Absolute and Relative Ambiguity Aversion: A Preferential Approach*

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Abstract

We study from a preferential viewpoint absolute and relative attitudes toward ambiguity determined by wealth effects. We provide different characterizations of these attitudes for a large class of preferences: monotone and continuous preferences which satisfy risk independence. We specify our results for different subclasses of preferences. Our results and characterizations provide alternative ways to experimentally test the validity of some of the models of choice under uncertainty.

1 Introduction

Beginning with the seminal work of David Schmeidler, several choice models have been proposed in the past thirty years in the large literature on choice under uncertainty that deals with ambiguity, that is, with Ellsberg-type phenomena.[1] At the same time, many papers have investigated the economic consequences of ambiguity. Our purpose in this paper is to study a basic economic problem: How the ambiguity attitudes of a decision maker change as his wealth changes. In other words, our purpose is to study absolute and relative ambiguity attitudes.

To fix ideas and understand our main motivation, one should think of how central is in many fields of Economics the relationship between wealth and agents' attitudes toward risk (for example, portfolio allocation problems and insurance demand). In his seminal work [3, p. 96], Arrow, in discussing measures of absolute and relative risk attitudes, mentions

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[1] See Gilboa and Marinacci [21] for a survey.

that "The behaviour of these measures as wealth changes is of the greatest importance for prediction of economic reactions in the presence of uncertainty".

To the best of our knowledge, no systematic study has been done in exploring a similar relation between *wealth* and *ambiguity* attitudes, despite the large and growing use in applications of models that are nonneutral toward ambiguity. In this context, Arrow's comment would seem to apply all the more. The challenge of our work, compared to the analysis done under risk by Arrow and Pratt, is that in the latter case their study has been restricted to the expected utility model. Conversely, under ambiguity there are by now several alternative models, thus moving the analysis well beyond expected utility. In characterizing how ambiguity attitudes change with wealth, our results provide some guidance in choosing between these models, as the standard theory of absolute risk aversion of Arrow and Pratt provides guidance in the choice of the von Neumann-Morgenstern utility function. For example, our results will show that a researcher who believes that agents are not constant absolute ambiguity averse – be that due to experimental evidence and/or personal introspection as for Arrow's assumption of decreasing absolute risk aversion – can rule out the use of some models: for example, α-maxmin, Choquet expected utility, and variational preferences under risk neutrality. Similarly, for a researcher relying on the smooth ambiguity model, behavioral assumptions on absolute and relative ambiguity attitudes translate into corresponding choices of the model's parameters. For instance, if risk attitudes are assumed to be CRRA and risk averse, as common in Macroeconomics,[2] and relative ambiguity attitudes are assumed to be constant as well (irrespective of the prior μ), then our results yield that ϕ must be either CARA or CRRA, depending on the von Neumann-Morgenstern function being either the logarithm or the power function.

Finally, our work provides alternative and useful methods to falsify models of choice under ambiguity as well as testable implications. For example, on the one hand, under the assumption agents are CARA,[3] falsifying our preferential notion of constant absolute ambiguity attitudes yields that preferences cannot be invariant biseparable preferences (e.g., α-maxmin and Choquet expected utility). On the other hand, in looking at portfolio composition data,[4] observing that the share invested in the uncertain asset is not constant with wealth yields again that preferences cannot be invariant biseparable.

A preferential viewpoint We study absolute and relative attitudes toward ambiguity from a purely preferential viewpoint, starting from a preferential first principle: a prefer-

[2] More formally, consequences are elements of $(0, \infty)$ and the von Neumann-Morgenstern utility function over consumption/money is often set to be $\bar{v}_\gamma(c) = c^\gamma$ if $\gamma \in (0,1)$ and $\bar{v}_\gamma(c) = \log c$ if $\gamma = 0$.

[3] For a portfolio-choice experiment estimating ambiguity aversion in a CARA setup, see e.g., Ahn et al. [1].

[4] For an empirical study of constant relative risk attitudes using portfolio composition data, see Chiappori and Paiella [14].

ence is, say, decreasing absolute ambiguity averse if, at a higher wealth level, it becomes comparatively less averse to ambiguity. This first principle implies that a proper analysis of absolute attitudes toward ambiguity requires that the underlying risk preference on lotteries be constant absolute risk averse, so that absolute risk attitudes do not intrude in wealth effects. In turn, this implies that different classes of preferences characterize absolute attitudes toward ambiguity, depending on the risk attitude (in terms of aversion or love) that the underlying risk preference exhibits. For instance, among uncertainty averse preferences, variational preferences characterize constant absolute ambiguity aversion under risk neutrality, but homothetic preferences characterize it under risk nonneutrality. Therefore, the two quite different properties, both conceptually and mathematically, of constant additivity (e.g., variational preferences) and positive homogeneity (e.g., homothetic preferences) may happen to characterize the preference functionals that are constant absolute ambiguity averse.

