On the Optimal Design of Lottery Contests*

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July 28, 2018

Abstract

This paper develops a novel technique that allows us to obtain optimal multiplicative biases for asymmetric Tullock contests—i.e., the weights placed on contestants' effort entries in the contest success function. Generally, Tullock contests with asymmetric valuations have no closed-form solutions when the impact functions are nonlinear. This prevents us from obtaining the optimum by the usual implicit programming approach, which requires an explicit solution to the equilibrium effort profile. We provide an alternative that allows us to circumvent the difficulty without solving for the equilibrium explicitly. Our approach is not limited to total effort maximization, and applies to contest design problems with noncanonical objective functions. Using this approach, we further establish that linear impact functions with zero headstarts are optimal under a broad class of contest objectives when the contest designer is able to choose any form of regular concave impact functions. In another application, we reexamine the classical issue of comparing all-pay auctions and lottery contests under alternative design objectives.

Keywords: Contest Design; Optimal Biases; Tullock Contest; All-Pay Auction

JEL Classification Codes: C72, D72.

*We are grateful to Jörg Franke, Kai Konrad, Wolfgang Leininger, and Junjie Zhou for helpful comments and suggestions. We thank Shanglyu Deng for excellent research assistance. Wu thanks the School of Economics, Peking University for research support. Any remaining errors are our own.

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1 Introduction

Contest-like competitive activities are ubiquitous in economic and social landscapes: Contenders sink irreversible effort or costly bids to vie for limited prizes, while their competitive outlays are nonrefundable regardless of the outcome. Such competitions can be exemplified by a plethora of observations, ranging from electoral competitions, lobbying, R&D races, and college admissions to sporting events. An enormous amount of research effort has been devoted to exploring economic agents' strategic interactions in contests and how these interactions depend on the institutional rules that govern the competition. This sparks extensive interest in contest design that varies the structural elements of a contest to achieve the designer's stated goal.

Our study focuses on optimal contest design when contestants differ in their strength—i.e., prize valuations—or bidding efficiency. This heterogeneity affords the designer the flexibility to administer biased treatments that are tailored to contestants' individual characteristics: She can strategically favor or handicap contestants to alter the competitive balance of the contest, in order to favor her own interests. The contest literature has conventionally espoused the merits of a more level playing field for fueling competition. Consider, for instance, the preferential government policies that favor small and medium-sized enterprises (SMEs) in public procurement to support local entrepreneurship in various countries (Che and Gale, 2003; Epstein et al., 2011). Prestigious colleges often allocate bonus points to minority applicants when practicing affirmative action in admissions (Fu, 2006; Franke, 2012). In competitions for a vacant position, existing workers are often *ex ante* preferred to external candidates to incentivize productive efforts (Chan, 1996). Alternatively, the International Table Tennis Federation (ITTF) substantially enlarged the ball's size to limit Chinese players' dominance (Wang, 2010).

An evolving literature has been developed to explore the optimal biased contest rules that maximally exploit contestants' heterogeneity. This paper develops a tractable and lucid approach to this classical optimization problem in a substantially generalized setting. Its novelty, compared with previous literature can be seen in two main respects.

First, it renders a closed-form solution to the optimum in a broader setting and allows for an enriched design space. The application of existing techniques is limited by parametric restrictions, while our approach largely relaxes these constraints.

Second, it allows for optimization regarding a wider array of objectives. The literature has typically assumed the maximization of total effort, while alternative objectives often cause technical difficulties when existing techniques are adopted. Our approach provides an avenue to bypass the complications.

We now provide a brief account of our setting and the logic underlying our approach. Specifically, we adopt the frequently used framework of lottery contests to model the stochastic mechanism that converts contestants' effort entries into their winning odds. Imagine an asymmetric n-player winner-take-all contest in which the n players differ in both their valuations of the prize and their contest technologies. For a given set of effort entries $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, a contestant wins the prize with a probability

$$p_i(\boldsymbol{x}) = \frac{f_i(x_i)}{\sum_{j=1}^{n} f_j(x_j)},$$

where $f_i(\cdot)$ is conventionally called the impact function of contestant $i \in \{1, \ldots, n\}$ mapping one's effort outlays into his effective entry. This form of contest success function (CSF) has been axiomatized by Skaperdas (1996) and Clark and Riis (1998).[1]

The literature often assumes that the impact function $f_i(\cdot)$ takes the form $f_i(x_i) = \alpha_i \cdot x_i^{r_i}$, in which case the lottery CSF boils down to the popularly adopted Tullock contest. The set of weights $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n)$ measures the fairness of the competitive environment, which arguably depends on the prevailing institutional rules of the contest. The weight is thus often viewed as a design variable: A contest designer chooses the set of weights $\boldsymbol{\alpha}$ before the contest begins, and contestants subsequently compete for the prize in response to the prevailing rules.

The optimization problem entails a mathematical programming with equilibrium constraints (MPEC). The typical approach requires one to solve for contestants' equilibrium bidding strategies under every possible weight $\boldsymbol{\alpha}$, then plug the solution to equilibrium efforts—as a function of $\boldsymbol{\alpha}$—into the objective function. The objective function can thus be rewritten as a function of $\boldsymbol{\alpha}$, which, in turn, allows us to optimize by searching for the maximizing $\boldsymbol{\alpha}$. The search for the optimal weights $\boldsymbol{\alpha}$ has been a notoriously difficult problem in the literature. Although the optimum has been fully characterized in two-player contests (see, for example, Wang, 2010; Epstein et al., 2011), the solution has long been missing when more than two contestants are involved. First, as pointed out by Franke et al. (2013), a complex nonsmooth optimization problem occurs because a contestant can choose to stay inactive, and the set of active contestants depends on the weights $\boldsymbol{\alpha}$, which complicates the search for a global optimum. Second, the solution to the equilibrium effort strategies is, in general, unavailable.

Franke et al. (2013) make a pioneering contribution by providing a solution to the optimal biases for an n-player Tullock contest. However, the technique that solves for the equilibrium in the n-player asymmetric contest game requires linear impact functions, i.e., $r_i = 1$ for all $i \in \{1, \ldots, n\}$. Further, Franke et al. focus on the maximization of total effort; however, the technique may not apply when other objective functions are considered, even if r_i is set to one. The pursuit of alternative objectives is not uncommon in practice. Consider, for

[1] Jia (2008) and Fu and Lu (2012) uncover its statistical foundation.

instance, that a college presumably cares about only the academic quality of its admitted student body (see Fu, 2006). In a crowdsourcing competition for a technical solution, the buyer would only value the quality of the winning entry. More suspense regarding the outcome of a sporting event makes promoting it easier (see Chan et al., 2008). Alternatively, in public procurement, a government could care about both domestic suppliers' efforts as a buyer and their welfare as a social planner (see Epstein et al., 2011).

Our analysis takes up the challenge to generalize the optimization problem in two dimensions: (i) a broader range of contest settings and (ii) a larger set of design objectives. The weights placed on effort entries determine the competitive balance of the contest, which ultimately determines the distribution of winning probabilities among contestants. We identify a correspondence among contestants' equilibrium winning probabilities, their equilibrium efforts, and the weights assigned to their effort entries. This allows us to adopt an indirect approach to the optimization problem. Specifically, we do not optimize over the choice of the weights $\boldsymbol{\alpha}$. Instead, we borrow a perspective from the mechanism design literature and allow the designer to assign winning probabilities directly to contestants. That is, the winning probability distribution is treated as a design variable; we first search for the most desirable winning probability distribution that maximizes the given objectives, then recover the the optimal weights through the aforementioned correspondence.

This simple transformation allows us to bypass the technical difficulty of solving explicitly for equilibrium effort strategies. Without an explicit solution to the equilibrium, this approach precisely predicts the equilibrium outcome—i.e., the distribution of contestants' winning probabilities—in the optimum. It applies broadly to a wide spectrum of contest design, including contest design problems with various noncanonical objective functions, and simplifies enormously the search for the optimum.

In addition, based on this approach, we further generalize the contest design problem beyond the restriction of optimizing over the choice of multiplicative bias. Instead, the designer is allowed to choose an arbitrary form for concave impact functions under minimum regularity restrictions. It is shown that linear impact functions with zero headstarts—i.e., $f(x_i) = \alpha_i \cdot x_i$— are optimal for a broad class of contest objectives. It is thus without loss of generality to search for the set of optimal weights $\boldsymbol{\alpha}$. Examples are provided to demonstrate the flexibility and efficiency of our approach to the search for optimal contest rules in a broad context.

As mentioned above, our paper is closely related to the seminal study of Franke et al. (2013) that takes the first step in providing a complete solution to the optimization of n-player asymmetric lottery contests. Our paper allows for an enriched design space that includes nonlinear impact functions and optimization toward alternative objectives.[2] The

[2]In an earlier version of Franke et al. (2018), they show that linear impact functions outperform any

approach can be applied in a wide array of contexts. For example, we are able to explore further the classical issue of the comparison between lottery contests and all-pay auctions, which has been studied extensively in the literature. Assuming fair contests without biases, Fang (2002) finds that an n-player Tullock contest can outperform an all-pay auction when contestants are heterogeneous. Applying the approach developed by Franke et al. (2013), Franke et al. (2014) demonstrate that when the contest designer is able to place multiplicative biases on contestants' effort entries, the optimal all-pay auction always outperforms. Franke et al. (2018) further allow for additive headstarts in the designer's toolkit and develop a ranking in total revenue between all-pay auctions and lottery contests when different design instruments are used, which reaffirms the dominance of all-pay auctions. All of these studies, however, focus on comparison in terms of the total effort. Epstein et al. (2011), in contrast, assume that the designer cares about both effort supply and contestants' welfare, but the comparison is conducted in a two-player setting. Enabled by our approach, the designer in our setting is given substantially more freedom in manipulating the impact functions; a ranking between all-pay and lottery contests is concluded in n-player settings for the expected winner's effort, as shown in this paper.

Our paper is also related to Nti (2004), who considers optimal contest success functions in two-player asymmetric contests. Dasgupta and Nti (1998) study optimal contest success functions in n-player contests with symmetric contestants.

The rest of the paper proceeds as follows. Section 2 describes the baseline setting and the contest design problem; it furthers develops a novel optimization approach that characterizes the optimal asymmetric Tullock contests. Section 3 generalizes our analysis by allowing for an enriched design space and a larger set of design objectives. Section 4 provides two applications of our approach. Section 5 concludes. All proofs missing from the text are relegated to the Appendix.

2 Optimization of Asymmetric Tullock Contests: Baseline Setting

In this part, we first depict the contest model and set up the design problem. Next, we provide a brief review of existing analytical approaches, which demonstrates the challenges for further studies. Finally, we lay out our analysis to develop an alternative approach that takes the challenge.

concave alternatives. However, the optimization problem focuses on the maximization of total effort.

2.1 Preliminaries: Baseline Model and Contest Design Problem

There are n risk-neutral contestants competing for a prize. The prize bears a value $v_i > 0$ to each contestant $i \in \mathcal{N} \equiv \{1, \ldots, n\}$, which is commonly known. To win the prize, contestants simultaneously submit their effort entries $x_i \geq 0$. One's bid incurs a unity marginal effort cost.

For a given effort profile $\boldsymbol{x} \equiv (x_{1,}, \ldots, x_n)$, a contestant i wins the prize with a probability

$$p_i(\boldsymbol{x}) = \begin{cases} \frac{f_i(x_i)}{\sum_{j=1}^n f_j(x_j)} & \text{if } \sum_{j=1}^n f_j(x_j) > 0, \\ \frac{1}{n} & \text{if } \sum_{j=1}^n f_j(x_j) = 0, \end{cases} \tag{1}$$

where the function $f_i(\cdot)$, known as the impact function in the contest literature, converts one's effort entry into his effective bid in the lottery contest. It is clear that contestant $i \in \mathcal{N}$ is excluded from the contest if $f_i(x_i) = 0$ for all $x_i \geq 0$. In the extreme case in which one contestant has an increasing impact function, while every other contestant's impact function is a zero constant, we assume that the contestant wins automatically.[3]

The set of impact functions $\left\{f_i(\cdot)\right\}_{i=1}^n$, together with contestants' valuations $\boldsymbol{v} \equiv (v_1, \ldots, v_n)$, defines a simultaneous-move contest game.

Definition 1 ***(Regular Concave Contests)*** *A contest $(\boldsymbol{v}, \left\{f_i(\cdot)\right\}_{i=1}^n)$ is called a regular concave contest if the impact function for contest $i \in \mathcal{N}$ is either a nonnegative constant or a twice differentiable function, with $f_i'(x_i) > 0$, $f_i''(x_i) \leq 0$ and $f_i(0) \geq 0$.*

The above definition simply says that the impact function is concave. This restriction guarantees that a contestant's payoff function is concave in effort, and is commonly assumed in the literature. Szidarovszky and Okuguchi (1997) prove the existence and uniqueness of equilibria in the above game, assuming that $f_i(0) = 0$ for all $i \in \mathcal{N}$. Therefore, their result cannot be applied directly to contests in which headstarts are in place, i.e., $f_i(0) > 0$ for some $i \in \mathcal{N}$. The following theorem generalizes their result by relaxing this zero-headstart assumption.

Theorem 1 ***(Existence and Uniqueness of Equilibrium)*** *There exists a unique pure strategy Nash equilibrium in a regular concave contest game $(\boldsymbol{v}, \left\{f_i(\cdot)\right\}_{i=1}^n)$.*

Contest Design: Mathematical Program with Equilibrium Constraints (MPEC) We now set up the contest design problem. We consider a two-stage structure for the game.

[3] This assumption is imposed simply for the sake of expositional convenience, and is not crucial to our result.

In the first stage, the contest designer sets the contest rule and announces it publicly. In the second stage, contestants commit to their effort entries in the competition.

For now, we focus on a baseline setting and will relax the restrictions in Section 3. First, we focus on Tullock contests, assuming that the impact function taking the form

$$f_i(x_i) = \alpha_i \cdot x_i^{r_i}, \text{ for all } i \in \mathcal{N}. \tag{2}$$

In the competition stage, each contestant sinks his effort x_i to maximize his expected payoff

$$\pi_i(\boldsymbol{x}, \boldsymbol{\alpha}) = \begin{cases} \frac{\alpha_i \cdot x_i^{r_i}}{\sum_{j=1}^n \alpha_j \cdot x_j^{r_j}} v_i - x_i & \text{if} \quad \boldsymbol{x} \neq \boldsymbol{0}, \\ \frac{1}{n} v_i & \text{if} \quad \boldsymbol{x} = \boldsymbol{0}. \end{cases}$$

The parameter $r_i \in (0, 1]$ indicates the contestant's efficiency in converting his effort into an effective bid. It is worth noting that in contrast to the usual setups, we allow r_i to be heterogeneous among contestants. Contestants are therefore heterogeneous in two dimensions: valuations v_i and bidding efficiency r_i. We assume that r_i is exogenously given, while the weight on a contestant's effective bid, $\alpha_i \geq 0$, is a design variable. The contest designer chooses the vector of weights $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n)$ prior to the competition, which is commonly known when contestants sink their bids. We allow for a larger design space in Section 3.

