B_i} is transitive (i.e., $\omega' \in b_{B_i}(\omega)$ implies $b_{B_i}(\omega') \subseteq b_{B_i}(\omega)$); and B_i satisfies Negative Introspection iff b_{B_i} is Euclidean (i.e., $\omega' \in b_{B_i}(\omega)$ implies $b_{B_i}(\omega) \subseteq b_{B_i}(\omega')$).

Third, various models of probabilistic and qualitative beliefs and knowledge take different sets of axioms. The framework accommodates possibility correspondence models of qualitative beliefs and knowledge when B_i satisfies the Kripke property. A partitional model of knowledge corresponds to the case when B_i satisfies Truth Axiom, Positive Introspection, and Negative Introspection.[9] A reflexive and transitive (non-partitional) possibility correspondence model is characterized by Truth Axiom and Positive Introspection.[10] When it comes to fully-introspective qualitative beliefs, b_{B_i} is serial, transitive, and Euclidean iff B_i satisfies Consistency, Positive Introspection, and Negative Introspection.

The probability 1-belief operator B_i^1 (that maps each event E to the event that player i believes E with probability one) would satisfy Necessitation, Countable Conjunction (thus Finite Conjunction), Consistency, and possibly Positive Introspection and Negative Introspection. With the framework defined in this section, for a model of (qualitative or probabilistic) belief or knowledge, I will study the formal sense in which the players are certain of the model.

3 When Is a Player Certain of Her Belief-Generating Mapping?

The previous section has defined a belief model, in which the objects of beliefs are events. Here, Section 3.1 first extends an object of beliefs in a model from an event to a function ("signal") defined on the state space. That is, the subsection formulates the statement that a player is certain of a function defined on the state space. Next, Section 3.2 represents a player's "belief-generating mapping" as a signal which associates, with each state, whether she believes each event or not. Then, Section 3.3 asks the sense in which she is certain of her own belief-generating mapping in terms of the introspective properties. Finally, Section 3.4 relates the introspective

[9]In fact, Truth Axiom, Negative Introspection, and the Kripke property yield all the other properties defined in this section.

[10]The literature on non-partitional possibility correspondence models studies information processing errors that lead to the failure of Negative Introspection. See, for example, Bacharach (1985), Binmore and Brandenburger (1990), Dekel and Gul (1997), Geanakoplos (1989), Lipman (1995), Pires (1994), Samet (1990), and Shin (1993).

properties to a notion of "informativeness" derived from a signal. Section 4, based on this formalization, provides a test by which the outside analysts can determine whether the players are (commonly) certain of a belief model.

3.1 Functions as Objects of Players' Beliefs

I start with defining a notion of a signal mapping. A signal mapping is any function x defined on the state space Ω with "observational" contents. For a non-empty set X and a non-empty subset $\mathcal{X}$ of $\mathcal{P}(X)$, call $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ a *signal* (mapping) if $x^{-1}(\mathcal{X}) \subseteq \mathcal{D}$. Mathematically, $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ is a signal if $x : (\Omega, \mathcal{D}) \to (X, \sigma(\mathcal{X}))$ is measurable. Examples include strategies, action/decision functions, random variables, state-contingent contracts, conditional expectations, and so on. A signal is interpreted as a mapping from the underlying state space Ω into the space of "observation" X endowed with "observational" contents $\mathcal{X}$. By observation, it means that each $F \in \mathcal{X}$ is deemed an object of reasoning. That is, each "observational" content $F \in \mathcal{X}$ can be regarded as an event $x^{-1}(F) \in \mathcal{D}$ through inverting the mapping.

The main purpose of this subsection is to define the statement that a player is certain of a signal. A player i is *certain of the value of* a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω, if she believes any observational content F (i.e., believes $x^{-1}(F)$) at ω whenever it is true: $x(\omega) \in F$. She is *certain of* the signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ if she is certain of its value at every ω. Likewise, the players are *commonly certain of the value of* the signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω, if the players commonly believe any observational content F at ω whenever it is true. The players are *commonly certain of* the signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ if they are certain of its value at every ω. Note that the word "certainty" is not necessarily related to probability-one belief. This terminology is used generically to refer to various probabilistic or non-probabilistic belief and knowledge (recall footnote 1). Formally:

Definition 1. Let $\overrightarrow{\Omega}$ be a belief model, and let $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ be a signal mapping.

1. (a) Player i is *certain of the value of the signal* $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω if $\omega \in B_i(x^{-1}(F))$ for any $F \in \mathcal{X}$ with $x(\omega) \in F$.

   (b) Player i is *certain of the signal* $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ if she is certain of the value of the signal x at any state.

2. (a) The players are *commonly certain of the value of the signal* $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω if $\omega \in C(x^{-1}(F))$ for any $F \in \mathcal{X}$ with $x(\omega) \in F$.

   (b) The players *are commonly certain of* the signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ if they are commonly certain of the value of the signal x is at every state.

For ease of terminology, player i is *certain of (the value of) the signal* $x : \Omega \to X$ (at ω) *with respect to* $\mathcal{X}$ if she is certain of (the value of) the signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ (at ω). Likewise, the players are *commonly certain of (the value of) the signal* $x : \Omega \to X$ (at ω) *with respect to* $\mathcal{X}$ if they are commonly certain of (the value of) the signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ (at ω).

Six remarks on Definition 1 are in order. First, Remark 1 below restates the fact that a player is certain of a signal in terms of self-evidence. Part (1) states that player i is certain of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ iff any observational content $F \in \mathcal{X}$ (i.e., any event $x^{-1}(F) \in \mathcal{D}$) is self-evident to i. Part (2) states that the players are commonly certain of the signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ iff any observational content $F \in \mathcal{X}$ is publicly-evident. Consequently, Part (3) says that the players are commonly certain of a signal iff every player is certain of it. Hence, for the outside analysts to assert that the players are commonly certain of a certain signal, it suffices to show that each player is certain of it.[11]

Remark 1. Let $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ be a signal.

1. Player i is certain of the signal x iff $x^{-1}(\mathcal{X}) \subseteq \mathcal{J}_{B_i}$.
2. The players are commonly certain of the signal x iff $x^{-1}(\mathcal{X}) \subseteq \mathcal{J}_{B_I}$.
3. The players are commonly certain of the signal x iff every player i is certain of the signal x.

Second, Remark 2 below shows that, when $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ is a player's strategy, Definition 1 formalizes the statement that the player is certain of the strategy in the literature on characterizations of solution concepts of games in state space models such as Brandenburger, Dekel, and Geanakoplos (1992) and Geanakoplos (1989). To see this, assume that $\mathcal{X}$ contains a singleton $\{x(\omega)\}$ to reason about the action taken at ω. That is, the set of states $[x(\omega)] := x^{-1}(\{x(\omega)\}) = \{\omega' \in \Omega \mid x(\omega') = x(\omega)\}$ at which player i takes the same action as she does at ω is an event. Since B_i satisfies Monotonicity, player i is certain of her action $x(\omega)$ (i.e., the value of the signal x) at ω iff $\omega \in B_i([x(\omega)])$, that is, player i believes that her action is $x(\omega)$ at ω. In fact:

Remark 2. Let X be a set of actions available to player i, and let $x : \Omega \to X$ be a strategy of player i with respect to $\mathcal{X} = \{\{x(\omega)\} \mid \omega \in \Omega\}$: the set of actions that could have been taken at each state. Then, Definition 1 states that player i is certain of her strategy iff $[x(\omega)]$ is self-evident at every $\omega \in \Omega$.

Definition 1 also subsumes the formulation of the certainty of the strategy by Aumann (1987a) in the (countable) partitional state space model of knowledge. Let

[11] In contrast, it is not necessarily the case that each player is certain of a signal at a state ω iff the players are commonly certain of the signal at ω. See Remark A.2 in Appendix A for an example.

$(b_{B_i}(\omega))_{\omega\in\Omega}$ be a countable partition on Ω. In Aumann (1987a), the player "knows" her own strategy x iff the strategy x is measurable with respect to the partition (which turns out to be equivalent to $b_{B_i}(\cdot) \subseteq [x(\cdot)]$). Since the partition is countable, the σ-algebra generated by the partition is equal to the self-evident collection: $\mathcal{J}_{B_i} = \sigma(\{b_{B_i}(\omega) \in \mathcal{D} \mid \omega \in \Omega\})$. Hence, player i is certain of her strategy $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ iff $x : (\Omega, \mathcal{J}_{B_i}) \to (X, \sigma(\mathcal{X}))$ is measurable.

Third, Remark 3 below states that if player i is certain of a signal $x : \Omega \to X$ with respect to the collection of singletons $\{\{a\} \mid a \in X\}$, then she is certain of x, in the strongest sense, with respect to $\mathcal{P}(X)$. To see this, it can be shown that if player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$, then she is certain of $x : \Omega \to X$ with respect to the collection of unions of $\mathcal{X}$: $\{\bigcup_{\lambda\in\Lambda} F_\lambda \in \mathcal{P}(X) \mid \{F_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{X}\}$.

Remark 3. Player i is certain of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{P}(X))$ iff she is certain of $x : \Omega \to X$ with respect to the collection of values of x: $\{\{x(\omega)\} \mid \omega \in \Omega\}$. For example, player i is certain of her strategy $x : (\Omega, \mathcal{D}) \to (X, \mathcal{P}(X))$ in the strongest sense iff she is certain of her strategy with respect to the actions $\{\{x(\omega)\} \mid \omega \in \Omega\}$ that she could have taken.

Fourth, Remark 4 provided conditions on beliefs under which a player is certain of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ iff she is certain of $x : (\Omega, \mathcal{D}) \to (X, \sigma(\mathcal{X}))$.

Remark 4. Under the following conditions on player i's belief operator B_i (in addition to Monotonicity), player i is certain of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ iff she is certain of $x : (\Omega, \mathcal{D}) \to (X, \sigma(\mathcal{X}))$.

1. B_i satisfies Consistency, Countable Conjunction, Positive Introspection, and Negative Introspection.

2. B_i satisfies Truth Axiom and Negative Introspection.

In Part (1), the collection of i's beliefs $\mathcal{B}_i := \{B_i(E) \in \mathcal{D} \mid E \in \mathcal{D}\}$ forms a sub-σ-algebra of $\mathcal{D}$. The conditions in Part (2) imply those in Part (1). In Part (2), $\mathcal{B}_i$ coincides with the self-evident collection $\mathcal{J}_{B_i}$. Interestingly, fully-introspective qualitative or probability-one beliefs satisfy the conditions in Part (1), and fully-introspective knowledge satisfies the conditions in Part (1).

Fifth, Remark 5 below shows that each player i satisfies Necessitation iff she is certain of any constant signal. Likewise, the common belief operator C satisfies Necessitation (equivalently, every B_i satisfies Necessitation) iff the players are commonly certain of any constant signal.

Remark 5.

1. Player i's belief operator B_i satisfies Necessitation iff she is certain of any constant signal.

2. The common belief operator C satisfies Necessitation iff the players are commonly certain of any constant signal iff she is certain of any constant signal.

In light of the certainty of a signal, Necessitation allows the players to be certain of any constant "random" variable that does not depend on the realization of a state. For example, consider whether player i is certain that an event $B_j(E)$ is equal to an event F in a belief model. The outside analysts determine whether player i believes that player j believes an event E at a state ω by examining whether $\omega \in B_i B_j(E)$ since player j's belief $B_j(E)$ itself is an event. The (implicit) assumption in any (semantic) belief model is that $E = F$ implies $B_i(E) = B_i(F)$. Thus, if two events are extensionally the same (e.g., E is the set of 1 and -1, and F is the set of real solutions of $x^2 = 1$) then each player's belief in the two events are the same.[12] To assess player i's belief about player j's belief about E, how can the outside analysts justify the fact that player i is able to equate $B_j(E)$ with another event (say, F)? Since either $B_j(E) = F$ or $B_j(E) \neq F$, player i is *certain that* $B_j(E)$ *is an event* F if player i is certain of the indicator function $\mathbb{I}_{B_j \leftrightarrow F}$, where $(B_j(E) \leftrightarrow F) := ((\neg B_j)(E) \cup F) \cap ((\neg F) \cup B_j(E))$. If player i's belief operator B_i satisfies Necessitation and if $B_j(E) = F$, then player i is certain of the constant indicator function $\mathbb{I}_{B_j \leftrightarrow F}$. Thus, under Necessitation, player i is certain that $B_j(E) = F$ if it is indeed the case. This argument justifies that, under Necessitation, the outside analysts can say that the players are certain of equating two extensionally equivalent events (say, $B_j(E)$ and F) if they are indeed extensionally equivalent. As discussed in footnote 12, one can construct a rich belief model in which the identification of two events is minimized. This argument states that, somewhat differently from such construction, the players are certain of the the identification of two events. Section 3.3.2 also provides another example of an implication of Necessitation.

Necessitation also follows from the fact that player i is certain of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ when $X = \bigcup_{F \in \mathcal{X}} F$. Thus, for example, if player i is certain of her strategy $x : (\Omega, \mathcal{D}) \to (X, \{\{a\} \mid a \in X\})$, then B_i satisfies Necessitation.

Sixth, Remark 6 below shows that player i is certain of a profile of signals (e.g., a strategy profile) iff she is certain of each of them. Observe that if $\varphi : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ satisfies $\varphi^{-1}(\mathcal{Y}) \subseteq \mathcal{X}$, then the composite $\varphi \circ x : (\Omega, \mathcal{D}) \to (Y, \mathcal{Y})$ is a signal. Then:

Remark 6. Let A be a non-empty set, and let $x_\alpha : (\Omega, \mathcal{D}) \to (X_\alpha, \mathcal{X}_\alpha)$ be a signal for each $\alpha \in A$. Let $X := \prod_{\alpha \in A} X_\alpha$, and let $\pi_\alpha : X \to X_\alpha$ be the projection. Every $x_\alpha : (\Omega, \mathcal{D}) \to (X_\alpha, \mathcal{X}_\alpha)$ is a signal iff $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ is a signal, where $\mathcal{X} := \bigcup_{\alpha \in A} \{(\pi_\alpha)^{-1}(F_\alpha) \in \mathcal{P}(X) \mid F_\alpha \in \mathcal{X}_\alpha\}$. It can be seen that player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ iff she is certain of every $x_\alpha : (\Omega, \mathcal{D}) \to (X_\alpha, \mathcal{X}_\alpha)$. Under either condition in Remark 4, player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \sigma(\mathcal{X}))$ iff she is certain of every $x_\alpha : (\Omega, \mathcal{D}) \to (X_\alpha, \mathcal{X}_\alpha)$, where $\sigma(\mathcal{X})$ is the product σ-algebra if each $\mathcal{X}_\alpha$ is a σ-algebra.

[12] Although such identification of events are implicitly assumed for any (semantic) belief model, one can construct a canonical ("universal") semantic model from a syntactic language which maximally distinguishes the denotations of events. In the canonical model, such identification of events can be minimized in a way such that two events are equated only when they are explicitly assumed to be equivalent by the outside analysts (see Fukuda (2019c) for a formal assertion).

Remarks 1 and 6 imply that the players are commonly certain of a profile of signals iff every player is certain of every signal.

3.2 A Qualitative-Type Mapping that Represents a Player's Beliefs

In order to formulate a test under which the outside analysts can examine whether the players are commonly certain of a belief model, I define the "belief-generating map," which I call the qualitative-type mapping, of a player. Given the belief operator of the player, the qualitative-type mapping associates, with each state, a binary value indicating whether the player believes each event in an analogous manner to the type mapping in the type-space literature.[13]

To that end, recall the notion of probabilistic types. A (probabilistic-)type is a σ-additive probability measure $\nu \in \Delta(\Omega)$. A (probabilistic-)type mapping is a measurable mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_{\Delta})$, where $\mathcal{D}_{\Delta(\Omega)}$ is the σ-algebra generated by $\beta_E^p := \{\nu \in \Delta(\Omega) \mid \nu(E) \geq p\}$ for all $(E, p) \in \mathcal{D} \times [0,1]$ (Heifetz and Samet, 1998). It associates, with each state ω, the player's probabilistic beliefs $\tau_i(\omega) \in \Delta(\Omega)$ at that state. Given the type mapping τ_i, define player i's p-belief operator $B_{\tau_i}^p : \mathcal{D} \to \mathcal{D}$ as $B_{\tau_i}^p(E) := \tau_i^{-1}(\beta_E^p)$. Thus, $\omega \in B_{\tau_i}^p(E)$ iff $\tau_i(\omega)(E) \geq p$. As in Samet (2000), the type mapping τ_i and the collection of p-belief operators $(B_{\tau_i}^p)_{p \in [0,1]}$ are equivalent, that is, a type space of the form $\langle (\Omega, \mathcal{D}), (\tau_i)_{i \in I} \rangle$ is equivalent to $\langle (\Omega, \mathcal{D}), (B_{\tau_i}^p)_{(i,p) \in I \times [0,1]} \rangle$.

