Games and Economic Behavior 108 (2018) 432–451

Contents lists available at ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

Money as minimal complexity ☆

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ARTICLE INFO

Article history:
Received 9 May 2016
Available online 14 September 2017

JEL classification:
C70
C72
C79
D44
D63
D82

Keywords:
Exchange mechanism
Minimal complexity
Money

ABSTRACT

We consider mechanisms that provide the *opportunity* to exchange commodity i for commodity j, for certain ordered pairs ij. Given any connected graph G of opportunities, we show that there is a unique "G-mechanism" that satisfies some natural conditions of "fairness" and "convenience". Next we define time and price complexity for any G-mechanism as (respectively) the time required to exchange i for j, and the information needed to determine the exchange ratio (each for the worst pair ij). If the number of commodities exceeds three, there are precisely three minimally complex G-mechanisms, where G corresponds to the star, cycle and complete graphs. The star mechanism has a distinguished commodity — the money — that serves as the sole medium of exchange and mediates trade between decentralized markets for the other commodities. Furthermore, for *any* weighted sum of complexities, the star mechanism is the *unique* minimizer of the sum for large enough m.

1. Introduction

The purpose of this paper is to study the transition from a barter economy to a money-based economy, a phenomenon that seems to have occurred in almost all primitive societies at some stage in their development. Our main result offers a novel rationale for this phenomenon, which is based solely on certain complexity considerations, and which is independent of any notion of economic equilibrium or, indeed, even of any behavioral assumptions — utilitarian or otherwise — regarding the traders.

The need for money in an exchange mechanism has, of course, been the topic of much discussion, and it would be impossible to summarize that literature here. We give some references that are indicative, but by no means exhaustive. (For a detailed survey, see Shubik, 1999 and Starr, 2012.)

Several search-theoretic models, involving random bilateral meetings between long-lived agents, have been developed following Jevons (1875) (see, *e.g.*, Bannerjee and Maskin, 1996; Iwai, 1996; Jones, 1976; Kiyotaki and Wright, 1989, 1993; Li and Wright, 1998; Ostroy, 1973; Trejos and Wright, 1995 and the references therein). These models turn on utility-maximizing behavior and beliefs of the agents in Nash equilibrium, and shed light on which commodities are likely to get adopted as money. A parallel, equally distinctive, strand of literature builds on partial or general equilibrium models with other kinds of frictions in trade, such as limited trading opportunities in each period, or transaction

☆ In honor of Lloyd Shapley. (The authors thank David Levine and two anonymous referees for suggestions that have improved the presentation.)
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http://dx.doi.org/10.1016/j.geb.2017.09.003

costs (see, *e.g.*, Foley, 1970; Hahn, 1971; Heller, 1974; Heller and Starr, 1976; Howitt and Clower, 2000; Norman, 1987; Ostroy and Starr, 1974, 1990; Starr, 2012; Starr and Stinchcombe, 1999; Starret, 1973; Wallace, 1980). In many of these models, a specific trading mechanism is fixed exogenously, and the focus is on activity within the mechanism that is induced by equilibrium, based again on the optimal behavior of utilitarian individuals.

Our approach complements this literature in two salient ways, and brings to light a new rationale for money that is different from those proposed earlier, but not at odds with them, in that the door is left open to incorporate their concerns within our framework. First and foremost, our focus is purely on mechanisms of trade with no regard to the characteristics of the individuals such as their endowments, production technologies, preferences or beliefs. Second, no specific mechanism is specified *ex ante* by us. We start with a welter of mechanisms and cut them down by four natural conditions and certain complexity criteria, ultimately ending up with the "star" mechanism in which money plays the central role.

It is worth reiterating that ours is a purely "mechanistic", as opposed to a "utilitarian" or "behavioral", approach to the emergence of money.[1] In the technical parlance of game theory, we are concerned with the "game form" behind the game or — to be more precise — with the mechanism that underlies the game form itself. Indeed, with the same mechanism as the foundation, several different game forms can be constructed by introducing other considerations, such as whether netting of commodities is permitted or not, and if so to what extent; or whether certain commodities can be borrowed prior to trade and on what terms, along with rules for the settlement of debt in the event of default.[2] These are no doubt important economic issues, bearing on the "liquidity" in the system and the efficiency of its equilibria. They have been discussed at length, often in terms of the star mechanism which conforms to the well-known Walrasian model once there is perfect competition and "sufficient liquidity" (see, *e.g.*, Dubey and Shapley, 1994; Dubey and Geanakoplos, 2003). However, to even raise these issues, we first *need a mechanism in the background*. It is this background *alone* that forms the domain of our inquiry.

The bulk of our analysis is carried out in the oligopolistic setting of finitely many traders. However, in Section 4 we show that it readily extends to the case of "perfect competition", where there is a continuum of traders and our mechanisms induce "price-taking" behavior as in the Walrasian model.

This paper would not have been possible without the pioneering work of Lloyd Shapley in the area of strategic market games, especially his elaboration (in Shapley, 1976; Shapley and Shubik, 1977; Dubey and Shapley, 1994) of the decentralized "trading posts" model due to Shubik (see Shubik, 1973), i.e. the star mechanism; as well as his sharply juxtaposed model of centralized "windows", i.e. the complete mechanism. Our analysis here (and in Dubey et al., 2015), builds squarely upon (Dubey and Sahi, 2003), which in turn was inspired by these two models.[3] The windows model has been thoroughly examined in Sahi and Yao (1989) (see also the closely related model in Sorin, 1996).[4] Various strategic market games, based upon trading posts, have been analyzed, with commodity or fiat money in Dubey and Shubik (1978), Peck et al. (1992), Peck and Shell (1991), Postlewaite and Schmeidler (1978), Shapley (1976), Shapley and Shubik (1977), Shubik (1973), Shubik and Wilson (1977); most of these papers also discuss the convergence of Nash equilibria (NE) to Walras equilibria (WE) under replication of traders.[5] For a continuum-of-traders version, with details on explicit properties of the commodity money (its distribution and desirability) or of fiat money (its availability and the harshness of default penalties), under which we obtain equivalence (or near-equivalence) of NE and WE, see Dubey and Shapley (1994), Dubey and Geanakoplos (2003); and, for an axiomatic approach to the equivalence phenomenon, see Dubey et al. (1980).

Strategic market games differ in a fundamental sense from the Walrasian model, despite the equivalence of NE and WE. In the WE framework, agents always optimize generating supply and demand, but markets do not clear except at equilibrium. We are left in the dark as to what happens outside of equilibrium. In sharp contrast, in the NE framework, markets always clear, producing prices and trades based on agents' strategies; but agents do not optimize except at equilibrium. The very formulation of a game demands that the "game form", *i.e.*, the map from strategies to outcomes, be defined prior to the introduction of agents' preferences on outcomes; thus disentangling the physics of trade from its psychology.[6] Our mechanisms are firmly in this genre, and indeed form the bases upon which many market games are built. To be precise: game forms arise from our mechanisms by introducing private endowments and the rules of trade (including the degree of netting or borrowing permitted); and strategic market games then arise by further introducing preferences.

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[1] There *is* a faint touch of rationality that we assume regarding the traders, but it an order-of-magnitude milder than utilitarian (or other behavioral) considerations. See Remark 16.

[2] Netting means that if an individual *ex ante* offers x units of commodity i to the mechanism, and is *ex post* entitled to receive y units of i from it, then he is deemed to owe $\max\{0, x-y\}$ or else to receive $\max\{0, y-x\}$. In this scenario, one may think that "offers" consist of *promises* to deliver commodities, rather than commodities themselves; and that the mechanism calls upon traders to make (take) net deliveries (receipts) of actual commodities. But note the *a priori* need for a mechanism with respect to which netting can be formulated (or, for that matter, borrowing and default, or any other trade regulation). Also note that these regulations do not come without a cost (see Remark 8).

[3] Indeed Dubey and Sahi (2003) provide an axiomatic characterization of the finite set of "G-mechanisms" (see Section 2) that include both trading posts and windows as special cases.

[4] The windows model was described verbally by Shapley to Sahi and Yao, which led to Sahi and Yao (1989). It is also referenced by Mertens (2003) as "personal communication" from Shapley.

[5] As for the windows model, the convergence of NE to CE under replication is immediate, as is their equivalence in the presence of a continuum of traders.

[6] To put it bluntly, the insistence on a game form pertains to the following situation in the real world. People exercise choice all the time through their actions; and the world goes merrily on, by well-defining the outcome of those actions — it does not come to a standstill until they can explain why they have acted as they did!

Finally, the authors feel a real sense of privilege to be able to dedicate this paper, and its companion (Dubey et al., 2015), to the fond memory of Lloyd Shapley — friend, philosopher, and guide. Lloyd's work ranged over a wide array of topics in Game Theory, and he had a profound impact on everything he touched. However his seminal contributions to the theory of strategic market games, seem to be relatively less known. We hope that these papers will serve to bring them more into the light.

2. The emergence of money

The mechanisms we consider are *Cournotian* in spirit,[7] and the setting for them is simple, in keeping with our aim of showing that the need for money can arise at a very rudimentary level.

Let $\{1, \ldots, m\}$ be a *fixed* set of commodities and $\{1, \ldots, n\}$ a *variable* set of traders, $n = 1, 2, 3, \ldots$.

A mechanism M on $\{1, \ldots, m\}$ operates as follows. For certain ordered pairs ij, pre-specified by M, each trader in $\{1, \ldots, n\}$ may offer any quantity of commodity i in order to obtain commodity[8] j. Once all offers are in, the mechanism redistributes to the traders the commodities it has received, holding back nothing. The returns to the traders are calculated by an algorithm[9] that is common knowledge. Not only the population $\{1, \ldots, n\}$ of traders, but also the offers they might bring, can be arbitrary: *any* n-tuple of offers that goes as input to the mechanism M will be converted to an output consisting of a corresponding n-tuple of returns, by the algorithm of M, for $n = 1, 2, 3, \ldots$. (See Section 3 for an algebraic description of such an abstract mechanism.)

Let us emphasize that, in our scenario, a mechanism M only knows, and only needs to know, the offer of commodities that each trader brings to it. It is irrelevant *how* he brings them or *why* he brings them; the mechanism simply accepts every offer and — based upon the conglomeration of all offers — computes the return to each one who made an offer. (Thus a "trader" is simply tantamount to his offer, from the point-of-view of M.)

To sum up, one may think of the mechanism M as an institution for the exchange of commodities, which operates on a universal domain of offers, so as to be of service for "generations to come".

2.1. The graph of opportunities G

When a mechanism M permits the offer of commodity i in order to obtain j, we shall refer to the ordered pair ij as an (exchange) *opportunity* in M. Any such opportunity may be visualized as a directed edge ij of a graph whose vertices correspond to the commodity set $\{1, \ldots, m\}$. Thus the collection of opportunities in M give rise to a directed graph G. We assume throughout that G is *connected*, i.e., M permits *iterative* exchange of any i for any j.