Second, we focus on two specific objective functions for contest design in this section. To put it formally, the designer chooses weights $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n)$ to maximize an objective function $\Lambda(\boldsymbol{x}, \boldsymbol{\alpha})$—which is a function of contestants' effort profile $\boldsymbol{x} \equiv (x_{1,}, \ldots, x_n)$ and the designer's choice of weights $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n)$—for a given contest $(\boldsymbol{\alpha}, \boldsymbol{r}, \boldsymbol{v})$, with $\boldsymbol{r} \equiv (r_1, \ldots, r_n)$ and $\boldsymbol{v} \equiv (v_1, \ldots, v_n)$.

The optimal contest design problem yields a mathematical program with equilibrium constraints (MPEC): Contestants' equilibrium effort profile, $\boldsymbol{x}$, is endogenously determined in the equilibrium as a function of the bias rule, $\boldsymbol{\alpha}$, set by the designer. Theorem 1 has established the existence and uniqueness of a pure-strategy equilibrium in the underlying contest game for given $\boldsymbol{\alpha}$. That is, the designer chooses $\boldsymbol{\alpha} \in \mathbb{R}_+^n \setminus \boldsymbol{0}$ for the optimization problem

$$\max_{\boldsymbol{x}, \boldsymbol{\alpha} \in \mathbb{R}_+^n \setminus \boldsymbol{0}} \Lambda(\boldsymbol{x}, \boldsymbol{\alpha}), \text{ subject to } x_i = \arg\max_{x_i \geq 0} \pi_i(\boldsymbol{x}, \boldsymbol{\alpha}).$$

The two objective functions are formally given by (i) maximization of aggregate effort—i.e., $\Lambda(\boldsymbol{x}, \boldsymbol{\alpha}) := \sum_{i=1}^n x_i$—and (ii) maximization of the expected winner's effort—i.e., $\Lambda(\boldsymbol{x}, \boldsymbol{\alpha}) := \sum_{i=1}^n p_i(\boldsymbol{x}) \cdot x_i$, where $p_i(\boldsymbol{x})$ is a contestant i's winning odds. We focus on the two objective functions in this part. It is noteworthy, however, that the technical approach to be developed subsequently applies to a substantially larger set of design problems, which will be shown in

Section 3.

2.2 Existing Techniques and Their Challenges

In this section, we provide a brief review of the existing approach to the optimization of asymmetric Tullock contests and elaborate on the gap in the literature our analysis aims to fill.

2.2.1 A Brief Review of the Existing Approach

The aforementioned MPEC typically requires an implicit programming approach. It first solves for the unique equilibrium solution $x_i = x_i(\boldsymbol{\alpha})$ and then rewrites the mathematical program as

$$\max_{\boldsymbol{\alpha}\in\mathcal{A}} \Lambda\left(\boldsymbol{x}(\boldsymbol{\alpha}), \boldsymbol{\alpha}\right),$$

where $\boldsymbol{x}(\boldsymbol{\alpha}) \equiv \big(x_1(\boldsymbol{\alpha}), \ldots, x_n(\boldsymbol{\alpha})\big)$. The seminal study of Franke et al. (2013) provides a remarkable generalization of optimizing asymmetric lottery contests by setting optimal $\boldsymbol{\alpha}$ for the case of $r_1 = \ldots = r_n = 1$. Their approach proceeds in the following two steps.

Step I: Fix an arbitrary strictly positive bias rule $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{++}$.[4] Without loss of generality, we assume that the players are ordered as $\frac{1}{\alpha_1 v_1} \leq \ldots \leq \frac{1}{\alpha_n v_n}$. Denote the set of active contestants by $\mathcal{K}(\boldsymbol{\alpha}) \subseteq \mathcal{N}$. Stein (2002) shows that the set $\mathcal{K}(\boldsymbol{\alpha})$ can be fully characterized as

$$\mathcal{K}(\boldsymbol{\alpha}) = \left\{ i \in \mathcal{N} \,\middle|\, (i-1)\frac{1}{\alpha_i v_i} < \sum_{j=1}^{i} \frac{1}{\alpha_j v_j} \right\}.$$

The equilibrium requires the first-order conditions $\frac{\partial \pi_i(\boldsymbol{x},\boldsymbol{\alpha})}{\partial x_i} = 0$, which can be simplified as

$$\frac{\sum_{j\in\mathcal{K}(\boldsymbol{\alpha})\backslash i} \alpha_j x_j}{\left(\sum_{j\in\mathcal{K}(\boldsymbol{\alpha})} \alpha_j x_j\right)^2} = \frac{1}{\alpha_i v_i}, \text{ for } i \in \mathcal{K}(\boldsymbol{\alpha}). \tag{3}$$

Summing up all the first-order conditions in (3) yields

$$\sum_{j\in\mathcal{K}(\boldsymbol{\alpha})} \alpha_j x_j = \frac{k(\alpha) - 1}{\sum_{j\in\mathcal{K}(\boldsymbol{\alpha})} \frac{1}{\alpha_j v_j}}, \tag{4}$$

[4] Franke et al. (2013) assume that the weight $\alpha_i > 0$ for all $i \in \mathcal{N}$, whereas we allow α_i to be zero. Note that these two specifications yield exactly the same optimum. Allowing the contest designer to set $\alpha_i = 0$ corresponds to explicitly excluding the contestant in our setup. This outright elimination can also be achieved by placing excessively small weights on the same contestant.

where $k(\alpha) := \left|\mathcal{K}(\alpha)\right|$. Combining (3) and (4), we can solve for the equilibrium effort effort profile in closed form as

$$x_i(\boldsymbol{\alpha}) = \begin{cases} \frac{1}{\alpha_i} \times \frac{k(\alpha)-1}{\sum_{j\in\mathcal{K}(\alpha)} \frac{1}{\alpha_j v_j}} \times \left(1 - \frac{1}{\alpha_i v_i} \times \frac{k(\alpha)-1}{\sum_{j\in\mathcal{K}(\alpha)} \frac{1}{\alpha_j v_j}}\right) & , \text{if } i \in \mathcal{K}(\boldsymbol{\alpha}), \\ 0 & , \text{if } i \notin \mathcal{K}(\boldsymbol{\alpha}). \end{cases} \tag{5}$$

Step II: An effort-maximizing contest designer thus chooses the bias rule $\boldsymbol{\alpha}$ from the feasible set $\mathbb{R}^n_{++}$ to maximize

$$\theta(\boldsymbol{\alpha}) := \sum_{i\in\mathcal{K}(\boldsymbol{\alpha})} x_i(\boldsymbol{\alpha}).$$

It is worth noting that the existence of a maximizer for the above program is a priori unclear, and Franke et al. (2013) provide a rigorous and elegant proof for that.[5]

Denote one optimal bias rule by $\boldsymbol{\alpha}^* \equiv (\alpha_1^*, \ldots, \alpha_n^*)$. Clearly, the optimum requires that $\nabla\theta(\boldsymbol{\alpha}^*) = 0$, which is equivalent to

$$-\sum_{j\in\mathcal{K}(\boldsymbol{\alpha}^*)} \frac{1}{\alpha_j^*} + 2(k^*-1) \sum_{j\in\mathcal{K}(\boldsymbol{\alpha}^*)} \frac{1}{(\alpha_j^*)^2 v_j} = -v_i + 2(k^*-1)\frac{1}{\alpha_i^*}, \text{ for } i \in \mathcal{K}(\boldsymbol{\alpha}^*), \tag{6}$$

where $k^* := \left|\mathcal{K}(\boldsymbol{\alpha}^*)\right|$. Franke et al. (2013) show, in the proof of their Theorem 4.2, that the above nonlinear system of equations can be transformed into a linear one. Specifically, summing up (6) over all $i \in \mathcal{K}(\boldsymbol{\alpha}^*)$ yields

$$-\sum_{j\in\mathcal{K}(\boldsymbol{\alpha}^*)} \frac{1}{\alpha_j^*} + 2(k^*-1) \times \sum_{j\in\mathcal{K}(\boldsymbol{\alpha}^*)} \frac{1}{(\alpha_j^*)^2 v_j} = -\frac{1}{k^*} \times \sum_{j\in\mathcal{K}(\boldsymbol{\alpha}^*)} v_j + \frac{2(k^*-1)}{k^*} \times \sum_{j\in\mathcal{K}(\boldsymbol{\alpha}^*)} \frac{1}{\alpha_j^*}. \tag{7}$$

Combining (6) and (7), we obtain

$$\frac{1}{\alpha_i^*} - \frac{1}{k^*} \times \sum_{j\in\mathcal{K}(\boldsymbol{\alpha}^*)} \frac{1}{\alpha_j^*} = \frac{1}{2(k^*-1)} \times \left(v_i - \frac{1}{k^*} \times \sum_{j\in\mathcal{K}(\boldsymbol{\alpha}^*)} v_j\right), \text{ for } i \in \mathcal{K}(\boldsymbol{\alpha}^*). \tag{8}$$

Note that the above equations now constitute a linear system of equations of $\left(\frac{1}{\alpha_1^*}, \ldots, \frac{1}{\alpha_{k^*}^*}\right)$, from which a candidate optimal bias $\boldsymbol{\alpha}^*$ can be solved explicitly once $\mathcal{K}(\boldsymbol{\alpha}^*)$ is known.

[5]Note that first, the continuity of the objective function $\theta(\cdot)$ is nontrivial because the set $\mathcal{K}(\cdot)$ may change at points arbitrarily close to the optimum. Second, the feasible set for $\boldsymbol{\alpha}$ is unbounded and is not closed.

2.2.2 The Challenge for Further Generalization

Two remarks are in order concerning further generalization of the MPEC. First, as pointed out by Franke et al. (2013), the implicit programming approach relies on the assumption of $r_i = 1$ for all $i \in \mathcal{N}$. To see that more clearly, let us consider the case in which all impact functions share the same bidding efficiency, i.e., $r_1 = \ldots = r_n =: r$. It is straightforward to verify that the first-order conditions in (3) become

$$\frac{\sum_{j \in \mathcal{K}(\boldsymbol{\alpha}) \setminus i} \alpha_j (x_j)^r}{\left[\sum_{j \in \mathcal{K}(\boldsymbol{\alpha})} \alpha_j (x_j)^r\right]^2} \times r(x_i)^{r-1} = \frac{1}{\alpha_i v_i}, \text{ for } i \in \mathcal{K}(\boldsymbol{\alpha}). \tag{9}$$

The term $r(x_i)^{r-1}$ on the left-hand side degenerates to a constant in the case of $r = 1$, which is crucial to transform the FOCs into a system of linear equations, as Step I illustrates. However, this term is a nonlinear function of x_i for $r < 1$, and thus complicates the derivation of the equilibrium effort profile. In fact, although the existence of equilibrium has been well established by Szidarovszky and Okuguchi (1997) and Cornes and Hartley (2005), in general a closed-form solution cannot be obtained.

Second, note that in Step II, the system of first-order conditions (6) can again be transformed into a linear system (8). This property plays a critical role in solving for the optimal $\boldsymbol{\alpha}^*$ that maximizes total effort. It is absent, however, when alternative objectives are to be considered. Imagine, for instance, that the contest designer aims to maximize the expected winner's effort. Then the objective function becomes

$$\tilde{\theta}(\boldsymbol{\alpha}) := \sum_{i=1}^{n} \Big[p_i\big(\boldsymbol{x}(\boldsymbol{\alpha})\big) \cdot x_i(\boldsymbol{\alpha})\Big] \equiv \frac{\sum_{i \in \mathcal{K}(\boldsymbol{\alpha})} \big[x_i(\boldsymbol{\alpha})\big]^{r+1}}{\sum_{i \in \mathcal{K}(\boldsymbol{\alpha})} \big[x_i(\boldsymbol{\alpha})\big]^{r}}.$$

It can be verified that a linear system can no longer be obtained, even for the case $r = 1$.

Therefore, an alternative approach is required in demand to tackle the more general MPEC studied in this paper; the following analysis addresses this issue.

2.3 Optimal Contest Design: Our Approach

We now conduct formal analysis to introduce our approach. We first reformulate the designer's MPEC, and then present the optimization result obtained by the approach.

2.3.1 Reformulation of the Designer's Problem

With the impact functions specified in expression (2), the first-order condition $\frac{\partial \pi_i(\boldsymbol{x},\boldsymbol{\alpha})}{\partial x_i} = 0$ for an *active* contestant i is

$$\frac{\sum_{j \neq i} \alpha_j (x_j)^{r_j}}{\left[\sum_{j=1}^{n} \alpha_j (x_j)^{r_j}\right]^2} \times r_i (x_i)^{r_i - 1} = \frac{1}{\alpha_i v_i}.$$

The above condition, together with the winning probability $p_i(\boldsymbol{x})$ specified in Equation (1), implies immediately that

$$p_i(1-p_i)v_i = \frac{(x_i)^{r_i}}{r_i (x_i)^{r_i - 1}} = \frac{x_i}{r_i},$$

and thus

$$x_i = p_i(1-p_i)v_i r_i. \tag{10}$$

Note that the above condition also holds for an *inactive* contestant, as $x_i = 0$ is associated with $p_i = 0$. In other words, if a contestant stands zero chance of winning in a Tullock contest, he must have exerted zero effort. From Equation (10), an equilibrium effort profile is uniquely associated with a distribution of contestants' winning probabilities. We further establish the following.

Lemma 1 *Fix any distribution of equilibrium winning probabilities $\boldsymbol{p} \equiv (p_1, \ldots, p_n) \in \Delta^{n-1}$. Then:*

i. If $p_j = 1$ for some $j \in \mathcal{N}$, then $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$ can be induced by the following set of biases $\boldsymbol{\alpha}(\boldsymbol{p}) \equiv \big(\alpha_1(\boldsymbol{p}), \ldots, \alpha_n(\boldsymbol{p})\big)$:

$$\alpha_i(\boldsymbol{p}) = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

ii. If there exist at least two active contestants, then $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$ can be induced by the following set of biases $\boldsymbol{\alpha}(\boldsymbol{p}) \equiv \big(\alpha_1(\boldsymbol{p}), \ldots, \alpha_n(\boldsymbol{p})\big)$:

$$\alpha_i(\boldsymbol{p}) = \begin{cases} \frac{(p_i)^{1-r_i}}{\left[(1-p_i)\hat{v}_i\right]^{r_i}} & \text{if} \quad p_i > 0, \\ 0 & \text{if} \quad p_i = 0. \end{cases} \tag{11}$$

The lemma formally states that the contest designer can properly set the weights $\boldsymbol{\alpha}$ in the Tullock contest success function to induce any desirable distributions of winning probabilities in equilibrium.