With this in mind, let $M(\Omega)$ be the set of binary set functions $\mu : \mathcal{D} \to \{0,1\}$ (i.e., $M(\Omega) \subseteq \{0,1\}^{\mathcal{D}}$) that satisfy a given set of logical properties of beliefs defined in Section 2.2 (these properties will be shortly expressed in terms of μ). Call each $\mu \in M(\Omega)$ a *qualitative-type*. Interpret $\mu(E) = 1$ as the belief in an event $E \in \mathcal{D}$. Once $M(\Omega) \subseteq \{0,1\}^{\mathcal{D}}$ is defined as the set of qualitative-types that satisfy the given set of logical properties of beliefs, I represent player i's beliefs by a *qualitative-type mapping* $t_i : \Omega \to M(\Omega)$ satisfying a certain measurability condition specified below. It is a measurable mapping which associates, with each state $\omega \in \Omega$, player i's qualitative-type $t_i(\omega) \in M(\Omega)$ at ω. Thus, player i believes an event E at ω if $t_i(\omega)(E) = 1$.

Now, I define the logical properties of μ in an analogous way to the corresponding logical properties of belief operators. Fix $\mu \in \{0,1\}^{\mathcal{D}}$.

0. *Monotonicity*: $E \subseteq F$ implies $\mu(E) \leq \mu(F)$.
1. *Necessitation*: $\mu(\Omega) = 1$.
2. *Countable Conjunction*: $\min_{n \in \mathbb{N}} \mu(E_n) \leq \mu(\bigcap_{n \in \mathbb{N}} E_n)$.
3. *Finite Conjunction*: $\min(\mu(E), \mu(F)) \leq \mu(E \cap F)$.
4. The *Kripke property*: $\mu(E) = 1$ iff $\bigcap \{F \in \mathcal{D} \mid \mu(F) = 1\} \subseteq E$.

[13] Fukuda (2017, Section 6) constructs a universal knowledge space consisting of hierarchies of qualitative-types that dictate players' interactive knowledge.

5. *Consistency*: $\mu(E) \leq 1 - \mu(E^c)$.

The interpretations of the above properties are similar to those in Section 2.2. For Countable Conjunction, if E_n is believed (i.e., $\mu(E_n) = 1$) for every $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} E_n$ is believed (i.e., $\mu(\bigcap_{n\in\mathbb{N}} E_n) = 1$). The Kripke property characterizes the condition under which player i's beliefs are induced by a possibility correspondence when every qualitative-type $t_i(\omega)$ satisfies it. Whether all of these properties are assumed or not depend on the model that the outside analysts study. For example, if the outside analysts examine a partitional possibility correspondence model, then since B_i satisfies all the logical properties, $M(\Omega)$ is the set of qualitative-types that satisfy all the logical properties. In contrast, if the outside analysts study a belief model in which only Monotonicity is assumed, then $M(\Omega)$ is the set of qualitative-types that satisfy Monotonicity.

I formally define the measurability condition of a qualitative-type mapping. A *qualitative-type mapping* is a measurable mapping $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ which satisfies given (logical and) introspective properties of beliefs, where $\mathcal{D}_M$ is the σ-algebra generated by the sets of the form $\beta_E := \{\mu \in M(\Omega) \mid \mu(E) = 1\}$ for all $E \in \mathcal{D}$. Note that $t_i : \Omega \to M(\Omega)$, by construction, satisfies given logical properties because any element in $M(\Omega)$ satisfies them. For example, if every $\mu \in M(\Omega)$ satisfies the Kripke property, then every $t_i(\omega)$ satisfies it. Denote $b_{t_i}(\omega) := \bigcap\{E \in \mathcal{D} \mid t_i(\omega)(E) = 1\}$ for each $\omega \in \Omega$.

The measurablity condition of t_i requires each $t_i^{-1}(\beta_E) = \{\omega \in \Omega \mid t_i(\omega)(E) = 1\}$ to be the event that player i believes E. Next, I define Truth Axiom and the introspective properties of t_i.

6. *Truth Axiom*: $t_i(\omega)(E) = 1$ implies $\omega \in E$.

7. *Positive Introspection*: $t_i(\omega)(E) = 1$ implies $t_i(\omega)(\{\omega' \in \Omega \mid t_i(\omega')(E) = 1\}) = 1$ (i.e., $t_i(\omega)(t_i^{-1}(\beta_E)) = 1$).

8. *Negative Introspection*: $t_i(\omega)(E) = 0$ implies $t_i(\omega)(\{\omega' \in \Omega \mid t_i(\omega')(E) = 0\}) = 1$ (i.e., $t_i(\omega)(\neg t_i^{-1}(\beta_E)) = 1$).

So far, a qualitative-type mapping $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ is introduced. Finally, I demonstrate that a belief operator and a qualitative-type mapping are equivalent. A given belief operator B_i induces the qualitative-type mapping t_{B_i} by

$$t_{B_i}(\omega)(E) := \begin{cases} 1 & \text{if } \omega \in B_i(E) \\ 0 & \text{otherwise} \end{cases}.$$

Conversely, a given qualitative-type mapping t_i induces the belief operator B_{t_i} defined as $B_{t_i}(E) := t_i^{-1}(\beta_E)$. It can be seen that $B_{t_{B_i}} = B_i$ and $t_i = t_{B_{t_i}}$.

3.3 Certainty of Own Type Mapping

I apply the certainty of a signal to a qualitative- and probabilistic-type mapping. The main results of this subsection are Propositions 1A and 1B. Roughly, they state that a player is certain of her own qualitative- and probabilistic-type mapping iff her beliefs are introspective, respectively. Hereafter, "A" and "B" in Proposition, Theorem, and Remark refer to the case with qualitative and probabilistic beliefs, respectively.

3.3.1 Certainty of Own Qualitative-Type Mapping

I start with the certainty of a qualitative-type mapping.

Proposition 1A. *Let $\overrightarrow{\Omega}$ be a belief model, and let $t_{B_i} : \Omega \to M(\Omega)$ be player i's qualitative-type mapping.*

1. (a) *Player i is certain of t_{B_i} with respect to $\{\beta_E \mid E \in \mathcal{D}\}$ iff B_i satisfies Positive Introspection $B_i(\cdot) \subseteq B_i B_i(\cdot)$.*
   (b) *Player i is certain of t_{B_i} with respect to $\{\neg\beta_E \mid E \in \mathcal{D}\}$ iff B_i satisfies Negative Introspection $(\neg B_i)(\cdot) \subseteq B_i(\neg B_i)(\cdot)$.*
   (c) *If player i is certain of $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$, then B_i satisfies Positive Introspection and Negative Introspection.*
2. (a) *Let B_i satisfy Truth Axiom. Player i is certain of $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ iff B_i satisfies (Positive Introspection and) Negative Introspection.*
   (b) *Let B_i satisfy Consistency and Countable Conjunction. Player i is certain of $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ iff B_i satisfies Positive Introspection and Negative Introspection.*

While Part (1) characterizes the certainty of the qualitative-type mapping t_{B_i} with respect to the possession or lack of beliefs, Part (2) examines the sense in which player i is certain of her qualitative-type mapping $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ in a model of knowledge (i.e., Part (2a)) and belief (i.e., Part (2b)).

Part (1a) states that player i is certain of her qualitative-type mapping t_{B_i} with respect to the possession of beliefs iff her belief operator B_i satisfies Positive Introspection. Parts (1a) and (1b) jointly state that B_i satisfies Positive Introspection and Negative Introspection iff player i is certain of her qualitative-type mapping t_{B_i} with respect to $\{\beta_E \mid E \in \mathcal{D}\} \cup \{\neg\beta_E \mid E \in \mathcal{D}\}$.

I discuss three additional implications of Proposition 1A. First, the proposition sheds light on the literature of non-partitional knowledge models in which Negative Introspection fails. The question is, when a player commits an information-processing error leading to the failure of Negative Introspection, is she certain of her own possibility correspondence?[14] The dichotomous answer leads to the following issue. If

[14] See, for instance, Brandenburger, Dekel, and Geanakoplos (1992), Dekel and Gul (1997), and Geanakoplos (1989)

the player is certain of her own possibility correspondence, then she is "certain" that she commits the information-processing error and yet she fails to overcome the lack of Negative Introspection. If she is not certain of her own possibility correspondence, where do her beliefs come from?

Part (1) provides the following eclectic answer: player i is not fully certain of her qualitative-type mapping. That is, without imposing Negative Introspection, player i is not certain of her own qualitative-type mapping with respect to $\mathcal{D}_M$ (or $\{\beta_E \mid E \in \mathcal{D}\} \cup \{\neg\beta_E \mid E \in \mathcal{D}\}$). Rather, she takes her own information at face value in the sense that she is only certain of her qualitative-type mapping with respect to her own beliefs $\{\beta_E \mid E \in \mathcal{D}\}$. Proposition 1A formalizes the very sense in which "she takes her own information at face value."

In contrast, Proposition 1A (2a) shows that, in a partitional possibility correspondence model of knowledge, the axioms of Truth Axiom, (Positive Introspection) and Negative Introspection characterize the sense in which a player is fully certain of her possibility correspondence. While the proposition does not necessarily require B_i to satisfy the Kripke property, consider a model of knowledge in which B_i satisfies Truth Axiom and the Kripke property, i.e., B_i is induced by the reflexive possibility correspondence b_{B_i}. Then, player i is certain of her "knowledge-generating" mapping iff B_i satisfies (Positive Introspection and) Negative Introspection.

Likewise, Proposition 1A (2b) demonstrates that, in a serial possibility correspondence model, the axioms of Consistency, Positive Introspection and Negative Introspection characterize the sense in which a player is fully certain of her possibility correspondence (note that the Kripke property implies Countable Conjunction).

In the above arguments, I have identified the statement that player i is certain of her possibility correspondence b_{B_i} with the one that she is certain of her qualitative-type mapping t_{B_i}. Since the belief operator B_i and the possibility correspondence b_{B_i} are equivalent (under the Kripke property) and since the belief operator B_i and the qualitative-type mapping t_{B_i} are equivalent representations of beliefs, b_{B_i} and t_{B_i} are equivalent. More directly, one can restate Proposition 1A in terms of player i's possibility correspondence $b_i : \Omega \to \mathcal{P}(\Omega)$ (where b_i is induced from either t_i or B_i), under the Kripke property: b_i satisfies $b_i^{-1}(\{F \in \mathcal{D} \mid F \subseteq E\}) \in \mathcal{D}$ for all $E \in \mathcal{D}$. For example, player i is certain of $b_i : (\Omega, \mathcal{D}) \to (\mathcal{P}(\Omega), \{\{F \in \mathcal{P}(\Omega) \mid F \subseteq E\} \mid E \in \mathcal{D}\})$ iff B_{b_i} satisfies Positive Introspection. Likewise, player i is certain of $b_i : (\Omega, \mathcal{D}) \to (\mathcal{P}(\Omega), \{\{F \in \mathcal{P}(\Omega) \mid E^c \cap F \neq \emptyset\} \mid E \in \mathcal{D}\})$ iff B_{b_i} satisfies Negative Introspection.

Second, Proposition 1A also sheds light on the identification of events discussed in Section 3.1. The belief operator B_i of player i associates, with each event E, the event $B_i(E)$ that she believes E. Since the players' beliefs themselves are events, player i can reason about player j's belief in E: $B_i B_j(E)$. However, the question is how does player i evaluate another player j's belief in E? The implicit assumption is again that "the belief model is commonly certain among the players."

In Proposition 1A, Positive Introspection and Negative Introspection pertain to every event E including $E = B_j(F)$ for some $F \in \mathcal{D}$. This means that if player i is certain of her qualitative-type mapping t_{B_i} with respect to $\{\beta_E \mid E \in \mathcal{D}\}$ then she is also certain of such qualitative-type mapping as $t_{B_iB_j}$ with respect to $\{\beta_E \mid E \in \mathcal{D}\}$, where $t_{B_iB_j}$ is the qualitative-type mapping associated with the operator B_iB_j (i.e., $t_{B_iB_j}(\omega)(E) = 1$ iff $\omega \in B_iB_j(E)$). That is, if player i is certain of her own belief-generating mapping, then she is also certain of the mapping that generates i's belief about j's belief about events. Although it is implicitly assumed in the belief model that player i figures out what B_j is,[15] the fact that each i is certain of $t_{B_iB_j}$ could possibly be a justification for why the outside analysts can assume that "i is certain of j's belief operator in i's mind." Section 4 studies the question whether the outside analysts can assume that each player i is certain of each other's qualitative-type mapping t_{B_j} (note that here I ask whether each player i can be certain of how she herself can evaluate an opponent's belief-generating process through studying whether player i is certain of the mapping $t_{B_iB_j}$ that generates the beliefs of player i about player j's beliefs).

Third, I can Proposition 1A to the case where a player has qualitative belief and knowledge. Consider a belief model $\langle(\Omega, \mathcal{D}), (K_i)_{i\in I}\rangle$ where $K_i : \mathcal{D} \to \mathcal{D}$ is player i's (monotone) knowledge operator (for simplicity, omit the common-knowledge operator). Now, for each player i, let $B_i : \mathcal{D} \to \mathcal{D}$ be her (monotone) qualitative-belief operator. Let t_{B_i} be player i's qualitative-type mapping that represents B_i, and ask whether player i is certain of her qualitative-type mapping $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$. Proposition 1A (2a) implies that player i is certain of $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ iff K_i satisfies *Positive Certainty* (with respect to B_i): $B_i(\cdot) \subseteq K_iB_i(\cdot)$ and *Negative Certainty* (with respect to B_i): $(\neg B_i)(\cdot) \subseteq K_i(\neg B_i)(\cdot)$. Whenever player i believes an event, she knows that she believes it. Whenever player i does not believe an event, she knows that she does not believe it. In fact, these two properties are often assumed in a model of belief and knowledge, and Proposition 1A (2a) justifies the assumptions in terms of the certainty of one's knowledge about her own beliefs.[16]

3.3.2 Certainty of Own Probabilistic-Type Mapping

Next, I study when a player is certain of her own (probabilistic-)type mapping. As in the previous discussion, a belief operator B_i satisfies *Positive Certainty* (with respect to $B^p_{\tau_i}$) if $B^p_{\tau_i}(\cdot) \subseteq B_iB^p_{\tau_i}(\cdot)$. Likewise, B_i satisfies *Negative Certainty* (with respect to $B^p_{\tau_i}$) if $(\neg B^p_{\tau_i})(\cdot) \subseteq B_i(\neg B^p_{\tau_i})(\cdot)$.

[15] Recalling footnote 12, this pertains to the assumption in any (semantic) belief model that if $E = F$ then $B_i(E) = B_i(F)$. Such identification of events can be minimized in the "universal" belief model, which is constructed from a syntactic language in a way such that two events are identified only when they are explicitly assumed to be equivalent by the outside analysts.

[16] In a model of knowledge and belief, often Entailment $K_i(\cdot) \subseteq B_i(\cdot)$ is also assumed. However, it is well-known that if the knowledge and belief operators K_i and B_i fully introspective, i.e., if K_i satisfies Truth Axiom, (Positive Introspection), and Negative Introspection, and if B_i satisfies Consistency, Positive Introspection, and Negative Introspection, then Positive Certainty, Negative Certainty, and Entailment yield $K_i = B_i$ (e.g., Lenzen (1978)). Proposition 1A does not yield or presuppose Entailment $K_i(\cdot) \subseteq B_i(\cdot)$ in a model of knowledge and qualitative belief.

Proposition 1B. *Let $\overrightarrow{\Omega}$ be a belief model, and let $\tau_i : \Omega \to \Delta(\Omega)$ be player i's type mapping.*

1. (a) *Player i is certain of her type mapping τ_i with respect to $\{\beta_E^p \mid (p, E) \in [0,1] \times \mathcal{D}\}$ iff B_i satisfies Positive Certainty: $B_{\tau_i}^p(\cdot) \subseteq B_i B_{\tau_i}^p(\cdot)$.*

   (b) *Player i is certain of her type mapping τ_i with respect to $\{\neg\beta_E^p \mid (p, E) \in [0,1] \times \mathcal{D}\}$ iff B_i satisfies Negative Certainty: $(\neg B_{\tau_i}^p)(\cdot) \subseteq B_i(\neg B_{\tau_i}^p)(\cdot)$.*

   (c) *If player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$, then B_i satisfies Positive Certainty and Negative Certainty.*

2. (a) *Let B_i satisfy Truth Axiom and Negative Introspection. Player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ iff B_i satisfies Positive Certainty iff B_i satisfies Negative Certainty.*

   (b) *Let B_i satisfy Consistency, Countable Conjunction, Positive Introspection, and Negative Introspection. Player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ iff B_i satisfies Positive Certainty and Negative Certainty.*

   (c) *Let B_i satisfy Entailment: $B_i(\cdot) \subseteq B_{\tau_i}^1$. Player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ iff B_i satisfies Positive Certainty and Negative Certainty.*

Part (1) characterizes the statement that player i is certain of her type mapping τ_i with respect to the possession of p-beliefs $\{\beta_E^p \mid (E, p) \in \mathcal{D} \times [0,1]\}$ or the lack of p-beliefs $\{\neg\beta_E^p \mid (E, p) \in \mathcal{D} \times [0,1]\}$. It also states that if player i is certain of the type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ then the belief operator B_i satisfies Positive Certainty and Negative Certainty: $B_{\tau_i}^p(\cdot) \subseteq B_i B_{\tau_i}^p(\cdot)$ and $(\neg B_{\tau_i}^p)(\cdot) \subseteq B_i(\neg B_{\tau_i}^p)(\cdot)$.