At this level of generality, there are infinitely many mechanisms (algorithms) for any given graph G. However, we shall show that only one of them satisfies some natural conditions of "fairness" and "convenience" (see Section 3). This special mechanism is denoted M_G and will be described precisely in the next Section 2.2.

Remark 2 *(Unrestricted domain).* At first glance, the restriction that one may offer i only for some (and not all) j may appear artificial.[10] However these restrictions only apply *within* a particular mechanism. Our domain is made up of mechanisms corresponding to *every* possible (directed, connected) graph G. In particular, we admit the "complete" mechanism (see below) in which each commodity can be exchanged for every other. It is left to our complexity criteria to select among the plethora of mechanisms in this unrestricted domain.

2.2. The mechanism M_G

Let G be a directed and connected graph[11] with vertex set $\{1, \ldots, m\}$, corresponding to a fixed set of commodities. We define a particular mechanism M_G which operates over any set of traders $\{1, \ldots, n\}$, where n may be arbitrary, as follows. Each trader α can use every opportunity in M_G, i.e., place arbitrary weights a_{ij}^{α} on the edges ij of G, representing his

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[7] It is our purpose to see how far matters may develop in an elementary Cournot framework. In particular, note that *ex ante* there are no "prices" to refer to, upon which a trader may condition his offers. We do show that prices can be "admitted" (see Remark 3 below) but this happens *ex post* once unconditional offers of commodities have come into the mechanism. Our mechanisms are thus a far cry from the more complex Bertrand mechanisms, in which traders use prices alongside quantities in order to make contingent statements to protect themselves against vagaries of the market (see, *e.g.*, Dubey, 1982 and Mertens, 2003 for Bertrandian analogues of the "star" and "complete" mechanisms). An analysis analogous to ours might well be possible in the Bertrand setting, but that is a topic for future exploration.

[8] Thus the offer of a trader consists of non-negative quantities (possibly zero) placed on every permissible pair ij in M.

[9] There is no presumption that the algorithm be "informationally decentralized". Indeed even the return to a simple offer of i, made only via the pair ij, may well depend on all the offers at *every* $kl \in G$; and may thus require a lot of information for its computation.

[10] Such restrictions do often occur in practice. Think, for example, of a currency exchange, where not all currencies are directly convertible to one another.

[11] In this paper by a graph we mean a *directed simple graph*. Such a graph G consists of a finite *vertex* set V_G, together with an *edge* set $E_G \subseteq V_G \times V_G$ that does not contain any loops, *i.e.,* edges of the form ii. For simplicity we shall often write $i \in G$, $ij \in G$ in place of $i \in V_G$, $ij \in E_G$ but there should be no confusion. By a *path* $ii_1i_2 \ldots i_kj$ from i to j we mean a nonempty sequence of edges in G of the form $ii_1, i_1i_2, \ldots, i_{k-1}i_k, i_kj$. If $k = 0$ then the path consists of the single edge ij, otherwise we insist that the *intermediate* vertices $i_1, \ldots, i_k$ be distinct from each other and from the endpoints i, j. However we do allow $i = j$, in which case the path is called a *cycle*. We say that G is *connected* if for any two vertices $i \neq j$ there is a *path* from i to j.

offer of i for j. Let $b_{ij} = \sum_\alpha a^\alpha_{ij}$ denote the total weight on ij (i.e., the aggregate amount of commodity i offered for j by all traders). We shall specify what happens when $b_{ij} > 0$ for every edge ij in G, i.e., when there is sufficient diversity in the population of traders so that each opportunity is active. Denote $b = (b_{ij})_{ij\in G}$ and let Λ denote the set of rays in $\mathbb{R}^m_{++}$ representing prices.[12] It is well-known that (with b_{ij} understood to be 0 if ij is not an edge in G) there is a unique ray[13] $p = p(b)$ in Λ satisfying

$$\sum_i p_i b_{ij} = \sum_i p_j b_{ji} \quad \text{for all } j. \tag{1}$$

Note that the left side of (1) is the total value of all the commodities "chasing" j, while the right side is the total value of commodity j on offer; thus (1) is tantamount to "*value conservation*".

It turns out that (1) has an explicit combinatorial solution, which we now describe. Let $\mathcal{T}_i$ be the collection of all "spanning" trees in G that are rooted at i (*i.e.*, subgraphs of G in which there is a unique directed path to i from *every* $j \neq i$); and for any subgraph H, define $b_H = \prod_{ij\in H} b_{ij}$; then we have[14]

$$p_i = \sum_{T\in\mathcal{T}_i} b_T. \tag{2}$$

The principle of value conservation, which determines prices, also determines trade. A trader who offers a^α_{ij} units of i via opportunity ij (and nothing on all other edges) gets back $r(a^\alpha_{ij}, b)$ units of j (and nothing of the other commodities), where $p_i a^\alpha_{ij} = p_j r(a^\alpha_{ij}, b)$. More generally, if a trader offers $a^\alpha = (a^\alpha_{ij})_{ij\in G} \geq 0$ across all edges of G, he gets a return $r(a^\alpha, b) \in \mathbb{R}^m_+$ whose components are given by

$$r_j(a^\alpha, b) = \sum_i (p_i/p_j) a^\alpha_{ij} \quad \text{for all } j \tag{3}$$

(which incidentally implies $p \cdot r(a^\alpha, b) = p \cdot a^\alpha$).

This specifies a map from any n-tuple $(a^\alpha)_{\alpha=1,\dots,n}$ of offers to a corresponding n-tuple $\left(r(a^\alpha, b)\right)_{\alpha=1,\dots,n}$ of returns, for arbitrary $n = 1, 2, 3, \dots$; and thereby completes the definition of the G-mechanism M_G.

Note that commodities are conserved by the map, i.e., $\sum_{\alpha,j} a^\alpha_{ij} = \sum_\alpha r_i(a^\alpha, b)$ (with, recall, a^α_{ij} understood to be zero if $ij \notin G$).

Remark 3 *(Price mediation).* At a formal level — which will be emphasized in Section 3 below — the mechanism M_G is a family of maps from n-tuples of offers to n-tuples of returns, one map for each n, with no prices in the middle. Indeed one may substitute from equation (2) into (3) and obtain an explicit formula for each such map, eliminating prices altogether. However were we to "conjure" prices, in accordance with (2), this would give economic meaning to the map. Note that prices are determined uniquely from the aggregate offers of the traders on the various edges of G, and they *mediate trade* in the following strong sense: first, the return to any trader depends only on his own offers and the prices[15]; second, the total value — under the prevailing prices — of every trader's offers is equal to that of his returns. The upshot of price mediation is that the returns to any trader can be calculated in a transparent manner from the prices and his own offers. We may therefore think of prices as being "admitted", or "implicit", in M_G.

By (2) and (3), the return to a trader depends only on his offer a and the *price ratios* p_i/p_j, which are well-defined functions of b (unlike the price vector $p = (p_i)$ which is only defined up to a scalar multiple). It might be instructive to see the formulae for price ratios (and thereby also for returns, thanks to equation (3)) for specific mechanisms. Let us, from now on, identify two mechanisms if one can be obtained from the other by relabeling commodities. There are three mechanisms of special interest to us called the *star, cycle,* and *complete mechanisms;* with the following edge-sets and price ratios:

Graph G	Star	Cycle	Complete
Edge-set E_G	$\{mi, im : i < m\}$	$\{12, 23, \dots, m1\}$	$\{ij : i \neq j\}$
Price ratio p_i/p_j	$b_{mi}b_{jm}/b_{im}b_{mj}$	$b_{j,j+1}/b_{i,i+1}$	$*$

For the star and cycle mechanisms, the right-hand side of (2) involves a *single* tree and, in the ratio p_i/p_j, several factors cancel leading to the simple expressions in the table above. However, for the complete mechanism there is no cancellation and in fact here each price ratio depends on *every* b_{ij}.

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[12] Prices are to be thought of as consistent exchange rates between commodities, i.e. the ratios p_i/p_j. Thus they correspond to rays in $\mathbb{R}^m_{++}$, each of which is represented by a vector p in $\mathbb{R}^m_{++}$ (and identified with all its scalar multiples λp for $\lambda > 0$).

[13] To avoid notational clutter, we write p instead of $p(b)$ throughout this section, with the understanding that b is held fixed. Later p will be thought of as a function (defined on a large domain of variable b). There should be no confusion; the meaning will always be clear from the context.

[14] Formula (2) expresses each p_i as a polynomial in the components of b. It has a short proof Sahi (2014) but a long history. It seems to be originally due to Hill (1966) but has been rediscovered several times (see the discussion in Anantharam and Tsoucas, 1989). It often goes by the name of "The Markov Tree Theorem" because in the context of a Markov chain, where the b_{ij} are transition probabilities from i to j, formula (2) gives the unique steady state distribution.

[15] This is not to say that he does not affect prices by his offers; he invariably does, since we are in an oligopolistic framework.

The class of *G-mechanisms* is the set

$$\mathfrak{M}(m) = \{M_G : G \text{ is a directed, connected graph on } \{1, \ldots, m\}\}. \tag{4}$$

Although finite, $\mathfrak{M}(m)$ is rather large, indeed super-exponential in m.

2.3. Time and price complexity

We shall show that if one invokes natural complexity considerations, based on the time needed to exchange any commodity i for j and the information needed to determine the exchange ratio p_i/p_j, then the welter of mechanisms in $\mathfrak{M}(m)$ is eliminated and we are left with only three mechanisms of minimal complexity, namely those that arise from the star, cycle and complete graphs (Theorem 4). Indeed, provided m is large enough, just the star mechanism remains (Theorem 5) in which one commodity emerges endogenously as money and mediates trade across decentralized markets for the other commodities.[16]

Consider a trader who interfaces with $M \in \mathfrak{M}(m)$ in order to exchange i for j. A natural concern for him would be: what is the minimum number of time periods $\tau_{ij}(M)$ needed to accomplish this exchange? We define the *time-complexity* of M to be

$$\tau(M) = \max_{i \neq j} \tau_{ij}(M). \tag{5}$$

It is evident that $\tau_{ij}(M)$ is the length of the shortest path in G from i to j and $\tau(M)$ is the *diameter*[17] of the graph G.

The other concern of our trader would be: how much of commodity j can he get per unit of i? It follows from equations (2) and (3) that he can calculate this from the "market state" b of the mechanism which determines the *price ratio* p_i/p_j. Thus the question can be rephrased: how many components of b does he need to know[18] in order to calculate p_i/p_j? The table above indicates that it is easier to compute p_i/p_j for the star and cycle mechanisms than, say, the complete mechanism.