By Equation (10) and Lemma 1, we are ready to reformulate the designer's optimization problem. Instead of setting $\boldsymbol{\alpha}$ directly, we treat the distribution of winning probabilities $\boldsymbol{p}$ as the design variable: She maximizes the objective function over feasible distributions of equilibrium winning probabilities. Upon obtaining the maximizer, the optimal weights can then be obtained by invoking (11).

To put it formally, an effort-maximizing contest designer chooses the winning probabilities $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$ to maximize

$$TE(\boldsymbol{p}; \boldsymbol{v}, \boldsymbol{r}) := \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} p_i(1-p_i)v_i r_i, \tag{12}$$

subject to the constraints:

$$\sum_{i=1}^{n} p_i = 1, \text{ and } p_i \geq 0, \text{ for } i \in \mathcal{N}. \tag{13}$$

It can be seen from the reformulated objective function (12) that the effort-maximizing contest design problem boils down to a simple quadratic programming problem.

Similarly, if the contest designer aims to maximize the expected winner's effort, then she faces the same constraints (13) as an effort-maximizing contest designer does; the objective function can be rewritten as the following:

$$WE(\boldsymbol{p}; \boldsymbol{v}, \boldsymbol{r}) := \sum_{i=1}^{n} p_i \cdot x_i = \sum_{i=1}^{n} p_i^2(1-p_i)v_i r_i. \tag{14}$$

2.3.2 Characterization of the Optimal Contest Rule

We now characterize the optimal contest for a contest designer under each stated objective. Reformulation enormously simplifies the MPEC. The above objective functions (12) and (14) are continuous in all arguments. Furthermore, the constraints (13) form a unit $(n-1)$-simplex, which is closed and bounded. Therefore, a maximizer must exist for either objective function. Treating the distribution of equilibrium winning probabilities as the design variable, together with the condition of Equation (10), allows us to transform the well-known nonsmooth optimization problem[6] into a smooth one.

The reformulated MPEC demonstrates several useful properties. First, it follows immediately from (12) and (14) that the optimal contest design problem with two-dimensional heterogeneity—i.e., bidding efficiency r_i and valuations v_i—can effectively be reduced to one with one-dimensional heterogeneity:[7] The heterogeneity is subsumed by the product $v_i r_i$; the

[6] The reader is referred to Franke et al. (2013) for more detailed discussion.

[7] It is clear that dimensionality reduction of contestant heterogeneity is not restricted to these two specific

program is thus equivalent to one in which all contestants are equally efficient with $r_i = 1$, while each contestant has a valuation $\hat{v}_i \equiv v_i \cdot r_i$ and contestants differ only in terms of $\hat{v}_i$s. This is formalized by the following lemma.

Lemma 2 ***(Dimensionality Reduction)*** *Suppose that the contest designer aims to maximize the total effort or the expected winner's effort. Then the optimal contest under* $(\boldsymbol{r}, \boldsymbol{v})$ *yields the same equilibrium effort profile, the same equilibrium winning probabilities, and the same payoff to the contest designer as that under* $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{v}}) \equiv \big((\hat{r}_1, \ldots, \hat{r}_n), (\hat{v}_1, \ldots, \hat{v}_n)\big)$*, where* $\hat{r}_i \equiv 1$ *and* $\hat{v}_i \equiv v_i r_i$ *for all* $i \in \mathcal{N}$*.*

The adjusted valuation $\hat{\boldsymbol{v}} \equiv (\hat{v}_1, \ldots, \hat{v}_n)$ provides a simple measure of a contestant's overall strength. Denote, respectively, by $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$ and $\boldsymbol{p}^{**} \equiv (p_1^{**}, \ldots, p_n^{**})$ the optimal distribution of winning probabilities that maximizes the total effort and expected winner's effort. The second useful property is established as follows.

Lemma 3 ***(Bottom-up Exclusion)*** *Suppose that* $\hat{v}_i > \hat{v}_j$*. Then* $p_i^* \geq p_j^*$ *and* $p_i^{**} \geq p_j^{**}$*.*

Lemma 3 states that in the optimal contest that maximizes either total effort or the expected winner's effort, an *ex ante* stronger contestant—i.e., a larger $\hat{v}_i$—must win with a (weakly) higher probability. This implies that when the designer excludes contestants—i.e., by assigning zero or excessively small weights to discourage participation—it must target the weakest.

We are now ready to characterize the optimal contest for each of the objectives.

Maximization of Total Effort The reformulated MPEC is a simple quadratic program. We lay out a sketch of the proof in the text, as it demonstrates the simplicity and efficiency of this approach and the roles played by the useful properties established by Lemmata 2 and 3. It should be noted, however, that Lemma 2 implies that we can transform our optimization problem to an alternative one with $r_i = 1$ and heterogeneous valuations, as in Franke et al. (2013) and Franke et al. (2014). Their results can then be converted and revived in this broader context.

To characterize the optimal equilibrium winning probabilities $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$, we consider the following sequence of auxiliary problems ($\mathcal{P}$-m): For each $m = 2, \ldots, n$, the contest designer maximizes $TE(\boldsymbol{p}; \hat{\boldsymbol{v}}, \hat{\boldsymbol{r}})$ in (12) subject to the plausibility constraint $\sum_{i=1}^n p_i = 1$, ignoring the nonnegativity constraint $p_i \geq 0$ for $i \in \{1, \ldots, m\}$ and setting $p_i = 0$ for $i \in \mathcal{N} \setminus \{1, \ldots, m\}$. The solution to the auxiliary equality constrained optimization problem

contest objectives in (12) and (14). In fact, as long as the designer's objective does not directly include $\boldsymbol{v} \equiv (v_1, \ldots, v_n)$, the dimensionality of contestant heterogeneity can be reduced to one.

($\mathcal{P}$-m), which we denote by $\check{\boldsymbol{p}}^m \equiv (\check{p}_1^m, \ldots, \check{p}_n^m)$, can be solved explicitly by computing the first-order conditions, and is given by

$$\check{p}_i^m = \begin{cases} \frac{1}{2}\left(1 - \frac{1}{\hat{v}_i} \times \frac{m-2}{\sum_{j=1}^m \frac{1}{\hat{v}_j}}\right) & \text{if} \quad i \in \{1, \ldots, m\}, \\ 0 & \text{if} \quad i \in \mathcal{N} \setminus \{1, \ldots, m\}. \end{cases}$$

Lemma 3 implies immediately that the maximizer of the original problem must be the solution to one of the above $n-1$ auxiliary problems. It is straightforward to see that $\check{p}_1^m \geq \ldots \geq \check{p}_m^m$. Therefore, if $\check{p}_m^m > 0$, then $\check{\boldsymbol{p}}^m \equiv (\check{p}_1^m, \ldots, \check{p}_n^m)$ is a candidate maximizer to the original maximization problem under the contest $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{v}})$. Define

$$\kappa := \max\left\{m = 2, \ldots, n \,\middle|\, \check{p}_m^m > 0\right\} \equiv \max\left\{m = 2, \ldots, n \,\middle|\, \frac{m-2}{\sum_{j=1}^m \frac{1}{\hat{v}_j}} < \hat{v}_m\right\}. \tag{15}$$

It can be verified that κ is well defined and unique.[8] Next, note that $TE(\check{\boldsymbol{p}}^m; \hat{\boldsymbol{v}}, \hat{\boldsymbol{r}})$ is increasing in m. Therefore, $\check{\boldsymbol{p}}^\kappa \equiv (\check{p}_1^\kappa, \ldots, \check{p}_\kappa^\kappa, 0, \ldots, 0)$ is the unique solution to the original maximization problem under contest $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{v}})$, which also constitutes the optimal winning probabilities under contest $(\boldsymbol{r}, \boldsymbol{v})$ by Lemma 2.

The above discussions are summarized below.

Theorem 2 ***(Effort-maximizing Contests)*** *Assume without loss of generality that contestants are ordered such that $\hat{v}_1 \geq \hat{v}_2 \geq \ldots \geq \hat{v}_n > 0$. Suppose that the contest designer aims to maximize total effort in the contest. Then the equilibrium winning probabilities $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$ are given by*

$$p_i^* = \begin{cases} \frac{1}{2}\left(1 - \frac{1}{\hat{v}_i} \times \frac{\kappa-2}{\sum_{j=1}^\kappa \frac{1}{\hat{v}_j}}\right) & \text{if} \quad i \in \{1, \ldots, \kappa\}, \\ 0 & \text{if} \quad i \in \mathcal{N} \setminus \{1, \ldots, \kappa\}, \end{cases} \tag{16}$$

where κ is given by Equation (15). Moreover, the corresponding weights, denoted by $\boldsymbol{\alpha}^* \equiv (\alpha_1^*, \ldots, \alpha_n^*)$, that induce $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$ are given by

$$\alpha_i^* = \begin{cases} \frac{(p_i^*)^{1-r_i}}{\left[(1-p_i^*)\hat{v}_i\right]^{r_i}} & \text{if} \quad p_i^* > 0, \\ 0 & \text{if} \quad p_i^* = 0. \end{cases}$$

The variable κ indicates the number of active contestants in the optimal contest. Theo-

[8] To see this more clearly, note that $\check{p}_2^2 = \frac{1}{2} > 0$. Therefore, the set $\left\{m = 2, \ldots, n \,\middle|\, \check{p}_n^n > 0\right\}$ is finite and nonempty.

rem 2 provides a complete characterization of the effort-maximizing contest for the case of $r_i \leq 1$.

Optimization of the Expected Winner's Effort As stated above, our approach also allows us to explore contest design with alternative objective functions, such as maximization of the expected winner's effort. Although the program involves cubic polynomials [see Equation (14)], which incurs additional complexity, the optimum can again be obtained in closed form. Our next result fully characterizes the optimal contest that maximizes the expected winner's effort.

Theorem 3 ***(Optimal Contest that Maximizes the Expected Winner's Effort)*** *Assume without loss of generality that contestants are ordered such that $\hat{v}_1 \geq \hat{v}_2 \geq \ldots \geq \hat{v}_n > 0$. Suppose that the contest designer aims to maximize the expected winner's effort. Then the equilibrium winning probabilities $\boldsymbol{p}^{**} \equiv (p_1^{**}, \ldots, p_n^{**})$ under the optimal contest are*

$$p_1^{**} = \begin{cases} \frac{\hat{v}_1 - 2\hat{v}_2 + \sqrt{\hat{v}_1^2 - \hat{v}_1\hat{v}_2 + \hat{v}_2^2}}{3(\hat{v}_1 - \hat{v}_2)} & \text{if } \hat{v}_1 > \hat{v}_2 \\ \frac{1}{2} & \text{if } \hat{v}_1 = \hat{v}_2 \end{cases}, \; p_2^{**} = 1 - p_1^{**}, \text{ and } p_3^{**} = \ldots = p_n^{**} = 0. \tag{17}$$

Moreover, the corresponding weights, denoted by $\boldsymbol{\alpha}^{**} \equiv (\alpha_1^{**}, \ldots, \alpha_n^{**})$, that induce $\boldsymbol{p}^{**} \equiv (p_1^{**}, \ldots, p_n^{**})$ are given by

$$\alpha_i^{**} = \begin{cases} \frac{\left(p_i^{**}\right)^{1-r_i}}{\left[\left(1-p_i^{**}\right)\hat{v}_i\right]^{r_i}} & \text{if } i \in \{1, 2\}, \\ 0 & \text{if } i \geq 3. \end{cases}$$

2.3.3 Total Effort vs. the Expected Winner's Effort

The optimal contests differ substantially under the two objectives. The following result regarding the number of active players can be immediately obtained from Theorems 2 and 7.

Corollary 1 *Suppose that $n \geq 3$. Then the optimal contest that maximizes total effort allows for at least three active contestants—i.e., $\kappa \geq 3$—while the optimal contest that maximizes the expected winner's effort allows for only two active contestants.*

Corollary 1 shows that a minimum number of contestants, i.e., two, are allowed to remain active in the contest when maximizing the expected winner's effort. This result is intuitive: This objective stresses individual incentive, while a larger contest dilutes it. In contrast, if the contest designer aims to maximize total effort, she engages at least three active contestants,[9] although exclusion could happen in the optimum (i.e., $\kappa < n$) when contestants are heterogeneous for $n \geq 4$.

In both scenarios, we observe (i) that the optimal contests balance the playing field among active contestants, and (ii) the assigned weights do not entirely offset the initial asymmetry.[10] Franke et al. (2013) state (i) in the context of $r = 1$, using a numeric example for the maximization of total effort. A similar conclusion can be expected in our setting, given the fact that the problem with nonlinear impact functions can be transformed to an equivalent problem with linear impact functions. A formal proof for (i) is infeasible in general for effort-maximizing contests: Equilibrium winning probabilities cannot be obtained in the benchmark context with $\alpha_1 = \ldots = \alpha_n = 1$, because there is no explicit solution to the equilibrium effort profile. It is straightforward, however, to verify part (ii) of the statement: Theorem 2 implies that in the optimum, two active contestants, i, j, have the same winning probabilities if and only if they have the same adjusted valuation, i.e., $p_i^* = p_j^* = \frac{1}{\kappa}$ if and only if $\hat{v}_i = \ldots = \hat{v}_j$.

More formal analysis can be conducted for the maximization of the expected winner's effort because only two active contestants remain in the optimum. We conclude the following.

Corollary 2 *When maximizing the expected winner's effort, the optimal contest handicaps the strongest contestant, but does not fully level the playing field, i.e.,* $\frac{\hat{v}_2}{\hat{v}_1+\hat{v}_2} < p_2^{**} < p_1^{**} < \frac{\hat{v}_1}{\hat{v}_1+\hat{v}_2}$ *for* $\hat{v}_1 > \hat{v}_2$.

If the two active contestants are equally treated (i.e., $\alpha_1 = \alpha_2$), it is well known from the literature that the favorite wins with probability $\frac{\hat{v}_1}{\hat{v}_1+\hat{v}_2}$ in equilibrium, while the underdog does with probability $\frac{\hat{v}_2}{\hat{v}_1+\hat{v}_2}$. The fact that $p_1^{**} < \frac{\hat{v}_1}{\hat{v}_1+\hat{v}_2}$ and $p_2^{**} > \frac{\hat{v}_2}{\hat{v}_1+\hat{v}_2}$ indicates that the favorite is handicapped in the optimal contest. The handicap is incomplete, though, because $p_2^{**} < p_1^{**}$.

3 Optimization in a Broader Setting

In this section, we generalize our analysis in two dimensions. First, we relax our restriction on the functional form of the impact functions. Instead of focusing on Tullock contests, we allow the designer to choose an arbitrary form for each contestant's impact function $f_i(\cdot)$, as long as it satisfies the requirements stated in Section 2, which are formally summarized as follows.