The difference between Propositions 1A and 1B emerges in Part (2). In Proposition 1B, player i may have two kinds of beliefs: B_i from the given belief model and p-beliefs $B_{\tau_i}^p$ from her type mapping. If $B_i = B_i^1$ then Proposition 1B (2c) asks whether player i is certain of her type mapping within the type space $\langle(\Omega, \mathcal{D}), (\tau_i)_{i \in I}\rangle$ itself. If B_i is either a knowledge (i.e., Part (2a)) or qualitative belief operator (i.e., Part (2b)), then the outside analysts consider players' knowledge or qualitative beliefs about their probabilistic beliefs.

Part (2a) corresponds to the case when B_i is a fully-introspective knowledge operator in addition to her type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$. Similarly, Part (2b) corresponds to the case in which B_i is a fully-introspective qualitative belief operator in addition to her type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$. When probabilistic beliefs and knowledge (or qualitative belief) are present, the introspective properties of Positive Certainty $B_{\tau_i}^p(\cdot) \subseteq B_i B_{\tau_i}^p(\cdot)$ and Negative Certainty $(\neg B_{\tau_i}^p)(\cdot) \subseteq B_i(\neg B_{\tau_i}^p)(\cdot)$ are the standard assumptions (e.g., Aumann (1999)). Whenever player i believes an event E with probability at least p, she knows that she believes E with probability at least p. Whenever player i does not believe an event E with probability at least p, she knows that she does not believe E with probability at least p. In this environment, Parts (2a) and (2b) formalize the sense in which player i is certain of her probabilistic beliefs (her type mapping).

Part (2c) sheds light on the certainty of a type mapping in the type space (i.e., purely probabilistic model) when B_i is taken as the probability 1-belief operator $B^1_{\tau_i}$. The introspective properties of probabilistic beliefs are now formulated in terms of probability-one belief about own beliefs: $B^p_{\tau_i}(\cdot) \subseteq B^1_{\tau_i} B^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq B^1_{\tau_i}(\neg B^p_{\tau_i})(\cdot)$. That is, if player i believes an event E with probability at least p, then she believes with probability one that she p-believes E; and if player i does not believe an event E with probability at least p, then she believes with probability one that she does not p-believe E. These two introspective properties are essential in the syntactic formulation of type spaces such as Heifetz and Mongin (2001) and Meier (2012). Part (2c) justifies the statement that player i is certain of her own type mapping in a type space.

Three remarks on Part (2c) are in order. First, in the type space literature, the informal assumption that each player is certain of her own type is represented as the condition on the type mapping to put probability one on the set of types indistinguishable from its own (Mertens and Zamir, 1985; Vassilakis and Zamir, 1993). While Proposition 3B in Section 3.4.2 examines this condition in terms of the notion of "informativeness" of a signal, this condition implies the above two introspective properties and is indeed equivalent to them under a technical assumption on the state space.

Second, Part (2c) shows that, in order for player i to be certain of her type mapping, her beliefs in her p-beliefs have to be at least as strong as probability-one belief $B^1_{\tau_i}$. While Entailment ($B_i(\cdot) \subseteq B^1_{\tau_i}$) is often interpreted as the axiom stating that qualitative-belief (or knowledge) is at least as strong as probability-one belief, Part (2c) provides an alternative interpretation of Entailment: probability-one belief is the weakest form of beliefs under which player is certain of her probabilistic beliefs.

Third, Part (2c) also justifies the structural assumption in a product type space: player i's type is a probability measure on an underlying set of nature states S and the types of the opponents $(T_j)_{j \in I \setminus \{i\}}$. Informally, this structural assumption means that each type of each player is certain of her own type and thus each player is certain of her own type mapping. Formally, Part (2c) implies that, in the product type space, each player is certain of her own type mapping. In the framework of this paper, it is well-known that such a product type space is subsumed as a non-product type space (referred to as a belief space by Mertens and Zamir (1985)). In the non-product type space, the marginal of each type on the player's own type space is the Dirac measure, which, in turn, implies that each player's probability-one belief operator satisfies Positive Certainty and Negative Certainty. Hence, Part (2c) implies that, in the formal sense, player i is certain of her own type mapping. See Appendix A.3 for a formal discussion.

I remark on two additional implications of Proposition 1B. First, Proposition 1B and Remark 5 allow one to formalize the sense in which each player is certain of her "prior." Consider a tuple $\langle(\Omega,\mathcal{D}),(\tau_i)_{i\in I},(\mu_i)_{i\in I}\rangle$ with the following properties: $(\Omega,\mathcal{D})$ is a measurable space, $\tau_i:(\Omega,\mathcal{D})\to(\Delta(\Omega),\mathcal{D}_\Delta)$ is player i's (probabilistic-)type mapping, and $\mu_i\in\Delta(\Omega)$ is a *prior* satisfying

$$\mu_i(E)=\int_\Omega \tau_i(\omega)(E)\mu_i(d\omega) \text{ for all } E\in\mathcal{D}.$$

That is, the prior belief $\mu_i(E)$ is equal to the expectation of the posterior beliefs $t_i(\cdot)(E)$ with respect to the prior μ_i (see, for instance, Mertens and Zamir (1985)). The model admits a *common prior* if $\mu_i=\mu_j$ for all $i,j\in I$. If each μ_i would be identified as a constant mapping $\mu_i:(\Omega,\mathcal{D})\to(\Delta(\Omega),\mathcal{D}_\Delta)$, then

$$\mu_i^{-1}(\beta_E^p)=\begin{cases}\emptyset & \text{if } \mu_i(E)<p\\ \Omega & \text{if } \mu_i(E)\geq p\end{cases}.$$

Hence, each player i is certain of every player j's prior. In fact, the players are commonly certain of the priors.

Second, if the players are certain of their own type mappings, then the common p-belief operator reduces to the iteration of mutual p-beliefs and is well-defined.

Remark 7. Let $(\Omega,\mathcal{D})$ be a measurable space, and let $\tau_i:(\Omega,\mathcal{D})\to(\Delta(\Omega),\mathcal{D}_\Delta)$ be player i's type mapping for each $i\in I$. Let $B_{\tau_i}^1$ be player i's probability-one belief operator. If each player i is certain of her type mapping $\tau_i:(\Omega,\mathcal{D})\to(\Delta(\Omega),\mathcal{D}_\Delta)$ (according to her probability-one belief), then the common p-belief operator reduces to the iteration of mutual p-beliefs and is well-defined: $C^p(\cdot)=\bigcap_{n\in\mathbb{N}}(B_I^p)^n(\cdot)\in\mathcal{D}$.

To see this, Negative Certainty $(\neg B_{\tau_i}^p)(\cdot)\subseteq B_{\tau_i}^1(\neg B_{\tau_i}^p)(\cdot)$ follows from the assumption. Then, player i's p-belief operator satisfies $B_{\tau_i}^p B_{\tau_i}^p(\cdot)\subseteq B_{\tau_i}^p(\cdot)$ (in fact, it holds with equality since $B_{\tau_i}^p(\cdot)\subseteq B_{\tau_i}^1 B_{\tau_i}^p(\cdot)\subseteq B_{\tau_i}^p B_{\tau_i}^p(\cdot)$). If this is the case for every player, so does the mutual p-beliefs: $B_I^p B_I^p(\cdot)\subseteq B_I^p(\cdot)$, where $B_I^p(\cdot):=\bigcap_{i\in I}B_{\tau_i}^p(\cdot)$ as in Section 2.1. It means that the chain of mutual p-beliefs is decreasing. Since mutual p-beliefs are preserved for a decreasing sequence of events (i.e., if $E_n\downarrow E$ then $B_I^p(E_n)\downarrow B_I^p(E)$), the common p-belief operator $C^p:\mathcal{D}\to\mathcal{D}$ defined according to Expression (1) reduces to the iteration of mutual p-beliefs.

3.4 Informativeness, Possibility, and Certainty

If a player is certain of a signal $x:(\Omega,\mathcal{D})\to(X,\mathcal{X})$, then, for each state ω, she would be able to conceive the collection of observational contents $F\in\mathcal{X}$ which hold at ω. Comparing such collections among all states, she would be able to rank the states according to "informativeness." Thus, for the informational ranking on the states induced by the given signal, if the player is certain of the signal then the information derived from the signal must have already been incorporated in her beliefs.

Section 3.4.1 defines the informativeness of a signal (Definition 2), and applies the informativeness criteria to qualitative- and probabilistic-type mappings to characterize the sense in which a player is certain of her type mapping in terms of informativeness (Propositions 2A and 2B). Section 3.4.2 studies the assumption in a Harsanyi (1967-1968) type space in terms of the informativeness (Proposition 3A and 3B): at each state, a player assigns probability-one to the set of states indistinguishable from (i.e., equally informative to) the state.

3.4.1 Informativeness, Possibility, and Certainty

I start with defining the informativeness of a signal:

Definition 2. For states ω and ω' in Ω, ω is *at least as informative as ω' according to a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$* if

$$\{F \in \mathcal{X} \mid \omega' \in x^{-1}(F)\} \subseteq \{F \in \mathcal{X} \mid \omega \in x^{-1}(F)\}. \tag{2}$$

States ω and ω' are *equally informative according to $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$* if

$$\{F \in \mathcal{X} \mid \omega' \in x^{-1}(F)\} = \{F \in \mathcal{X} \mid \omega \in x^{-1}(F)\}. \tag{3}$$

The ideas behind Definition 2 are (i) that the informational content of a signal mapping $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω is expressed as the collection of observational contents $\{F \in \mathcal{X} \mid x(\omega) \in F\}$ true at ω and (ii) that informational contents are ranked by the implication in the form of set inclusion.[17] While the notion of informativeness (i.e., the relation induced by Expression (2)) is reflexive and transitive, the notion of equal informativeness (i.e., the relation induced by Expression (3)) is an equivalence relation.

If player i is certain of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$, then the notion of possibility derived from her beliefs is incorporated in the notion of informativeness derived from the signal: suppose player i is certain of the signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$. If she considers a state ω' possible at a state ω (i.e., $\omega' \in b_{B_i}(\omega)$) then ω' is at least as informative as ω according to $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$. Hence, possibility implies informativeness according to the signal, when the player is certain of the signal. In fact:

Remark 8. Assume the Kripke property for B_i. Player i is certain of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ iff possibility implies informativeness (i.e., if $\omega' \in b_{B_i}(\omega)$ then ω' is at least as informative as ω according to $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$).

Three additional remarks are in order. First, when $\mathcal{X}$ is not necessarily closed under complementation, Definition 2 does not take into account the collection of

[17] The notion of informativeness is closely related to that of information studied by Bonanno (2002). Ghirardato (2001), Lipman (1995), and Mukerji (1997) also study (not-fully-introspective) information processing in which informational contents are ranked by the implication in the form of set inclusion.

observational contents $\{F \in \mathcal{X} \mid x(\omega) \notin F\}$ that do not hold at ω. In contrast, when $\mathcal{X}$ is closed under complementation, if ω is at least as informative as ω' according to a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$, then ω and ω' are equally informative. Second, somewhat similarly, suppose that B_i satisfies Consistency, Positive Introspection, and Negative Introspection. If ω is at least as informative as ω' according to a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$, then ω and ω' are equally informative. Third, under the assumption that $x^{-1}(\{\omega\}) \in \mathcal{D}$ for each $\omega \in \Omega$, the equivalence relation of equal informativeness coincides with the one induced by the partition $\{x^{-1}(\{x(\omega)\}) \mid \omega \in \Omega\}$: ω and ω' are equally informative iff $x(\omega) = x(\omega')$.

Next, I apply the notion of informativeness to i's qualitative-type mapping $t_i : \Omega \to M(\Omega)$ with respect to $\{\beta_E \mid E \in \mathcal{D}\}$. That is, suppose that player i is reasoning about the underlying states based on her possession of beliefs. For states ω and ω' in Ω, ω is at least as informative as ω' to i (precisely, according to $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \{\beta_E \mid E \in \mathcal{D}\}))$ iff $t_i(\omega')(\cdot) \leq t_i(\omega)(\cdot)$ (i.e., $t_i(\omega')(E) \leq t_i(\omega)(E)$ for all $E \in \mathcal{D}$). Likewise, states ω and ω' are equally informative according to i iff $t_i(\omega) = t_i(\omega')$.

Fix $\omega \in \Omega$, and let $(\uparrow t_i(\omega)) := \{\omega' \in \Omega \mid t_i(\omega)(\cdot) \leq t_i(\omega')(\cdot)\}$ be the set of states that are at least as informative to i as ω. Also, define $(\downarrow t_i(\omega)) := \{\omega' \in \Omega \mid t_i(\omega')(\cdot) \leq t_i(\omega)(\cdot)\}$ and $[t_i(\omega)] := \{\omega' \in \Omega \mid t_i(\omega) = t_i(\omega')\}$. If $\omega' \in [t_i(\omega)]$ then ω and ω' are indistinguishable to player i in that her qualitative-types (and thus the collections of events that she believes) are exactly the same at these states. Put differently, the equal informativeness is translated into the indistinguishability. Thus, the collection $\{[t_i(\omega)] \mid \omega \in \Omega\}$ forms a partition of Ω generated by the qualitative-type mapping t_i. Note that $(\uparrow t_i(\omega))$, $(\downarrow t_i(\omega))$, and $[t_i(\omega)]$ may not necessarily be events.

Before characterizing the statement that a player is certain of her qualitative-type mapping in terms of informativeness, I remark that if ω' is at least as informative to i as ω according to t_i (i.e., $\omega' \in (\uparrow t_i(\omega))$), then

$$b_{t_i}(\omega') = \bigcap\{E \in \mathcal{D} \mid t_i(\omega')(E) = 1\} \subseteq \bigcap\{E \in \mathcal{D} \mid t_i(\omega)(E) = 1\} = b_{t_i}(\omega).$$

That is, the informativeness relation implies the set-inclusion between possibility sets. Note that if t_i satisfies the Kripke property, then the converse also holds: $b_{t_i}(\omega') \subseteq b_{t_i}(\omega)$ implies $\omega' \in (\uparrow t_i(\omega))$. This is simply because, if $t_i(\omega)(E) = 1$ then $b_{t_i}(\omega') \subseteq b_{t_i}(\omega) \subseteq E$ and thus $t_i(\omega')(E) = 1$. In other words, under the Kripke property, the possibility set can alternatively be used to define the notion of informativeness: $\omega' \in (\uparrow t_i(\omega))$ iff $b_{t_i}(\omega') \subseteq b_{t_i}(\omega)$.

Now, I examine the sense in which a player is certain of her qualitative-type mapping by studying how introspective properties imply the relations between informativeness and possibility.

Proposition 2A. *Let $(\Omega, \mathcal{D})$ be a measurable space, and let $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \{\beta_E \mid E \in \mathcal{D}\})$ be player i's qualitative-type mapping.*

1. *(a) B_i satisfies Truth Axiom iff $(\omega \in [t_{B_i}(\omega)] \subseteq)(\uparrow t_{B_i}(\omega)) \subseteq b_{B_i}(\omega)$ (for all $\omega \in \Omega$).*

(b) *If B_i satisfies Positive Introspection, then $b_{B_i}(\cdot) \subseteq (\uparrow t_{B_i}(\cdot))$. If B_i satisfies the Kripke property, the converse also holds.*

(c) *If B_i satisfies Negative Introspection, then $b_{B_i}(\cdot) \subseteq (\downarrow t_{B_i}(\cdot))$. If B_i satisfies the Kripke property, the converse also holds.*

2. (a) *If B_i satisfies Truth Axiom and Positive Introspection, then $(\uparrow t_{B_i}(\cdot)) = b_{B_i}(\cdot)$. If t_{B_i} satisfies the Kripke property, the converse also holds.*

   (b) *If B_i satisfies Truth Axiom, (Positive Introspection), and Negative Introspection, then $(\uparrow t_{B_i}(\cdot)) = (\downarrow t_{B_i}(\cdot)) = [t_{B_i}(\cdot)] = b_{B_i}(\cdot)$. If B_i satisfies the Kripke property, the converse also holds.*

Part (1a) states that, under Truth Axiom, informativeness implies possibility. In Part (1b), since she is certain of her qualitative-type mapping t_i with respect to $\{\beta_E \mid E \in \mathcal{D}\}$, the notion of possibility that comes from her beliefs is already encoded in the notion of informativeness. That is, Part (1b) states that possibility implies informativeness when player i is certain of her qualitative-type mapping $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \{\beta_E \mid E \in \mathcal{D}\})$.

Hence, when player i's qualitative-type mapping satisfies Truth Axiom and Positive Introspection as in a reflexive-and-transitive possibility correspondence model (see footnote 10), the notions of informativeness and possibility coincide: $b_{B_i}(\cdot) = (\uparrow t_{B_i}(\cdot))$. A simple corollary of this argument is that, as with possibility correspondence models, if B_i satisfies Truth Axiom and Positive Introspection as well as the Kripke property, then b_{B_i} is reflexive and transitive.