Our results thus underscore the importance of keeping track of risk attitudes that may, otherwise, confound the analysis of ambiguity attitudes. They also underscore the importance of keeping track of the unit of account: Absolute and relative attitudes are, indeed, properly modelled via properties of the monetary certainty equivalents (which are in the same unit of account of wealth).

Wealth effects We consider a standard Anscombe-Aumann set up.[5] This choice is motivated by our aim to study how wealth effects change *ambiguity* attitudes, thus we want to control for the effects due to risk attitudes. We denote by $\mathcal{F}$ the set of all Anscombe-Aumann acts $f : S \to \Delta_0(\mathbb{R})$, where S is a state space and $\Delta_0(\mathbb{R})$ is the set of simple monetary lotteries. As usual, preferences over final wealth levels are modelled by a binary relation $\succsim$. Given a wealth level w and an act f, we define by f^w the act whose final monetary outcomes are the outcomes of f shifted by w (see Section 2.1, for a formal definition). Given this, we define preferences at wealth level w by

$$f \succsim^w g \overset{def}{\iff} f^w \succsim g^w.$$

We say that $\succsim$ is decreasing absolute ambiguity averse if at lower wealth levels ambiguity aversion is higher, that is $w' > w$ yields that $\succsim^w$ is more ambiguity averse than $\succsim^{w'}$ – in the sense of Ghirardato and Marinacci [19].[6] This definition is an adaptation to the ambiguity setting of the classic definition of decreasing absolute *risk* aversion. In a similar fashion, we also define the notions of increasing and constant absolute ambiguity aversion (see Definition 3).

In the paper, we characterize absolute ambiguity attitudes for the class of rational preferences. This class of preferences is large and contains several models of choice which are

[5] The relevant decision theoretic and mathematical notions are introduced in Section 2 and Appendix A.

[6] See Epstein [15] for a different comparative notion of ambiguity attitudes.

common in the literature (e.g., maxmin, α-maxmin, smooth ambiguity, and variational preferences). Rational preferences are known to admit a representation of the form $V : \mathcal{F} \to \mathbb{R}$ such that

$$V(f) = I(u(f)) \qquad \forall f \in \mathcal{F}, \tag{1}$$

where u is a von Neumann-Morgenstern expected utility functional over $\Delta_0(\mathbb{R})$ and I is a normalized and monotone functional that maps utility profiles $s \mapsto u(f(s))$ into the real line. This decomposition of the utility function V dates back to Schmeidler [31].[7] From a behavioral point of view, this decomposition is particularly useful since the pair (u, I), other than representing $\succsim$ as in (1), characterizes the attitudes of the decision maker toward risk and ambiguity: Namely, u characterizes the risk attitudes of the decision maker, while I describes the ambiguity attitudes. This specific feature of this decomposition has been emphasized by Ghirardato and Marinacci [18] and exploited several times in the literature.[8] Also in this work the two functions u and I will play a key role.

Classifiable preferences As in the risk case, it is not hard to show that absolute attitudes do not provide an exhaustive class of categories with which we can classify rational preferences. In other words, there exist rational preferences that are neither decreasing, nor increasing, nor constant absolute ambiguity averse. When a rational preference relation $\succsim$ exhibits one of these three absolute ambiguity attitudes, we will say that $\succsim$ is *classifiable*. Our first result (Proposition 3) states that if $\succsim$ is a classifiable rational preference, then $\succsim$ must be constant absolute risk averse (henceforth, CARA). Conceptually, this is important because, in this way, absolute risk attitudes do not intrude in wealth effects and all the differences in terms of attitudes toward uncertainty can be then rightfully attributed to attitudes toward ambiguity. Below in the Introduction and in Section 4, we further elaborate on the CARA restriction and relax this assumption.

With this in mind, we proceed by characterizing absolute ambiguity attitudes using the decomposition (u, I) (Theorem 2 and Corollary 1). The following table provides an informal summary of our characterization for a classifiable $\succsim$:

	Risk averse	Risk loving	Risk neutral
DAAA	I superhomogeneous	I subhomogeneous	I constant superadditive
IAAA	I subhomogeneous	I superhomogeneous	I constant subadditive
CAAA	I homogeneous	I homogeneous	I constant additive

[7] In [31] it plays a key role in characterizing Choquet expected utility preferences (the functional I is indeed a Choquet integral).

[8] For example, it has been useful in characterizing comparative ambiguity attitudes, as in Ghirardato and Marinacci [19], as well as in exploring the relation between ambiguity attitudes and preference for the timing of resolution of uncertainty, as in Strzalecki [34].