[9] Franke et al. (2013) obtain the same result in their Theorem 4.6.

[10] In contrast, it can be verified that $p_1^* = p_2^* = \frac{1}{2}$ if $n = 2$ for all pairs of adjusted values $(\hat{v}_1, \hat{v}_2)$. In other words, an effort-maximizing contest designer would completely level the playing field in a two-player contest.

Assumption 1 ***(Concave Impact Function)*** *$f_i(\cdot)$ is a nonnegative constant function or is twice differentiable, with $f_i'(x_i) > 0$, $f_i''(x_i) \leq 0$ and $f_i(0) \geq 0$.*

With Assumption 1 in place, we consider a regular concave contest as defined previously. A unique bidding equilibrium is shown to exist in Theorem 1. Optimal design is thus feasible.

Second, we consider a general objective function that addresses a broad spectrum of interests. Let the objective function $\Lambda(\cdot)$ be written as a function of effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, the profile of winning probability $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$, and the profile of winning values $\boldsymbol{v} \equiv (v_1, \ldots, v_n)$. Without loss of generality, we assume that contestants are *ex ante* ranked in descending order in terms of their valuations, i.e., $v_1 \geq \ldots \geq v_n > 0$. We make the following assumption on the contest designer's objective function.

Assumption 2 ***(Objective Function)*** *Fixing $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$ and $\boldsymbol{v} \equiv (v_1, \ldots, v_n)$, $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$ is weakly increasing in x_i for all $i \in \mathcal{N}$.*

Clearly, Assumption 2 is satisfied if the contest designer maximizes the aggregate effort of the contest (i.e., $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) = \sum_{i=1}^n x_i$) or maximizes the expected winner's effort (i.e., $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) = \sum_{i=1}^n p_i \cdot x_i$). The MPEC can thus be stated as follows:

$$\begin{aligned} & \max_{\{f_i(\cdot)\}_{i=1}^n} \Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) \\ \text{subject to } x_i &= \arg\max \pi_i(\boldsymbol{x}, \boldsymbol{p}), \\ p_i(\boldsymbol{x}) &= \begin{cases} \frac{f_i(x_i)}{\sum_{j=1}^n f_j(x_j)} & \text{if } \sum_{j=1}^n f_j(x_j) > 0, \\ \frac{1}{n} & \text{if } \sum_{j=1}^n f_j(x_j) = 0. \end{cases} \end{aligned}$$

Recall that our approach uses the profile of winning probabilities, $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$, as the direct choice variable for the contest design problem. However, it is noteworthy that the objective function addresses broad concerns and allows the designer to have a direct preference not only on contestants' effort profiles, but also the distribution of winning probabilities.

For example, in sports competitions, spectators often not only appreciate contenders' efforts, but also demand more suspense about the outcome of the game (see Chan et al., 2008). The preference for a closer race can be represented by the following noncanonical objective function:

$$\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) := \sum_{i=1}^n x_i - \gamma \sum_{i=1}^n \left(p_i - \frac{\sum_{j=1}^n p_j}{n} \right)^2.$$

The contest design serves to both elicit efforts and maintain the competitive balance. The latter is measured by the term $-\sum_{i=1}^n \left(p_i - \frac{\sum_{j=1}^n p_j}{n} \right)^2$, which is the variance (or standard deviation) of the winning probabilities. Such a preference is also assumed by Fort and Quirk (1995), Szymanski (2003), and Runkel (2006). However, these studies focus on two-player settings. It is straightforward to observe that Assumption 2 is satisfied. We show below that our approach enables analysis of such an objective function in an n-player setting, in which case the model yields substantially different implications than in a two-player environment.

In contrast, consider the objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) := \sum_{i=1}^{n} p_i \cdot x_i$, in which case the designer maximizes the expected winner's effort. The designer cares about the distribution of contestants' efforts, which implicitly depends on the distribution of winning probabilities through the channel of contest success functions. However, she does not have a direct preference on the distribution of winning probabilities.

Alternatively, consider a scenario in which the contest designer cares about both effort supply and contestants' welfare (e.g., Epstein et al., 2011). Define u_i to be a contestant i's expected utility. This preference can be formally expressed as

$$\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) := \tau \sum_{i=1}^{n} u_i + (1-\tau) \sum_{i=1}^{n} x_i. \tag{18}$$

Given the fact that $u_i = p_i \cdot v_i - x_i$, the objective function can be rewritten as

$$\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) := \tau \sum_{i=1}^{n} p_i v_i + (1-2\tau) \sum_{i=1}^{n} x_i.$$

This objective function satisfies the requirement of Assumption 2 whenever $\tau \leq \frac{1}{2}$.

Based on our approach, we obtain the following important result.

Theorem 4 ***(Optimality of Linear Impact Functions with Zero Headstarts)*** *Suppose that Assumptions 1 and 2 are satisfied. Then the optimum can be achieved by setting each contestant's impact function $f_i(x_i)$ in the form of $f_i(x_i) = \alpha_i \cdot x_i$, with $\alpha_i \geq 0$.*[11]

The proof of Theorem 4 is collated in the Appendix. The proof proceeds in two steps. First, we establish an upper bound for the functional value of $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$ that can possibly be achieved with any set of concave impact functions. Second, we show that to achieve that upper bound, it is sufficient to properly set the set of weights $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n)$, i.e., linear impact functions with zero headstarts.

This result substantially simplifies the search for the optimal contest structure, as we can focus, without loss of generality, on setting the linear coefficient α_i. Consider, for instance, the optimization problem studied by Epstein et al. (2011) in a two-player Tullock

[11] In the working paper version, Franke et al. (2018) establish the same result under the objective of effort maximization.

contest setting. Let us allow for an n-player setting and let the designer choose any form of concave impact functions that satisfy Assumption 1. The objective function (18), by the correspondence between equilibrium probability and equilibrium effort, can be rewritten as

$$\begin{aligned}\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) : &= \tau \sum_{i=1}^{n} p_i v_i + (1 - 2\tau) \sum_{i=1}^{n} x_i \\ &= \sum_{i=1}^{n} p_i \left[(1 - \tau) - (1 - 2\tau) p_i)\right] v_i.\end{aligned}$$

A standard quadratic program results, and a technique similar to the proof of Theorem 2 can be applied to obtain the solution.

4 Applications

In this section, we consider two scenarios to demonstrate the flexibility of our optimization approaches. We first consider the optimization of the aforementioned noncanonical objective function. In the second example, we reexamine the comparison between lottery contests and all-pay auctions (e.g., Epstein et al., 2011; Franke et al., 2014; Franke et al., 2018).

4.1 Contest Design with Closeness Concerns

We now examine the case in which the designer maximizes the following objective function:

$$\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) := \sum_{i=1}^{n} x_i - \gamma \sum_{i=1}^{n} \left(p_i - \frac{\sum_{j=1}^{n} p_j}{n} \right)^2 .$$

The designer cares about both the effort supply and the closeness of the competition. When $\gamma = 0$, the closeness concern fades away, and her objective degenerates to effort maximization.

The solution to the optimization problem can be obtained in a few simple steps. Denote the optimal distribution of winning probabilities by $\boldsymbol{p}^*_{1c} \equiv (p^*_{1c}, \ldots, p^*_{nc})$, where we use the subscript c to indicate "closeness concerns." First, it is straightforward to verify that $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$ strictly increases in x_i for all $i \in \mathcal{N}$, which satisfies Assumption 2. Theorem 4 states that it suffices to search for the optimum by focusing on linear impact functions without positive headstarts, i.e., $f_i(x_i) = \alpha_i \cdot x_i$. Second, recall by the analysis in Section 2.3 that the correspondence between equilibrium efforts and winning probabilities is

$$x_i = p_i(1 - p_i)v_i,$$

which in turn allows us to rewrite the objective function as the following:

$$\Lambda\left(\boldsymbol{x}(\boldsymbol{p}), \boldsymbol{p}, \boldsymbol{v}\right) = \sum_{i=1}^{n} p_i(1-p_i)v_i - \gamma \sum_{i=1}^{n} p_i^2 - \frac{3\gamma}{n}.$$

Despite the noncanonical form of the objective function, a standard quadratic program results. Third, the proof of Lemma 3 can be applied to establish the bottom-up exclusion rule in this context, which states $p_{1c}^* \geq \ldots \geq p_{nc}^*$ in the optimum. Finally, a similar approach to that in the proof of Theorem 2 allows us to obtain the optimum. Define the variable κ_c as follows:

$$\kappa_c := \max\left\{ m = 2, \ldots, n \,\middle|\, \frac{\sum_{j=1}^{m} \frac{v_j}{v_j+\gamma} - 2}{\sum_{j=1}^{m} \frac{1}{v_j+\gamma}} < v_m \right\}. \tag{19}$$

It can be verified that κ_c is well defined and unique. We are then ready to obtain the optimum.

Theorem 5 ***(Optimal Contest with Closeness Concerns)*** *Suppose that contestants are ordered such that $v_1 \geq \ldots \geq v_n > 0$. Then the equilibrium winning probabilities $\boldsymbol{p}_{1c}^* \equiv (p_{1c}^*, \ldots, p_{nc}^*)$ under the optimal contest are given by*

$$p_{ic}^* = \begin{cases} \frac{1}{2}\left(\frac{v_i}{v_i+\gamma} - \frac{1}{v_i+\gamma} \times \frac{\sum_{j=1}^{\kappa_c} \frac{v_j}{v_j+\gamma} - 2}{\sum_{j=1}^{\kappa_c} \frac{1}{v_j+\gamma}} \right) & \text{for} \quad i \leq \kappa_c, \\ 0 & \text{for} \quad i > \kappa_c. \end{cases} \tag{20}$$

The proof is similar to that in Theorem 2 and is omitted for brevity. By Theorem 5, κ_c indicates the number of active contestants in the optimum. Consider this as a function of γ, with $\kappa_c := \kappa_c(\gamma)$. A closer look at Equation (19) leads to the following result.

Corollary 3 *Suppose that $\gamma_H > \gamma_L \geq 0$, then $\kappa_c(\gamma_H) \geq \kappa_c(\gamma_L)$.*

Corollary 3 formally establishes the comparative statics of κ_c with respect to γ. With closeness concerns in place, the contest designer is given extra incentives to balance the playing field. Underdogs are favored more to ensure a more even distribution of winning probabilities. Exclusion is thus less likely, which leads to the observation of Corollary 3.

It is worth noting that the implications in the general setting run in sharp contrast to those obtained in a two-player case. In the latter scenario, the designer's concerns about effort supply and closeness are perfectly aligned. Recall the correspondence $x_i = p_i(1-p_i)v_i$, which implies that in a two-player case, total effort can be expressed as

$$\begin{aligned} \sum_{i=1}^{2} x_i &= p_1(1-p_1)v_1 + p_2(1-p_2)v_2 \\ &= p_1(1-p_1)(v_1+v_2), \end{aligned}$$

given the fact that $p_2 = 1 - p_1$. Maximization of total effort requires fully balancing the playing field, i.e., $p_1^* = p_2^* = \frac{1}{2}$, which is in line with maximizing the uncertainty of the contest outcomes. However, the two concerns diverge when n exceeds two: A fully balanced playing field is no longer optimal, which leads to the above observation that closeness concerns require the designer to further favor underdogs.

4.2 Lottery Contests vs. All-pay Auctions

We now apply our technique to address the classical question in the contest literature: the comparison between an optimally designed all-pay auction and an optimally designed lottery contest. As stated previously, the literature mainly focuses on comparison in terms of total effort.[12] Our approach allows us to provide a ranking in terms of the expected winner's effort. This objective is often adopted for optimal design in all-pay auction models (see Moldovanu and Sela, 2006; Fu, 2006), while studies in lottery-contest settings have been relatively scarce, partly due to the technical difficulty described in Section 2.2.2.

All-Pay Auctions We first consider the optimal design of all-pay auctions. As in Franke et al. (2018), we allow the designer to choose a combination of multiplicative bias and additive headstart. For an all–pay auction with biases $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n) \geq \mathbf{0}$ and headstarts $\boldsymbol{\beta} \equiv (\beta_1, \ldots, \beta_n) \geq \mathbf{0}$, the winning rule can be written as

$$p_i^a(\boldsymbol{x}) := \begin{cases} 1, & \text{if } \alpha_i \cdot x_i + \beta_i > \alpha_j \cdot x_j + \beta_j \text{ for all } j \neq i, \\ \frac{1}{d+1}, & \begin{array}{l} \text{if } \alpha_i \cdot x_i + \beta_i = \alpha_j \cdot x_j + \beta_j \text{ for } d \text{ contestants,} \\ \text{and } \alpha_i \cdot x_i + \beta_i > \alpha_\ell \cdot x_\ell + \beta_\ell \text{ for all other contestants } \ell \neq i, \end{array} \\ 0, & \text{if } \alpha_i \cdot x_i + \beta_i < \alpha_j \cdot x_j + \beta_j \text{ for some } j \neq i, \end{cases}$$

where we use the superscript a to denote "all-pay auctions." Each contestant $i \in \mathcal{N}$ receives a score of $\alpha_i \cdot x_i + \beta_i$ and the prize is awarded to the contestant with the highest score. In the case in which a tie occurs, the prize is distributed randomly with equal probability among the contestants with the highest scores. We now limit our attention to a design problem with only two instruments, i.e., multiplicative biases and headstarts. However, as will be shown later, this suffices for our purpose.

The optimal all-pay auction with two players has been well established in the literature (see Fu, 2006; Epstein et al., 2011; Li and Yu, 2012). The optimum with more than two players involves substantially more subtlety when headstart is involved as a design instrument.

[12] Epstein et al. (2011) provide a notable exception, but the analysis is placed in a two-player setting.

Franke et al. (2018) argue that the optimum could involve more than two active players, which stands in sharp contrast to the design with multiplicative biases as the sole design variable.

As in Franke et al. (2018), we do not derive the optima, but establish a lower bound for the expected winner's effort an optimal all-pay auction can possibly induce.

Proposition 1 ***(Lower Bound for the Expected Winner's Effort in All-pay Auctions)*** *Fixing* $\boldsymbol{v} \equiv (v_1, \ldots, v_n)$ *with* $v_1 \geq \ldots \geq v_n > 0$*, an optimally designed all-pay contest generates the expected highest effort of no less than* $\frac{v_1+v_2}{3}$*.*

The lower bound for the expected winner's effort is obtained by the contest rule $\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = (\frac{v_2}{v_1}, 1, 0, \ldots, 0)$, $\boldsymbol{\beta} \equiv (\beta_1, \ldots, \beta_n) = (0, \ldots, 0)$. That is, the contest designer uses only the multiplicative biases but not the headstarts, and includes only the two strongest contestants.