Part (1b) and (1c) jointly state that if player i considers ω' possible at ω then the states ω and ω' are equally informative. The intuition is as follows. Suppose player i considers ω' possible at ω. Since player i is certain of her own qualitative-type mapping t_i with respect to the possession of her beliefs $\{\beta_E \mid E \in \mathcal{D}\}$, the state ω' is at least as informative as ω. However, since player i is also certain of her own qualitative-type mapping t_i with respect to the lack of her beliefs $\{\neg\beta_E \mid E \in \mathcal{D}\}$, that is, since she is able to reason about the informational contents $F \in \mathcal{X}$ that do not realize at each state, she must be able to compare the collections $\{\beta_E \mid \omega' \not\in t_i^{-1}(\beta_E)\}(= \{F \in \mathcal{X} \mid \omega' \not\in x_i^{-1}(F)\})$ and $\{\beta_E \mid \omega \not\in t_i^{-1}(\beta_E)\}(= \{F \in \mathcal{X} \mid \omega \not\in x_i^{-1}(F)\})$. Since she is positively introspective, the former collection is a sub-collection of the latter. However, since she is able to reason about the lack of her beliefs, if she is certain of her qualitative-type mapping with respect to the lack of beliefs, she must be able to infer that these two collections have to coincide. Otherwise, she must be able to update her beliefs based on reasoning about her own qualitative-type mapping. Thus, when x is taken as a player's qualitative-type mapping, the conceptual distinction between Expressions (2) and (3) is analogous to the difference in introspective abilities coming from non-partitional (reflexive and transitive) and partitional models of knowledge studied in Proposition 1A.

In a model of knowledge in which player i's qualitative-type mapping satisfies Truth Axiom, (Positive Introspection,) and Negative Introspection, either notion of informativeness or possibility induces the same partition $\{b_{B_i}(\omega) \mid \omega \in \Omega\} = \{[t_{B_i}(\omega)] \mid \omega \in \Omega\}$ of Ω with the following property: if $\omega' \in [t_i(\omega)] = b_{t_i}(\omega)$, then, for any event, she knows it at ω iff she knows it at ω'. In a model of qualitative belief in which player i's qualitative-type mapping satisfies Consistency, Positive Introspection, and Negative Introspection, $\emptyset \neq b_{B_i}(\cdot) \subseteq [t_{B_i}(\cdot)](= (\uparrow t_{B_i}(\cdot)) = (\downarrow t_{B_i}(\cdot)))$.

Next, I apply the notion of informativeness to player i's probabilistic-type mapping $\tau_i : \Omega \to \Delta(\Omega)$ with respect to $\{\beta^p_E \mid (E, p) \in \mathcal{D} \times [0, 1]\}$. That is, player i is reasoning about the underlying states based on her possession of p-beliefs. However, note that since the notion of possibility comes from qualitative beliefs, I consider a model that has both qualitative and probabilistic beliefs (implications to probabilistic-type spaces will be discussed in Section 3.4.2).

A state ω is at least as informative as a state ω' to i (precisely, according to $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \{\beta^p_E \mid (E, p) \in \mathcal{D} \times [0, 1]\}))$ iff $\tau_i(\omega')(\cdot) \leq \tau_i(\omega)(\cdot)$. However, since each $\tau_i(\cdot)$ is (countably-)additive, it follows that ω is at least as informative as ω' to i iff ω and ω' are equally informative: $\omega' \in [\tau_i(\omega)] := \{\omega'' \in \Omega \mid \tau_i(\omega'') = \tau_i(\omega)\}$. If player i does not believe an event E with probability at least p at a state, then she does believe E^c with probability at least $1 - p$. Since player i is able to reason about the possession of beliefs for any event and any probability, she is also able to reason about the lack of beliefs when her probabilistic beliefs are (countably-)additive. While one can obtain a nuanced understanding of the relation between the informativeness and certainty of a type mapping τ_i when each $\tau_i(\cdot)$ is a non-additive measure (indeed, one can analyze the statement that player i is certain of such non-additive type mapping τ_i in this framework), I focus on studying the sense in which player i is certain of her (countably-additive) type mapping τ_i (recall footnote 7).[18]

Proposition 2B. *Let $\overrightarrow{\Omega}$ be a belief model, and let $\tau_i : \Omega \to \Delta(\Omega)$ be player i's type mapping.*

1. *Either Positive Certainty $B^p_{\tau_i}(\cdot) \subseteq B_i B^p_{\tau_i}(\cdot)$ or Negative Certainty $(\neg B^p_{\tau_i})(\cdot) \subseteq B_i(\neg B^p_{\tau_i})(\cdot)$ yields $b_{B_i}(\cdot) \subseteq [\tau_i(\cdot)]$: possibility implies (equal) informativeness.*

2. *Under the Kripke property of B_i, all are equivalent: $B^p_{\tau_i}(\cdot) \subseteq B_i B^p_{\tau_i}(\cdot)$ iff $(\neg B^p_{\tau_i})(\cdot) \subseteq B_i(\neg B^p_{\tau_i})(\cdot)$ iff $b_{B_i}(\cdot) \subseteq [\tau_i(\cdot)]$.*

3.4.2 Harsanyi Property

Next, I move on to studying the notion of informativeness in a type space. To that end, player i's type mapping $\tau_i : \Omega \to \Delta(\Omega)$ satisfies the *Harsanyi property* if

[18] In the decision theory literature, Ghirardato (2001) and Mukerji (1997) connect non-additive beliefs and a player's limited introspection, where informational contents of an act at each state are ranked by set inclusion as in Expression (2). Fukuda (2019b) studies the interaction between qualitative and probabilistic beliefs without assuming properties on beliefs. It is shown that the additivity of a type mapping plays a role in disentangling Positive and Negative Certainty of qualitative beliefs with respect to probabilistic beliefs.

$[\tau_i(\omega)] \subseteq E$ implies $\omega \in B^1_{\tau_i}(E)$ for any $(\omega, E) \in \Omega \times \mathcal{D}$. That is, whenever an event E is implied by the set of states $[\tau_i(\omega)]$ indistinguishable from ω, player i believes E with probability one at ω (e.g., Meier (2008, 2012) and Mertens and Zamir (1985)).

Before I will characterize the Harsanyi property in terms of certainty and informativeness in a type space, two remarks on the relation between the Harsanyi property and the Kripke property are in order. To that end, consider a qualitative belief model, and recall Proposition 2B.

First, suppose qualitative and probabilistic beliefs satisfy Entailment $B_i(\cdot) \subseteq B^1_{\tau_i}(\cdot)$. Suppose also that player i is certain of her type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \{\beta^p_E \mid (E, p) \in \mathcal{D} \times [0, 1]\})$. Thus, $b_{B_i}(\cdot) \subseteq [\tau_i(\cdot)]$ holds as in Proposition 2B. Then, the Kripke property yields the Harsanyi property: if $[\tau_i(\omega)] \subseteq E$ then it follows from $b_{B_i}(\omega) \subseteq E$ that $\omega \in B_i(E) \subseteq B^1_{\tau_i}(E)$.

Second, to examine the distinction between probability-one and qualitative beliefs in terms of the Kripke property, especially, the conjunction property given by the Kripke property, apply $B_i = B^1_{\tau_i}$ to Proposition 2B (2). If the probability-one belief operator $B^1_{\tau_i}$ satisfies the Kripke property, the Kripke property for the probability-one belief operator may be too strong because it gives an unlimited reasoning ability about the conjunction of events: for any collection $(E_\lambda)_{\lambda \in \Lambda}$ of events with $\bigcap_{\lambda \in \Lambda} E_\lambda \in \mathcal{D}$, $\bigcap_{\lambda \in \Lambda} B^1_{\tau_i}(E_\lambda) \subseteq B^1_{\tau_i}(\bigcap_{\lambda \in \Lambda} E_\lambda)$. For example, if player i believes an event $\Omega \setminus \{\omega\}$ with probability one for every $\omega \in \Omega$, then she must believe the interaction, which is the empty set, with probability one.

In a type space, I show that the Harsanyi property characterizes the idea that a player is certain of her own type mapping with respect to the beliefs that she could have been able to possess (i.e., in the strongest sense).

Proposition 3B. *Let $(\Omega, \mathcal{D})$ be a measurable space, and let $\tau_i : \Omega \to \Delta(\Omega)$ be player i's type mapping.*

1. *Suppose that $[\tau_i(\cdot)] \in \mathcal{D}$. The type mapping τ_i satisfies the Harsanyi property iff player i is certain of $\tau_i : \Omega \to \Delta(\Omega)$ with respect to her realized beliefs $\{\{\tau_i(\omega)\} \mid \omega \in \Omega\}$.*

2. *Let $\mathcal{D}$ be generated from a countable algebra. The following are all equivalent.*

   (a) *The type mapping τ_i satisfies the Harsanyi property.*

   (b) *Player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \{\{\tau_i(\omega)\} \mid \omega \in \Omega\})$.*

   (c) *Player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$.*

   (d) *Player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \{\beta^p_E \mid (E, p) \in \mathcal{D} \times [0, 1]\})$.*

   (e) *Player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \{\neg\beta^p_E \mid (E, p) \in \mathcal{D} \times [0, 1]\})$.*

In Part (1), under the regularity condition $[\tau_i(\cdot)] \in \mathcal{D}$, the Harsanyi property is equivalent to $\tau_i(\omega)([\tau_i(\omega)]) = 1$ for each $\omega \in \Omega$. It states that, at each state, player i assign probability one to the set of states indistinguishable from that state.

The technical condition in Part (2) yields the technical condition in Part (1). It states that the Harsanyi property is equivalent to $B^{p}_{\tau_i}(\cdot) \subseteq B^{1}_{\tau_i} B^{p}_{\tau_i}(\cdot)$ or $(\neg B^{p}_{\tau_i})(\cdot) \subseteq B^{1}_{\tau_i}(\neg B^{p}_{\tau_i})(\cdot)$. Generally, without any technical condition on $\mathcal{D}$, the Harsanyi property implies $B^{p}_{\tau_i}(\cdot) \subseteq B^{1}_{\tau_i} B^{p}_{\tau_i}(\cdot)$ and $(\neg B^{p}_{\tau_i})(\cdot) \subseteq B^{1}_{\tau_i}(\neg B^{p}_{\tau_i})(\cdot)$.

One can define the analogue of the Harsanyi property for qualitative belief: player i believes an event E at ω if E is implied by the set $[t_{B_i}(\omega)]$ of states indistinguishable from ω. The proposition below shows that the analogue of the Harsanyi property characterizes the certainty of the qualitative-type mapping in the strongest sense.

Proposition 3A. *Let* $\overrightarrow{\Omega}$ *be a belief model such that* $[t_{B_i}(\cdot)] \in \mathcal{D}$.

1. *The following are equivalent.*
   (a) *Player* i *is certain of her qualitative-type mapping* $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \{\{t_{B_i}(\omega)\} \mid \omega \in \Omega\})$.
   (b) *For any* $(\omega, E) \in \Omega \times \mathcal{D}$ *with* $[t_{B_i}(\omega)] \subseteq E$, $\omega \in B_i(E)$.
2. *If* B_i *satisfies the Kripke property, Positive Introspection, and Negative Introspection, then player* i *is certain of her qualitative-type mapping* $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \{\{t_{B_i}(\omega)\} \mid \omega \in \Omega\})$.
3. *Under either condition in Part (1), Truth Axiom yields the Kripke property.*

Proposition 3A hinges on the regularity condition $[t_{B_i}(\cdot)] \in \mathcal{D}$, as in Proposition 3B. Since $[t_{B_i}(\omega)] = \bigcap_{E \in \mathcal{D}: \omega \in B_i(E)} B_i(E) \cap \bigcap_{E \in \mathcal{D}: \omega \in (\neg B_i)(E)} (\neg B_i)(E)$, if $\mathcal{D}$ is a complete algebra then the regularity condition holds. Part (1) is similar to Proposition 3B: under the regularity condition, the Harsanyi property states that a player is certain of her own type mapping in the strongest sense. Part (2) states that, in a possibility correspondence model, if a player's belief is fully introspective then she is certain of her qualitative-type mapping (or her possibility correspondence) in the strongest sense. In Part (3), if either condition in Part (1) holds and if B_i satisfies Truth Axiom, then B_i also satisfies Positive Introspection and Negative Introspection. Then, $b_{B_i}(\cdot) = [t_{B_i}(\cdot)]$.

4 When are the Players Commonly Certain of a Bleief Model?

With the analyses in Section 3 in mind, I formalize the sense in which the players are commonly certain of a belief model itself: the players are commonly certain of the profile of their (qualitative- or probabilistic-)type mappings. By Remark 2 and 6, it is sufficient to ask when every player i is certain of each player j's (qualitative- or probabilistic-)type mapping.

I start with a qualitative belief model. Proposition 1A applies to the case in which player i is certain of player j's qualitative-type mapping. For example, if player i is certain of player j's qualitative-type mapping $t_j : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$, then $B_{t_j}(\cdot) \subseteq B_i B_{t_j}(\cdot)$ and $(\neg B_{t_j})(\cdot) \subseteq B_i(\neg B_{t_j})(\cdot)$ hold. Proposition 1A implies:

Remark 9A. Let $\overrightarrow{\Omega}$ be a belief model, and let $t_{B_j} : \Omega \to M(\Omega)$ be player j's qualitative-type mapping.

1. (a) Player i is certain of t_{B_j} with respect to $\{\beta_E \mid E \in \mathcal{D}\}$ iff $B_j(\cdot) \subseteq B_i B_j(\cdot)$.
   (b) Player i is certain of t_{B_j} with respect to $\{\neg\beta_E \mid E \in \mathcal{D}\}$ iff $(\neg B_j)(\cdot) \subseteq B_i(\neg B_j)(\cdot)$.
   (c) If player i is certain of $t_{B_j} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$, then $B_j(\cdot) \subseteq B_i B_j(\cdot)$ and $(\neg B_j)(\cdot) \subseteq B_i(\neg B_j)(\cdot)$.
2. (a) Let B_i satisfy Truth Axiom. Player i is certain of $t_{B_j} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ iff ($B_j(\cdot) \subseteq B_i B_j(\cdot)$ and) $(\neg B_j)(\cdot) \subseteq B_i(\neg B_j)(\cdot)$.
   (b) Let B_i satisfy Consistency and Countable Conjunction. Player i is certain of $t_{B_j} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ iff $B_j(\cdot) \subseteq B_i B_j(\cdot)$ and $(\neg B_j)(\cdot) \subseteq B_i(\neg B_j)(\cdot)$.

Roughly, Remark 9A states that player i is certain of player j's qualitative-type mapping t_{B_j} if and only if (i) whenever player j believes an event E at ω, player i believes player j believes E at ω; and (ii) whenever player j does not believe an event E at ω, player i believes player j does not believe E at ω.

Now, I move to one of the main questions of this paper: I ask when the players are commonly certain of the qualitative-type mappings in a belief model.

Theorem 1A. *Let $\overrightarrow{\Omega}$ be a belief model, and let $t_{B_i} : \Omega \to M(\Omega)$ be player i's qualitative-type mapping for each $i \in I$.*

1. *Assume Truth Axiom for every B_i. The players are commonly certain of the profile of qualitative-type mappings $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ iff $B_i = B_j$ for every $i, j \in I$, (Positive Introspection $B_i(\cdot) \subseteq B_i B_i(\cdot)$), and Negative Introspection $(\neg B_i)(\cdot) \subseteq B_i(\neg B_i)(\cdot)$. In particular, $B_i = C$ for each $i \in I$.*

2. *Assume Consistency and Countable Conjunction for every B_i. The players are commonly certain of the profile of qualitative-ytpe mappings $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ iff $B_i(\cdot) \subseteq C B_i(\cdot)$ and $(\neg B_i)(\cdot) \subseteq C(\neg B_i)(\cdot)$ for every $i \in I$. In particular, $C = B_I$.*

While Part (1) studies a knowledge model, Part (2) does a belief model. I start with discussing three implications of Part (2). In words, Part (2) states that the players are commonly certain of their qualitative-type mappings iff (i) for any event E which some player i believes at some state ω, it is commonly believed that player

i believes E at ω; and (ii) for any event E which some player i does not believe at some state ω, it is commonly believed that player i does not believe E at ω.

First, one sufficient condition for Part (2) is met in a "universal" belief model. In the universal belief model, the differences in the players' beliefs are not described by the differences in their belief operators (or their qualitative-type mappings), but are incorporated within the underlying states. In the universal belief model in which the players are introspective, the players are commonly certain of the model. Hence, Theorem 1A formally justifies the statement that the players are commonly certain of a universal belief model.

Second, Part (2) sheds light on the interpretation of a state as a "complete" description of the world. If, in a given belief model, the differences in the players' beliefs are not described by their belief operators but are incorporated within the states themselves, then the players with introspective beliefs are commonly certain of the model itself. If, by "complete" descriptions, it means that each state describes the possible ways in which the players may have possibly different beliefs, then the statement that the players are commonly certain of the model is formally guaranteed by Theorem 1A.[19] This formalizes the informal argument of Aumann (1976) that the players in a model are commonly certain of the model (their information partitions) if the underlying states describe how their beliefs are given within themselves.