To make this notion precise, if f is a function of several variables $x = (x_1, \ldots, x_l)$, let us say that the component i of x is *influential* if there are two inputs x, x', differing only in the i-th place, such that $f(x) \neq f(x')$. Define $\pi_{ij}(M)$ to be the number of influential components of b in the price ratio function p_i/p_j. For example, from the expression for p_i/p_j for the star mechanism in the previous table, it is clear that $\pi_{ij}(M)$ is 4 unless one of i or j is m, in which case it is 2. We define the *price complexity* of M to be[19]

$$\pi(M) = \max_{i \neq j} \pi_{ij}(M). \tag{6}$$

See Section 2.6 for further elaboration of this notion.

2.4. The main results

We now define a *quasiorder* $\preceq$ (reflexive and transitive) on $\mathfrak{M}(m)$ by

$$M \preceq M' \iff \tau(M) \leq \tau'(M') \text{ and } \pi(M) \leq \pi'(M'). \tag{7}$$

We are ready to state our main result.[20]

Theorem 4. *If*[21] $m > 3$ *then the three special mechanisms are* precisely *the $\preceq$-minimal*[22] *elements of $\mathfrak{M}(m)$. Their complexities are as follows*:

[16] To be precise: the price of any commodity $i = 1, \ldots, m-1$, in terms of money m, depends only on the aggregate offers on edges im and mi; and therefore so does the trade between i and m. Hence the pair $\{im, mi\}$ of edges may be viewed as a *decentralized* market (or, "trading post") for i and m. See Section 3.1 for more discussion.

[17] Any commodity exchange in M must occur on an edge of its underlying graph G of opportunities, and each such exchange may be thought of as taking one unit of (transaction) time. Thus the length of a path from i to j in the graph serves a proxy for the time taken for successive exchanges along that path in order to convert i to j. Thus if a trader always chooses the most efficient (shortest) path to exchange any i for j, "diameter G" is the minimum time needed to effect the exchange for an *arbitrary* ordered pair ij.

[18] And, since he always knows his own offer, this is the same as asking: how many components does he need to know of the aggregate offer of the *others*?

[19] Recall that prices are *not* announced by the mechanism, but only implicit in it, as was pointed out in Remark 3. The trader has to "explicate" the ratio $p_i(b)/p_j(b)$ by computing it from b, and for this purpose he must inform himself about the relevant components of b. Looking at the worst case scenario across all market states b, the numbers $\pi_{ij}(M)$ and $\pi(M)$ reflect the "informational complexity" in the mechanism M that is encountered by a trader who interfaces with M. (This term could well be used in lieu of "price complexity".)

[20] A word about the numbering system used in this paper: all theorems, remarks, conditions, lemmas etc. are arranged in a *single* grand sequence. Thus the reader shall see, in order of appearance: Theorem 1, Theorem 2, Remark 3, Condition 4, ... This does *not* mean that Condition 4 is the fourth condition; in fact it is the first condition, but it has fourth place in the grand sequence (and, the marker 4 makes the condition easy to locate).

[21] When $m = 3$, we get a fourth mechanism with complexities 4, 2 identical to the star mechanism. And when $m = 2$, we must change 4 to 2 in the table (the three graphs become identical with complexities 2, 2 for each).

[22] M is said to be $\preceq$-*minimal* in $\mathfrak{M}(m)$ if there is no $M' \in \mathfrak{M}(m)$ for which $\tau(M') \leq \tau(M)$ and $\pi(M') \leq \pi(M)$, with strict inequality in at least one place.

	Star	Cycle	Complete
$\pi(M)$	4	2	$m(m-1)$
$\tau(M)$	2	$m-1$	1

This has the following immediate consequence.

Theorem 5. *Given any choice of strictly positive weights $\lambda, \mu > 0$, there exists an integer m_0 such that for $m \geq m_0$ the star mechanism is the* unique *minimizer in $\mathfrak{M}(m)$ of $\lambda\pi(M) + \mu\tau(M)$.*

Theorem 5 says that, so long as traders ascribe positive weight to *both* time and price complexity, the star mechanism with money is the unique optimal mechanism as soon as the number of commodities is sufficiently large.

Remark 6 *(Sufficient commodities).* In fact m_0 does not have to be too large. We only require $4\lambda + 2\mu < 2\lambda + (m-1)\mu$ and $4\lambda + 2\mu < m(m-1)\lambda + \mu$ for the star to beat the cycle and complete mechanisms, respectively; which may be rearranged

$$m > 2\left(\frac{\lambda}{\mu}\right) + 3 \text{ and } m^2 - m > \frac{\mu}{\lambda} + 4$$

So, for example, if at least 10% weight is accorded to both π and μ, then λ/μ and μ/λ can each be at most 9 and the above inequalities will hold if $m > 18 + 3$ and $m^2 - m > 9 + 4$; thus $m_0 = 22$ does the job.

Remark 7 *(Computational complexity).* Our notion $\pi_{ij}(M)$ of price complexity counts the number of components of the market state b that are needed for the computation of price ratios p_i/p_j. The difficulty of that computation is not taken into account. However, even if it were, the star would perform well relative to the other G-mechanisms. To see this, recall (from the first table in Section 2) that $p_i/p_j = b_{mi}b_{jm}/b_{im}b_{mj}$ in the star mechanism, a simple formula whose "computational complexity" is hardly worth the mention. (In contrast, most graphs G have multiple spanning trees rooted at some of their vertices, which leads — via (2) — to more complicated expressions for p_i/p_j.)

Remark 8 *(Netting of deliveries).* If *promises* to deliver commodities — rather than commodities themselves — are being traded and "netting" of deliveries is permitted (recall footnote 3), an individual could trade i for j in one go, instead of trading iteratively along the path that connects i to j. This reduction of time complexity is quite illusory, however. It requires bookkeeping — carried out by a centralized clearing-house? — to determine the net due to, or owed by, any individual across his many trades. Thus netting simply transfers time complexity to the complexity of bookkeeping.

Remark 9 *(Contingent commodities).* Once promises can be traded, the door is open for *contingent commodities* "*ilts*" *a la* Arrow–Debreu, namely promises which call for the delivery of i in location l at time t if state of nature s occurs. This, of course, creates a bewildering number m of promises (quadruples *ilts*) to be traded. Our result shows that *the star mechanism alone scales well with m*, since its time and price complexities remain 2 and 4 regardless of m, making it the most felicitous among all G-mechanisms (in the sense of Theorems 4 and 5).

Remark 10 *(Only time complexity).* If price complexity is dropped, and only time complexity retained (i.e., $\lambda = 0$ and $\tau > 0$ in Theorem 5), it is evident that the complete mechanism will be the unique minimal mechanism in $\mathfrak{M}(m)$. To see this, consider any (directed, connected) graph G on vertex set $\{1, \ldots, m\}$ and suppose G is not complete. Then there must exist two vertices i and j such that there is no directed edge ij in G. In this case, the shortest path from i to j has length at least 2, and hence the diameter of G is not less than 2. On the other hand the diameter of the complete graph is obviously 1, establishing its minimality. Thus price complexity is crucial for arriving at the star (money) mechanism. (If one were to try to restrict the domain of mechanisms so that time complexity by itself leads to the star, one would need to exclude not only the complete mechanism but also all G-mechanisms — except for "star" — whenever G has diameter 2. We do not see any natural way of doing this.)

2.5. Outline of the proof

Let us give a brief outline of the proof of Theorem 4, which immediately implies Theorem 5.

We focus first on price complexity, which plays the decisive role in our argument. (The key to this was provided for us by Sahi (2014), in which a general version of (2) is established.)

Since any (directed, connected) graph G gives rise to a *unique* mechanism M_G (see Section 3), we may write $\pi(M_G) = \pi(G)$. Using equation (2), it is straightforward to show that

$$\pi(G) = 2, \text{ if } G \text{ is a cycle}$$

and

$$\pi(G) = 4, \text{ if } G \text{ is either a chorded cycle or a } k\text{-rose}$$

(A chorded cycle is a cycle with a path between two of its nodes that is distinct from the cycle, and a k-rose is a set of k cycles with which have a common vertex but are otherwise disjoint. The formal definitions are given in Section 5.1. Note that the star is an m-rose.)

We establish the following fundamental Theorem 21, that may be of independent mathematical interest:

If G is not a cycle or a chorded cycle or a k-rose, then $\pi(G) \geq 5$

The proof of Theorem 21 is intricate, and involves several ideas from graph theory. (Its outline is given just after the statement of Theorem 21 in Section 5.1.) Based on Theorem 21, the rest of the proof of Theorem 4 proceeds as follows. It is evident that $\tau(G) = 1$ if, and only if, G is the complete graph; and in this event (again using equation (2)) it is easy to check that $\pi(G) = m(m-1)$, i.e., $(\tau(G), \pi(G)) = (1, m(m-1))$. Next, as is obvious, $\tau(G) = 2$ if G is a star, so — from the second display above — $(\tau(G), \pi(G)) = (2, 4)$. Thus for any graph G which is not complete, and which therefore has $\tau(G) \geq 2$, we must require $\pi(G) \leq 4$ for G to remain in the reckoning, otherwise G will be worse than star according to $\preceq$. If $\pi(G) \leq 4$, Theorem 21 implies that G must be either a cycle or a k-rose. If G is a cycle, it is immediate that $\tau(G) = m-1$ and hence — from the first display above — $(\tau(G), \pi(G)) = (m-1, 2)$. If G is a k-rose that is not a star, it can be easily verified that $\tau(G) \geq 3$ whenever $m > 3$, so that — again from the second display above — $(\tau(G), \pi(G)) = (3^+, 4)$, showing that G is worse than star in the quasi-order $\preceq$. This establishes Theorem 4.

2.6. Literal vs marginal exchange rates

Consider a trader who wants to convert x units of i to j along some path $P_{ij} = ii_1 \ldots i_k j$ from i to j in the graph G of the mechanism $M = M_G$. Let b denote the aggregate offer (market state) *before* x is brought on the market and assume throughout that others' offers stay fixed according to b. If x is small, then it will have negligible effect on the prices and we may treat prices $p = p(b)$ as fixed. So — to a first order of approximation — the trader will get $x_{i_1} = (p_i/p_{i_1})x$ units of i_1 on the edge ii_1, then $x_{i_2} = (p_{i_1}/p_{i_2})x_{i_1}$ units of i_2 on the edge $i_1 i_2$ and so on, winding up with $(p_i/p_{i_1})(p_{i_1}/p_{i_2}) \ldots (p_{i_k}/p_j) = (p_i/p_j)x$ units of j. Thus p_i/p_j represents the *marginal exchange rate*[23] between commodity i and j, and it is independent of the path P_{ij}.