Lottery Contests We now consider optimally designed lottery contests. A lottery contest is resolved with a contest success function

$$p_i(\boldsymbol{x}) = \frac{f_i(x_i)}{\sum_{j=1}^{n} f_j(x_j)}.$$

The designer is allowed to choose arbitrary forms of impact functions $\{f_i(\cdot)\}_{i=1}^n$ that satisfy Assumption 1.

By Theorem 4, it is sufficient to focus our attention on linear impact functions $f_i(x_i) = \alpha_i \cdot x_i$, with $\alpha_i \geq 0$. Denote by $WE_{max}(\boldsymbol{v})$ the maximal expected winner's effort optimal contests can achieve when contestants have the valuations $\boldsymbol{v} \equiv (v_1, \ldots, v_n)$. Theorem 7 states that the optimum requires that only the two strongest contestants remain active, and optimal weights depend purely on the two top dogs' characteristics. The following can be obtained by standard techniques based on Theorem 7.

Proposition 2 *Consider a lottery contest in which contestants have the valuations* $\boldsymbol{v} \equiv (v_1, \ldots, v_n)$*, with* $v_1 \geq \ldots \geq v_n$*. Let the designer choose any form of impact functions* $\{f_i(x_i)\}_{i=1}^n$ *that satisfy Assumption 1 to maximize the expected winner's effort. The maximum expected winner's effort is given by*

$$WE_{max}(\boldsymbol{v}) = \begin{cases} \frac{\left[v_1+v_2+\sqrt{v_1^2-v_1v_2+v_2^2}\right]\times\left[(v_1^2-4v_1v_2+v_2^2)+(v_1+v_2)\sqrt{v_1^2-v_1v_2+v_2^2}\right]}{27(v_1-v_2)^2} & \text{if } v_1 > v_2, \\ \frac{v_1}{4} & \text{if } v_1 = v_2. \end{cases}$$

Comparison between Lottery Contests and All-pay Auctions We are now ready to conclude the following dominance results.

Theorem 6 ***(Concave Lottery Contests vs. All-Pay Auctions)*** *An optimally designed all-pay auction strictly dominates any Tullock contest with impact functions that satisfy Assumption 1 in terms of the expected winner's effort, i.e.,*

$$WE_{max}(\boldsymbol{v}) < \frac{v_1 + v_2}{3}.$$

Theorem 6 states that the optimally designed all-pay auction strictly outperforms the optimally designed lottery contest in terms of maximizing the expected winner's effort. The result reaffirms all-pay auctions' dominance when an alternative design objective—i.e., the maximization of the expected winner's effort—is considered and the designer is endowed with a larger design space.

5 Concluding Remarks

In this paper, we study the optimal contest design of asymmetric lottery contests. We first develop a technical approach to address the open question raised by Franke et al. (2013): The search for the optimal multiplicative biases—i.e., the weights placed on contestants' effort entries—in n-player asymmetric Tullock contests. In contrast to the previous studies, our approach allows us to obtain the optimum in Tullock contests with nonlinear impact functions. Moreover, the objective for contest design is no longer limited to total effort maximization. Based on this approach, we take a further step in generalizing our exercise of optimal contest design: The designer is given full discretion in choosing the form of (regular concave) impact functions in lottery contests, and it is shown that linear impact functions with zero headstarts are optimal for a wide array of design objectives.

We demonstrate that our technique can be applied broadly to simplify the search for optimal contest rules, including design problems with noncanonical objective functions. We also apply the technique to reexamine the classical issue of comparing all-pay auctions and lottery contests when alternative design objectives are in place. Our paper considers a static contest; this approach can also be applied in dynamic settings. For instance, Fu and Wu (2018) consider a two-stage contest in which the designer assigns weights on contestants' second-stage effort entries based on their first-stage ranking.

Several interesting questions arise that can be pursued in future studies. First, this paper considers lottery contests with complete information, and it would be interesting to exercise this approach in an environment with incomplete information. Second, our study provides a general technique to solve for optimal contest rules when contestants have asymmetric valuations. It would be interesting to explore how optimal contest rules specifically depend on the distribution of contestants' valuations. We leave the exploration of these possibilities to future research.

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Appendix: Proofs

Proof of Theorem 1

Proof. Note that $x_i = 0$ is a strictly dominant strategy for contestant i if $f_i(\cdot)$ is a constant. Therefore, it suffices to prove the theorem for the case in which $f_i(\cdot)$ satisfies $f_i'(x_i) > 0$, $f_i''(x_i) \leq 0$ and $f_i(0) \geq 0$ for all $i \in \mathcal{N}$.

For notational convenience, define $y_i := f_i(x_i)$, $\delta_i := f_i(0)$, $\tilde{f}_i(x_i) := f_i(x_i) - \delta_i$, and $g_i(y_i) := \tilde{f}_i^{-1}(y_i - \delta_i)/v_i$. It follows immediately that $x_i = g_i(y_i) \cdot v_i$. Moreover, we have that $g_i' > 0$ and $g_i'' \geq 0$. The expected payoff of contestant $i \in \mathcal{N}$ choosing $y_i \geq \delta_i$ is equal to

$$\left[\frac{y_i}{\sum_{j=1}^n y_j} - g_i(y_i)\right] \cdot v_i.$$

It remains to show that there exists a unique equilibrium $\boldsymbol{y}^* \equiv (y_1^*, \ldots, y_n^*)$ that satisfies $y_i^* \geq \delta_i$ for all $i \in \mathcal{N}$. Let $s := \sum_{j=1}^n y_j$. It is clear that $s \geq \sum_{j=1}^n \delta_j \equiv \underline{\delta}$. The first-order condition of the above expected utility with respect to y_i yields the following:

$$\frac{s - y_i}{s^2} - g_i'(y_i) \leq 0, \text{ with equality if } y_i > \delta_i.$$

Fixing s, let us define $y_i(s)$ as the following:

$$y_i(s) := \begin{cases} \delta_i & \text{if } s^2 g_i'(\delta_i) - s + \delta_i \geq 0, \\ \text{The unique solution to } s - y_i = s^2 g_i'(y_i) & \text{otherwise.} \end{cases} \tag{21}$$

It is straightforward to verify that $y_i(s)$ is well defined and continuous in $s \in [\delta_i, \infty]$. Moreover, we must have that $y_i(s) \in (\delta_i, s)$ if $s^2 g_i'(\delta_i) - s + \delta_i < 0$.

Suppose that there exists an interval of s such that $y_i(s) > \delta_i$. It follows immediately from the implicit function theorem that

$$y_i'(s) = \frac{1 - 2s g_i'(y_i)}{1 + s^2 g_i''(y_i)} = \frac{2y_i(s) - s}{\left[1 + s^2 g_i''(y_i)\right] s}, \tag{22}$$

where the second equality follows from $s - y_i = s^2 g_i'(y_i)$. Therefore, $y_i(s)$ is strictly decreasing in this interval if $2y_i < s$ and strictly increasing otherwise. By Equation (21), the latter case occurs if and only if

$$s - \frac{1}{2}s > s^2 g_i'\left(\frac{s}{2}\right) \Leftrightarrow 2s g_i'\left(\frac{s}{2}\right) < 1.$$

Note that $2sg_i'\left(\frac{s}{2}\right)$ is strictly increasing in s, which implies that there exists at most one solution to $2sg_i'\left(\frac{s}{2}\right) = 1$. Denote the solution by $\hat{s}_i$ whenever it exists.

Next, we denote the two different real number solutions of $s^2 g_i'(\delta_i) - s + \delta_i = 0$ by $s_i^\dagger$ and $s_i^{\dagger\dagger}$ respectively, with $s_i^\dagger < s_i^{\dagger\dagger}$, whenever they exist. The above analysis, together with the fact that the expression $s^2 g_i'(\delta_i) - s + \delta_i$ in Equation (21) is quadratic in s, implies that the function $y_i(s)$ must fall into one of the following four cases:

Case I: There exist no different real number solutions of $s^2 g_i'(\delta_i) - s + \delta_i = 0$ for $s \in [\underline{\delta}, \infty]$. Then we must have that $s^2 g_i'(\delta_i) - s + \delta_i \geq 0$ for all $s \geq \underline{\delta}$, which in turn implies that $y_i(s) = \delta_i$ for all $s \geq \underline{\delta}$ by Equation (21). To slightly abuse the notation, we let $s_i^{\dagger\dagger} := \underline{\delta}$ for this case.

Case II: $s_i^\dagger \leq \underline{\delta} \leq s_i^{\dagger\dagger}$ and $y_i(\underline{\delta}) \leq \frac{1}{2}\underline{\delta}$. Then $y_i(s)$ is strictly decreasing in s for $s \in [\underline{\delta}, s_i^{\dagger\dagger}]$, and $y_i(s) = \delta_i$ for $s \in [s_i^{\dagger\dagger}, \infty]$.

Case III: $s_i^\dagger \leq \underline{\delta} \leq s_i^{\dagger\dagger}$ and $y_i(\underline{\delta}) > \frac{1}{2}\underline{\delta}$. It can be verified that $\underline{\delta} < \hat{s}_i < s_i^{\dagger\dagger}$. Therefore, $y_i(s)$ is strictly increasing in s for $s \in [\underline{\delta}, \hat{s}_i]$, is strictly decreasing in s for $s \in [\hat{s}_i, s_i^{\dagger\dagger}]$, and $y_i(s) = \delta_i$ for $s \in [s_i^{\dagger\dagger}, \infty]$.

Case IV: $\underline{\delta} < s_i^\dagger < s_i^{\dagger\dagger}$. It can be verified that $s_i^\dagger < \hat{s}_i < s_i^{\dagger\dagger}$. Moreover, $y_i(s)$ is strictly increasing in s for $s \in [s_i^\dagger, \hat{s}_i]$; is strictly decreasing in s for $s \in [\hat{s}_i, s_i^{\dagger\dagger}]$; and $y_i(s) = \delta_i$ for $s \in [\underline{\delta}, s_i^\dagger] \cup [s_i^{\dagger\dagger}, \infty]$.

The four different cases are depicted in Figure 1 graphically. For Case I and Case II, we define $s_i := \underline{\delta}$; for Case III and Case IV, we define $s_i := \hat{s}_i \geq \underline{\delta}$. It is straightforward to verify that $y_i(s) > \frac{1}{2}s$ holds if $s < s_i$ for all four cases. Without loss of generality, we order the contestants such that

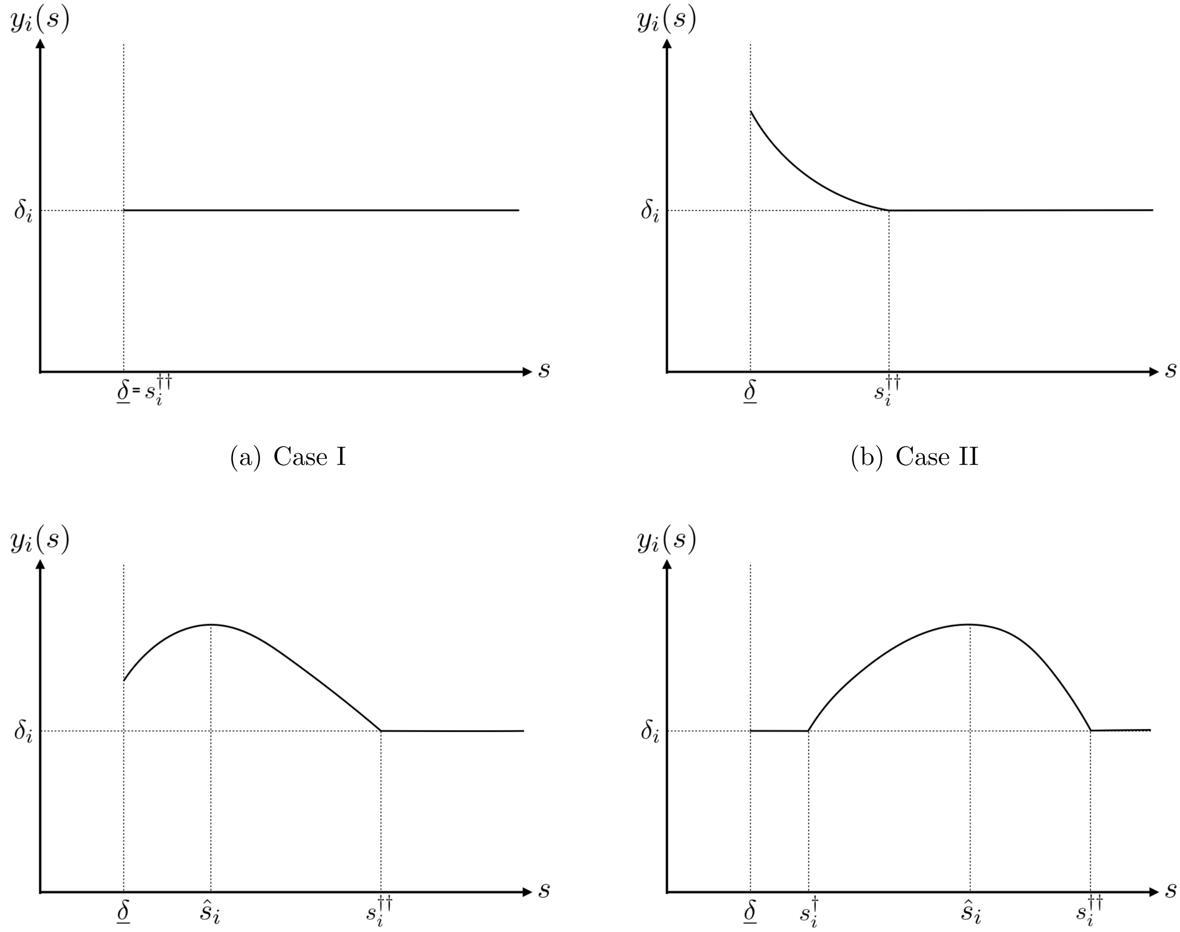
$$s_1 \geq s_2 \geq \ldots \geq s_n \geq \underline{\delta}.$$

Define $Y(s) := \sum_{i=1}^n y_i(s) - s$. It remains to show that $Y(s) = 0$ has a unique positive solution. First, note that no solution exists for $s < s_2$ because

$$Y(s) := \sum_{i=1}^n y_i(s) - s \geq y_1(s) + y_2(s) - s > \frac{1}{2}s + \frac{1}{2}s - s = 0, \text{ for } s < s_2.$$

Next, we claim that $Y(s)$ is strictly decreasing in s for $s \geq s_2$. Clearly, $Y(s)$ is strictly decreasing in s for $s \geq s_1$. Moreover, for $s \in [s_2, s_1]$, $Y(s)$ can be rewritten as

$$Y(s) = \underbrace{\sum_{i=2}^n y_i(s)}_{\text{first term}} + \underbrace{[y_1(s) - s]}_{\text{second term}}.$$



(a) Case I
(b) Case II
(c) Case III
(d) Case IV

Figure 1: $y_i(s)$.