The table should be read as follows: Under the assumption of classifiability, the rows specify the absolute ambiguity attitudes while the columns specify the risk attitudes, be they averse, loving, or neutral;[9] each cell then provides a full characterization in terms of the functional I. For example, consider a preference relation which is decreasing absolute ambiguity averse (DAAA) and risk averse. By Theorem 2, I is superhomogeneous. On the other hand, if I is assumed to be superhomogeneous, the table shows that there are only two possibilities for a classifiable preference: either $\succsim$ is risk averse and DAAA or $\succsim$ is risk loving and IAAA.

The table also shows that (Corollary 3) invariant biseparable preferences – so in particular α-maxmin and Choquet expected utility preferences – are classifiable if and only if they are constant absolute ambiguity averse (CAAA). The reason is simple: For this class of preferences, the functional I is both positively homogeneous and constant additive.

The dichotomic properties of the functional I, which characterize absolute attitudes toward ambiguity in the risk neutral and nonneutral cases and are most evident for CAAA preferences, are the outcome of a unit of account problem. In fact, though wealth effects are in monetary units (as traditional in Economics), for each act f the number $I(u(f))$ is in von Neumann-Morgenstern utils.[10] In contrast, if v denotes the von Neumann-Morgenstern utility function on monetary outcomes of u, then the map $c : \mathcal{F} \to \mathbb{R}$ defined by

$$c(f) = v^{-1}(I(u(f))) \qquad \forall f$$

is a *monetary* certainty equivalent. Clearly, c is expressed in the same unit of account of the wealth w. We show that monetary certainty equivalents emerge as the proper representation for absolute attitudes (Proposition 4); for example, $\succsim$ is DAAA if and only if $\succsim$ is CARA and c is wealth superadditive, that is,

$$c(f^w) \geq c(f) + w \qquad \forall w \geq 0$$

for every act f. To sum up, a consistent use of the unit of account allows for a clear-cut characterization of absolute ambiguity attitudes.

We then proceed to characterize absolute attitudes toward ambiguity by focusing on the subclass of uncertainty averse preferences. For this class, we provide a characterization of absolute attitudes in terms of their dual representation, that is, in terms of properties of their ambiguity aversion index (Theorem 3). For this particular class, we are able to show how constant absolute ambiguity attitudes are characterized by two radically different models: variational preferences, under risk neutrality, and homothetic preferences under risk nonneutrality (Corollaries 4 and 7).

[9] Being classifiable, $\succsim$ must be CARA (Proposition 3). Thus, the von Neumann-Morgenstern utility function over monetary outcomes can be normalized to be either $v(c) = -\frac{1}{\alpha}e^{-\alpha c}$ with $\alpha \neq 0$ or $v(c) = c$.

[10] Since I is normalized, if an act f is such that, for some scalar k, $u(f(s)) = k$ for all $s \in S$, then $I(u(f)) = k$.

In Section 3.5, we also study some portfolio implications of absolute attitudes toward ambiguity. Our portfolio application is the adaptation of Arrow's portfolio exercise to our setting.

General risk attitudes and absolute uncertainty attitudes We conclude the part on absolute attitudes toward ambiguity by allowing for more general attitudes over risk. The initial part of our paper focuses on ambiguity attitudes in a CARA setup. At the same time, two questions come up naturally: 1) Are the mathematical properties found in the previous results useful only in a CARA world or are they portable to more general settings? 2) Do absolute risk and ambiguity attitudes compound? In particular, addressing the first question allows us to discuss the conceptual role played by CARA preferences. We will do so by reminding the reader about the specificities of the Anscombe and Aumann framework and by drawing a parallel with a standard comparative static exercise done in consumer theory.

In an Anscombe and Aumann setting, there are two sources of uncertainty: risk (i.e., the lotteries in $\Delta_0(\mathbb{R})$) and ambiguity (i.e., the events in S). In studying how changes in wealth affect ambiguity attitudes, our analysis rested on the comparative notion of *being more ambiguity averse* of Ghirardato and Marinacci [19]. This notion has the desirable feature of equalizing risk attitudes (cf. Proposition 3). Thus, all the differences between $\succsim^w$ and $\succsim^{w'}$ can be rightfully attributed to attitudes toward ambiguity, since the decision maker over risk is necessarily CARA (cf. Proposition 11).[11]

This is in line with the standard *ceteris paribus* approach adopted in comparative statics exercises in Economics. Indeed, when the change of the variable of interest (wealth, in our case) generates changes through different channels, say two (risk and ambiguity, in our case), typically one of the two channels has to be "shut down" to fully grasp the effects of a change of the variable of interest only due to the active channel. In consumer theory, for example, comparative statics is often done in terms of change of the price of one good keeping all the other variables equal, that is, other prices and income. It is well known that a change in price affects demand via two channels that need to be separated: substitution effect and wealth effect. Hicksian demand is the tool that allows for an analysis that "shuts down" the wealth channel and allows for comparative statics to be carried out only in terms of substitution effects. It is only at a second stage that price changes are studied without separating the two channels. Nevertheless, one should keep recall that this is done only when the two effects move in the same direction. For example, the celebrated Law of Demand holds for normal goods which are exactly the ones for which substitution effects and wealth effects move in the same direction.