Third, the mutual belief and common belief operators coincide if the players are commonly certain of their qualitative-type mappings, under the mild conditions of Consistency and Countable Conjunction.[20] This is because, for any event E which everybody believes at some state ω, it is commonly believed that everybody believes E at ω: $B_I(\cdot) \subseteq CB_I(\cdot)$. Intuitively, in a model of which the players are commonly certain, if everybody believes an event E then it is common belief that everybody believes E. Thus, if everybody believes E then everybody believes that everybody believes E. Hence, the first-order mutual belief itself implies any higher-order mutual beliefs, and thus the mutual and common beliefs coincide. In a "universal" space in which the players' beliefs are identical, i.e., $B_i = B_j$ for all $i, j \in I$, and in which each B_i satisfies at least Positive Introspection in addition to Monotonicity, $B_i = B_I = C$ for each $i \in I$.

Next, I discuss Part (1). This part provides a contrast between knowledge and belief. In a knowledge model with Truth Axiom, for the players to be commonly certain of the model, it is *necessary* that their knowledge coincides with each other. In contrast, in a belief model without Truth Axiom, it may be the case that the

[19] In a syntactic model of qualitative beliefs and knowledge such as Aumann (1999), Fukuda (2019c), and Gilboa (1988), each state consists of the set of syntactic formulas, intending to express players' beliefs, that hold at that state. Hence, each state encodes the statements (events) which the players believe at that state within itself.

[20] The converse does not hold, i.e., $C = B_I$ does not necessarily imply that the players are certain of the profile of their qualitative-type mappings. As a simple example, let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2\}, \mathcal{P}(\Omega))$, $B_1(E) = E$, and $B_2(E) = E^c$. Then, $B_I(\cdot) = C(\cdot) = \emptyset$, and the players are not commonly certain of their qualitative-type mappings.

players' beliefs are different but they are commonly certain of their qualitative-type mappings.[21]

Finally, I briefly discuss the implicit assumption that the players are commonly certain of the qualitative-type mappings in a particular space in which the players' beliefs may not be homogeneously given. Consider the collection of belief models satisfying assumptions on the players' beliefs specified in Theorem 1A (1) or (2), and take a belief model $\overrightarrow{\Omega}$ from the collection. Since there exists a "universal" belief model $\overrightarrow{\Omega^*}$, there exists a unique "structure-preserving" map $D : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$. Under certain conditions on the belief model $\overrightarrow{\Omega}$ (namely, "non-redundancy" and "minimality;" see Fukuda (2019c)), the image space $D(\Omega)$ is endowed with the belief structure so that $\overrightarrow{D(\Omega)}$ is a belief model isomorphic to $\overrightarrow{\Omega}$. For the new belief model $\overrightarrow{D(\Omega)}$, the players are commonly certain of their qualitative-type mappings.

Moving on to probabilistic beliefs, Proposition 1B applies to the case in which player i is certain of player j's probabilistic-type mapping:

Remark 9B. Let $\overrightarrow{\Omega}$ be a belief model, and let $\tau_j : \Omega \to \Delta(\Omega)$ be player j's type mapping.

1. (a) Player i is certain of τ_j with respect to $\{\beta_E^p \mid (E, p) \in \mathcal{D} \times [0, 1]\}$ iff $B_{\tau_j}^p(\cdot) \subseteq B_i B_{\tau_j}^p(\cdot)$.
   (b) Player i is certain of τ_j with respect to $\{\neg\beta_E^p \mid (E, p) \in \mathcal{D} \times [0, 1]\}$ iff $(\neg B_{\tau_j}^p)(\cdot) \subseteq B_i(\neg B_{\tau_j}^p)(\cdot)$.
   (c) If player i is certain of $\tau_j : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$, then $B_{\tau_j}^p(\cdot) \subseteq B_i B_{\tau_j}^p(\cdot)$ and $(\neg B_{\tau_j}^p)(\cdot) \subseteq B_i(\neg B_{\tau_j}^p)(\cdot)$.
2. (a) Let B_i satisfy Truth Axiom and Negative Introspection. Player i is certain of $\tau_j : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ iff $B_{\tau_j}^p(\cdot) \subseteq B_i B_{\tau_j}^p(\cdot)$ iff $(\neg B_{\tau_j}^p)(\cdot) \subseteq B_i(\neg B_{\tau_j}^p)(\cdot)$.
   (b) Let B_i satisfy Consistency, Countable Conjunction, Positive Introspection, and Negative Introspection. Player i is certain of $\tau_j : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ iff $B_{\tau_j}^p(\cdot) \subseteq B_i B_{\tau_j}^p(\cdot)$ and $(\neg B_{\tau_j}^p)(\cdot) \subseteq B_i(\neg B_{\tau_j}^p)(\cdot)$.

As in Remark 9A, Remark 9B roughly states that player i is certain of player j's probabilistic-type mapping iff (i) whenever player j believes an event E with probability at least p at ω, player i believes that player j believes an event E with probability at least p at ω; and (ii) whenever player j does not believe an event E with probability at least p at ω, player i believes that player j does not believe an event E with probability at least p at ω.

[21] As a simple example, take $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2\}, \mathcal{P}(\Omega))$. Let $B_i(\emptyset) = B_i(\{\omega_{3-i}\}) = \emptyset$ and $B_i(\{\omega_i\}) = B_i(\Omega) = \{\omega_i\}$ for each $i \in I = \{1, 2\}$ (that is, $b_{B_i}(\cdot) = \{\omega_i\}$). The belief operators satisfy the conditions in Theorem 1A (2).

Now, I ask one of the main questions of this paper in the context of probabilistic-type mappings: when are the players in a belief model commonly certain of their probabilistic-type mappings?

Theorem 1B. *Let $\overrightarrow{\Omega}$ be a belief model, and let $\tau_i : \Omega \to \Delta(\Omega)$ be player i's type mapping for each $i \in I$. Assume Consistency and Countable Conjunction for every B_i. The players are commonly certain of the profile of type mappings $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ iff $B^p_{\tau_i}(\cdot) \subseteq CB^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq C(\neg B^p_{\tau_i})(\cdot)$ for every $(i, p) \in I \times [0, 1]$. If $B_i = B^1_{\tau_i}$ is taken for every $i \in I$, then $C^1 = B^1_I$, where $B^1_I(\cdot) := \bigcap_{i \in I} B^1_{\tau_i}(\cdot)$ is the mutual 1-belief operator.*

Theorem 1B roughly states: the players are commonly certain of their probabilistic-type mappings iff (i) for any event E which some player i believes with probability at least p at some state ω, it is commonly believed that player i believes E with probability at least p at ω; and (ii) for any event E which some player i does not believe with probability at least p at some state ω, it is commonly believed that player i does not believe E with probability at least p at ω. If the belief operators in the belief model is taken as probability-one belief operator, then the probability-one common belief operator reduces to the probability-one mutual belief operator.

Since the implications of Theorems 1A (2) and 1B are conceptually similar, I discuss Theorem 1B for only one point (however, the other points also apply here). Namely, the theorem also formally justifies the statement that the players are commonly certain of a "universal" belief model (or a "universal" type space), provided the players are introspective.

To conclude this section, I provide two discussions on Theorems 1A and 1B. First, I study an implication of Theorems 1A and 1B (the "common meta-certainty" of a belief model) to the certainty of a signal. Suppose that the players are commonly certain of a belief model. If player i is certain of a signal $x : \Omega \to X$, then is player j certain of the signal x, too? While the players' beliefs may not be homogeneous, the proposition below shows that this is the case.

Proposition 4. *Let $\overrightarrow{\Omega}$ be a belief model such that each B_i satisfies Consistency. Let $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ be a signal such that, for any $F \in \mathcal{X}$, there exists a sub-collection $(F_\lambda)_{\lambda \in \Lambda}$ of $\mathcal{X}$ with $F^c = \bigcup_{\lambda \in \Lambda} F_\lambda$.*

A.
- *i. If player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ and if player j is certain of player i's qualitative-type mapping $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$, then player j is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$.*
- *ii. Suppose that the players are commonly certain of the profile of their qualitative-type mappings $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$. Then, player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ iff player j is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$.*

B.
- *i. Let $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ be player i's (probabilistic)-type mapping, and assume Entailment: $B_i(\cdot) \subseteq B^1_{\tau_i}(\cdot)$. If player i is certain of $x : (\Omega, \mathcal{D}) \to$*

$(X, \mathcal{X})$ and if player j is certain of player i's type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$, then player j is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$.

ii. *Suppose that the players are commonly certain of the profile of their type mappings $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$. Suppose Entailment for every player i: $B_i(\cdot) \subseteq B^1_{\tau_i}(\cdot)$ Then, player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ iff player j is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$.*

While Part (A) asks the certainty of qualitative-type mappings, Part (B) does that of probabilistic-type mappings. The meta-common-certainty assumption states that if player i is certain of her own strategy and if player j is certain of player i's type mapping then player j is certain of player i's strategy. In particular, if the players are commonly certain of the profile of their type mappings and if each player is certain of her own strategy, then it follows that the players are commonly certain of the strategy profile. In the next section, I clarify the role of such meta-certainty assumptions on game-theoretic solution concepts.

Second, one can also ask whether the players are commonly certain of the qualitative-type mapping t_C that represents common belief. Since the common belief operator C satisfies Positive Introspection, the players are commonly certain of $t_C : (\Omega, \mathcal{D}) \to (M(\Omega), \{\beta_E \mid E \in \mathcal{D}\})$, equivalently, $C(\cdot) \subseteq B_i C(\cdot)$. Now:

Remark 10. Let $\overrightarrow{\Omega}$ be a belief model, and let $t_C : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ be the qualitative-type mapping that represents the common belief operator C. Suppose that each belief operator B_i satisfies Consistency and Countable Conjunction. The following are equivalent.

1. The players are commonly certain of $t_C : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$.
2. C satisfies Negative Introspection.
3. $C(\cdot) \subseteq B_i C(\cdot)$ for each $i \in I$.

Since each B_i satisfies Consistency and Countable Conjunction, so does the common belief operator C. Then, the players are commonly certain of $t_C : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ iff $C(\cdot) \subseteq B_i C(\cdot)$ and $(\neg C)(\cdot) \subseteq B_i(\neg C)(\cdot)$ for every $i \in I$. It can be seen that the second condition is equivalent to Negative Introspection of C (establishing the equivalence between Parts (1) and (2)). Also, Parts (1) and (2) are equivalent (see, for example, Bonanno and Nehring (1998) Fukuda (2020)).

5 What Role Does the "Meta-Certainty" of a Model Play in Game-theoretic Analyses?

Section 4 has examined when the players are commonly certain of a belief model. Moving on to the second objective of the paper, I examine the role that the "meta-certainty" assumption plays in game-theoretic analyses of solution concepts.

5.1 Iterated Elimination of Strictly Dominated Actions

Specifically, I consider the solution concept of iterated elimination of strictly dominated actions (IESDA) in a strategic game. Informally, an epistemic characterization of IESDA states that, in a strategic game, if the (i) "logical" players are (ii) "commonly (meta-)certain of the game" and if they (iii) commonly believe their rationality, then their resulting actions survive IESDA. Formally, in the context of the framework of this paper, Fukuda (2020) shows that if the players commonly believe each player's rationality and if each of them correctly believes their own rationality, then their resulting actions survive IESDA, without assuming any property on individual players' beliefs. This paper connects these two statements as follows: first, suppose that the players are logical in that their beliefs satisfy Consistency and Finite Conjunction in addition to Monotonicity. Second, suppose that each of them is certain of their own qualitative-type mapping and strategy. Third, suppose that the players commonly believe their rationality. Then, their resulting actions survive IESDA.

Here I show that the certainty (of her own strategy and type mapping) allows her to correctly believe her own rationality. In other words, if a player is able to reason about informativeness of her own beliefs, she is able to correctly believe her own rationality.

5.1.1 A Strategic Game, a Model of a Game, and Rationality

To define the notion of rationality in a game, define a *(strategic) game* as a tuple $\Gamma = \langle (A_i)_{i \in I}, (\succcurlyeq_i)_{i \in I} \rangle$: A_i is a non-empty at-most-countable set of player i's actions, and $\succcurlyeq_i$ is i's (complete and transitive) preference relation on $A := \bigtimes_{i \in I} A_i$.[22] Denote by $\sim_i$ and $\succ_i$ the indifference and strict relations, respectively.

A (belief) *model* of the game Γ is a tuple $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, C, (\sigma_i)_{i \in I} \rangle$ (abusing the notation, denote it by $\overrightarrow{\Omega}$) with the following two properties. First, $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, C \rangle$ is a belief model. Second, $\sigma_i : \Omega \to A_i$ is a *strategy* of player i satisfying the measurability condition that $\sigma_i^{-1}(\{a_i\}) \in \mathcal{D}$ for all $a_i \in A_i$. Denote $[\sigma_i(\omega)] := \sigma_i^{-1}(\{\sigma_i(\omega)\})$ for each $\omega \in \Omega$.

Denote by $[a_i' \succcurlyeq_i a_i] := \{\omega' \in \Omega \mid (a_i', \sigma_i(\omega')) \succcurlyeq_i (a_i, \sigma_{-i}(\omega'))\} \in \mathcal{D}$ for any $a_i, a_i' \in A_i$. In words, $[a_i' \succcurlyeq_i a_i]$ is the event that player i prefers taking action a_i' to a_i given the opponents' strategies σ_{-i}. The set $[a_i' \succcurlyeq_i a_i]$ is an event because $[a_i' \succcurlyeq_i a_i] = \sigma_{-i}^{-1}(\{a_{-i} \in A_{-i} \mid (a_i', a_{-i}) \succ_i (a_i, a_{-i})\}) \in \mathcal{D}$. Define $[a_i' \succ_i a_i]$ and $[a_i' \sim_i a_i]$ analogously.

[22] The assumption on the cardinality of each action set A_i is to simplify the analysis. For example, it guarantees that each player is able to reason about any subset of action profiles and that the rationality of each player is an event. More generally, fix a game without imposing any cardinal restriction on I and $(A_i)_{i \in I}$. Take a (regular) infinite cardinal κ with $\max(|A|, |I|) < \kappa$. In a belief space in which the collection of events $\mathcal{D}$ is a κ-complete algebra (recall footnote 6), the players are able to reason about any subset of action profiles.

Denote by RAT_i the event that player i is *rational* (see, e.g., Bonanno (2008, 2015) and Chen, Long, and Luo (2007)):

$$\begin{aligned}\mathrm{RAT}_i :=& \{\omega \in \Omega \mid \omega \in B_i([a_i' \succ_i \sigma_i(\omega)]) \text{ for no } a_i' \in A_i\} \\ =& \bigcap_{a_i \in A_i} \left((\Theta_i^{-1}(\{a_i\}))^c \cup \bigcap_{a_i' \in A_i} (\neg B_i)([a_i' \succ_i a_i]) \right) \in \mathcal{D}\end{aligned}$$

Let $\mathrm{RAT}_I := \bigcap_{i \in I} \mathrm{RAT}_i$. Player i is *rational* at $\omega \in \Omega$ if there is no action $a_i' \in A_i$ such that player i believes that playing a_i' is strictly better than playing $\sigma_i(\omega)$ given the opponents' strategies σ_{-i}. In other words, player i is rational at ω if, for any action a_i', she always considers it possible that playing $\sigma_i(\omega)$ is at least as good as playing a_i' given the opponents' strategies σ_{-i}: $\omega \in (\neg B_i)(\neg[\sigma_i(\omega) \succcurlyeq_i a_i'])$ for any $a_i' \in A_i$.

Now, the epistemic characterization of IESDA is stated as follows. Suppose that each player i correctly believes her own rationality: $B_i(\mathrm{RAT}_i) \subseteq \mathrm{RAT}_i$ for every $i \in I$. If every player's rationality is common belief at ω, i.e., $\omega \in \bigcap_{i \in I} C(\mathrm{RAT}_i)$, then the resulting actions $(\sigma_i(\omega))_{i \in I} \in A$ survives any process of IESDA.[23]

Finally, in this section, player i is *certain of her own strategy* σ_i if she is certain of $\sigma_i : (\Omega, \mathcal{D}) \to (A_i, \{\{a_i\} \mid a_i \in A_i\})$, equivalently, $[\sigma_i(\cdot)] \subseteq B_i([\sigma_i(\cdot)])$. Note that, under Consistency in addition to Monotonicity, if player i is certain of her own strategy then $B_i([\sigma_i(\cdot)]) = [\sigma_i(\cdot)]$, $[\sigma_i(\cdot)]^c = B_i([\sigma_i(\cdot)]^c)$, and $B_i(\Omega) = \Omega$.[24]

5.1.2 The Role of Meta-certainty in Correctly Believing One's Own Rationality

I ask under what conditions player i correctly believes her own rationality: $B_i(\mathrm{RAT}_i) \subseteq \mathrm{RAT}_i$. For qualitative belief, the standard assumptions on qualitative belief (i.e., Consistency, Positive Introspection, Negative Introspection, and the Kripke property) guarantee that $B_i(\mathrm{RAT}_i) = \mathrm{RAT}_i$ (e.g., Bonanno (2008, 2015)).[25] Here, I provide a compatibility condition on belief with informativeness, under which a player correctly believes her own rationality. The compatibility condition does not hinge on a particular form of belief, i.e., whether it is qualitative or probabilistic.

[23] Since each player's belief operator B_i satisfies Monotonicity, $B_i(\mathrm{RAT}_I) \subseteq \bigcap_{i \in I} B_i(\mathrm{RAT}_i)$ and $C(\mathrm{RAT}_I) \subseteq \bigcap_{i \in I} C(\mathrm{RAT}_i)$. Thus, if every player i correctly believes the rationality of the players, then each player correctly believes her own rationality. Likewise, if it is common belief that the players are rational, then, for every $i \in I$, it is common belief that player i is rational. Hence, I examine the weaker condition that each player i correctly believes her own rationality.