Because $s \geq s_2 \geq \ldots \geq s_n$, the first term is weakly decreasing in s. Taking derivative of the second term with respect to s yields

$$y_1'(s) - 1 = \frac{2y_1(s) - s}{[1 + s^2 g_1''\big(y_1(s)\big)]s} - 1 \leq \frac{2y_1(s) - s}{s} - 1 = \frac{2}{s}\big[y_1(s) - s\big] < 0,$$

where the first equality follows from Equation (22); the first inequality follows from $g_i'' \geq 0$ and $y_1(s) \geq \frac{s}{2}$; and the second inequality follows from $y_i(s) < s$ [see Equation (21)]. Therefore, the second term is strictly decreasing in s, which in turn implies that $Y(s)$ is strictly decreasing for $s \in [s_2, \infty]$.

It is straightforward to see that for all four cases, we have that $y_i(s) = \delta_i$ for $s \geq s_i^{\dagger\dagger}$. Let

$s^{\dagger\dagger} := s_2 + \sum_{i=1}^{n} s_i^{\dagger\dagger} + \sum_{i=1}^{n} \delta_i$. It is clear that $s^{\dagger\dagger} \geq s_2$. Moreover, we have that

$$Y(s^{\dagger\dagger}) = \sum_{i=1}^{n} y_i(s^{\dagger\dagger}) - s^{\dagger\dagger} = \sum_{i=1}^{n} \delta_i - \left(s_2 + \sum_{i=1}^{n} s_i^{\dagger\dagger} + \sum_{i=1}^{n} \delta_i \right) = -s_2 - \sum_{i=1}^{n} s_i^{\dagger\dagger} \leq 0.$$

Therefore, there exists a unique positive solution to $Y(s) = 0$ for $s \in [s_2, s^{\dagger\dagger}]$. This completes the proof. ■

Proof of Lemma 1

Proof. Part (i) of the proposition is trivial, and it remains to show part (ii). It is clear that $x_i = 0$ is a strictly dominant strategy if $\alpha_i = 0$. For $(p_i, p_j) > (0, 0)$, we must have $(x_i, x_j) > (0, 0)$. Therefore, the following first-order conditions must be satisfied by Equation (10):

$$\begin{aligned} x_i &= p_i(1-p_i)v_i r_i \equiv p_i(1-p_i)\hat{v}_i, \\ x_j &= p_j(1-p_j)v_j r_j \equiv p_j(1-p_j)\hat{v}_j, \end{aligned}$$

Therefore, it remains to show that the weights $(\alpha_1(\boldsymbol{p}), \ldots, \alpha_n(\boldsymbol{p}))$ specified in Equation (11) satisfy

$$\frac{x_i}{x_j} = \frac{p_i(1-p_i)\hat{v}_i}{p_j(1-p_j)\hat{v}_j}. \tag{23}$$

Note that Equation (1) implies that

$$\frac{p_i}{p_j} = \frac{\frac{\alpha_i \cdot x_i^{r_i}}{\sum_{j=1}^{n} \alpha_j \cdot x_j^{r_j}}}{\frac{\alpha_j \cdot x_j^{r_j}}{\sum_{j=1}^{n} \alpha_j \cdot x_j^{r_j}}} = \frac{\alpha_i \cdot x_i^{r_i}}{\alpha_j \cdot x_j^{r_j}}.$$

Therefore, Equation (23) is equivalent to

$$\begin{aligned} \frac{x_i}{x_j} = \frac{\alpha_i \cdot x_i^{r_i}}{\alpha_j \cdot x_j^{r_j}} \times \frac{(1-p_i)\hat{v}_i}{(1-p_j)\hat{v}_j} &\Leftrightarrow 1 = \frac{\alpha_i \cdot [p_i(1-p_i)\hat{v}_i]^{r_i - 1}}{\alpha_j \cdot [p_j(1-p_j)\hat{v}_j]^{r_j - 1}} \times \frac{(1-p_i)\hat{v}_i}{(1-p_j)\hat{v}_j} \\ &\Leftrightarrow \frac{\alpha_i}{\alpha_j} = \frac{(p_i)^{1-r_i}\left[(1-p_i)\,\hat{v}_i\right]^{-r_i}}{(p_j)^{1-r_j}\left[(1-p_j)\,\hat{v}_j\right]^{-r_j}}. \end{aligned}$$

The last equation clearly holds for the set of weights specified in Equation (11). This completes the proof. ■

Proof of Lemma 3

Proof. We first prove the result for total effort maximization. Suppose to the contrary that there exists a pair (i, j) such that $\hat{v}_i > \hat{v}_j$ and $p_i^* < p_j^*$ in the optimal contest. We consider the following two cases depending on $p_i^* + p_j^*$ relative to 1.

Case I: $p_i^* + p_j^* < 1$**.** Switching the winning probability between these two contestants generates a strictly higher amount of total efforts to the contest designer. To see this, it suffices to show that

$$\begin{aligned} &\hat{v}_i p_i^*(1-p_i^*) + \hat{v}_j p_j^*(1-p_j^*) < \hat{v}_i p_j^*(1-p_j^*) + \hat{v}_j p_i^*(1-p_i^*) \\ \Leftrightarrow\ &(\hat{v}_i - \hat{v}_j) \cdot (p_i^* - p_j^*) \cdot (1 - p_i^* - p_j^*) < 0. \end{aligned}$$

It is evident that the last inequality holds due to the postulated $\hat{v}_i > \hat{v}_j$, $p_i^* < p_j^*$ and $p_i^* + p_j^* < 1$.

Case II: $p_i^* + p_j^* = 1$**.** Then only two contestants are active in equilibrium. It is clear that $p_i^* \in (0, 1)$ and $p_j^* \in (0, 1)$. Together with the postulated $p_j^* > p_i^*$, we have that $p_j^* > \frac{1}{2} > p_i^*$. Next we show that increasing p_i^* by a sufficiently small $\epsilon > 0$ and decreasing p_j^* by the same amount lead to a strictly higher payoff to the contest designer. Define $TE(\epsilon)$ as

$$TE(\epsilon) := (p_i^* + \epsilon)(1 - p_i^* - \epsilon)\hat{v}_i + (p_j^* - \epsilon)(1 - p_j^* + \epsilon)\hat{v}_j,$$

Simple algebra yields that

$$TE'(0) = (1 - 2p_i^*)\hat{v}_i - (1 - 2p_j^*)\hat{v}_j = (p_j^* - p_i^*)(\hat{v}_i + \hat{v}_j) > 0,$$

where the second equality follows from $p_i^* + p_j^* = 1$ and the strict inequality follows the postulated $p_j^* > p_i^*$.

The proof for the expected winner's effort maximization is similar. Suppose to the contrary that there exists a pair (i, j) such that $\hat{v}_i > \hat{v}_j$ and $p_i^{**} < p_j^{**}$ in the optimal contest. Again, we consider the following two cases depending on $p_i^{**} + p_j^{**}$ relative to 1.

Case I: $p_i^{**} + p_j^{**} < 1$**.** It suffices to show that

$$\begin{aligned} &\hat{v}_i(p_i^{**})^2(1-p_i^{**}) + \hat{v}_j(p_j^{**})^2(1-p_j^{**}) < \hat{v}_i(p_j^{**})^2(1-p_j^{**}) + \hat{v}_j(p_i^{**})^2(1-p_i^{**}) \\ \Leftrightarrow\ &(\hat{v}_i - \hat{v}_j) \cdot (p_i^{**} - p_j^{**}) \cdot \Big[p_i^{**} + p_j^{**} - (p_i^{**})^2 - (p_j^{**})^2 - p_i^{**}p_j^{**}\Big] < 0. \end{aligned} \tag{24}$$

Because $p_i^{**} + p_j^{**} < 1$, we have that

$$p_i^{**}+p_j^{**}-(p_i^{**})^2-(p_j^{**})^2-p_i^{**}p_j^{**} = p_i^{**}(1-p_i^{**})+p_j^{**}(1-p_j^{**})-p_i^{**}p_j^{**} > 2p_i^{**}p_j^{**}-p_i^{**}p_j^{**} = p_i^{**}p_j^{**} \geq 0.$$

Therefore, the strict inequality (24) holds.

Case II: $p_i^{**} + p_j^{**} = 1$. In this case only two contestants remain active. Fixing $\hat{v}_i > \hat{v}_j$, it can be verified that $p_i^{**} = \frac{\hat{v}_i - 2\hat{v}_j + \sqrt{\hat{v}_i^2 - \hat{v}_i\hat{v}_j + \hat{v}_j^2}}{3(\hat{v}_i - \hat{v}_j)} > \frac{1}{2}$, which in turn implies that $p_i^{**} > \frac{1}{2} > 1 - p_i^{**} = p_j^{**}$. This completes the proof. ■

Proof of Theorem 7

Proof. It is useful to state an intermediary result.

Lemma 4 *The following must hold in the optimal contest that maximizes the expected winner's effort: (i) if $\hat{v}_i > \hat{v}_j$, $p_i^{**} > 0$ and $p_j^{**} > 0$, then $p_i^{**} + p_j^{**} > \frac{2}{3}$; (ii) if $\hat{v}_i = \hat{v}_j$, $p_i^{**} > 0$ and $p_j^{**} > 0$, then $p_i^{**} + p_j^{**} \geq \frac{2}{3}$.*

Proof. Fix $a \in (0, 1]$. Consider the following function $\varphi(x) := x^2(1-x)\hat{v}_i + (a-x)^2\left[1-(a-x)\right]\hat{v}_j$ with domain $x \in [0, a]$. To prove the lemma, it suffices to show that $\varphi(x)$ is maximized at either $x = 0$ or $x = a$ if: (i) $\hat{v}_i > \hat{v}_j$ and $a \leq \frac{2}{3}$; or (ii) $\hat{v}_i = \hat{v}_j$ and $a < \frac{2}{3}$.

Note that $\varphi(x)$ can be rewritten as

$$\varphi(x) = -(\hat{v}_i - \hat{v}_j)x^3 + \left[\hat{v}_i + (1-3a)\hat{v}_j\right]x^2 + a(3a-2)\hat{v}_j x + a^2(1-a)\hat{v}_j.$$

Taking derivative of $\varphi(x)$ with respect to x yields

$$\varphi'(x) = -3(\hat{v}_i - \hat{v}_j)x^2 + 2\left[\hat{v}_i + (1-3a)\hat{v}_j\right]x + a(3a-2)\hat{v}_j.$$

Case I: $\hat{v}_i > \hat{v}_j$ and $a \leq \frac{2}{3}$. Note that

$$\hat{v}_i + (1-3a)\hat{v}_j \geq \hat{v}_i + \left(1 - 3 \times \frac{2}{3}\right)\hat{v}_j = \hat{v}_i - \hat{v}_j > 0.$$

Moreover, it can be verified that

$$\varphi'(0) = a(3a-2)\hat{v}_j \leq 0,$$

and

$$\varphi'(a) = -3(\hat{v}_i - \hat{v}_j)a^2 + 2\left[\hat{v}_i + (1-3a)\hat{v}_j\right]a + a(3a-2)\hat{v}_j = (2-3a)a\hat{v}_i \geq 0.$$

Therefore, $\varphi(x)$ is first decreasing and then increasing in x for $x \in [0, a]$, indicating that $\varphi(x)$ is maximized at either $x = 0$ or $x = a$.

Case II: $\hat{v}_i = \hat{v}_j$ and $a < \frac{2}{3}$. Then $\varphi'(x)$ can be simpled as

$$\varphi'(x) = 2(2 - 3a)\hat{v}_i x + a(3a - 2)\hat{v}_i = (2 - 3a)(2x - a)\hat{v}_i.$$

It is straightforward to see that $\varphi'(x) < 0$ for $x < \frac{a}{2}$ and $\varphi'(x) > 0$ for $x > \frac{a}{2}$. Therefore, $\varphi(x)$ is again maximized at either $x = 0$ or $x = a$. This completes the proof. ■

Now we can prove Theorem 7. It is clear that a contest with only one active contestant in equilibrium is never optimal to the contest designer. Next we show that it is suboptimal to induce three or more active contestants in the optimum. Suppose to the contrary that there exist three contestants i, j and k with $\hat{v}_i \geq \hat{v}_j \geq \hat{v}_k$ such that $p_i^{**} > 0$, $p_j^{**} > 0$, and $p_k^{**} > 0$. Then we have the following two cases:

Case I: $\hat{v}_i > \hat{v}_j > \hat{v}_k$, $\hat{v}_i > \hat{v}_j = \hat{v}_k$, or $\hat{v}_i = \hat{v}_j > \hat{v}_k$. Lemma 4 implies instantly that $p_i^{**} + p_j^{**} \geq \frac{2}{3}$, $p_i^{**} + p_k^{**} \geq \frac{2}{3}$ and $p_j^{**} + p_k^{**} \geq \frac{2}{3}$, with one of the inequalities at least being strict. Summing up the three inequalities yields that

$$p_i^{**} + p_j^{**} + p_k^{**} > 1,$$

which is a contradiction to $p_i^{**} + p_j^{**} + p_k^{**} \leq 1$.

Case II: $\hat{v}_i = \hat{v}_j = \hat{v}_k$. Lemma 4 implies that $p_i^{**} + p_j^{**} \geq \frac{2}{3}$, $p_i^{**} + p_k^{**} \geq \frac{2}{3}$ and $p_j^{**} + p_k^{**} \geq \frac{2}{3}$. Therefore, we must have that $p_i^{**} = p_j^{**} = p_k^{**} = \frac{1}{3}$, and the corresponding expected winner's effort is

$$(p_i^{**})^2(1 - p_i^{**})\hat{v}_i + (p_j^{**})^2(1 - p_j^{**})\hat{v}_j + (p_k^{**})^2(1 - p_k^{**})\hat{v}_k = \frac{2}{9}\hat{v}_i.$$

However, by inducing $(p_i, p_j, p_k) = (\frac{1}{2}, \frac{1}{2}, 0)$, the contest designer can obtain

$$(p_i)^2(1 - p_i)\hat{v}_i + (p_j)^2(1 - p_j)\hat{v}_j + (p_k)^2(1 - p_k)\hat{v}_k = \frac{1}{4}\hat{v}_i > \frac{2}{9}\hat{v}_i,$$

which is again a contradiction.