[11] At the same time, any analysis of absolute ambiguity attitudes necessarily must have, as important subcase, the one where the decision maker is CARA.

With this in mind, in Section 4 we allow for more general absolute risk attitudes. In fact, a researcher could be interested in considering more general absolute attitudes toward risk, for example decreasing absolute risk aversion (henceforth, also DARA). We identify absolute attitudes toward ambiguity with the functional characterizations found in the previous part of the paper and show that, for a large class of preferences, absolute attitudes toward risk and ambiguity indeed compound (cf. Proposition 12), answering positively to the above question 2). For example, decreasing absolute risk aversion plus decreasing absolute ambiguity aversion yields decreasing absolute uncertainty aversion. This confirms the portability of the characterizations found in the previous part of the paper (answering question 1) and conceptually follows the scheme described above for a comparative statics exercise. Finally, the results of this part of the paper provide an alternative way to test models of choice under ambiguity as we argue right after Corollary 8.

Relative attitudes Finally, in Section 5, we conduct a similar analysis for relative ambiguity aversion. Our analysis rests on the same arguments and intuitions used for the absolute case. At the same time, due to the relevance of relative attitudes in applied work where uncertain returns are studied, we report the main definitions and characterizations (see, e.g., Example 2). For example, a preference is decreasing relative ambiguity averse if, at a higher *proportional* wealth level, it becomes comparatively less averse to ambiguity. Similarly to the absolute case, we obtain that a proper analysis of *relative* attitudes toward ambiguity requires that the underlying risk preference on lotteries be constant relative risk averse (CRRA, a popular assumption in Macroeconomics and Finance), so that *relative* risk attitudes do not intrude in proportional wealth effects. Our analysis of relative attitudes reinforces our main message: It is fundamental to keep track of risk attitudes (i.e., risk aversion/love) in studying ambiguity attitudes, be they absolute or relative. Also for relative attitudes, we perform a portfolio exercise. In a two asset allocation problem, we obtain that constant relative ambiguity attitudes yield that the share of wealth invested in the non risk free asset does not vary with wealth. Thus, the empirical evidence on individuals' portfolio allocations in favor of CRRA preferences might be consistent with both CRRA and constant relative ambiguity attitudes (see Section 5.3).

Empirical evidence on risk attitudes Our first set of results regarding absolute and relative ambiguity attitudes deals with a class of preferences that under risk satisfy, respectively, CARA and CRRA.[12] Apart from the conceptual appeal of this class, one is left to wonder how limited these classes are. They should be of sufficient interest for applications. From a theoretical point of view, CARA preferences as well as CRRA preferences are very

[12] We are thankful to Aurélien Baillon and Peter Wakker for some very helpful discussion on the topic of this section.

standard assumptions in many areas of Economics such as Macroeconomics, Finance, Auction Theory, and Experimental Economics (see, e.g., Blanchard and Fischer [5, pp. 43-44], Chiappori and Paiella [14], Holt and Laury [25],[13] and Wakker [36]). From an experimental point of view, consistent with Arrow's suggestion [3, p. 96], the common thinking and also finding is that preferences under risk are both decreasing absolute risk averse and increasing relative risk averse. At the same time, there have been several studies and exceptions to this claim,[14] some providing evidence that CARA and CRRA preferences might not be too narrow classes (see, e.g., for some evidence in favor of CARA preferences, Holt and Laury [25], Post et. al. [30], and Baillon and Placido [4];[15] for evidence in favor of CRRA preferences, Friend and Blume [16], Szpiro [35], Brunnermeier and Nagel [7], and Chiappori and Paiella [14]).[16] Finally, our results dealing with general absolute risk attitudes dispense with the CARA assumption and can accommodate both increasing as well as decreasing absolute risk attitudes.

Related literature Absolute attitudes toward uncertainty have been previously studied in a few insightful papers. On the one hand, Cherbonnier and Gollier [13] propose and characterize a preferential definition of absolute attitudes toward uncertainty (being the sum of risk and ambiguity) within the α-maxmin and the smooth ambiguity models (see Section 4 for more details as well as Remarks 1 and 2) while Wakker and Tversky [37, Propositions 9.5 and 9.6] characterize constant attitudes over gains within the prospect theory model. The latter paper shows that constant attitudes, be those either absolute or relative, within the prospect theory model, translate into the same properties (i.e., either CARA or CRRA) of the corresponding von Neumann-Morgenstern utility v. This is perfectly in line with our Corollary 3, despite having been derived in a different setting and for a specific model. Instead, for the former paper, the key differences with our work are that Cherbonnier and Gollier focus on the portfolio implications of their characterizations and, since they do not operate in an Anscombe and Aumann setup, they are not able to disentangle risk and ambiguity attitudes, which is essential to our preferential analysis. Moreover, their analysis

[13] In Auction Theory, a standard assumption is risk neutrality, that is, constant absolute and relative risk aversion.