When x is not small, its influence on prices is not negligible, and matters are no longer so simple. Each time the trader makes an offer on an edge, he perturbs the market state, and thereby the prices as well as his returns, which in turn constitute his offer on the next edge of the path P. Thus the *literal exchange rate* from i to j is a complicated function of the vector (b, x), as well the path $P = P_{ij}$. Holding P fixed, let us denote this function by F^P_{ij} and further denote[24]

$$\pi^*_{ij}(M_G, P) = \text{number of influential components of } b \text{ in the function } F^P_{ij}(b, x);$$

$$\pi^*_{ij}(M_G) = \min\left\{\pi^*_{ij}(P) : P \text{ is a path from } i \text{ to } j \text{ in } G\right\};$$

and

$$\pi^*(M_G) = \max\left\{\pi^*_{ij}(M_G) : 1 \leq i, j \leq m \text{ and } i \neq j\right\}.$$

In view of equations (2) and (3), F^P_{ij} is a rational function (a ratio of two polynomials in b and x) whose denominator remains positive as x goes to zero.[25] Thus F^P_{ij} is continuous at $x = 0$ for any fixed b, i.e., $F^P_{ij}(b, x) \longrightarrow p_i(b)/p_j(b)$ as $x \longrightarrow 0$ for all b. This implies that the number of influential components of b in the function F^P_{ij} cannot jump up at the limit when $x \longrightarrow 0$. To see this, suppose that μ is an influential component of the function p_i/p_j, i.e., $p_i(b)/p_j(b) \neq p_i(d)/p_j(d)$ where b and d differ only in component μ. Since $F^P_{ij}(b, x) \longrightarrow p_i(b)/p_j(b)$ and $F^P_{ij}(d, x) \longrightarrow p_i(d)/p_j(d)$ as $x \longrightarrow 0$, it follows that $F^P_{ij}(b, x) \neq F^P_{ij}(d, x)$ for all small x, showing that μ is also an influential component of b in the function $F^P_{ij}(b, x)$. Thus we have shown that

$$\pi_{ij}(M_G) \leq \pi^*_{ij}(M_G); \text{ and hence } \pi(M_G) \leq \pi^*(M_G)$$

for *all* G-mechanisms, i.e., the literal exchange rate is never less than the marginal exchange rate. On the other hand, it is readily verified that

$$\pi_{ij}(M_G) = \pi^*_{ij}(M_G) \text{ if } G \text{ is a star}$$

(This equality essentially stems from the fact that the star mechanism is made up of decentralized markets.) From the last two displays and our main Theorem 5, it is immediate that the variant of Theorem 5, with π^* in place of π, also holds.

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[23] This is a rate that pertains solely to the input–output map of our mechanism and has nothing to do with marginal utilities of the traders (indeed recall that such utilities do not exist in our model).

[24] The trader already knows his own offer x. What he needs to find out, in order to compute how much j he will get against x via the path P, are all the components of b on which the function $F^P_{ij}(b, x)$ depends. (Recall that F^P_{ij} is defined on the domain of all (b, x) such that $x > 0$ and such that $b_{ij} > 0$, if and only if, ij is an edge in G.)

[25] This is so since b is positive on all the edges of G.

In short, the variant Theorem for the complicated literal rates π^* is an easy corollary of our current Theorem 5 for the simple marginal rates π, not the other way round!

3. Characterization of G-mechanisms

Our analysis above was carried out on the domain $\mathfrak{M}(m)$. We now show how to derive $\mathfrak{M}(m)$ from a more general standpoint. To this end, let us first define an *abstract* exchange mechanism on commodity set $\{1, \ldots, m\}$ and with trading opportunities given by a directed, connected graph G on $\{1, \ldots, m\}$. Such a mechanism allows individuals in $\{1, \ldots, n\}$ to trade by means of quantity offers in each commodity i across all edges ij in G. (Here m is fixed and n can be arbitrary.) The offer of any trader can thus be (conveniently) viewed as an $m \times m$ non-negative matrix in the space

$$S = \left\{a : a_{ij} = 0 \text{ if } ij \notin G, a_{ij} \geq 0 \text{ otherwise}\right\}$$

Define

$$S_+ = \left\{a \in S : a_{ij} > 0 \text{ if } ij \in G\right\}$$

Also define

$$\overline{a} = \left(\overline{a_1}, \ldots, \overline{a_m}\right)$$

where $\overline{a_i} = \sum_j a_{ij}$ is the i-th row sum of a and denotes the total amount of commodity i involved in sending offer a_i. Let S^n be the n-fold Cartesian product of S with itself, and (with $\boldsymbol{a} = (\boldsymbol{a}^1, \ldots, \boldsymbol{a}^n)$) let

$$S(n) = \left\{\boldsymbol{a} \in S^n : \sum_{\alpha=1}^{n} \boldsymbol{a}^\alpha \in S_+\right\}$$

denote the n-tuples of offers that are positive on aggregate. Also let $C = \mathbb{R}^m_+$ denote the *commodity space*; and C^n its n-fold product.

An *exchange mechanism* M, for a given set $\{1, \ldots, m\}$ of commodities and with trading opportunities in accordance with the graph G, is a collection of maps (one for each positive integer n) from $S(n)$ to C^n such that, if $\boldsymbol{a} \in S(n)$ leads to returns $\mathbf{r} \in C^n$, then we have

$$\sum_{\alpha=1}^{n} \overline{\boldsymbol{a}}^\alpha = \sum_{\alpha=1}^{n} \boldsymbol{r}^\alpha,$$

i.e., there is *conservation of commodities*. It is furthermore understood, in keeping with our concept of opportunity ij, that for an offer $a \in S$ whose only non-zero components are $\left\{a_{ij} : j = \ldots\right\}$, the return will consist exclusively of commodity j.

We shall impose four conditions on the mechanisms which reflect "convenience" and "fairness" in trade. The first condition is that the mechanism must be blind to all other characteristics of a trader except for his offer (and rules out discrimination on irrelevant grounds):

Condition 11 *(Anonymity). Let $(\boldsymbol{r}^1, \ldots, \boldsymbol{r}^n) \in C^n$ denote the returns from $(\boldsymbol{a}^1, \ldots, \boldsymbol{a}^n) \in S(n)$. Then for any permutation σ the returns from $(\boldsymbol{a}^{\sigma(1)}, \ldots, \boldsymbol{a}^{\sigma(n)})$ are $(\boldsymbol{r}^{\sigma(1)}, \ldots, \boldsymbol{r}^{\sigma(n)})$.*

The second condition is that if any trader pretends to be two different persons by splitting his offer, the returns to the others is unaffected. In its absence, traders would be faced with the complicated task of tracking everyone's offers. It is easier (and sufficient!) to state this condition for the "last" trader.

Condition 12 *(Aggregation). Suppose $\boldsymbol{a} \in S(n)$ and $\boldsymbol{b} \in S(n+1)$ are such that $\boldsymbol{a}^\alpha = \boldsymbol{b}^\alpha$ for $\alpha < n$ and $\boldsymbol{a}^n = \boldsymbol{b}^n + \boldsymbol{b}^{n+1}$. Let $\mathbf{r}, \mathbf{s}$ denote the returns that accrue from $\boldsymbol{a}, \boldsymbol{b}$ respectively. Then $\mathbf{r}^\alpha = \boldsymbol{s}^\alpha$ for $\alpha < n$.*

Anonymity and *Aggregation* immediately imply that, regardless of the size n of the population, the return to any trader may be written $r(a, b)$,where $a \in S$ is his own offer and $b \in S_+$ is the aggregate of all offers.

Let ν denote his *net trade*:

$$\nu(a, b) = r(a, b) - \overline{a}$$

The third condition is *Invariance*. Its main content is that the *maps* which comprise M are invariant under a change of units in which commodities are measured. This makes the mechanism much simpler to operate in: one does not need to keep track of seven pounds or seven kilograms or seven tons, just the numeral 7 will do.

In what follows, we will consistently use a for an individual's offer and b for the positive aggregate offer; so, when we refer to the pair a, b it will be implicit that $a \in S$, $b \in S_+$ and $a \leq b$.

Condition 13 *(Invariance).* $\nu(\lambda a, \lambda b) = \lambda\nu(a, b)$ *for all* a, b *and any* $m \times m$ *strictly positive diagonal matrix* λ.

The fourth, and last, condition is that no trader can get strictly less than his offer (otherwise, such unfortunate traders would tend to abandon the mechanism).

Condition 14 *(Non-dissipation). If* $\nu(a, b) \neq 0$*, then* $\nu_i(a, b) > 0$ *for some component* i.

It turns out that these four conditions categorically determine a unique mechanism.

Theorem 15. *Let* M *be an exchange mechanism on commodity set* $\{1, \ldots, m\}$ *and let* G *be the (directed, connected) graph induced by the trading opportunities in* M*. If* M *satisfies* Anonymity, Aggregation, Invariance *and* Non-dissipation*, then* $M = M_G$.

3.1. Comments on the conditions

Aggregation does not imply that if two individuals were to merge, they would be unable to enhance their "oligopolistic power". For despite the *Aggregation* condition, the merged individuals are free to *coordinate* their actions by jointly picking a point in the Cartesian product of their action spaces. Indeed all the mechanisms we obtain display this "oligopolistic effect", even though they also satisfy *Aggregation*.

It is worthy of note that the cuneiform tablets of ancient Sumeria, which are some of the earliest examples of written language and arithmetic, are in large part devoted to records and receipts pertaining to economic transactions. *Invariance* postulates the "numericity" property of the maps $r(a, b)$ (equivalently, $\nu(a, b)$) making them independent of the underlying choice of units, and this goes to the very heart of the quantitative measurement of commodities. In its absence, one would need to figure out how the maps are altered when units change, as they are prone to do, especially in a dynamic economy. This would make the mechanism cumbersome to use.

Non-dissipation (in conjunction with *Aggregation, Anonymity*, and the conservation of commodities) immediately implies *No-arbitrage:* for any a, b neither $\nu(a, b) \gneqq 0$ nor $\nu(a, b) \lneqq 0$. To check this, we need consider only the case $a \leq b$ and rule out $\nu(a, b) \gneqq 0$. Denote $c = b - a$. Then $\nu(a, b) + \nu(c, b) = \nu(a + c, b) = \nu(b, b) = 0$, where the first equality follows from *Aggregation*, and the last from conservation of commodities. But then $\nu(a, b) \gneqq 0$ implies $\nu(c, b) \lneqq 0$, contradicting *Non-dissipation.*

Our four conditions are *tight*: dropping any one of them will allow for mechanisms other than our graphical M_G and cause Theorem 15 to break down. For instance, the mechanism $\widetilde{M}$ studied in Amir et al. (1990), with the underlying complete graph of exchange opportunities, satisfies *Aggregation, Anonymity* and *Invariance* but not *Non-dissipation.*[26] There are $m(m-1)/2$ "trading posts" in $\widetilde{M}$, one for each unordered pair ij of commodities, which function as follows: all ij-offers and ji-offers exchange proportionately against each other, unaffected by the offers on edges other than ij or ji. In this sense, a trading post ij in $\widetilde{M}$ does represent a *decentralized* market for i and j. However, while $\widetilde{M}$ admits prices p_{ij} (the ratio of aggregate j to aggregate i at trading post ij), they are rarely consistent : indeed $p_{ij}(b)p_{jk}(b) \neq p_{jk}(b)$ for almost all market states b of the mechanism.