Therefore, an optimally designed contest must induce exactly two active contestants in equilibrium, and Lemma 3 indicates instantly that it is without loss of generality to assume that contestants 1 and 2 are active. Therefore, the contest designer's maximization problem can be simplified as the following:

$$\max_{\{p_1,p_2\}} p_1^2(1-p_1)\hat{v}_1 + p_2^2(1-p_2)\hat{v}_2,$$

subject to $p_1 + p_2 = 1$, $p_1 \geq 0$, and $p_2 \geq 0$. It is straightforward to verify that $p_1^{**} = \frac{1}{2}$ if $\hat{v}_1 = \hat{v}_2$, and $p_1^{**} = \frac{\hat{v}_1 - 2\hat{v}_2 + \sqrt{\hat{v}_1^2 - \hat{v}_1\hat{v}_2 + \hat{v}_2^2}}{3(\hat{v}_1 - \hat{v}_2)}$ if $\hat{v}_1 > \hat{v}_2$. This completes the proof. ■

Proof of Corollary 1

Proof. It suffices to show that $\kappa \geq 3$ for the case $n \geq 3$, or equivalently, $\check{p}_3^3 > 0$ from Equation (15). Simple algebra yields that

$$\check{p}_3^3 = \frac{1}{2} \times \left(1 - \frac{1}{\hat{v}_3} \times \frac{1}{\sum_{j=1}^{3} \frac{1}{\hat{v}_j}}\right) = \frac{1}{2} \times \left(1 - \frac{1}{\frac{\hat{v}_3}{\hat{v}_1} + \frac{\hat{v}_3}{\hat{v}_2} + 1}\right) > 0.$$

This completes the proof. ■

Proof of Corollary 2

Proof. It follows from $\hat{v}_1 > \hat{v}_2$ and Equation (28) that

$$p_1^{**} = \frac{\hat{v}_1 - 2\hat{v}_2 + \sqrt{\hat{v}_1^2 - \hat{v}_1\hat{v}_2 + \hat{v}_2^2}}{3(\hat{v}_1 - \hat{v}_2)},$$

and

$$p_2^{**} = \frac{2\hat{v}_1 - \hat{v}_2 - \sqrt{\hat{v}_1^2 - \hat{v}_1\hat{v}_2 + \hat{v}_2^2}}{3(\hat{v}_1 - \hat{v}_2)}.$$

It is straightforward to show that $p_2^{**} < p_1^{**}$. Moreover, $p_1^{**} < \frac{\hat{v}_1}{\hat{v}_1 + \hat{v}_2}$ is equivalent to

$$(\hat{v}_1 + \hat{v}_2) \times \sqrt{\hat{v}_1^2 - \hat{v}_1\hat{v}_2 + \hat{v}_2^2} < 2\left(\hat{v}_1^2 - \hat{v}_1\hat{v}_2 + \hat{v}_2^2\right),$$

which can be further simplified as

$$(\hat{v}_1 + \hat{v}_2)^2 < 4\left(\hat{v}_1^2 - \hat{v}_1\hat{v}_2 + \hat{v}_2^2\right) \Leftrightarrow 3\left(\hat{v}_1 - \hat{v}_2\right)^2 > 0.$$

The last inequality holds clearly. Therefore, we have $p_1^{**} < \frac{\hat{v}_1}{\hat{v}_1 + \hat{v}_2}$, and thus $p_2^{**} = 1 - p_1^{**} > 1 - \frac{\hat{v}_1}{\hat{v}_1 + \hat{v}_2} = \frac{\hat{v}_2}{\hat{v}_1 + \hat{v}_2}$. This completes the proof. ■

Proof of Theorem 4

Proof. The proof proceeds in two steps. We first establish an upper bound of $\Lambda(\cdot)$ for any combination of equilibrium winning probability $\boldsymbol{p}^\star \equiv (p_1^\star, \ldots, p_n^\star)$ that the contest designer would like to induce. In a second step, we construct a set of linear impact functions, with $f_i(x_i) = \alpha_i \cdot x_i$, to achieve this upper bound.

Step I: Denote the set of active players the designer would like to induce by $\mathcal{N}^\star \subseteq \mathcal{N}$, and the equilibrium effort profile by $x^\star \equiv (x_1^\star, \ldots, x_n^\star)$. Then the following first-order condition must be satisfied:

$$\frac{\sum_{j\neq i} f_j(x_j^\star) \cdot f_i'(x_i^\star)}{\left[\sum_{j=1}^n f_j(x_j^\star)\right]^2} = \frac{1}{v_i}, \text{ for all } i \in \mathcal{N}^\star.$$

The above condition can be rewritten as,

$$\frac{\sum_{j\neq i} f_j(x_j^\star)}{\sum_{j=1}^n f_j(x_j^\star)} \times \frac{f_i(x_i^\star)}{\sum_{j=1}^n f_j(x_j^\star)} = \frac{1}{v_i} \times \frac{f_i(x_i^\star)}{f_i'(x_i^\star)}, \text{ for all } i \in \mathcal{N}^\star,$$

which is equivalent to,

$$p_i^\star(1-p_i^\star)v_i = \frac{f_i(x_i^\star)}{f_i'(x_i^\star)}, \text{ for all } i \in \mathcal{N}^\star. \tag{25}$$

Next, note that

$$\frac{f_i(x_i^\star)}{f_i'(x_i^\star)} \geq \frac{f_i(x_i^\star) - f_i(0)}{f_i'(x_i^\star)} = \frac{\int_0^{x_i^\star} f_i'(t)dt}{f_i'(x_i^\star)} \geq \frac{f_i'(x_i^\star)x_i^\star}{f_i'(x_i^\star)} = x_i^\star, \text{ for all } i \in \mathcal{N}^\star. \tag{26}$$

where the first inequality follows directly from $f_i(0) \geq 0$ and the second inequality follows from $f_i''(x_i) \leq 0$. Define $\overline{x}_i^\star := p_i^\star(1-p_i^\star)v_i$ and $\overline{\boldsymbol{x}}^\star := (\overline{x}_1^\star, \ldots, \overline{x}_n^\star)$. The above inequality, together with Equation (25), implies that

$$x_i^\star \leq p_i^\star(1-p_i^\star)v_i \equiv \overline{x}_i^\star, \text{ for all } i \in \mathcal{N}^\star.$$

For $i \in \mathcal{N} \setminus \mathcal{N}^\star$, it is clear that

$$x_i^\star = 0 \leq p_i^\star(1-p_i^\star)v_i \equiv \overline{x}_i^\star, \text{ for all } i \in \mathcal{N} \setminus \mathcal{N}^\star.$$

Because the contest designer's objective function is weakly increasing in x_i by Assumption 2, we have that

$$\Lambda(\boldsymbol{x}^\star, \boldsymbol{p}^\star, \boldsymbol{v}) \leq \Lambda(\overline{\boldsymbol{x}}^\star, \boldsymbol{p}^\star, \boldsymbol{v}).$$

Step II: Next, we construct a set of linear impact functions with zero headstarts to induce the vector of the desired winning probability $\boldsymbol{p}^\star \equiv (p_1^\star, \ldots, p_n^\star)$ and the equilibrium effort profile $\overline{\boldsymbol{x}}^\star \equiv (\overline{x}_1^\star, \ldots, \overline{x}_n^\star)$ simultaneously. It is straightforward to verify that the set of biases $\boldsymbol{\alpha}(\boldsymbol{p}^\star) \equiv \big(\alpha_1(\boldsymbol{p}^\star), \ldots, \alpha_n(\boldsymbol{p}^\star)\big)$ constructed in Lemma 1, after setting $r_i = 1$ for all $i \in \mathcal{N}$ in Equation (11), induces the desired equilibrium effort profile $\overline{\boldsymbol{x}}^\star \equiv (\overline{x}_1^\star, \ldots, \overline{x}_n^\star)$ and the distribution of winning probabilities $\boldsymbol{p}^\star \equiv (p_1^\star, \ldots, p_n^\star)$. This completes the proof. ■

Proof of Corollary 3

Proof. Carrying out the algebra, κ_c in Equation (19) can be written as

$$\kappa_c := \max\left\{ m = 2, \ldots, n \,\middle|\, \sum_{j=1}^{m} \frac{v_j - v_m}{v_j + \gamma} < 2 \right\}.$$

Define $\mathcal{G}(m, \gamma)$ as

$$\mathcal{G}(m, \gamma) := \sum_{j=1}^{m} \frac{v_j - v_m}{v_j + \gamma}.$$

It follows immediately that

$$\begin{aligned}
\mathcal{G}(m+1, \gamma) - \mathcal{G}(m, \gamma) &= \sum_{j=1}^{m+1} \frac{v_j - v_{m+1}}{v_j + \gamma} - \sum_{j=1}^{m} \frac{v_j - v_m}{v_j + \gamma} \\
&= \sum_{j=1}^{m} \frac{v_j - v_{m+1}}{v_j + \gamma} - \sum_{j=1}^{m} \frac{v_j - v_m}{v_j + \gamma} = \sum_{j=1}^{m} \frac{v_m - v_{m+1}}{v_j + \gamma} \geq 0.
\end{aligned}$$

Therefore, $\mathcal{G}(m, \gamma)$ is weakly increasing in m. Moreover, it is straightforward to see that $\mathcal{G}(m, \gamma)$ is weakly decreasing in γ. These two facts imply instantly that $\kappa_c(\gamma)$ is weakly increasing in γ. This completes the proof. ■

Proof of Proposition 1

Proof. We first establish the following intermediate result.

Lemma 5 ***(Epstein et al., 2011 and Franke et al., 2014)*** *Suppose that $\alpha_1 = v_2/v_1$, $\alpha_2 = 1$, $\alpha_3 = \ldots = \alpha_n = 0$ and $\beta_i = 0$ for all $i \in \mathcal{N}$. Then in equilibrium, contestant $i \in \{3, \ldots, n\}$ is inactive. Moreover, contestant 1 bids uniformly between $[0, v_1]$, and contestant 2 bids uniformly between $[0, v_2]$. That is, $x_1^* = \mathcal{U}[0, v_1]$ and $x_2^* = \mathcal{U}[0, v_2]$.*

It can be verified that the expected winner's effort under the set of contest biases constructed in Lemma 5 is

$$\begin{aligned}
&\Pr(\alpha_1 x_1^* \geq \alpha_2 x_2^*) \times \mathbb{E}\left(x_1^* | \alpha_1 x_1^* \geq \alpha_2 x_2^*\right) + \Pr(\alpha_1 x_1^* \leq \alpha_2 x_2^*) \times \mathbb{E}\left(x_2^* | \alpha_1 x_1^* \leq \alpha_2 x_2^*\right) \\
=&\frac{1}{v_1 v_2} \times \left[\int_0^{v_1} \int_0^{\alpha_1 b_1} b_1 db_2 db_1 + \int_0^{v_1} \int_{\alpha_1 b_1}^{v_2} b_2 db_2 db_1\right] \\
=&\frac{1}{v_1 v_2} \times \left[\int_0^{v_1} \alpha_1 b_1^2 db_1 + \int_0^{v_1} \frac{1}{2}\left(v_2^2 - \alpha_1^2 b_1^2\right) db_1\right] = \frac{1}{3}\left(v_1 + v_2\right),
\end{aligned}$$

which is the lower bound of the expected winner's effort from an optimally designed all-pay auction. This completes the proof. ■

Proof of Proposition 2

Proof. The expression of $WE_{max}(\boldsymbol{v})$ follows immediately from Equations (10), (14), and (28). This completes the proof. ■

Proof of Theorem 6

Proof. It follows from Theorem 7 that $p_1^{**} > 0$, $p_2^{**} > 0$, and $p_3^{**} = \ldots = p_n^{**} = 0$. Therefore, we have that $p_1^{**} + p_2^{**} = 1$ and

$$\begin{aligned}
WE_{max}(\boldsymbol{v}) &= p_1^{**}\left[p_1^{**}(1 - p_1^{**})\right] v_1 + p_2^{**}\left[p_2^{**}(1 - p_2^{**})\right] v_2 \\
&\leq \frac{1}{4} p_1^{**} v_1 + \frac{1}{4} p_2^{**} v_2 \leq \frac{v_1 + v_2}{4} < \frac{v_1 + v_2}{3},
\end{aligned}$$

where the equality follows from Equation (14); the first inequality follows from $0 \leq p_1^{**}(1 - p_1^{**}) \leq \frac{1}{4}$ and $0 \leq p_2^{**}(1 - p_2^{**}) \leq \frac{1}{4}$; and the second inequality follows from $0 \leq p_1^{**} \leq 1$ and $0 \leq p_2^{**} \leq 1$. This completes the proof. ■

6 Limited Resources and Optimal Contest Design

Suppose that the contest designer has amount of resource (e.g., capital) that is normalized to one. The weighting rule $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \geq \mathbf{0}$ is interpreted as the resource allocation. Contestant i has the following production technology: $f_i(x_i) = \alpha_i x_i^r$, where α_i is the resource allocated to contestant i and $r \leq 1$ reflects the diminishing marginal return. Each contestant receives a score $s_i = f_i(x_i) + \epsilon_i$, where ϵ_i follows the Gumbel distribution. The prize is awarded to the contestant with the highest score.

Instead of total effort $\sum_{i\in\mathcal{N}} x_i$, the contestant designer aims to maximize the total output from the contest, that is, $\sum_{i\in\mathcal{N}} \alpha_i x_i^r$, subject to constraint $\sum_{i\in\mathcal{N}} \alpha_i \leq 1$. Because the contest bias α_i is interpreted as resources and has direct impact on the output, the analysis becomes more complicated. Clearly, we have $\sum_{i\in\mathcal{N}} \alpha_i = 1$. For notational convenience, let us denote the set of contestants with a strictly positive equilibrium winning probability by $\mathcal{N}_+(\boldsymbol{p})$:

$$\mathcal{N}_+(\boldsymbol{p}) := \Big\{ i = 1, \ldots, n \,|\, p_i > 0 \Big\}.$$

A lemma that is in parallel to Lemma 1 can then be established.

Lemma 6 *Fix any distribution of equilibrium winning probabilities $\boldsymbol{p} \equiv (p_1, \ldots, p_n) \in \Delta^{n-1}$. Then:*

i. If $r = 1$, then $\alpha_i(\boldsymbol{p}) = \frac{1}{(1-p_i)\hat{v}_i} \times \frac{1}{\eta(1)}$, where $\eta(\boldsymbol{p}, 1) := \sum_{j\in\mathcal{N}_+(\boldsymbol{p})} \frac{1}{(1-p_j)\hat{v}_j}$.

ii. If $r < 1$, then $\alpha_i(\boldsymbol{p}) = \frac{(p_i)^{1-r_i}}{\left[(1-p_i)\hat{v}_i\right]^{r_i}} \times \frac{1}{\eta(r)}$, where $\eta(\boldsymbol{p}, r) := \sum_{j\in\mathcal{N}} \frac{\left(p_j\right)^{1-r_j}}{\left[\left(1-p_j\right)\hat{v}_j\right]^{r_j}}$.