[14] For a comprehensive review of more than 45 papers studying absolute and relative risk attitudes, we refer the reader to the annotated bibliography of Peter Wakker. The reader will find that there are several experimental violations of Arrow's conjecture – for example, either finding increasing absolute risk aversion or decreasing relative risk aversion. Our goal is simply to mention that also from an empirical point of view CARA and CRRA preferences are relevant classes.

[15] In [25], the authors find that CARA is approximately correct for low payoffs. In [30], the authors estimate risk aversion using an expo-power specification for the contestants of the famous "Deal No Deal" game. They find that Dutch participants exhibit a DARA and IRRA behavior, German participants are approximately CARA, while US participants are approximately CRRA. In [4], roughly 60% of the agents is CARA.

[16] These studies use data regarding individual portfolio composition. The only exception is [35] which uses data on property/liability insurance.

is limited to two particular classes of preferences. On the other hand, Grant and Polak [23] start from the following observation: "Constant absolute risk aversion says that if we add or subtract the same constant both to a random variable and to a sure outcome to which it is preferred, then the preference is maintained". They consider an Anscombe and Aumann setting where lotteries are not necessarily monetary. They identify random variables with acts and constants with constant acts (i.e. lotteries). Then, they observe that in such a setting formal standard additions are not allowed,[17] but convex combinations are. Hence, they replace the former with the latter. In this way, constant absolute ambiguity aversion becomes the following property: for any act f in $\mathcal{F}$ and any three lotteries x, y, and z, and any α in $(0,1)$,

$$\alpha f + (1-\alpha)\, x \succsim \alpha z + (1-\alpha)\, x \implies \alpha f + (1-\alpha)\, y \succsim \alpha z + (1-\alpha)\, y. \tag{2}$$

For rational preferences, (2) turns out to be equivalent to the Weak C-Independence Axiom (e.g., variational preferences as in [27] and vector expected utility preferences as in [32]), which in turn is equivalent to the constant additivity of I, irrespective of any property of u and its risk attitudes. From a comparative point of view, their analysis would be equivalent to the following approach. Consider a rational preference with representation as in (1). As in [23], assume that $\operatorname{Im} u = \mathbb{R}$. Define a preference relation $\succcurlyeq$ over utility profiles by

$$u(f) \succcurlyeq u(g) \overset{def}{\iff} f \succsim g.$$

It turns out that the binary relation $\succcurlyeq$ is a well defined monotone preference over simple real-valued random variables. For, since $\operatorname{Im} u = \mathbb{R}$, for each simple real-valued random variable φ there exists an act $f \in \mathcal{F}$ such that $u(f) = \varphi$. This fact and the definition of $\succcurlyeq$ allow for defining a derived preference $\succcurlyeq^k$ over utility profiles by imposing that

$$u(f) \succcurlyeq^k u(g) \overset{def}{\iff} u(f) + k \succcurlyeq u(g) + k.$$

The binary relation $\succcurlyeq^k$ is interpreted as the preference of the decision maker at a utility level $k \in \mathbb{R}$. In other words, in this analysis, adding or subtracting the same constant is done at a utility level. With this in mind, constant absolute ambiguity aversion of Grant and Polak [23] would be equivalent to say that $\succcurlyeq^k$ is as ambiguity averse as $\succcurlyeq^{k'}$ for any two utility levels k and k'. Xue [38] and [39] considers more general attitudes, namely decreasing and increasing absolute attitudes, by suitably weakening (2) and by axiomatizing a constant superadditive version of variational preferences as well as two equivalent representations of uncertainty averse preferences. Independently of Xue, Ghirardato and Siniscalchi [20] studied a similar notion of absolute ambiguity attitudes in a general class of symmetric preferences. Relative to these papers, the key difference with our work is that we directly address the effect of

[17] In other words, in order to define the sum of an act f and a lottery x, we would need to define the sum, state by state, of two lotteries, namely $f(s)$ and x, which is clearly something nonstandard.

baseline monetary shifts. As mentioned, in the latter four papers instead absolute ambiguity attitudes are defined in terms of utility shifts rather than wealth shifts.