The mechanism $\widetilde{M}$ with $m(m-1)/2$ trading posts was also the center of attention in Starr and Stinchcombe (1999), though in a context which is quite different from, and complementary to, ours. In Starr and Stinchcombe (1999), a Walras Equilibrium (WE) is specified exogenously, and the problem is to execute the WE trades (at WE prices) via the trading posts. There is a set up cost for each trading post (also specified exogenously) and it is shown that the star mechanism will be most cost-efficient.

In our framework, the star is the *only* mechanism which is composed exclusively of trading posts. Indeed, for any graph G other than the star, there will exist at least one edge ij where the exchange (between i and j in M_G) is affected by offers on some edge lk that is different from both ij and ji (in the event that ji exists in G), exhibiting lack of "decentralization". In contrast to Amir et al. (1990) and Starr and Stinchcombe (1999), the decentralized trading posts are a matter of *deduction* for us (via the star mechanism), and we do not postulate them.

Returning to the backdrop of a pre-specified WE (as in Starr and Stinchcombe, 1999), there is an equally striking paper of Norman (1987) in which a broker is introduced, whose task is to arrange a sequence of commodity transfers between individuals in order to achieve their WE trades. An in-depth analysis is presented of the computational complexity of the broker's task. One of the central results in Norman (1987) is that the presence of money minimizes this complexity. Roughly speaking, money enables the broker to create interim credit in the sequence, which need net to zero only by the end, considerably simplifying his task. (Computational complexity, in the context of our mechanisms, is discussed in Remark 7.)

[26] We leave it to the reader to construct the easy examples which show that none of *Aggregation, Anonymity*, *Invariance* can be dropped unilaterally without violating Theorem 15.

It is worth noting that the approach taken in both Starr and Stinchcombe (1999) and Norman (1987) focuses on the interaction between traders (as in much of the literature[27]). In contrast, our primary focus is on the interaction between commodities, once they are offered to the mechanism; from which we *derive,* in turn, the interaction between traders.[28]

Remark 16 *(Touch of rationality).* There is a "touch" of rationality, imputed to the traders, in these conditions. *Non-dissipation* implies that commodities are liked and an uncompensated loss of them is not tolerable. (This is compatible with *any* monotonic utility function and hardly very restrictive.) *Anonymity* rules out discrimination among traders on extra-economic grounds. *Aggregation* and *Invariance,* as well our notion of the complexity of a mechanism, reflect the fact that traders find complicated computations inconvenient. These requirements are minimalistic and an order-of-magnitude milder than the standard utilitarian (or other behavioral) assumptions. In fact, our mechanisms permit *arbitrary* utility functions to be ascribed to the traders in order to build a game (see, *e.g.*, Shapley, 1976; Dubey and Shubik, 1978; Sahi and Yao, 1989 and the references therein).

3.2. Alternative characterizations of G-mechanisms

The formula (3) for the return function of a G-mechanism immediately implies

$$p(b) = p(c) \Longrightarrow r(a,b) = r(a,c) \text{ for all } a \geq 0 \text{ and } b, c > 0 \tag{8}$$

In Dubey and Sahi (2003), a mechanism was supposed to produce both trades and prices, based upon everyone's offers; and the property (8) was referred to as *Price Mediation*. It was shown in Dubey and Sahi (2003) that $\mathfrak{M}(m)$ is characterized by *Anonymity, Aggregation, Invariance, Price Mediation* and *Accessibility* (the last representing a weak form of continuity).

An alternative characterization of $\mathfrak{M}(m)$, which assumes — as we do here — that a mechanism produces only trades (and no prices), was given in Dubey et al. (2014). Here we have presented a simplified version of the analysis in Dubey et al. (2014), and established that M_G arises "naturally" *once* we assume that trading opportunities are restricted to pairwise exchange of commodities, i.e., correspond to the edges of a connected graph G. In contrast, in both Dubey and Sahi (2003) and Dubey et al. (2014), the opportunity structure G was *itself* an object of deduction, starting from a more abstract viewpoint; and is the theme of our companion paper (Dubey et al., 2015) (extracted from Dubey et al., 2014).

4. The case of perfect competition

When the set of traders is finite, each trader invariably influences the market state b via his offer, and thereby the price vector $p = p(b)$. (See equation (2), and recall that b is the aggregate of everone's offers.) This is but to be expected in an oligopolistic framework. However we may easily accommodate the Walrasian world of perfect competition by considering a continuum of traders. In this event, b corresponds to an integral which no trader can influence by unilateral variations of his own offers, with the upshot that he is rendered a "price taker". The literal and marginal exchange rates of Section 2.6 now coincide, leading to a simpler economic interpretation of price complexity. Moreover, *all* our results remain intact, by essentially the same arguments. Indeed, the one change we need to make is to combine the *Aggregation* and *Anonymity* conditions into a single condition. (See Section 8 for the details.) The rest of the paper can be reread *word for word*, interpreting b throughout as an integral instead of a finite sum.

Finally, as was already mentioned in the introduction, the star mechanism leads to equivalence (or, near-equivalence) of Nash and Walras equilibria under suitable postulates regarding the commodity or fiat money. (See Dubey and Shapley, 1994 for a detailed discussion.) In the complete mechanism — where every commodity is akin to money in that it can trade with all other commodities — there are no liquidity constraints, and the equivalence is automatic in the presence of a continuum.

5. Proofs

5.1. Graphs with complexity ≤ 4

Let G be a connected graph on $\{1, \ldots, m\}$ as in Section 2.3, and write

$$\pi_{ij}(G) = \pi_{ij}(M_G) \text{ and } \pi(G) = \pi(M_G)$$

If G consists of a single vertex then $\pi(G) = 0$ by definition.

Lemma 17. *If G is a cycle then $\pi(G) = 2$.*

[27] See the references in the third paragraph of introduction (on models involving "search", "random matching", etc.).

[28] We are grateful to an anonymous referee for the eloquence of this distinction.

Proof. Each vertex i in a cycle has a unique outgoing edge, and we denote its weight by[29] a_i. For each i we have $p_i = b_G/a_i$ where $b_G = \prod_{ij \in G} b_{ij} = \prod_i a_i$ as in (2); hence $p_i/p_j = a_j/a_i$ and the result follows. □

By a *chorded cycle* we mean a graph that is a union $G = C \cup P$ where C is a cycle and P, the chord, is a path that connects two distinct vertices of C, but which is otherwise disjoint from C.

Lemma 18. *If $G = C \cup P$ is a chorded cycle then $\pi(G) = 4$.*

Proof. Let i be the initial vertex of the path P, then i has two outgoing edges, ij and ik say, on the cycle and path respectively. Any vertex $l \neq i$ has a unique outgoing edge, and we denote its weight by a_l as before. Let x be the terminal vertex of the path P. If $x = j$ then G has two j-trees, otherwise there is a unique j-tree; similarly if $x = k$ then there are two k-trees, otherwise there is a unique k-tree. Thus we get the following table[30]:

	$x = j$	$x = k$	$x \neq j, k$
p_j/b_G	$a_j^{-1}\left(b_{ik}^{-1} + b_{ij}^{-1}\right)$	$a_j^{-1} b_{ik}^{-1}$	$a_j^{-1} b_{ik}^{-1}$
p_k/b_G	$a_k^{-1} b_{ij}^{-1}$	$a_k^{-1}\left(b_{ik}^{-1} + b_{ij}^{-1}\right)$	$a_k^{-1} b_{ij}^{-1}$

In every case, the ratio p_j/p_k depends on all 4 variables a_j, a_k, b_{ij}, b_{ik}, thus $\pi(G) \geq 4$.

On the other hand, since all vertices other than i have a unique outgoing edge, it follows that if x is any vertex then every x-tree contains all the outgoing edges except perhaps the edges b_{ij}, b_{ik} and a_x (if $x \neq i$); thus p_x is divisible by all other weights. It follows that for any two vertices x, y the ratio p_x/p_y can only depend on the variables b_{ij}, b_{ik}, a_x, a_y. Thus we get $\pi(G) \leq 4$ and hence $\pi(G) = 4$ as desired. □

Remark 19. A special case of a chorded cycle is a graph T_0 with three vertices that we call a chorded triangle.

3		
$\uparrow\downarrow$	$\nwarrow$	
1	$\longrightarrow$	2

p_1	$b_{23} b_{31}$
p_2	$b_{12} b_{31}$
p_3	$b_{23}(b_{12} + b_{13})$

p_1/p_2	b_{23}/b_{12}
p_2/p_3	$b_{12} b_{31}/b_{23}(b_{12} + b_{13})$
p_3/p_1	$(b_{12} + b_{13})/b_{31}$

For future use we note that for each index j there is an i such that $\pi_{ij} \geq 3$.

By a *k-rose* we mean a graph that is a union $C_1 \cup \cdots \cup C_k$, where the C_i are cycles that share a single vertex j, but which are otherwise disjoint. Thus a 0-rose is a single vertex and a 1-rose is a cycle. If G is a k-rose for some $k \geq 2$ then we will simply say that G is a *rose*.

If each cycle in a rose G has exactly two vertices, *i.e.*, is a bidirected edge, then we say that G is a *star*.

Lemma 20. *If G is a rose then $\pi(G) = 4$.*

Proof. Let G be the union of cycles $C_1 \cup \cdots \cup C_k$ with common vertex j as above. Let $a_1, \ldots, a_k$ be the weights of the outgoing edges from j in cycles $C_1, \ldots, C_k$ respectively, and for all other vertices x let b_x denote the weight of the unique outgoing edge at x. It is easy to see that there for each vertex v of G there is a unique v-tree, and thus the price vectors are given as follows:

$$p_j = \prod_{x \neq j} b_x, \quad p_x = \frac{a_i p_j}{b_x} \text{ if } x \neq j \text{ is a vertex of } C_i$$

Thus we get

$$p_j/p_x = b_x/a_i, \quad p_y/p_x = b_x a_l / b_y a_i \text{ if } y \neq j \text{ is a vertex of } C_l$$

Taking $i \neq l$, we see that p_y/p_x depends on 4 variables, and $\pi(G) = 4$. □

Our main result is a classification of connected graphs with $\pi(G) \leq 4$.