Given the resource allocation established in Lemma 6 and Equation (10), we can derive the total output as the following:

$$\sum_{i\in\mathcal{N}} \alpha_i x_i^r = \sum_{i\in\mathcal{N}} \alpha_i \left[p_i(1-p_i)\hat{v}_i\right]^r = \frac{1}{\eta(\boldsymbol{p}, r)}, \text{ for } r \in (0, 1]. \tag{27}$$

Therefore, the contest designer chooses $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$ to *minimize* $\eta(\boldsymbol{p}, r)$, subject to constraints (13).

Theorem 7 ***(Optimal Contest that Maximizes the Total Output)*** *Assume without loss of generality that contestants are ordered such that $\hat{v}_1 \geq \hat{v}_2 \geq \ldots \geq \hat{v}_n > 0$. Suppose that the contest designer aims to maximize the total output. Fixing $r \in (0, 1]$, then the equilibrium winning probabilities $\boldsymbol{p}^{***} \equiv (p_1^{***}, \ldots, p_n^{***})$ under the optimal contest are*

$$p_1^{***} = \frac{\sqrt{\mathcal{M}^2 + 4\chi} + \mathcal{M}}{\sqrt{\mathcal{M}^2 + 4\chi} + \mathcal{M} + 2}, \; p_2^{***} = 1 - p_1^{***}, \; \textit{and } p_3^{***} = \ldots = p_n^{***} = 0, \tag{28}$$

where $\chi := \left(\frac{v_1}{v_2}\right)^r \geq 1$ *and* $\mathcal{M} := \frac{(\chi-1)(1-r)}{r}$. *Moreover, the corresponding resource allocation, denoted by* $\boldsymbol{\alpha}^{***} \equiv (\alpha_1^{***}, \ldots, \alpha_n^{***})$, *that induces* $\boldsymbol{p}^{***} \equiv (p_1^{***}, \ldots, p_n^{***})$ *are given by*

$$\alpha_i^{***} = \begin{cases} \frac{\left(p_i^{***}\right)^{1-r}}{\left[\left(1-p_i^{***}\right)v_i\right]^r} \Bigg/ \left[\frac{\left(p_1^{***}\right)^{1-r}}{\left[\left(1-p_1^{***}\right)v_1\right]^r} + \frac{\left(p_2^{***}\right)^{1-r}}{\left[\left(1-p_2^{***}\right)v_2\right]^r}\right] & \textit{if} \quad i \in \{1, 2\}, \\ 0 & \textit{if} \quad i \geq 3. \end{cases}$$

Proof. We first show that only the two strongest contestants would remain active in the optimal contest. Similar to Lemma 3, we can show that $p_1 \geq \ldots \geq p_n$ in the optimal contest. We consider the following two cases:

Case I: $r = 1$**.** Consider the following sequence of auxiliary problems ($\mathcal{P}$-m): For each $m = 2, \ldots, n$, the contest designer minimizes $\eta(\boldsymbol{p}, r)$ in (12) subject to the plausibility constraint $\sum_{i=1}^n p_i = 1$, ignoring the nonnegativity constraint $p_i \geq 0$ for $i \in \{1, \ldots, m\}$ and setting $p_i = 0$ for $i \in \mathcal{N} \setminus \{1, \ldots, m\}$. The solution to the auxiliary equality constrained optimization problem ($\mathcal{P}$-m), which we denote by $\ddot{\boldsymbol{p}}^m \equiv (\ddot{p}_1^m, \ldots, \ddot{p}_n^m)$, can be solved explicitly by computing the first-order conditions, and is given by

$$\ddot{p}_i^m = \begin{cases} 1 - \frac{1}{\sqrt{\hat{v}_i}} \times \frac{n-1}{\sum_i^m \frac{1}{\sqrt{\hat{v}_i}}} & \text{if} \quad i \in \{1, \ldots, m\}, \\ 0 & \text{if} \quad i \in \mathcal{N} \setminus \{1, \ldots, m\}. \end{cases}$$

The corresponding η can be derived as the following:

$$\eta(\ddot{\boldsymbol{p}}^m, r) = \frac{\left(\sum_{i=1}^m \frac{1}{\sqrt{\hat{v}_i}}\right)^2}{m-1}.$$

Next, we show that $\eta(\ddot{\boldsymbol{p}}^m, r) < \eta(\ddot{\boldsymbol{p}}^{m+1}, r)$ for $m \in \{2, \ldots, n-1\}$, which is equivalent to

$$\frac{\sum_{i=1}^m \frac{1}{\sqrt{\hat{v}_i}}}{\sum_{i=1}^m \frac{1}{\sqrt{\hat{v}_i}} + \frac{1}{\sqrt{\hat{v}_{m+1}}}} < \frac{\sqrt{m-1}}{\sqrt{m}}.$$

Note that $\sum_{i=1}^m \frac{1}{\sqrt{\hat{v}_i}} \leq \frac{m}{\sqrt{\hat{v}_{m+1}}}$ due to the fact that $\hat{v}_1 \geq \ldots \geq \hat{v}_n$. Therefore, we have that $\frac{\sum_{i=1}^m \frac{1}{\sqrt{\hat{v}_i}}}{\sum_{i=1}^m \frac{1}{\sqrt{\hat{v}_i}} + \frac{1}{\sqrt{\hat{v}_{m+1}}}} \leq \frac{m}{m+1}$ and hence it remains to show that $\frac{m}{m+1} < \frac{\sqrt{m-1}}{\sqrt{m}}$, which can be easily shown to hold after some algebra. Therefore, only two contestants remain active in the optimal contest if $r = 1$.

Case II: $r < 1$**.** It is useful to state an intermediary result.

Lemma 7 *Suppose* $0 < \omega < \frac{2}{3}$, $\xi \geq 1$ *and* $p \in (0, \omega)$, *then*

$$\frac{(p)^{1-r}}{(1-p)^r} + \xi \times \frac{(\omega - p)^{1-r}}{(1-\omega+p)^r} > \frac{\omega^{1-r}}{(1-\omega)^r}.$$

Proof. Define $\mathcal{G}(\omega, p)$ as

$$\mathcal{G}(\omega, p) := \frac{(p)^{1-r}}{(1-p)^r} + \xi \times \frac{(\omega - p)^{1-r}}{(1-\omega+p)^r} - \frac{\omega^{1-r}}{(1-\omega)^r}.$$

We want to show $\mathcal{G}(\omega, p) > 0$ for $\omega > p$. Fixing p, let us view $\mathcal{G}(\cdot)$ as a function of ω. Clearly, we have that $\mathcal{G}(p, p) = 0$. Moreover, we have that

$$\mathcal{G}\left(\frac{2}{3}, p\right) = \frac{(p)^{1-r}}{(1-p)^r} + \xi \times \frac{\left(\frac{2}{3} - p\right)^{1-r}}{\left(\frac{1}{3}+p\right)^r} - \frac{\frac{2}{3}^{1-r}}{\frac{1}{3}^r} \geq \frac{(p)^{1-r}}{(1-p)^r} + \frac{\left(\frac{2}{3} - p\right)^{1-r}}{\left(\frac{1}{3}+p\right)^r} - \frac{\frac{2}{3}^{1-r}}{\frac{1}{3}^r},$$

where the inequality follows from the postulated $\xi \geq 1$. It can be verified that $\frac{(p)^{1-r}}{(1-p)^r} + \frac{\left(\frac{2}{3}-p\right)^{1-r}}{\left(\frac{1}{3}+p\right)^r} - \frac{\frac{2}{3}^{1-r}}{\frac{1}{3}^r} \geq 0$ for $(p, r) \in [0, \frac{2}{3}] \times (0, 1)$. Therefore, to prove the lemma, it suffices to show that $\mathcal{G}(\omega, p)$ is single-peaked or increasing in ω.

Carrying out the algebra, we have that

$$\frac{\partial \mathcal{G}(\omega, p)}{\partial \omega} = \xi \times \frac{(1-\omega+p) - (1-2\omega+2p)r}{(\omega-p)^r (1-\omega+p)^{1+r}} - \frac{(1-\omega) - (1-2\omega)r}{\omega^r (1-\omega)^{1+r}}.$$

It can be verified that $\frac{\partial \mathcal{G}(\omega,p)}{\partial \omega} > 0$ is equivalent to

$$\begin{aligned}\mathcal{Z}(\omega, p, r) := &\log(\xi) + \log\big((1-\omega+p) - (1-2\omega+2p)r\big) - \log\big((1-\omega) - (1-2\omega)r\big) \\ &- r\log\left(\frac{\omega - p}{\omega}\right) - (1+r)\log\left(\frac{1-\omega+p}{1-\omega}\right) > 0.\end{aligned}$$

Note that $\mathcal{Z}(p, p, r) = \infty$. To prove that $\mathcal{G}(\omega, p)$ is single-peaked or increasing in ω, it suffices to show that $\mathcal{Z}(\omega, p, r)$ is strictly decreasing in ω, that is,

$$\begin{aligned}\frac{\partial \mathcal{Z}(\omega, p, r)}{\partial \omega} = &\frac{2r-1}{(1-\omega+p) - (1-2\omega+2p)r} - \frac{2r-1}{(1-\omega) - (1-2\omega)r} + r\left(\frac{1}{\omega} - \frac{1}{\omega - p}\right) \\ &+ (1+r)\left(\frac{1}{1-\omega+p} - \frac{1}{1-\omega}\right) < 0, \text{ for } (\omega, p, r) \in (0, \frac{2}{3}) \times (0, \omega) \times (0, 1).\end{aligned}$$

Carrying out the algebra, $\frac{\partial \mathcal{Z}(\omega,p,r)}{\partial \omega} < 0$ is equivalent to

$$\mathcal{W}(r) := \frac{(2r-1)^2}{\big[(1-\omega+p)-(1-2\omega+2p)r\big]\times\big[(1-\omega)-(1-2\omega)r\big]} - \frac{r}{(\omega-p)\omega} - \frac{1+r}{(1-\omega+p)(1-\omega)} < 0.$$

For $r \leq (0,\frac{1}{2}]$, $\mathcal{W}(r)$ can be bounded above by

$$\begin{aligned}
\mathcal{W}(r) &= \frac{(2r-1)^2}{\big[(1-\omega+p)-(1-2\omega+2p)r\big]\times\big[(1-\omega)-(1-2\omega)r\big]} - \frac{r}{(\omega-p)\omega} - \frac{1+r}{(1-\omega+p)(1-\omega)} \\
&< \frac{(2r-1)^2}{\big[(1-\omega+p)-(1-2\omega+2p)r\big]\times\big[(1-\omega)-(1-2\omega)r\big]} - \frac{1}{(1-\omega+p)(1-\omega)} \\
&\leq \frac{(1-2r)^2}{(1-\omega+p)(1-\omega)(1-r)^2} - \frac{1}{(1-\omega+p)(1-\omega)} \\
&= \frac{1}{(1-\omega+p)(1-\omega)} \times \left[\frac{(1-2r)^2}{(1-r)^2} - 1\right] < 0,
\end{aligned}$$

where the first inequality follows from $r > 0$; the second inequality follows from $(1-\omega)-(1-2\omega)r = (1-\omega)(1-r)+\omega r \geq (1-\omega)(1-r)$ and $(1-\omega+p)-(1-2\omega+2p)r = (1-\omega+p)(1-r)+(\omega-p)r \geq (1-\omega+p)(1-r)$; and the last inequality follows from $\frac{1-2r}{1-r} < 1$.

Similarly, for $r \in (\frac{1}{2}, 1)$, we have that

$$\begin{aligned}
\mathcal{W}(r) &= \frac{(2r-1)^2}{\big[(1-\omega+p)-(1-2\omega+2p)r\big]\times\big[(1-\omega)-(1-2\omega)r\big]} - \frac{r}{(\omega-p)\omega} - \frac{1+r}{(1-\omega+p)(1-\omega)} \\
&\leq \frac{(2r-1)^2}{(\omega-p)\omega r^2} - \frac{r}{(\omega-p)\omega} - \frac{1+r}{(1-\omega+p)(1-\omega)} \\
&= \frac{1}{(\omega-p)\omega} \times \frac{1-r}{r^2} \times \big(r^2-3r+1\big) - \frac{1+r}{(1-\omega+p)(1-\omega)} < 0,
\end{aligned}$$

where the first inequality follows from $(1-\omega)-(1-2\omega)r = (1-\omega)(1-r)+\omega r \geq \omega r$ and $(1-\omega+p)-(1-2\omega+2p)r = (1-\omega+p)(1-r)+(\omega-p)r \geq (\omega-p)r$; and the second inequality follows from $r^2-3r+1<0$ for $r \in [\frac{1}{2}, 1)$. This completes the proof. ■

By Lemma 7, if $p_i^{***} > 0$ and $p_j^{***} > 0$, then we must have that $p_i^{***} + p_j^{***} \geq \frac{2}{3}$. Applying the same argument in the proof of Theorem 7, we can show that there are at most three active players in the optimum. Moreover, in the case that three contestants remain active, we must have that $p_1^{***} = p_2^{***} = p_3^{***} = \frac{1}{3}$, which can be easily proved to be suboptimal. Therefore, only the two strongest players would remain active in the optimum.

Next, we characterize the optimal winning probabilities $p^{***} \equiv (p_1^{***}, \dots, p_n^{***})$. Because $p_i^{***} = 0$ for $i \in \{3, \dots, n\}$, we must have $p_2^{***} = 1 - p_1^{***}$. Therefore, the contest designer's optimization problem can be simplified as the following:

$$\min_{p_1 \in (0,1)} \frac{(p_1)^{1-r}}{(1-p_1)^r} + \chi \frac{(1-p_1)^{1-r}}{(p_1)^r},$$

where $\chi := \left(\frac{v_1}{v_2}\right)^r \geq 1$. The first-order condition with respect to p_1 yields

$$\frac{p_1}{1-p_1} - \chi \frac{1-p_1}{p_1} = \frac{(\chi-1)(1-r)}{r} =: \mathcal{M},$$

from which p_1^{***} can be solved as the following:

$$p_1^{***} = \frac{\sqrt{\mathcal{M}^2 + 4\chi} + \mathcal{M}}{\sqrt{\mathcal{M}^2 + 4\chi} + \mathcal{M} + 2}.$$

This completes the proof. ■

Corollary 4 ***("National Champion" vs. Handicapping)*** *Suppose that $v_1 > v_2 > 0$. Then $\alpha_1 \gtreqless \alpha_2$ if and only if $r \lesseqgtr \frac{1}{2}$.*

Proof. It is straightforward to verify that $\alpha_1 > \alpha_2$ if and only if $p_1^{***} > \frac{\chi}{\chi+1}$, which can be simplified as

$$\mathcal{M} > \chi - 1 \Leftrightarrow \frac{(\chi-1)(1-r)}{r} > \chi - 1 \Leftrightarrow r < \frac{1}{2}.$$

This completes the proof. ■