A similar approach is also present in Klibanoff, Marinacci, and Mukerji [26] as well as in Strzalecki [34]. As a consequence, our analysis is consistent with their results under risk neutrality: this is the only case when additive wealth shifts coincide with additive utility shifts. In general, standard shifts in wealth considered in Economics do not generate well behaved shifts in utility and, apart from the risk neutral case, our analysis leads to strikingly different results and a richer picture (given the characterizations being dependent on risk aversion/love). For example, homothetic preferences in this literature would be classified as constant *relative* ambiguity averse while in our case they turn out to be constant *absolute* ambiguity averse under risk nonneutrality. Finally, to the best of our knowledge, the only experimental paper testing absolute/relative ambiguity attitudes is the one of Baillon and Placido [4]. They discuss their findings using both definitions: the one based on utility shifts as well as the one based on wealth shifts. They observe that roughly 60% of their subjects are CARA. Within this group, the majority of risk neutral subjects was constant absolute ambiguity averse, followed by decreasing absolute ambiguity averse. Risk averse agents were, instead, mostly increasing absolute ambiguity averse.[18]

2 Preliminaries

2.1 Setup

We consider a generalized version of the Anscombe and Aumann [2] setup with a nonempty set S of *states of the world*, an algebra Σ of subsets of S called *events*, and a nonempty convex set X of *consequences*. We denote by $\mathcal{F}$ the set of all *(simple) acts*: functions $f : S \to X$ that are Σ-measurable and take on finitely many values.

Given any $x \in X$, define $x \in \mathcal{F}$ to be the constant act that takes value x. Thus, with the usual slight abuse of notation, we identify X with the subset of constant acts in $\mathcal{F}$. Using the linear structure of X, we define a mixture operation over $\mathcal{F}$. For each $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, the act $\alpha f + (1-\alpha)g \in \mathcal{F}$ is defined to be such that $(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s) \in X$ for all $s \in S$. Given a binary relation $\succsim$ on $\mathcal{F}$ (a *preference*), for each $f \in \mathcal{F}$ we denote by $x_f \in X$ a certainty equivalent of f, that is, $x_f \sim f$.[19] Given a function $u : X \to \mathbb{R}$, we denote by $\operatorname{Im} u$ the set $u(X)$; in particular, observe that $u \circ f \in B_0(\Sigma)$ when

[18] Given our results, for a risk neutral agent, for example, this would be consistent with either having invariant biseparable or variational preferences. For a risk averse agent, instead, this would rule out risk averse variational preferences.

[19] In a monetary framework when X is either $\Delta_0(\mathbb{R})$ or $\Delta_0(\mathbb{R}_{++})$, note that given f, x_f is a lottery that, received with certainty in each state s, is indifferent to f. Thus, x_f is a *risky* prospect which is independent of the realization on S.

$f \in \mathcal{F}$. The mathematical notions used in the main text, but not defined there, are collected in Appendix A.

In what follows, we will consider affine maps $^\circ : X \to X$, that is, $(\alpha x + (1-\alpha) y)^\circ = \alpha x^\circ + (1-\alpha) y^\circ$ for all $x, y \in X$ and $\alpha \in [0,1]$. These maps can be naturally extended to $\mathcal{F}$ by defining $f \mapsto f^\circ$ where $f^\circ(s) = f(s)^\circ$ for all $s \in S$. We will confine our attention to sets X of monetary simple lotteries and affine maps induced by wealth shifts that are either additive (absolute ambiguity attitudes case and $X = \Delta_0(\mathbb{R})$) or multiplicative (relative ambiguity attitudes case and $X = \Delta_0(\mathbb{R}_{++})$). Since part of the analysis of these two cases is in common, some of the results only use the abstract notion of affine map (e.g., Propositions 1 and 2).

The paper relies on the following comparative notion of Ghirardato and Marinacci [19].

Definition 1 *Given two preferences $\succsim_1$ and $\succsim_2$ on $\mathcal{F}$, we say that $\succsim_1$ is more ambiguity averse than $\succsim_2$ if, for each $f \in \mathcal{F}$ and $x \in X$, $f \succsim_1 x$ implies $f \succsim_2 x$.*

An important example of a convex consequence set X is that of all *simple monetary lotteries*:

$$\Delta_0(\mathbb{R}) = \left\{ x \in [0,1]^{\mathbb{R}} : x(c) \neq 0 \text{ for finitely many } c \in \mathbb{R} \text{ and } \sum_{c \in \mathbb{R}} x(c) = 1 \right\}.$$

For our purposes, the most important bijective affine transformation on $\Delta_0(\mathbb{R})$ is the one induced by a scalar w, interpreted as a wealth level: for each x in $\Delta_0(\mathbb{R})$, x^w is the lottery such that $x^w(c) = x(c-w)$ for all $c \in \mathbb{R}$. We thus interpret the outcome of a lottery, $c \in \mathbb{R}$, as a final wealth level. Thus, given x in $\Delta_0(\mathbb{R})$, if the decision maker has wealth w, we interpret x^w as being the distribution on final wealth levels. In fact, lottery x yields a consequence $d \in \mathbb{R}$ (on top of w) with probability $x(d)$ and the probability of having as final wealth $w+d$, that is $x^w(w+d)$, is equal to $x(d)$. This implies that $x^w(w+d) = x(d)$ for all $d \in \mathbb{R}$ which is equivalent to our definition of x^w.