[24] Thus, under Consistency and Monotonicity of B_i, the certainty of own strategy implies that if player i is rational at ω, then she never takes a strictly dominated action at ω (if she takes a strictly dominated action, then her belief violates Necessitation).

[25] It can be seen that Consistency, Positive Introspection, and the Kripke property in addition to the certainty of i's own strategy yield $B_i(\mathrm{RAT}_i) \subseteq \mathrm{RAT}_i$. Likewise, Negative Introspection and the Kripke property in addition to the certainty of i's own strategy yield $\mathrm{RAT}_i \subseteq B_i(\mathrm{RAT}_i)$.

Definition 3. Player i's belief (operator B_i) is *compatible with informativeness* if $(\uparrow t_{B_i}(\omega)) \cap E \neq \emptyset$ for any $E \in \mathcal{D}$ with $\omega \in B_i(E)$.

In words, player i's beliefs are compatible with informativeness if, for any event E which player i believes at some ω, there exists a state ω' in E which is at least as informative as ω. In the context of qualitative beliefs, if player i's belief operator B_i satisfies the Kripke property, Consistency, and Positive Introspection, then B_i is compatible with informativeness.[26] The compatibility with informativeness does not necessarily imply the Kripke property (and vice versa).[27] In the context of probabilistic beliefs, player i's probability-one belief operator $B^1_{\tau_i}$ is compatible with informativeness under the Harsanyi property (see Proposition 5 (B) below). If player i's belief operator B_i is compatible with informativeness, then it satisfies $B_i(\emptyset) = \emptyset$. Thus, under Finite Conjunction, if B_i is compatible with informativeness, then it satisfies Consistency.

The following proposition states that the compatibility of beliefs with informativeness is implied by the certainty of a type mapping.

Proposition 5. *Let $\overrightarrow{\Omega}$ be a belief model.*

A. *Assume: (i) $(\uparrow t_{B_i}(\cdot)) \in \mathcal{D}$; (ii) B_i satisfies Consistency and Finite Conjunction; and that (iii) player i is certain of $t_{B_i} : \Omega \to M(\Omega)$ with respect to $\{\{\mu \in M(\Omega) \mid \mu(\cdot) \geq t_{B_i}(\omega)(\cdot)\} \mid \omega \in \Omega\}$. Then, B_i is compatible with informativeness.*

B. *Let $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ be player i's type probabilistic-mapping on the belief model $\overrightarrow{\Omega}$. Assume (i) $[\tau_i(\cdot)] \in \mathcal{D}$; the Harsanyi property; and (iii) Entailment: $B_i(\cdot) \subseteq B^1_{\tau_i}(\cdot)$. Then, B_i is compatible with informativeness.*

Part (A) states that, under the regularity condition (i), if player i is logical (in that her belief operator satisfies Consistency and Finite Conjunction) and if she is certain of her qualitative-type mapping, then her beliefs are compatible with informativeness. Theorem 2 below establishes that if player i's beliefs are compatible with informativeness then she correctly believes her rationality, which is a part of the preconditions of the epistemic characterization of IESDA.

Part (B) states that the Harsanyi property implies the compatibility with informativeness. Recall Proposition 2B: by the regularity condition (i), the Harsanyi property holds iff player i is certain of her probabilistic-type mapping.

Now, I present the main result of this subsection. The theorem says a player correctly believes her own rationality if: (i) she is certain of her own strategy; (ii) her belief is compatible with the informativeness; and if (iii) her belief is (finitely)

[26]The proof goes as follows. For any $E \in \mathcal{D}$ with $\omega \in B_i(E)$, $\emptyset \neq E \cap b_{B_i}(\omega) \subseteq E \cap (\uparrow t_{B_i}(\omega))$.

[27]Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2\}, \mathcal{P}(\Omega))$. Define B_i as $B_i(E) = E$ if $E \neq \Omega$; and $B_i(\Omega) = \emptyset$. While B_i is compatible with informativeness, it does not satisfy the Kripke property. Define B_j as $B_j(E) = E$ if $E \in \{\emptyset, \Omega\}$; and $B_j(E) = E^c$ if $E \in \{\{\omega_1\}, \{\omega_2\}\}$. While B_j satisfies the Kripke property, it is not compatible with informativeness.

conjunctive so that she can simultaneously reason about her own strategy and her own rationality.

Theorem 2. *Suppose that player i is certain of her own strategy (i.e., $[\sigma_i(\cdot)] \subseteq B_i([\sigma_i(\cdot)])$). Also, let B_i be compatible with informativeness and satisfy Finite Conjunction. Then, player i correctly believes her own rationality: $B_i(\mathrm{RAT}_i) \subseteq \mathrm{RAT}_i$.*

Proposition 5 and Theorem 2 imply that player i correctly believes her own rationality if she is logical in that her belief operator satisfies Consistency and Finite Conjunction and if she is certain of her own type mapping and strategy. Theorem 2 states that, for the role of the meta-certainty assumption of a belief model on IESDA, it is not necessary that each player is certain of the profile of type mappings but it is sufficient that each player is certain of her own type mapping. In fact, one can incorporate the assumptions that each player is certain of her own qualitative type-mapping and strategy into the condition that she is certain of the part of the model of a game $\langle(\Omega, \mathcal{D}), (t_{B_i}, \sigma_i)\rangle$ that dictates her beliefs and strategy.

Two remarks are in order. First, if the preconditions of Theorem 2 hold, then B_i satisfies Consistency and Necessitation. Second, in Theorem 2, the assumptions of the compatibility with informativeness and Finite Conjunction cannot be dropped. See Remark A.1 in Appendix A for counterexamples.

6 Conclusion

This paper asked two questions. First, what does it mean by the statement that the players in a belief model are commonly (meta-)certain of the model itself? Second, what role does such meta-certainty assumption play in epistemic characterizations of game-theoretic solution concepts? The paper started with expanding the objects of the players' beliefs from events to signals (functions) defined on the underling states. A player is certain of the value of a signal x at a state if, the player believes, at the state, any observational content that holds at the state. If she is certain of the value of the signal x at every state, then she is certain of the signal. The common certainty of the signal was analogously defined: the players are commonly certain of the value of the signal x if the players commonly believe, at the state, any observational content that holds at the state. The players are commonly certain of the signal x if they are commonly certain of its value at every state. Then, the players' belief-generating maps (i.e., type mappings) and strategies became objects of their beliefs. A player is certain of her own type mapping iff her belief satisfies the positive and negative introspective properties. For probabilistic beliefs, the Harsanyi property is the strongest form of the certainty of own type mapping.

The main result regarding the first question is: the players are commonly certain of the profile of the players' type mappings (i.e., the belief model) iff, for any event E which some player i believes at some state, it is common belief that player i believes

E at that state. I summarize three implications. First, in a "universal" space in which the differences in the players' beliefs are not represented through their belief operators but are incorporated into the underlying states themselves, the players are commonly certain of their type mappings in the formal sense. Thus, this paper justifies the statement that, in the universal space, the players are commonly certain of the model itself. Second, the common belief operator collapses into the mutual belief operator when the players are commonly certain of the model. This is because, whenever everybody believes an event, everybody believes that everybody believes the event. Third, if the players are commonly certain of their type mappings and if each player is certain of her own strategy, then the players are commonly certain of their strategies.

Using the formalization of certainty of signals, the second objective was to elucidate the role of the common meta-certainty assumption on epistemic characterizations of game-theoretic solution concepts. The paper studied the solution concept of iterated elimination of strictly dominated actions (IESDA). Informally, if the players are "logical," if they are (meta-)certain of a game, and if they commonly believe their rationality, then their resulting actions survive any process of IESDA. Formally, the paper showed: if the players' beliefs satisfy Consistency and Finite Conjunction, if each player is certain of her qualitative-type mapping (or if each player's beliefs are compatible with informativeness), and if the players commonly believe their rationality, then their resulting actions survive any process of IESDA.

A Appendix

A.1 Proofs

Proof of Remark 1.

1. Suppose that player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$. Take $F \in \mathcal{X}$. Since $\emptyset \in \mathcal{J}_{B_i}$, assume $x^{-1}(F) \neq \emptyset$. For any $\omega \in x^{-1}(F)$, $\omega \in B_i(x^{-1}(F))$. Thus, $x^{-1}(F) \subseteq B_i(x^{-1}(F))$, i.e., $x^{-1}(F) \in \mathcal{J}_{B_i}$. Hence, $x^{-1}(\mathcal{X}) \subseteq \mathcal{J}_{B_i}$. Conversely, assume $x^{-1}(\mathcal{X}) \subseteq \mathcal{J}_{B_i}$. For any $\omega \in \Omega$ and $F \in \mathcal{X}$ with $\omega \in x^{-1}(F)$, $\omega \in B_i(x^{-1}(F))$. Thus, player i is certain of x.

2. Suppose that the players are commonly certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$. Take $F \in \mathcal{X}$. Since $\emptyset \in \mathcal{J}_{B_I}$, assume $x^{-1}(F) \neq \emptyset$. For any $\omega \in x^{-1}(F)$, $\omega \in C(x^{-1}(F))$. Thus, $x^{-1}(F) \subseteq C(x^{-1}(F)) \subseteq B_I(x^{-1}(F))$, and hence $x^{-1}(F) \in \mathcal{J}_{B_i}$. Then, $x^{-1}(\mathcal{X}) \subseteq \mathcal{J}_{B_i}$. Conversely, assume $x^{-1}(\mathcal{X}) \subseteq \mathcal{J}_{B_I}$. For any $\omega \in \Omega$ and $F \in \mathcal{X}$ with $\omega \in x^{-1}(F)$, $\omega \in x^{-1}(F) \subseteq B_i(x^{-1}(F))$, and thus $\omega \in C(x^{-1}(F))$. Thus, the players are commonly certain of x.

3. This assertion follows because $\mathcal{J}_{B_I} = \bigcap_{i \in I} \mathcal{J}_{B_i}$.

□

Proof of Remark 4. As discussed in the main text, it suffices to show Part (1). First, it follows from Finite Conjunction, Consistency, and Monotonicity that $\emptyset = B_i(E) \cap B_i(E^c) = B_i(E \cap E^c) = B_i(\emptyset) \in \mathcal{B}_i$. Second, I show that $\mathcal{B}_i$ is closed under countable intersection. Countable Conjunction and Monotonicity imply $\bigcap_{n\in\mathbb{N}} B_i(E_n) = B_i(\bigcap_{n\in\mathbb{N}} E_n) \in \mathcal{B}_i$. Third, I show that $\mathcal{B}_i$ is closed under complementation by proving $(\neg B_i)(\cdot) = B_i(\neg B_i)(\cdot)$. Consistency, Positive Introspection, and Negative Introspection imply $(\neg B_i)(\cdot) \subseteq B_i(\neg B_i)(\cdot) \subseteq (\neg B_i)B_i(\cdot) \subseteq (\neg B_i)(\cdot)$. □

Proof of Remark 5. I only prove Part (1). Suppose $B_i(\Omega) = \Omega$. Take any constant signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$. Fix $\omega \in \Omega$. For any $F \in \mathcal{X}$ with $x(\omega) \in F$, $x^{-1}(F) = \Omega$. Thus, $x^{-1}(F) \subseteq B_i(x^{-1}(F))$. Conversely, take $\omega \in \Omega$, and consider the constant signal $x : (\Omega, \mathcal{D}) \to (\{\omega\}, \{\{\omega\}\})$. Since player i is certain of it, $\Omega = x^{-1}(\{\omega\}) \subseteq B_i(x^{-1}(\{\omega\})) = B_i(\Omega)$. □

Proof of Proposition 1A.

1. For (1a), i is certain of t_{B_i} with respect to $\{\beta_E \mid E \in \mathcal{D}\}$ iff $t_{B_i}^{-1}(\beta_E) = B_{t_{B_i}}(E)$ is self-evident, i.e., $B_{t_{B_i}}(E) \subseteq B_i B_{t_{B_i}}(E)$. Likewise, for (1b), i is certain of t_{B_i} with respect to $\{\neg\beta_E \mid E \in \mathcal{D}\}$ iff $\neg t_{B_i}^{-1}(\beta_E) = (\neg B_{t_{B_i}})(E)$ is self-evident, i.e., $(\neg B_{t_{B_i}})(E) \subseteq B_i(\neg B_{t_{B_i}})(E)$. Then, (1c) follows from the previous two parts.
2. It suffices to show the "if" part of (2b). It follows from Remark 4 that $\mathcal{B}_i = \{B_i(E) \in \mathcal{D} \mid E \in \mathcal{D}\}$ is a sub-σ-algebra of $\mathcal{D}$. Since B_i satisfies Positive Introspection, $\mathcal{B}_i \subseteq \mathcal{J}_{B_i}$. Since $t_i^{-1}(\beta_E) = B_i(E) \in \mathcal{B}_i$ and since $\mathcal{B}_i$ is a σ-algebra, $t_i^{-1}(\mathcal{D}_M) = \sigma(\{t_i^{-1}(\beta_E) \in \mathcal{D} \mid E \in \mathcal{D}\}) \subseteq \sigma(\mathcal{B}_i) = \mathcal{B}_i \subseteq \mathcal{J}_{B_i}$.

□

Proof of Proposition 1B.

1. For (1a), player i is certain of τ_i with respect to $\{\beta_E^p \mid (E,p) \in \mathcal{D} \times [0,1]\}$ iff $B_{\tau_i}^p(E) = \tau_i^{-1}(\beta_E^p) \subseteq B_i(\tau_i^{-1}(\beta_E^p)) = B_i B_{\tau_i}^p(E)$. For (1b), player i is certain of τ_i with respect to $\{\neg\beta_E^p \mid (E,p) \in \mathcal{D} \times [0,1]\}$ iff $(\neg B_{\tau_i}^p)(E) = \neg\tau_i^{-1}(\beta_E^p) \subseteq B_i(\neg\tau_i^{-1}(\beta_E^p)) = B_i(\neg B_{\tau_i}^p)(E)$. Then, (1c) follows from the previous two parts.
2. (a) If player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ then B_i satisfies Positive Certainty. Conversely, let B_i satisfy Positive Certainty. By (1a), $\tau_i^{-1}(\{\beta_E^p \mid (E,p) \in \mathcal{D} \times [0,1]\}) \subseteq \mathcal{J}_{B_i}$. Since B_i satisfies Truth Axiom and Negative Introspection, $\mathcal{J}_{B_i}$ is a sub-σ-algebra of $\mathcal{D}$. Thus, $\tau_i^{-1}(\mathcal{D}_\Delta) \subseteq \mathcal{J}_{B_i}$. Hence, player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$. Next, I show that, since B_i satisfies Truth Axiom and Negative Introspection, Positive Certainty is equivalent to Negative Certainty. By Positive Certainty, $(\neg B_{\tau_i}^p) = (\neg B_i)B_{\tau_i}^p = B_i(\neg B_i)B_{\tau_i}^p = B_i(\neg B_{\tau_i}^p)$. Conversely, by Negative Certainty, $B_{\tau_i}^p = (\neg B_i)(\neg B_{\tau_i}^p) = B_i(\neg B_i)(\neg B_{\tau_i}^p) = B_i B_{\tau_i}^p$.

   (b) It is sufficient to prove the "if" part. First, it follows from the assumptions and Remark 4 that $\mathcal{B}_i = \{B_i(E) \in \mathcal{D} \mid E \in \mathcal{D}\}$ is a sub-σ-algebra of $\mathcal{D}$. Second, since B_i satisfies Positive Introspection, $\mathcal{B}_i \subseteq \mathcal{J}_{B_i}$. Third, I show that Positive Certainty, Negative Certainty, and Consistency of B_i imply $B^p_{\tau_i}(E) = B_i B^p_{\tau_i}(E)$. Fourth, since $\tau_i^{-1}(\beta^p_E) = B^p_{\tau_i}(E) = B_i B^p_{\tau_i}(E) \in \mathcal{B}_i$ and since $\mathcal{B}_i$ is a σ-algebra, $\tau_i^{-1}(\mathcal{D}_\Delta) = \sigma(\{\tau_i^{-1}(\beta^p_E) \in \mathcal{D} \mid (E,p) \in \mathcal{D}\times[0,1]\}) \subseteq \sigma(\mathcal{B}_i) = \mathcal{B}_i \subseteq \mathcal{J}_{B_i}$.

It remains to show $B^p_{\tau_i}(E) = B_i B^p_{\tau_i}(E)$. The "$\subseteq$" part is Positive Certainty. Conversely, it follows from Negative Certainty and Consistency that $(\neg B^p_{\tau_i})(E) \subseteq B_i(\neg B^p_{\tau_i})(E) \subseteq (\neg B_i)B^p_{\tau_i}(E)$. Then, $B_i B^p_{\tau_i}(E) \subseteq B^p_{\tau_i}(E)$.