[29] This is a departure from our convention heretofore that a shall refer to an individual's offer, and b to the aggregate offer; but there should be no confusion.

[30] Let us explain the entry in row 1, column 1. (The argument for the other entries in this table, as well as entries in tables that come later, is in the same vein.) There are two spanning trees Γ_1 and Γ_2 in this case, coming into j, each of which omits exactly two edges of the graph. Both omit the unique edge going out of j; and, furthermore, Γ_1 omits ik while Γ_2 omits ij. Therefore the weight of Γ_1 (resp. Γ_1), divided by b_G, is $(a_j b_{ik})^{-1}$ (resp. $(a_j b_{ij})^{-1}$). The rest follows from the formula (2) for prices.

Theorem 21. *If G is not a chorded cycle or a k-rose, then* $\pi(G) \geq 5$.

We give a brief sketch of the proof of this theorem, which will be carried out in the rest of this section. The actual proof is organized somewhat differently, but the main ideas are as follows.

We say that a graph H is a *minor* of G, if H can be obtained from G by removing some edges and vertices, and collapsing certain kinds of edges. Our first key result is that the property $\pi(G) \leq 4$ is a *hereditary* property, in the sense that connected minors of such graphs also satisfy the property. The usual procedure for studying a hereditary property is to identify the *forbidden minors*, namely a set Γ of graphs such that G fails to have the property iff it contains one of the graphs from Γ. We identify a finite collection of such graphs. The final step is to show that if G is not a chorded cycle or a k-rose then it contains one of the forbidden minors.

We note the following immediate consequence of the results of this section.

Corollary 22. *If G is not a cycle then* $\pi_{ij}(G) \geq 4$ *for some* ij.

5.2. Subgraphs

Throughout this section G denotes a connected graph. We say that a graph H is a *subgraph* of G if H is obtained from G by deleting some edges and vertices.

Proposition 23. *If* G' *is a connected subgraph of* G *then* $\pi(G) \geq \pi(G')$.

Proof. For a vertex i in G' let p'_i and p_i denote its price in G' and G respectively; we first relate p'_i to a certain specialization of p_i.

Let E, E' be the edge sets of G, G' respectively, and let E_0 (resp. E_1) denote the edges in $E \setminus E'$ whose source vertex is inside (resp. outside) G'. Let $\bar{p}_i$ be the specialization of p_i obtained by setting the edge weights in E_0 and E_1 to 0 and 1 respectively. Then we claim that

$$p'_i = |F|\, \bar{p}_i, \tag{9}$$

where F is the set of directed forests ϕ in G such that

1. the root vertices of ϕ are contained in G',
2. the non-root vertices of ϕ consist of *all* G-vertices not in G'.

Indeed, consider the expression of p_i as a sum of i-trees in G. The specialization $\bar{p}_i$ assigns zero weight to all trees with an edge from E_0. The remaining i-trees in G are precisely of the from $\tau \cup \phi$ where τ is an i-tree in G' and $\phi \in F$, and these get assigned weight $wt(\tau)$. Formula (9) is an immediate consequence.

Now if i, j are vertices in G', then formula (9) gives

$$\frac{p'_i}{p'_j} = \frac{\bar{p}_i}{\bar{p}_j}$$

Thus the ij price ratio in G' is obtained by a *specialization* of the ratio in G. Consequently the former cannot involve *more* variables. Taking the maximum over all i, j we get $\pi(G) \geq \pi(G')$ as desired. □

5.3. Collapsible edges

We write $out(k)$ for the number of outgoing edges at the vertex k. In a connected graph we have $out(k) \geq 1$ for all vertices, and we will say k is *ordinary* if $out(k) = 1$ and *special* if $out(k) > 1$. Among special vertices, we will say that k is *binary* if $out(k) = 2$ and *tertiary* if $out(k) = 3$.

Definition 24. We say that an edge ij of a graph G is *collapsible* if

1. i is an ordinary vertex,
2. ji is not an edge of G,
3. there is no vertex k such that ki and kj are both edges of G.

Definition 25. If G has no collapsible edges we will say G is *rigid*.

If G is a connected graph with a collapsible edge ij, we define the *ij-collapse* of G to be the graph G' obtained by deleting the vertex i and the edge ij, and replacing any edges of the form li with edges lj. The assumptions on ij imply that the procedure does not introduce any loops or double edges, hence G' is also simple (and connected). Moreover each vertex $k \neq i$ has the same outdegree in G' as in G.

Lemma 26. *If G' is the ij-collapse of G as above, then $\pi(G) \geq \pi(G')$.*

Proof. Let k be any vertex of G' then k is also a vertex of G. Since i is ordinary every k-tree in G must contain the edge ij; collapsing this edge gives a k-tree in G' and moreover every k-tree in G' arises uniquely in this manner. Thus we have a factorization

$$p_k(G) = a_{ij} p_k(G').$$

Thus for any two vertices k, l of G' we get $p_k(G)/p_l(G) = p_k(G')/p_l(G')$ and the result follows. □

We will say that H is a *minor* of G if it is obtained from G by a *sequence* of steps of the following kind: a) passing to a connected subgraph, b) collapsing some collapsible edges. By Proposition 23 and Lemma 26 we get

Corollary 27. *If H is a minor of G then $\pi(H) \leq \pi(G)$.*

5.4. Augmentation

Throughout this section G denotes a connected graph.

Notation 28. *We write $H \trianglelefteq G$ if H is a* connected *subgraph of G, and write $H \triangleleft G$ to mean $H \trianglelefteq G$ and $H \neq G$.*

We say that $H \triangleleft G$ can be *augmented* if there is a path P in G whose endpoints are in H, but which is otherwise completely disjoint from H. We refer to P as an augmenting path of H, and to $K = H \cup P$ as an augmented graph of H; note that K is also connected, *i.e.* $K \trianglelefteq G$. It turns out that augmentation is always possible.

Lemma 29. *If $H \triangleleft G$ then H can be augmented.*

Proof. If G and H have the same vertex set then any edge in $G \setminus H$ comprises an augmenting path. Otherwise consider triples (k, P_1, P_2) where k is a vertex not in H, P_1 is a path from some vertex in H to k, and P_2 is a path from k to some vertex in H. Among all such triples choose one with $e(P_1) + e(P_2)$ as small as possible. Then P_1 and P_2 cannot share any *intermediate* vertices with H or with each other, else we could construct a smaller triple. It follows that $P = P_1 \cup P_2$ is an augmenting path. □

We are particularly interested in augmenting paths for H that consist of one or two edges; we refer to these as *short augmentations* of H.

Corollary 30. *If $H \triangleleft G$ then G has a minor that is a short augmentation of H.*

Proof. Let $K = H \cup P$ be an augmentation of H. If P has more than two edges, then we may collapse the first edge of P in K. The resulting graph is a minor of G, which is again an augmentation of H. The result follows by iteration. □

Lemma 31. *If $K = H \cup P$ with $P = \{jk, kl\}$, then for any vertex i of H we have $\pi_{ik}(K) = \pi_{ij}(H) + 2$.*

Proof. The edges (j,k) and (k,l) are the unique incoming and outgoing edges at k. It follows that every i-tree in K is obtained by adding the edge kl to an i-tree in H, and every k-tree in K is obtained by adding the edge jk to a j-tree in H. Thus if a_{jk} and a_{kl} are the respective weights of the two edges in the path P then we have

$$p_i(K) = a_{kl} p_i(H), \; p_k(K) = a_{jk} p_j(H) \Longrightarrow \frac{p_i(K)}{p_k(K)} = \frac{a_{kl}}{a_{jk}} \frac{p_i(H)}{p_j(H)}$$

Thus the price ratio in question depends on two additional variables, and the result follows. □

Corollary 32. *If G contains the chorded triangle T_0 as a proper subgraph then $\pi(G) \geq 5$.*

Proof. By Corollary 30, G has a minor $K = T_0 \cup P$, which is a short augmentation of T_0, and it is enough to show that $\pi(K) \geq 5$. If P consists of two edges $\{jk, kl\}$ then by Remark 19 we can choose i such that $\pi_{ij}(T_0) = 3$; now by Lemma 31, we have $c_{ik}(K) = 5$ and hence $\pi(K) \geq 5$. If P consists of a single edge then K is necessarily as below, and once again $\pi(K) \geq 5$.

2		
$\uparrow\downarrow$	$\searrow$	
1	$\leftrightarrows$	3

p_1/p_3
$\dfrac{b_{31}(b_{21}+b_{23})}{b_{23}b_{12}+b_{23}b_{13}+b_{21}b_{13}}$

□

5.5. The circuit rank

As usual G denotes a simple connected graph, and we will write $e(G)$ and $v(G)$ for the numbers of edges and vertices of G.

Definition 33. The circuit rank of G is defined to be

$$c(G) = e(G) - v(G) + 1$$

The circuit rank is also known as the *cyclomatic number*, and it counts the number of independent cycles in G, see *e.g.* (Berge, 2001).

Example 34. *If G is a k-rose then $c(G) = k$, and if G is a chorded cycle then $c(G) = 2$.*

We now prove a crucial property of $c(G)$.

Proposition 35. *If $H \lhd G$ then there is some $K \unlhd G$ such that $H \lhd K$ and $c(K) = c(H) + 1$.*

Proof. Let $K = H \cup P$ be an augmentation of H. If P consists of m edges, then K has $e(H) + m$ edges and $v(H) + m - 1$ vertices; hence $c(K) = c(H) + 1$. □

Corollary 36. *Let G be a connected graph.*

1. *If $H \lhd G$ then $c(H) < c(G)$.*
2. *$c(G) = 0$ iff G is a single vertex.*
3. *$c(G) = 1$ iff G is a cycle.*
4. *$c(G) = 2$ iff G is a chorded cycle or a 2-rose.*

Proof. The first part follows from Proposition 35, the other parts are completely straightforward. □

Lemma 37. *If G is not a rose and $c(G) > 3$, then there is some $K \lhd G$ such that K is not a rose and $c(K) = 3$.*

Proof. Let R be a k-rose in G with $c(R) = k$ as large as possible, then $R \lhd G$ by assumption. If $c(R) \leq 2$ then any $K \lhd G$ with $c(K) = 3$ is not a rose. Thus we may assume that $c(R) > 2$, and in particular R has a unique special vertex i and at least three loops. Since $R \neq G$, R can be augmented, and $S = R \cup P$ is an augmentation, then P cannot both begin and end at i, else $R \cup P$ would be a rose, contradicting the maximality of R. Since there are at most two endpoints of P, we can choose two *distinct* loops L_1 and L_2 of R, such that $L_1 \cup L_2$ contains these endpoints of P. Then $K = L_1 \cup L_2 \cup P$ is the desired graph. □

5.6. Covered vertices

Definition 38. Let i be an ordinary vertex of G with outgoing edge ij. We say that a vertex k *covers* i, if one of the following holds:

1. the edges ki and kj belong to G,
2. $j = k$ and the edge ki belongs to G.