2.2 Axioms and representations

We will consider the following classes of preferences $\succsim$ on $\mathcal{F}$: rational preferences (Cerreia-Vioglio et al. [8]), uncertainty averse preferences (Cerreia-Vioglio et al. [9]), invariant biseparable preferences (Ghirardato, Maccheroni, and Marinacci [17]), variational preferences (Maccheroni, Marinacci, and Rustichini [27]), and maxmin preferences (Gilboa and Schmeidler [22]). They rely on the following axioms, discussed in the original papers as well as in Gilboa and Marinacci [21].

Axiom A. 1 (Weak Order) *$\succsim$ is nontrivial, complete, and transitive.*

Axiom A. 2 (Monotonicity) *If $f, g \in \mathcal{F}$ and $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.*

Axiom A. 3 (Continuity) *If $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succsim h\}$ and $\{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha)g\}$ are closed.*

Axiom A. 4 (Risk Independence) *If $x, y, z \in X$ and $\alpha \in (0,1)$,*

$$x \sim y \Longrightarrow \alpha x + (1-\alpha) z \sim \alpha y + (1-\alpha) z.$$

Axiom A. 5 (Convexity) *If $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$,*

$$f \sim g \Longrightarrow \alpha f + (1-\alpha) g \succsim f.$$

Axiom A. 6 (Weak C-Independence) *If $f, g \in \mathcal{F}$, $x, y \in X$, and $\alpha \in (0,1)$,*

$$\alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x \Longrightarrow \alpha f + (1-\alpha)y \succsim \alpha g + (1-\alpha)y.$$

Axiom A. 7 (C-Independence) *If $f, g \in \mathcal{F}$, $x \in X$, and $\alpha \in (0,1)$,*

$$f \succsim g \iff \alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x.$$

Axiom A. 8 (Unboundedness) *There exist x and y in X such that $x \succ y$ and for each $\alpha \in (0,1)$ there exists $z \in X$ that satisfies*

$$\text{either } y \succ \alpha z + (1-\alpha)x \text{ or } \alpha z + (1-\alpha)y \succ x.$$

The following omnibus result collects some of the results that the above papers proved for the classes of preferences that they studied.

Theorem 1 (Omnibus) *A preference $\succsim$ on $\mathcal{F}$ satisfies Weak Order, Monotonicity, Continuity, and Risk Independence if and only if there exist a nonconstant and affine function $u : X \to \mathbb{R}$ and a normalized, monotone, and continuous functional $I : B_0(\Sigma, \operatorname{Im} u) \to \mathbb{R}$ such that the criterion $V : \mathcal{F} \to \mathbb{R}$, given by*

$$V(f) = I(u(f)) \qquad \forall f \in \mathcal{F} \tag{3}$$

represents $\succsim$. The function u is cardinally unique and, given u, I is the unique normalized, monotone, and continuous functional satisfying (3). In this case, we say that $\succsim$ is a rational preference. *A rational preference satisfies:*

(i) C-Independence if and only if I is constant linear; in this case, we say that $\succsim$ is an invariant biseparable preference.[20]

[20] Invariant biseparable preferences correspond to the general class of $\alpha(f)$-maxmin preferences of Ghirardato, Maccheroni, and Marinacci [17], which, inter alia, includes the Choquet expected utility preferences of Schmeidler [31].

(ii) Convexity if and only if I is quasiconcave; in this case, we say that $\succsim$ is an uncertainty averse preference.[21]

(iii) Convexity and Weak C-Independence if and only if I is quasiconcave and constant additive; in this case, we say that $\succsim$ is a variational preference*.*

(iv) Convexity and C-Independence if and only if I is quasiconcave and constant linear; in this case, we say that $\succsim$ is a maxmin preference*.*

(v) Unboundedness if and only if $\operatorname{Im} u$ *is unbounded.*

Given u and I as in Theorem 1, we call (u, I) a (*canonical*) *representation* of the rational preference $\succsim$.[22]

We say that $\succsim$ on $\mathcal{F}$ is a *homothetic* (*uncertainty averse*) *preference* if there exists a canonical representation (u, I), with $\operatorname{Im} u$ equal to either $(-\infty, 0)$ or $(0, \infty)$, such that

$$I(\varphi) = \min_{p \in \Delta} \int \varphi c(p)^{-\operatorname{sgn} \varphi} dp = \begin{cases} \min_{p \in \Delta} \frac{\int \varphi dp}{c(p)} & \text{if } \operatorname{Im} u = (0, \infty) \\ \\ \min_{p \in \Delta} c(p) \int \varphi dp & \text{if } \operatorname{Im} u = (-\infty, 0) \end{cases}$$

where $c : \Delta \to [0, 1]$ is normalized, upper semicontinuous, and quasiconcave.[23] Note that I is positively homogeneous. These preferences, proposed by Chateauneuf and Faro [12], are a natural counterpart to variational preferences with positive homogeneity in place of constant additivity. As [12] showed, positive homogeneity is implied by a form of homotheticity/independence with respect to a worst consequence, when such a consequence exists (something that in this paper we do not allow for; this is why these preferences are not included in the omnibus theorem).