(c) It suffices to prove the "if" part. First, I show that $\mathcal{B}^1_{\tau_i} := \{B^1_{\tau_i}(E) \in \mathcal{D} \mid E \in \mathcal{D}\}$ is a sub-σ-algebra of $\mathcal{D}$. Second, since B_i satisfies Positive Certainty, $\mathcal{B}^1_{\tau_i} \subseteq \mathcal{J}_{B_i}$. Third, I show that Positive Certainty, Negative Certainty, and Consistency of $B^1_{\tau_i}$ (i.e., $B^1_{\tau_i}(E) \subseteq (\neg B^1_{\tau_i})(E^c)$) imply $B^p_{\tau_i}(E) = B^1_{\tau_i} B^p_{\tau_i}(E)$. Fourth, since $\tau_i^{-1}(\beta^p_E) = B^p_{\tau_i}(E) = B^1_{\tau_i} B^p_{\tau_i}(E) \in \mathcal{B}^1_{\tau_i}$ and since $\mathcal{B}^1_{\tau_i}$ is a σ-algebra, $\tau_i^{-1}(\mathcal{D}_\Delta) = \sigma(\{\tau_i^{-1}(\beta^p_E) \in \mathcal{D} \mid (E,p) \in \mathcal{D} \times [0,1]\}) \subseteq \sigma(\mathcal{B}^1_{\tau_i}) = \mathcal{B}^1_{\tau_i} \subseteq \mathcal{J}_{B_i}$.

Hence, I first show that $\mathcal{B}^1_{\tau_i}$ is a sub-σ-algebra of $\mathcal{D}$. First, since $\tau_i(\cdot)(\emptyset) = 0$, $\emptyset = B^1_{\tau_i}(\emptyset) \in \mathcal{B}^1_{\tau_i}$. Second, since $B^1_{\tau_i}$ satisfies Monotonicity and Countable Conjunction, $\mathcal{B}^1_{\tau_i}$ is closed under countable intersection. Third, as in the proof of Remark 4, to prove that $\mathcal{B}^1_{\tau_i}$ is closed under complementation, it is sufficient to show $(\neg B^1_{\tau_i})(\cdot) = B^1_{\tau_i}(\neg B^1_{\tau_i})(\cdot)$. However, this property follows from $B^1_{\tau_i}(E) \subseteq (\neg B^1_{\tau_i})(E^c)$, $B^1_{\tau_i}(\cdot) \subseteq B^1_{\tau_i} B^1_{\tau_i}(\cdot)$, and $(\neg B^1_{\tau_i})(\cdot) \subseteq B^1_{\tau_i}(\neg B^1_{\tau_i})(\cdot)$. Indeed, $(\neg B^1_{\tau_i})(\cdot) \subseteq B^1_{\tau_i}(\neg B^1_{\tau_i})(\cdot) \subseteq (\neg B^1_{\tau_i}) B^1_{\tau_i}(\cdot) \subseteq (\neg B^1_{\tau_i})(\cdot)$.

Next, I show $B^p_{\tau_i}(E) = B^1_{\tau_i} B^p_{\tau_i}(E)$. It follows from Positive Certainty and Entailment that $B^p_{\tau_i}(E) \subseteq B_i B^p_{\tau_i}(E) \subseteq B^1_{\tau_i} B^p_{\tau_i}(E)$. Conversely, it follows from Negative Certainty and Entailment that $(\neg B^p_{\tau_i})(E) \subseteq B_i(\neg B^p_{\tau_i})(E) \subseteq B^1_{\tau_i}(\neg B^p_{\tau_i})(E)$. Then, it follows from Consistency of $B^1_{\tau_i}$ that $B^1_{\tau_i} B^p_{\tau_i}(E) \subseteq (\neg B^1_{\tau_i})(\neg B^p_{\tau_i})(E) \subseteq B^p_{\tau_i}(E)$.

$\square$

Proof of Proposition 2A. 1. (a) Since Truth Axiom yields $\omega' \in b_{B_i}(\omega')$ for all $\omega' \in \Omega$, $\omega' \in b_{B_i}(\omega') \subseteq b_{B_i}(\omega)$ for all $\omega' \in (\uparrow t_{B_i}(\omega))$. Conversely, Truth Axiom follows from $\omega \in b_{B_i}(\omega)$ for all $\omega \in \Omega$.

(b) Suppose $\omega' \in b_{B_i}(\omega)$. For any $F \in \mathcal{D}$ with $t_{B_i}(\omega)(F) = 1$, it follows from Positive Introspection that $t_{B_i}(\omega)(t^{-1}_{B_i}(\beta_F)) = 1$. By the supposition, $\omega' \in t^{-1}_{B_i}(\beta_F)$, i.e., $t_{B_i}(\omega')(F) = 1$. Thus, $\omega' \in (\uparrow t_{B_i}(\omega))$.

Conversely, let B_i satisfy the Kripke property, and assume $b_{B_i}(\omega) \subseteq (\uparrow t_{B_i}(\omega))$. Suppose $\omega \in B_i(E)$. In order to show $\omega \in B_i B_i(E)$, it is enough to prove $b_{B_i}(\omega') \subseteq E$ for all $\omega' \in b_{B_i}(\omega)$. Take any $\omega' \in b_{B_i}(\omega)$. Since $\omega' \in (\uparrow t_{B_i}(\omega))$ and $\omega \in B_i(E)$, it follows $\omega' \in B_i(E)$. Thus, $b_{B_i}(\omega') \subseteq E$.

(c) The proof is analogous to Part (1b). Suppose $\omega' \in b_{B_i}(\omega)$. Suppose to the contrary that $\omega' \notin (\downarrow t_{B_i}(\omega))$, i.e., $t_{B_i}(\omega)(F) = 0 < 1 = t_{B_i}(\omega')(F)$ for some $F \in \mathcal{D}$. By Negative Introspection, $t_{B_i}(\omega)(\neg t_{B_i}^{-1}(\beta_F)) = 1$, and thus $\omega' \in \neg t_{B_i}^{-1}(\beta_F)$, i.e., $t_{B_i}(\omega')(F) = 0$, a contradiction.

Conversely, let B_i satisfy the Kripke property, and suppose $b_{B_i}(\omega) \subseteq (\downarrow t_{B_i}(\omega))$. If $\omega \notin B_i(E)$, then $b_{B_i}(\omega) \cap E^c \neq \emptyset$. In order to establish $\omega \in B_i(\neg B_i)(E)$, it is enough to show that $b_{B_i}(\omega') \cap E^c \neq \emptyset$ for all $\omega' \in b_{B_i}(\omega)$. Take any $\omega' \in b_{B_i}(\omega)$. Since $\omega' \in (\downarrow t_{B_i}(\omega))$ and since $t_{B_i}(\omega)(E) = 0$, it follows $t_{B_i}(\omega')(E) = 0$, i.e., $b_{B_i}(\omega') \cap E^c \neq \emptyset$.

2. (a) The assertion follows from Parts (1a) and (1b).

   (b) By Part (1), $[t_{B_i}(\omega)] \subseteq (\uparrow t_{B_i}(\omega)) = b_{B_i}(\omega) \subseteq (\uparrow t_{B_i}(\omega)) \cap (\downarrow t_{B_i}(\omega)) = [t_{B_i}(\omega)]$. Then, $(\uparrow t_{B_i}(\omega)) \subseteq (\downarrow t_{B_i}(\omega))$ implies $(\downarrow t_{B_i}(\omega)) \subseteq (\uparrow t_{B_i}(\omega))$, i.e., $(\uparrow t_{B_i}(\omega)) = (\downarrow t_{B_i}(\omega))$. If B_i satisfies the Kripke property, then Part (1) implies that the converse also holds.

□

Proof of Proposition 2B. 1. It can be seen that

$$[\tau_i(\omega)] = \bigcap_{(E,p)\in\mathcal{D}\times[0,1]:\omega\in B^p_{\tau_i}(E)} B^p_{\tau_i}(E) = \bigcap_{(E,p)\in\mathcal{D}\times[0,1]:\omega\in(\neg B^p_{\tau_i})(E)} (\neg B^p_{\tau_i})(E). \quad \text{(A.1)}$$

Now, $B^p_{\tau_i}(\cdot) \subseteq B_i B^p_{\tau_i}(\cdot)$ implies $b_{B_i}(\omega) \subseteq B^p_{\tau_i}(E)$ for any $(E,p) \in \mathcal{D} \times [0,1]$ with $\omega \in B^p_{\tau_i}(E)$. Likewise, $(\neg B^p_{\tau_i})(\cdot) \subseteq B_i(\neg B^p_{\tau_i})(\cdot)$ implies $b_{B_i}(\omega) \subseteq (\neg B^p_{\tau_i})(E)$ for any $(E,p) \in \mathcal{D} \times [0,1]$ with $\omega \in (\neg B^p_{\tau_i})(E)$. In either case, $b_{B_i}(\omega) \subseteq [\tau_i(\omega)]$.

2. Assume the Kripke property. Suppose $b_{B_i}(\cdot) \subseteq [\tau_i(\cdot)]$. Take $(E,p) \in \mathcal{D} \times [0,1]$. Since $b_{B_i}(\omega) \subseteq [\tau_i(\omega)] \subseteq B^p_{\tau_i}(E)$ for any $\omega \in B^p_{\tau_i}(E)$, $B^p_{\tau_i}(E) \subseteq B_i B^p_{\tau_i}(E)$. Likewise, since $b_{B_i}(\omega) \subseteq [\tau_i(\omega)] \subseteq (\neg B^p_{\tau_i})(E)$ for any $\omega \in (\neg B^p_{\tau_i})(E)$, $(\neg B^p_{\tau_i})(E) \subseteq B_i(\neg B^p_{\tau_i})(E)$.

□

Proof of Proposition 3B. 1. Let τ_i satisfy the Harsanyi property. For any $\omega, \omega' \in \Omega$ with $\omega' \in [\tau_i(\omega)]$, $\tau_i(\omega')([\tau_i(\omega)]) = \tau_i(\omega)([\tau_i(\omega)]) = 1$, i.e., $\omega' \in B^1_{\tau_i}([\tau_i(\omega)])$. Thus, player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \{\{\tau_i(\omega)\} \mid \omega \in \Omega\})$. Conversely, since $\omega \in [\tau_i(\omega)]$, $\omega \in B^1_{\tau_i}([\tau_i(\omega)])$, i.e., $\tau_i(\omega)([\tau_i(\omega)]) = 1$.

2. Let $\mathcal{D}$ be generated by a countable algebra $\mathcal{A}$. Let $[0,1]_{\mathbb{Q}} := [0,1] \cap \mathbb{Q}$. Similarly to Expression (A.1),

$$[\tau_i(\omega)] = \bigcap_{(E,p)\in\mathcal{A}\times[0,1]_{\mathbb{Q}}:\omega\in B^p_{\tau_i}(E)} B^p_{\tau_i}(E) = \bigcap_{(E,p)\in\mathcal{A}\times[0,1]_{\mathbb{Q}}:\omega\in(\neg B^p_{\tau_i})(E)} (\neg B^p_{\tau_i})(E) \in \mathcal{D}.$$

Then, it follows from Part (1) that (2a) and (2b) are equivalent. Part (2b) implies (2c), which, in turn, implies (2d) and (2e).

Assume (2d). Fix $\omega \in \Omega$. For any $(E,p) \in \mathcal{A} \times [0,1]_{\mathbb{Q}}$ with $\omega \in B^p_{\tau_i}(E)$, $B^p_{\tau_i}(E) \subseteq B^1_{\tau_i} B^p_{\tau_i}(E)$. Since $\mathcal{A} \times [0,1]_{\mathbb{Q}}$ is countable, take the intersection over all $(E,p) \in \mathcal{A} \times [0,1]_{\mathbb{Q}}$ with $\omega \in B^p_{\tau_i}(E)$ to obtain:

$$[\tau_i(\omega)] = \bigcap_{(E,p)} B^p_{\tau_i}(E) \subseteq \bigcap_{(E,p)} B^1_{\tau_i} B^p_{\tau_i}(E) \subseteq B^1_{\tau_i}\left(\bigcap_{(E,p)} B^p_{\tau_i}(E)\right) = B^1_{\tau_i}([\tau_i(\omega)]).$$

Thus, (2a) holds. Likewise, assume (2e). Fix $\omega \in \Omega$. For any $(E,p) \in \mathcal{A}\times[0,1]_{\mathbb{Q}}$ with $\omega \in (\neg B^p_{\tau_i})(E)$, $(\neg B^p_{\tau_i})(E) \subseteq B^1_{\tau_i}(\neg B^p_{\tau_i})(E)$. Since $\mathcal{A} \times [0,1]_{\mathbb{Q}}$ is countable, take the intersection over all $(E,p) \in \mathcal{A} \times [0,1]_{\mathbb{Q}}$ with $\omega \in (\neg B^p_{\tau_i})(E)$ to obtain:

$$[\tau_i(\omega)] = \bigcap_{(E,p)} (\neg B^p_{\tau_i})(E) \subseteq \bigcap_{(E,p)} B^1_{\tau_i}(\neg B^p_{\tau_i})(E) \subseteq B^1_{\tau_i}\left(\bigcap_{(E,p)} (\neg B^p_{\tau_i})(E)\right) = B^1_{\tau_i}([\tau_i(\omega)]).$$

Hence, (2a) holds.

□

Proof of Proposition 3A. 1. Notice that (1b) is equivalent to $\omega \in B_i([t_{B_i}(\omega)])$ for all $\omega \in \Omega$. Suppose (1a), and fix $\omega \in \Omega$. Since $\omega \in [t_{B_i}(\omega)]$, $\omega \in B_i([t_{B_i}(\omega)])$. Conversely, suppose (1b), and fix $\omega \in \Omega$. For any $\omega' \in [t_{B_i}(\omega)]$, $\omega' \in B_i([t_{B_i}(\omega')]) = B_i([t_{B_i}(\omega)])$.

2. Suppose $[t_{B_i}(\omega)] \subseteq E$. Since B_i satisfies Positive Introspection and Negative Introspection, it follows from Proposition 3A that $b_{B_i}(\omega) \subseteq [t_{B_i}(\omega)] \subseteq E$. By the Kripke property, $\omega \in B_i(E)$.

3. Without loss, assume (1b). If $b_{B_i}(\omega) \subseteq E$, then it follows from Truth Axiom of B_i and Proposition 3A that $[t_{B_i}(\omega)] \subseteq b_{B_i}(\omega) \subseteq E$. Then, $\omega \in B_i(E)$.

□

Proof of Theorem 1A. 1. Suppose that the players are commonly certain of the profile of qualitative-type mappings. Since player i is certain of her own qualitative-type mapping, it follows from Proposition 1A that Positive Introspection and Negative Introspection hold: $B_i(\cdot) \subseteq B_i B_i(\cdot)$ and $(\neg B_i)(\cdot) \subseteq B_i(\neg B_i)(\cdot)$. Next, since player i is certain of player j's qualitative-type mapping, it follows from Remark 9A that $B_j(\cdot) \subseteq B_i B_j(\cdot)$. Since B_j satisfies Truth Axiom and since B_i satisfies Monotonicity, $B_j(\cdot) \subseteq B_i B_j(\cdot) \subseteq B_i(\cdot)$. Since i and j are arbitrary, $B_i = B_j$. Conversely, it follows from the suppositions that each player i is certain of every player j's qualitative-type mapping $\tau_{B_j} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$.

In fact, player i is certain of the profile of the qualitative-type mappings $\tau : (\Omega, \mathcal{D}) \to (M(\Omega)^I, \mathcal{D}^I_M)$, where $\mathcal{D}^I_M$ is the product σ-algebra on the product space $M(\Omega)^I$ (recall Remark 6). Then, the players are commonly certain of $\tau : (\Omega, \mathcal{D}) \to (M(\Omega)^I, \mathcal{D}^I_M)$.

Lastly, since $B_i = B_I$ for all $i \in I$ and since B_i satisfies Positive Introspection, it follows $B_i = C$ for each $i \in I$.

2. Suppose that the players are commonly certain of the profile of qualitative-type mappings. Since player j is certain of player i's qualitative-type mapping, it follows from Remark 9A that $B_i(\cdot) \subseteq B_j B_i(\cdot)$ and $(\neg B_i)(\cdot) \subseteq B_j(\neg B_i)(\cdot)$. Since j is arbitrary, $B_i(\cdot) \subseteq B_I B_i(\cdot)$ and $(\neg B_i)(\cdot) \subseteq B_I(\neg B_i)(\cdot)$. Then, $B_i(\cdot) \subseteq CB_i(\cdot)$ and $(\neg B_i)(\cdot) \subseteq C(\neg B_i)(\cdot)$. Conversely, it follows from the supposition that $B_i(\cdot) \subseteq CB_i(\cdot) \subseteq B_j B_i(\cdot)$ and $(\neg B_i)(\cdot) \subseteq C(\neg B_i)(\cdot) \subseteq B_j(\neg B_i)(\cdot)$. Thus, player j is certain of player i's qualitative-type mapping.

   In fact, player j is certain of the profile of the qualitative-type mappings $\tau : (\Omega, \mathcal{D}) \to (M(\Omega)^I, \mathcal{D}_M^I)$ (recall Remark 6). Consequently, the players are commonly certain of the profile of the qualitative-type mappings $\tau : (\Omega, \mathcal{D}) \to (M(\Omega)^I, \mathcal{D}_M^I)$.