If there is no such k then we say that i is an *uncovered* vertex.

We emphasize that the terminology covered/uncovered is only applicable to ordinary vertices in a graph G. The main point of this definition is the following simple observation.

Remark 39. An ordinary vertex is uncovered iff its outgoing edge is collapsible.

Lemma 40. *Suppose G is a connected graph.*

1. *If $v(G) \geq 3$ then an ordinary vertex cannot cover another vertex.*
2. *If $v(G) \geq 4$ then a binary vertex can cover at most one vertex.*
3. *A tertiary vertex can cover at most three vertices.*
4. *If G is a rigid graph with $c(G) = 3$, then $v(G) \leq 4$.*

Proof. If k is an ordinary vertex covering i then G must contain the edges ki and ik. Thus i and k do not have any other outgoing edges, and if G has a third vertex j then there is no path from k or i to j, which contradicts the connectedness of G, thereby proving the first statement.

If k is a binary vertex covering the ordinary vertices i and j then G must contain the edges ki, kj, ij, ji. The vertices i, j, k cannot have any other outgoing edges, so a fourth vertex would contradict the connectedness of G as before. This proves the second statement.

If a vertex k covers i then there must be an edge from k to i. Thus if $out(k) = 3$ then k can cover at most three vertices.

If $c(G) = 3$ then G has either 2 binary vertices or 1 tertiary vertex, with the remaining vertices being ordinary. If $v(G) > 4$ then by previous two paragraphs G would have an uncovered vertex, which is a contradiction. □

5.7. Proof of Theorem 21

Proposition 41. *If $c(G) \geq 3$ and G is not a rose, then $\pi(G) \geq 5$.*

Proof. By Proposition 23 and Lemma 37 we may assume that $c(G) = 3$. By Lemma 26, we may further assume that G is rigid, and thus by Lemma 40 that $v(G) \leq 4$. We now divide the argument into three cases.

First suppose that G contains a 3-cycle C. We claim that at least one of the edges of C must be a bidirected edge in G, so that G properly contains a chorded triangle T_0, whence $\pi(G) \geq 5$ by Corollary 32. Indeed if G has no other vertices outside C, then G must have 5 edges and 3 vertices and the claim is obvious. Thus we may suppose that there is an outside vertex l. We further claim that C contains two vertices i, j such that i covers j. Granted this, it is immediate that G contains either the bidirected edge ij and ji, or the bidirected edge jk and kj where k is the third vertex of C. To prove the "further" claim we note that the special vertices of G consist of either a) one tertiary vertex, or b) two binary vertices. In case a) the connectedness of G implies that the tertiary vertex must be in C, and hence it must cover both the ordinary vertices in C. In case b) either C contains both binary vertices, one of which must cover the unique ordinary vertex of C; or C contains one binary vertex, which must cover one of the two ordinary vertices of C.

Next suppose that G does not contain a 3-cycle, but does contain a 4-cycle labeled 1234, say. Now G has two additional edges, which cannot be the diagonals $13, 31, 24, 42$, since otherwise G would have a 3-cycle; therefore G must have two bidirected edges. The bidirected edges cannot be adjacent else G would have a collapsible vertex, therefore G must be the first graph below, which has $\pi(G) \geq 5$.

2	$\longrightarrow$	3
$\uparrow\downarrow$		$\uparrow\downarrow$
1	$\longleftarrow$	4

$$p_1/p_3 = \frac{b_{21}b_{34}b_{41}}{b_{23}b_{12}(b_{41}+b_{43})}$$

2	$\rightleftarrows$	3
$\uparrow\downarrow$		$\uparrow\downarrow$
1		4

$$p_1/p_4 = \frac{b_{21}b_{32}b_{43}}{b_{34}b_{23}b_{12}}$$

Finally suppose G has no 3-cycles or 4-cycles. Then every edge must be a bidirected edge, and G must be a tree with all bidirected edges. Since G is not a star, this only leaves the second graph above, which has $\pi(G) \geq 6$. □

We can now finish the proof of Theorem 21.

Proof of Theorem 21. If $c(G) \leq 2$ then, by Corollary 36, G is a single vertex, a cycle, chorded cycle or a 2-rose. If $c(G) \geq 3$ then the result follows by Proposition 41. □

6. Proof of Theorem 4

In this section, after a couple of preliminary results, we apply Theorem 21 to prove Theorem 4.

Lemma 42. *If G is a chorded cycle on 4 or more vertices, then $\tau(G) \geq 3$.*

Proof. We can express G as a union of two paths P, Q from 1 to 2, say and a third path R from 2 to 1. At least one of the first two paths, say P must have an intermediate vertex, say 3. Since $m \geq 4$ there is an additional intermediate vertex 4 on one of the paths.

If $m = 4$ then we get three possible graphs depending on the location of the vertex 4:

$$\boxed{\begin{array}{ccc} 3 & \rightarrow & 4 \\ \uparrow & & \downarrow \\ 1 & \leftrightarrows & 2 \end{array}} \quad \boxed{\begin{array}{ccc} 3 & \rightarrow & 2 \\ \uparrow & \nearrow & \downarrow \\ 1 & \leftarrow & 4 \end{array}} \quad \boxed{\begin{array}{ccc} 3 & \rightarrow & 2 \\ \uparrow & \swarrow & \uparrow \\ 1 & \rightarrow & 4 \end{array}}$$

For these graphs we have $\tau_{24} = 3$, $\tau_{42} = 3$ and $\tau_{34} = 3$, respectively. Thus $\tau(G) \geq 3$ in all three cases.

If $m > 4$ then G can be realized as one of these graphs, albeit with additional intermediate vertices on one or more of the paths P, Q, R. These additional vertices are ordinary uncovered vertices, with collapsible outgoing edges. Collapsing one of these edges does not increase time complexity, and produces a smaller chorded cycle G'. Arguing by induction on m we conclude $\tau(G) \geq \tau(G') \geq 3$. □

Lemma 43. *If G is the complete graph, then $\pi_{ij}(G) = m(m-1)$ for all $i \neq j$.*

Proof. Fix a pair of vertices $i \neq j$ in G. Then we claim that the price ratio $p_{ij}(G)$ depends on each of the $m(m-1)$ edge weights b_{kl}. Indeed if H is any “spanning” connected subgraph of G then $p_{ij}(H)$ is obtained from $p_{ij}(G)$ by specializing to 0 the weights of all edges outside H. Therefore it suffices to find a connected subgraph H such that $p_{ij}(G)$ depends on b_{kl}.

We consider two cases. If $\{i, j\} = \{k, l\}$ then exchanging i, j if necessary we may assume $i = k$, $j = l$. Let H be an m-cycle two of whose edges are ij and hi (say); then $p_i/p_j = b_{hi}/b_{ij}$ depends on $b_{kl} = b_{ij}$.

If $\{i, j\} \neq \{k, l\}$ then let H be an 2-rose with loops C_1 and C_2 such that

1. k is the special vertex, and kl is an edge in C_1
2. i belongs to C_1 and j belongs to C_2

Then p_i and p_j are each given by unique directed trees T_i and T_j. Moreover T_i involves kl while T_j does not. Hence $p_{ij}(H)$ depends on b_{kl}. □

Proof of Theorem 4. (**Completion**) Let $\mathfrak{S}$ denote the set consisting of the three special mechanisms: star, cycle and complete. We need to show that $\mathfrak{M}_{\preceq} = \mathfrak{S}$, where $\mathfrak{M}_{\preceq}$ denotes the set of $\preceq$-minimal elements of $\mathfrak{M} = \mathfrak{M}(\mathfrak{m})$.

Let us say that G is a minimal graph if M_G is a minimal mechanism of $\mathfrak{M}$. Now the star mechanism has complexity $(\tau, \pi) = (2, 4)$. Therefore if G is any minimal graph then either $\tau(G) = 1$ or $\pi(G) \leq 4$. For $\tau(G) = 1$ we get the complete graph, which has complexity $(\tau, \pi) = (1, m(m-1))$ by Lemma 43. The graphs with $\pi(G) \leq 4$ are characterized by Theorem 21, and we have three possibilities for G:

1. *Chorded cycle.* In this case we have $(\tau, \pi) = (3^+, 4)$ by Lemma 42, and so G is not minimal.
2. *Cycle.* In this case we have $(\tau, \pi) = (m-1, 2)$ by Lemma 17.
3. *k-rose, $k \geq 2$.* If each petal of G has exactly 2 edge then G is the star mechanism. Otherwise after collapsing edges, we obtain the following minor with $\tau_{12} = 3$

$$\boxed{\begin{array}{cccccc} 1 & & & & & \\ \downarrow & \nwarrow & & & & \\ \cdot & \rightarrow & \cdot & \leftrightarrows & 2 \end{array}}$$

Thus G has complexity $(\tau, \pi) = (3^+, 4)$ and so is not minimal.

Thus the three graphs in the statement of Theorem 4 are the only *possible* minimal graphs, and have the indicated complexities. Since they are incomparable with each other, each is minimal. Thus we conclude $\mathfrak{M}_{\preceq} = \mathfrak{S}$ as desired. □

Remark 44. For $m = 3$, Lemma 42 does not hold and we have an additional strongly minimal mechanism with $(\tau, \pi) = (2, 4)$, namely the chorded triangle

$$\boxed{\begin{array}{ccc} \cdot & & \\ \downarrow & \nwarrow & \\ \cdot & \leftrightarrows & \cdot \end{array}}$$

7. Proof of Theorem 15

Note that a mechanism is determined uniquely by its *net trade* function $\nu(a, b) := r(a, b) - \bar{a}$ which, although initially defined for $a \leq b$, admits a natural extension as follows.

Proposition 45. *The function* ν *admits a unique extension to* $S \times S_+$ *satisfying*

$$\nu(\lambda a + \lambda' a', b) = \lambda \nu(a, b) + \lambda' \nu(a', b), \quad \nu(a, \lambda b) = \nu(a, b) \text{ for } \lambda, \lambda' > 0$$

Proof. Since $\nu(a, b) := r(a, b) - \overline{a}$, it suffices to show

$$r(\lambda a + \lambda' a', b) = \lambda r(a, b) + \lambda' r(a', b), \quad r(a, \lambda b) = r(a, b) \text{ for } \lambda, \lambda' > 0 \tag{10}$$

But this is just Lemma 1 of Dubey and Sahi (2003), whose proof we now reproduce for the sake of completeness.

First observe that, by the conservation of commodities, $r(a, b) \leq \overline{b}$ for all $a \leq b$; moreover if a and a' in S are such that $a + a' \leq b$, then *Aggregation* implies the functional (Cauchy) equation $r(a + a', b) = r(a, b) + r(a', b)$.