   Lastly, since $B_I(\cdot) \subseteq B_i(\cdot) \subseteq CB_i(\cdot)$ for each $i \in I$, $B_I(\cdot) \subseteq \bigcap_{i \in I} CB_i(\cdot) \subseteq CB_I(\cdot)$, where C satisfies Countable Conjunction because each B_i satisfies it. Then, each $B_I(\cdot)$ itself is a publicly-evident event implying the mutual belief, and thus $C = B_I$.

□

Proof of Theorem 1B. Suppose that the players are commonly certain of the profile of type mappings. Since player j is certain of player i's type mapping, it follows from Remark 9B that $B^p_{\tau_i}(\cdot) \subseteq B_j B^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq B_j(\neg B^p_{\tau_i})(\cdot)$. Since j is arbitrary, $B^p_{\tau_i}(\cdot) \subseteq B_I B^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq B_I(\neg B^p_{\tau_i})(\cdot)$. Then, $B^p_{\tau_i}(\cdot) \subseteq CB^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq C(\neg B^p_{\tau_i})(\cdot)$. Conversely, it follows from the supposition that $B^p_{\tau_i}(\cdot) \subseteq CB^p_{\tau_i}(\cdot) \subseteq B_j B^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq C(\neg B^p_{\tau_i})(\cdot) \subseteq B_j(\neg B^p_{\tau_i})(\cdot)$. Thus, player j is certain of player i's type mapping.

In fact, player j is certain of the profile of the probabilistic-type mappings $\tau : (\Omega, \mathcal{D}) \to (\Delta(\Omega)^I, \mathcal{D}_\Delta^I)$, where $\mathcal{D}_\Delta^I$ is the product σ-algebra on the product space $\Delta(\Omega)^I$ (recall Remark 6). Thus, the players are commonly certain of the profile of the probabilistic-type mappings $\tau : (\Omega, \mathcal{D}) \to (\Delta(\Omega)^I, \mathcal{D}_\Delta^I)$.

Lastly, since $B^1_I(\cdot) \subseteq B^1_{\tau_i}(\cdot) \subseteq C^1 B^1_{\tau_i}(\cdot)$ for each $i \in I$, $B_I(\cdot) \subseteq \bigcap_{i \in I} C^1 B_{\tau_i}(\cdot) \subseteq C^1 B^1_I(\cdot)$, where C^1 satisfies Countable Conjunction because each $B^1_{\tau_i}$ satisfies it. Then, each $B^1_I(\cdot)$ itself is a publicly-1-evident event implying the mutual 1-belief, and thus $C^1 = B^1_I$. □

Proof of Proposition 4. A. I only prove (Ai). Take $F \in \mathcal{X}$. It suffices to show $x^{-1}(F) \subseteq B_j(x^{-1}(F))$. It follows from Remark 9A and Consistency of B_j that $B_i = B_j B_i$. Take $(F_\lambda)_{\lambda \in \Lambda}$ from $\mathcal{X}$ with $F^c = \bigcup_{\lambda \in \Lambda} F_\lambda$. Then,

$$\neg x^{-1}(F) = x^{-1}(F^c) = \bigcup_{\lambda \in \Lambda} x^{-1}(F_\lambda) \subseteq \bigcup_{\lambda \in \Lambda} B_i(x^{-1}(F_\lambda)) \subseteq B_i(x^{-1}(F^c)) \subseteq (\neg B_i)(x^{-1}(F)),$$

implying $x^{-1}(F) = B_i(x^{-1}(F))$. It follows from Remark 9A that

$$x^{-1}(F) = B_i(x^{-1}(F)) = B_j B_i(x^{-1}(F)) = B_j(x^{-1}(F)).$$

B. I only prove (Bi). Take $F \in \mathcal{X}$. It suffices to show $x^{-1}(F) \subseteq B_j(x^{-1}(F))$. It follows from Theorem 1B and Consistency of B_j that $B^p_{\tau_i} = B_j B^p_{\tau_i}$. Take $(F_\lambda)_{\lambda\in\Lambda}$ from $\mathcal{X}$ with $F^c = \bigcup_{\lambda\in\Lambda} F_\lambda$. Then,

$$\begin{aligned}\neg x^{-1}(F) = x^{-1}(F^c) = \bigcup_{\lambda\in\Lambda} x^{-1}(F_\lambda) &\subseteq \bigcup_{\lambda\in\Lambda} B_i(x^{-1}(F_\lambda)) \subseteq \bigcup_{\lambda\in\Lambda} B^1_{\tau_i}(x^{-1}(F_\lambda)) \\ &\subseteq B^1_{\tau_i}(x^{-1}(F^c)) \subseteq (\neg B^1_{\tau_i})(x^{-1}(F)),\end{aligned}$$

implying $x^{-1}(F) = B^1_{\tau_i}(x^{-1}(F))$. Now, it follows from Theorem 1B that

$$x^{-1}(F) = B^1_{\tau_i}(x^{-1}(F)) = B_j B^1_{\tau_i}(x^{-1}(F)) = B_j(x^{-1}(F)).$$

□

Proof of Proposition 5. A. By (i) and (iii) and by observing $(\uparrow t_{B_i}(\omega)) = t^{-1}_{B_i}(\{\mu \in M(\Omega) \mid \mu(\cdot) \geq t_{B_i}(\omega)(\cdot)\})$, $\omega \in (\uparrow t_{B_i}(\omega)) \subseteq B_i(\uparrow t_{B_i}(\omega))$. If $(\omega, E) \in \Omega \times \mathcal{D}$ satisfies $\omega \in B_i(E)$, then it follows from Finite Conjunction that $\omega \in B_i(E \cap (\uparrow t_{B_i}(\omega)))$. By Consistency and Monotonicity, $B_i(\emptyset) = \emptyset$ (note: $B_i(\emptyset) = B_i(E \cap E^c) \subseteq B_i(E) \cap B_i(E^c) = \emptyset$). Then, $E \cap (\uparrow t_{B_i}(\omega)) \neq \emptyset$.

B. Take $E \in \mathcal{D}$ with $\omega \in B_i(E)$. By Entailment, $\omega \in B^1_{\tau_i}$. If $E \cap [\tau_i(\omega)] = \emptyset$ (observe $(\uparrow \tau_i(\omega)) = [\tau_i(\omega)]$), then $[\tau_i(\omega)] \subseteq E^c$ and thus $\omega \in B^1_{\tau_i}(E^c)$, a contradiction to Consistency of $B^1_{\tau_i}$.

□

Proof of Theorem 2. Let $\omega \in B_i(\mathrm{RAT}_i)$. Since $\omega \in B_i([\sigma_i(\omega)])$, it follows from Finite Conjunction that $\omega \in B_i(\mathrm{RAT}_i \cap [\sigma_i(\omega)])$. Next, since B_i is compatible with informativeness, there is $\omega' \in \Omega$ such that $\omega' \in (\uparrow t_{B_i}(\omega)) \cap \mathrm{RAT}_i \cap [\sigma_i(\omega)]$. If there is $a'_i \in A_i$ such that $\omega \in (\neg B_i)(\neg[\sigma_i(\omega) \succcurlyeq_i a'_i]) = (\neg B_i)(\neg[\sigma_i(\omega') \succcurlyeq_i a'_i])$, then $\omega' \in (\neg B_i)(\neg[\sigma_i(\omega') \succcurlyeq_i a'_i])$, a contradiction. Thus, $\omega \in \mathrm{RAT}_i$. □

Remark A.1. To see simple counterexamples, consider the two-player coordination game represented by the left panel of Table 1. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Let $(\sigma_1(\omega))_{\omega\in\Omega} = (a, a, b)$ and $(\sigma_2(\omega))_{\omega\in\Omega} = (a, a, c)$. Suppose that B_1 and B_2 are given by the right panel of Table 1: B_1 violates Finite Conjunction, and B_2 is not consistent with informativeness. Then, $\mathrm{RAT}_i = \{\omega_1, \omega_2\}$ and thus $B_i(\mathrm{RAT}_i) \not\subseteq \mathrm{RAT}_i$ for each $i \in \{1, 2\}$.

	a	b	c
a	$1,1$	$0,0$	$0,0$
b	$0,0$	$1,1$	$0,0$
c	$0,0$	$0,0$	$1,1$

E	$B_1(E)$	$B_2(E)$
$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\emptyset$	$\emptyset$
$\{\omega_2\}$	$\emptyset$	$\emptyset$
$\{\omega_3\}$	$\emptyset$	$\emptyset$
$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_3\}$
$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2\}$
$\{\omega_2, \omega_3\}$	$\{\omega_2\}$	$\{\omega_1\}$
Ω	Ω	Ω

Table 1: A Counterexample for Theorem 2. The left panel depicts the coordination game while the right panel depicts the players' beliefs on $(\Omega, \mathcal{D})$.

E	$\emptyset$	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	Ω
$B_i(E)$	$\emptyset$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_1, \omega_2\}$
$C(E)$	$\emptyset$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\emptyset$	$\{\omega_1\}$

Table 2: Inidvidual and Common Beliefs B_i and C

A.2 Difference between Mutual and Common Certainty at a State

Remark 1 states that if every player is certain of (the value of) a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ (at every state) then the players are commonly certain of (the value of) x (at every state). This appendix shows through an example that the mutual and common certainty may differ if the players are certain of the value of the signal only at some state. This appendix also briefly discusses the higher-order certainty of a signal at a state.

Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ with $m \geq 3$, and let $\mathcal{D} = \mathcal{P}(\Omega)$. Introduce the natural order on Ω based on the indices: $\omega_k \leq \omega_\ell$ iff $k \leq \ell$. Define each player's belief operator B_i as follows:

$$B_i(E) := \begin{cases} \emptyset & \text{if } E = \emptyset \\ \{\omega_1\} & \text{if } |E| = 1 \\ E \setminus \{\max E\} & \text{if } |E| \geq 2 \end{cases}.$$

Then, the common belief operator C is written as:

$$C(E) = \begin{cases} \emptyset & \text{if } E = \emptyset \text{ or } |E| \geq 2 \text{ and } \omega_1 \notin E \\ \{\omega_1\} & \text{if } |E| = 1 \text{ or } |E| \geq 2 \text{ and } \omega_1 \in E \end{cases}.$$

For example, if $m = 3$ then the individual and common belief operators are depicted in Table 2.

Let $(X, \mathcal{X}) = (\{x_1, x_2\}, \mathcal{P}(X))$, and define $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ as follows:

$$x(\omega) = \begin{cases} x_1 & \text{if } \omega = \omega_1 \\ x_2 & \text{if } \omega \neq \omega_1 \end{cases}.$$

I show that (i) each player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω_2 and that (ii) the players are not commonly certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω_2. Observe $F \in \{\{x_2\}, X\}$ satisfies $\omega_2 \in x^{-1}(F)$. Indeed, $x^{-1}(\{x_2\}) = \{\omega_2, \ldots, \omega_m\}$ and $x^{-1}(X) = \Omega$. Then,

$$B_I^k(x^{-1}(\{x_2\})) = \begin{cases} \{\omega_2, \ldots, \omega_{m-k}\} & \text{if } k \leq m-2 \\ \emptyset & \text{if } k > m-2 \end{cases} \text{ and } C(x^{-1}(\{x_2\})) = \emptyset.$$

Also,

$$B_I^k(x^{-1}(X)) = \begin{cases} \{\omega_1, \ldots, \omega_{m-k}\} & \text{if } k \leq m-2 \\ \{\omega_1\} & \text{if } k \geq m-1 \end{cases} \text{ and } C(x^{-1}(X)) = \{\omega_1\}.$$

In fact, one can define higher-order certainty as follows. Player i is *certain that player j is certain of the value of a signal* $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at a state $\omega \in \Omega$ if, for any $F \in \mathcal{X}$ with $\omega \in x^{-1}(F)$, $\omega \in B_i B_j(x^{-1}(F))$. The players are *mutually certain of the value of* $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω if, for any $F \in \mathcal{X}$ with $\omega \in x^{-1}(F)$, $\omega \in B_I(x^{-1}(F))$. One can analogously define higher-order mutual certainty of the value of the signal x. If the mutual belief operator B_I satisfies Countable Conjunction in addition to Monotonicity, then the players are commonly certain of the value of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω iff they are mutually certain of the value of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω, they are mutually certain that they are mutually certain of the value of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω, and so forth *ad infinitum*.

A.3 Product Type Spaces

Fix a measurable space $(S, \mathcal{S})$ of nature states. A *product type space* is a tuple $\langle (T_i, \mathcal{T}_i)_{i \in I}, (m_i)_{i \in I} \rangle$ such that each $(T_i, \mathcal{T}_i)$ is a measurable space of player i's types and that each $m_i : (T_i, \mathcal{T}_i) \to (\Delta(T_{-i}), (\mathcal{T}_{-i})_\Delta)$ is a measurable mapping, where $T_{-i} = S \times \prod_{j \in I \setminus \{i\}} T_j$, $\mathcal{T}_{-i}$ is the product σ-algebra on T_{-i}, and $(\mathcal{T}_{-i})_\Delta$ is the σ-algebra generated by $\{\mu \in \Delta(T_{-i}) \mid \mu(E) \geq p\}$ for some $(E, p) \in \mathcal{T}_{-i} \times [0, 1]$.

I show that a product type space $\langle (T_i, \mathcal{T}_i)_{i \in I}, (m_i)_{i \in I} \rangle$ are identified as a type space $\langle (\Omega, \mathcal{D}), (\tau_i)_{i \in I} \rangle$ with certain properties. First, a given product type space induces a type space $\langle (\Omega, \mathcal{D}), (\tau_i)_{i \in I} \rangle$ as follows. Let the state space Ω be the product space $\Omega := S \times \prod_{i \in I} T_i$. Let $\mathcal{D}$ be the product σ-algebra on Ω. Define each player i's type mapping $\tau_i : \Omega \to \Delta(\Omega)$ as follows: for each state $(s, (\omega_i)_{i \in I}) \in \Omega$, let $\tau_i(\omega)$ be the product measure $\tau_i(\omega) = m_i(\omega_i) \times \delta_{\omega_i}$ induced by the type $m_i(\omega_i)$ and the Dirac measure δ_{ω_i}. Observe that $\delta_i : (T_i, \mathcal{T}_i) \ni \omega_i \mapsto \delta(\omega_i) = \delta_{\omega_i} \in (\Delta(T_i), (\mathcal{T}_i)_\Delta)$

and $m_i : (T_i, \mathcal{T}_i) \to (\Delta(T_{-i}), \mathcal{T}_{-i})$ are measurable, and hence $m_i \times \delta_i : (T_i, \mathcal{T}_i) \ni \omega_i \mapsto (m_i \times \delta_i)(\omega_i) = m_i(\omega_i) \times \delta_{\omega_i} \in (\Delta(\Omega), \mathcal{D}_\Delta)$ is measurable. Then, let $\tau_i : (\Omega, \mathcal{D}) \ni \omega \mapsto \tau_i(\omega) = (m_i \times \delta_i)(\omega_i) \in (\Delta(\Omega), \mathcal{D}_\Delta)$ is measurable.

Conversely, consider a type space $\langle (\Omega, \mathcal{D}), (\tau_i)_{i \in I} \rangle$ with the following properties: the state space $(\Omega, \mathcal{D})$ is the product measurable space of $(S, \mathcal{S})$ and $((T_i, \mathcal{T}_i))_{i \in I}$; and each $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ satisfies

1. $\tau_i(s, (\omega_j)_{j \in I}) = \tau_i(\tilde{s}, (\omega_i, \tilde{\omega}_{-i}))$ for all $s, \tilde{s}, \omega_i, \omega_{-i}, \tilde{\omega}_{-i}$; and
2. $\tau_i(\omega) \circ \pi_i^{-1} = \delta_{\omega_i}$, where $\pi_i : \Omega \to T_i$ is the projection.

Then, define $m_i : T_i \to \Delta(T_{-i})$ as $m_i(\omega_i) := \tau_i(\omega) \circ \pi_{-i}^{-1}$, where $\pi_{-i} : \Omega \to T_{-i}$ is the projection. It can be seen that $m_i : (T_i, \mathcal{T}_i) \to (\Delta(T_{-i}), (\mathcal{T}_{-i})_\Delta)$ is measurable.

Now, I formally show that a player is certain of her type mapping m_i in a product type space $\langle (T_i, \mathcal{T}_i)_{i \in I}, (m_i)_{i \in I} \rangle$ in the sense that she is certain of her type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ in the corresponding type space $\langle (\Omega, \mathcal{D}), (\tau_i)_{i \in I} \rangle$. Given a product type space $\langle (T_i, \mathcal{T}_i)_{i \in I}, (m_i)_{i \in I} \rangle$, take the corresponding type space $\langle (\Omega, \mathcal{D}), (\tau_i)_{i \in I} \rangle$. Since $\tau_i(\omega)(E) \geq p$ implies $\tau_i(\omega)(\{\omega' \in \Omega \mid \tau_i(\omega')(E) \geq p\}) = 1$ and since $\tau_i(\omega)(E) < p$ implies $\tau_i(\omega)(\{\omega' \in \Omega \mid \tau_i(\omega')(E) < p\}) = 1$, it follows that $B^p_{\tau_i}(\cdot) \subseteq B^1_{\tau_i} B^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq B^1_{\tau_i}(\neg B^p_{\tau_i})(\cdot)$. Hence, Proposition 1B (2c) implies that player i is certain of her own type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$.

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