From Corollary 2 in Aczel and Dhombres (1989) we conclude that, for all non-negative λ and λ' such that $\lambda a + \lambda' a' \leq b$, the first equality of (10) holds.

Next let $a \leq b$ and choose $\lambda \geq 1$. Then the argument just given shows that $r(\lambda a, \lambda b) = \lambda r(a, \lambda b)$. On the other hand, *Invariance* implies that the left side equals $\lambda r(a, b)$. Comparing these expressions we obtain the second inequality of (10).

Thus even for a *not* less than b, we may define $r(a, b)$ via (10) by choosing λ sufficiently large. This extends r to all of $S \times S_+$. □

In view of the above result, we drop the restriction $a \leq b$ when considering $\nu(a, b)$.

The net trade vector can have negative and positive components, and hence belongs to $\mathbb{R}^m$. The next definition pertains to such vectors in $\mathbb{R}^m$.

Definition 46. By an i-vector, we mean a vector whose ith component is positive and all other components are zero. By an $\bar{\imath} j$-vector we mean a vector that has a negative i-component, a positive j-component and zeros in all other components.

Proposition 47. *For* $b \in S_+$ *and any* $i \neq j$ *there is* $a \in S$ *such that* $\nu(a, b)$ *is an* $\bar{\imath} j$*-vector.*

Proof. Since the graph G underlying the mechanism is connected, there is a directed path from i to j. Denote the nodes on the path by $i = 1, \ldots, t = j$. Let w^1 be an i-vector which can be offered on edge 12 to get a return $w^2 \neq 0$ consisting only of commodity 2 (here $w^2 \neq 0$ by *Non-dissipation*); then w^2 can be offered on edge 23 to get $w^3 \neq 0$ consisting only of commodity 3, and so on. This yields a sequence $w^1, \ldots, w^t$ such that

$$w^i + \nu\left(w^i, b\right) = w^{i+1} \text{ for } i = 1, \ldots, t-1$$

If $w = \sum w^i$ then by Proposition 45 we have

$$\nu(w, b) = \sum \nu\left(w^i, b\right) = w^t - w^1$$

which is an $\bar{\imath} j$-vector. □

It will be convenient to write an $\bar{\imath} j$-vector in the form $(-x, y)$ after suppressing the other components. In the context of the above proposition if $\nu(a, b) = (-x, y)$ then by linearity $\nu(a/x, b) = (-1, y/x)$, and we will say that the offer a (or a/x) achieves an *ij-exchange ratio* of y/x at b.

Proposition 47 shows that there exists at least one offer a to achieve an $\bar{\imath} j$-vector in trade, at any given b. But a is by no means unique. There may be many paths from i to j, along which i can be exchanged exclusively for j; and, also, there may be more complicated trading strategies, that use edges no longer confined to any single path, to accomplish such an exchange. These could give rise to offers different from a and yield (for the fixed aggregate b) other $\bar{\imath} j$-vectors in trade. But, as the following lemma shows, the *same* exchange ratio obtains under all circumstances.

Lemma 48. *If* a', a'' *achieve* ij*-exchange ratios* α', α'' *at* b*, then* $\alpha' = \alpha''$.

Proof. By Proposition 47 there exists an a such that $\nu(a, b)$ is a $\bar{\jmath} i$-vector; if α is the corresponding exchange ratio then by rescaling a, a', a'' we may assume that

$$\nu(a, b) = (1, -\alpha), \nu\left(a', b\right) = \left(-1, \alpha'\right), \nu\left(a'', b\right) = \left(-1, \alpha''\right).$$

By Proposition 45 we get

$$\nu\left(a + a', b\right) = \left(0, \alpha' - \alpha\right)$$

Now by *Non-dissipation* we get $\alpha \leq \alpha'$, and exchanging the roles of i and j we conclude that $\alpha' \leq \alpha$ and hence[31] that $\alpha = \alpha'$. Arguing similarly we get $\alpha = \alpha''$ and hence that $\alpha' = \alpha''$ □

[31] Equivalently: *no-arbitrage* of subsection 3.1 directly implies that $\alpha = \alpha'$.

Lemma 49. *Denote the net trade function of M by ν. Then there is a unique map* $p\colon \mathbb{R}^K_{++} \to \mathbb{R}^m_{++}/\sim$ *satisfying* $p(b)\cdot \nu(a,b) = 0$.

Proof. Fix $b \in S_+$ and consider the vector

$$p = (1, p_2, \dots, p_m)$$

where p_j^{-1} is the $1j$-exchange ratio at b, as in Lemma 48. We will show that p satisfies the budget balance condition, *i.e.* that

$$p \cdot \nu(a,b) = 0 \text{ for all } a. \tag{11}$$

We argue by induction on the number $d(a,b)$ of non-zero components of $\nu(a,b)$ in positions $2, \dots, m$. If $d(a,b) = 0$ then $\nu(a,b) = 0$ by *Non-dissipation* (enhanced to *no-arbitrage*, see Subsection 3.1) and (11) is obvious. If $d(a,b) = 1$ then $\nu(a,b)$ is either an $\bar{1}j$-vector or a $\bar{j}1$ vector, which by the definition of p_j and Lemma 48 is necessarily of the form

$$\left(-x, xp_j^{-1}\right) \text{ or } \left(x, -xp_j^{-1}\right);$$

for such vectors (11) is immediate. Now suppose $d(a,b) = d > 1$ and fix j such that $\nu_j(a,b) \neq 0$. Then we can choose a' such that $\nu\left(a', b\right)$ is a $\bar{1}j$ or a $\bar{j}1$- vector such that $\nu_j(a,b) = -\nu_j\left(a',b\right)$. It follows that $d\left(a + a', b\right) < d$ and by linearity we get

$$p \cdot \nu(a,b) = p \cdot \nu\left(a + a', b\right) - p \cdot \nu\left(a', b\right).$$

By the inductive hypothesis the right side is zero, hence so is the left side.

Finally the uniqueness of the price function is obvious, because the return function of the mechanism dictates how many units of j may be obtained for one unit of i, yielding just one possible candidate for the exchange rate for every pair ij. □

We can now prove Theorem 15

Proof of Theorem 15. (**Completion**) To prove that $M = M_G$ it is enough to show that p and r satisfy (1) and (3).

Let us write, as before,

$$b = \sum a^\alpha,\ p = p(b) \text{ and } \nu(a,b) = r(a,b) - \bar{a}.$$

Consider replacing trader α by m traders $\alpha_1, \dots, \alpha_m$, where trader α_j makes only the offers $\left\{a^\alpha_{ij} : 1 \le i \le m\right\}$ in a^α that entitle α to the return of commodity j. By *Aggregation* this will have no effect on traders other than α; and hence α_j will get precisely the return $r_j(a^\alpha, b)$. By Lemma 49, applied to each such trader α_j, we have

$$p_j r_j(a^\alpha, b) = \sum_i p_i a^\alpha_{ij} \tag{12}$$

which is just (3).

Now (1) follows by summing (12) over all α. □

8. A continuum of traders

Our analysis easily extends to the case where the set of individuals T is the unit interval $[0,1]$, endowed with a nonatomic population measure.[32] Let $\mathcal{S}$ denote the collection of all integrable functions $\boldsymbol{a} : T \mapsto S$ such that $\int_T \boldsymbol{a} \in S_+$. (An element of $\mathcal{S}$ represents a choice of offers by the traders in T which are positive on aggregate.) In the same vein, let $\mathcal{R}$ denote the collection of all integrable functions from T to C, whose elements $\mathbf{r} : T \mapsto C$ represent returns to T. An *exchange mechanism M*, on a given set of m commodities, is a map from $\mathcal{S}$ to $\mathcal{R}$ such that, if M maps $\boldsymbol{a}$ to $\mathbf{r}$ then we have (reflecting conservation of commodities):

$$\int_T \boldsymbol{a} = \int_T \mathbf{r}$$

We wrap the *Aggregation* and *Anonymity* conditions into one, and directly postulate that the return to any individual depends only on his own offer and the integral of everyone's offers, and that this return function is the same for everyone. Thus we have a function r from $S \times S_+$ to C such that $\mathbf{r}(t) = r(a,b)$, where $a = \mathbf{a}(t)$ and $b = \int_T \boldsymbol{a}$. The following lemma is essentially from Dubey et al. (1980).

[32] Denote the measure μ. And since μ is to be held fixed throughout, we may suppress it, abbreviating $\int_T \mathbf{f}(t)d\mu(t)$ by $\int_T \mathbf{f}$ for any measurable function $\mathbf{f}$ on $[0,1]$.

Proposition 50. *$r(a,b)$ is linear in a (for fixed b) and $r(a,\lambda b) = r(a,b)$ for any a,b and positive scalar λ.*

Proof. We will first show that if $a, c \in S$ and $0 < \lambda < 1$, then

$$r(\lambda a + (1-\lambda)c, b) = \lambda r(a,b) + (1-\lambda) r(c,b)$$

There clearly exists an integrable map **d** from $T = [0,1]$ to space of offers S such that (i) positive mass of traders choose a in **d**; (ii) positive mass of traders choose c in **d**; and (iii) the integral of **d** on T is b. So $\int_T r(\mathbf{d}^\alpha, b) d\mu(\alpha) = \int_T r(\mathbf{d}, b) = \overline{b}$ since commodities are conserved. Shift $\varepsilon\lambda$ mass from a to $\lambda a + (1-\lambda)c$ and $(1-\lambda)\varepsilon$ mass from c to $\lambda a + (1-\lambda)c$, letting the rest be according to **d**. This yields a new function (from T to S) which we call **e**. Clearly the integral of **e** on T is also b. Therefore, once again by conservation of commodities, we must have $\int_T r(\mathbf{e}, b) = \overline{b}$, hence $\int_T r(\mathbf{d}, b) = \int_T r(\mathbf{e}, b)$. But this can only be true if the displayed equality holds, proving that (every coordinate of) r is affine in a for fixed b.

Now $r(0,b) \geq 0$ by assumption. Suppose $r(0,b) \gneqq 0$. Partition T into two non-null sets T_1 and T_2. Consider the case where all the individuals in T_1 offer 0, and all in T_2 offer $b/\mu(T_2)$. Then, since everone in T_1 gets the return $r(0,b) \gneqq 0$, by conservation of commodities everyone in T_2 gets $\overline{b} - \mu(T_1)\; r(0,b) \lneqq b/\mu(T_2)$, contradicting *non-dissipation.* So $r(0,b) = 0$, showing r is linear.

Finally $\lambda r(a,b) = r(\lambda a, \lambda b) = \lambda r(a, \lambda b)$, where the first equality comes from *Invariance* and the second from linearity. □

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