3 Results

3.1 Induced preferences

A preference $\succsim$ on $\mathcal{F}$ induces, through an affine and bijective transformation $^\circ$ on X, a preference $\succsim^\circ$ on $\mathcal{F}$ given by

$$f \succsim^\circ g \Longleftrightarrow f^\circ \succsim g^\circ.$$

The induced preference inherits some of the properties of the original preference.

[21] Uncertainty averse preferences, within the rational preferences class, are distinguished by further satisfying the Convexity axiom. A slightly stronger version of this axiom was termed Uncertainty Aversion by Schmeidler [31, p. 582], since it captures a preference for diversification/hedging. According to our terminology, these preferences should be called ambiguity averse preferences, yet we opted to use their original name as in [9].

[22] In Appendix B, we discuss more in detail the uniqueness features of canonical representations.

[23] The function c is normalized if and only if $\max_{p \in \Delta} c(p) = 1$. Observe also that since $\operatorname{Im} u$ is equal to either $(-\infty, 0)$ or $(0, \infty)$, then $-\operatorname{sgn} \varphi = 1$ or $-\operatorname{sgn} \varphi = -1$, yielding that $c(p)$ can be brought outside the integral.

Proposition 1 *Let $\succsim$ be a preference on $\mathcal{F}$ and $^\circ : X \to X$ an affine bijection. Then:*

(i) If $\succsim$ is a rational preference, so is $\succsim^\circ$.

(ii) If $\succsim$ is an uncertainty averse preference, so is $\succsim^\circ$.

Next, we compare the ambiguity aversion of different induced preferences.

Proposition 2 *Let $\succsim$ be a rational preference on $\mathcal{F}$ and $^\circ$ and $^\#$ two affine and bijective transformations on X. If $\succsim^\circ$ is more ambiguity averse than $\succsim^\#$, then u° is a positive affine transformation of $u^\#$.*[24]

In the rest of the paper (with the exception of Sections 4 and 5) we specialize the set of consequences X to be made of monetary lotteries, that is $X = \Delta_0(\mathbb{R})$, and the maps $^\circ$ and $^\#$ to be ^{w} and $^{w'}$. Moreover, note that an affine utility function $u : \Delta_0(\mathbb{R}) \to \mathbb{R}$ takes the form $u(x) = \sum_{c\in\mathbb{R}} v(c)\, x(c)$, where $v : \mathbb{R} \to \mathbb{R}$.

Throughout the paper we make the following assumption.

Assumption The function v is strictly increasing and continuous.

In this monetary setup, we have the following classic notion.

Definition 2 *A preference $\succsim$ on $\mathcal{F}$ is* constant absolute risk averse *(CARA) if, for any two levels w and w' of wealth, the induced preferences $\succsim^w$ and $\succsim^{w'}$ agree on $\Delta_0(\mathbb{R})$.*

This behavioral definition amounts to say that preferences over lotteries are unaffected by the level of wealth w. A routine argument shows that, if $\succsim$ (on lotteries) is represented by an affine utility function $u : \Delta_0(\mathbb{R}) \to \mathbb{R}$, then $\succsim$ is CARA if and only if there exist $\alpha \in \mathbb{R}$, $a > 0$, and $b \in \mathbb{R}$ such that

$$v(c) = v_\alpha(c) = \begin{cases} -a\frac{1}{\alpha}e^{-\alpha c} + b & \text{if } \alpha \neq 0 \\ ac + b & \text{if } \alpha = 0 \end{cases}, \tag{4}$$

that is, if v_α is either exponential or affine. In the former case, $\succsim$ is a CARA preference which is not risk neutral; in particular, it is (strictly) risk averse if $\alpha > 0$ and (strictly) risk loving if $\alpha < 0$.[25] Note that

$$\operatorname{Im} u = \begin{cases} (-\infty, b) & \text{if } \alpha > 0 \\ (b, +\infty) & \text{if } \alpha < 0 \\ (-\infty, +\infty) & \text{if } \alpha = 0 \end{cases}$$

and so $b = \sup \operatorname{Im} u$ when $\succsim$ is risk averse and $b = \inf \operatorname{Im} u$ when $\succsim$ is risk loving. Momentarily, this extremum role of b will play a key role in Theorem 2.

[24] Here, u° and $u^\#$ are part of a canonical representation for, respectively, $\succsim^\circ$ and $\succsim^\#$.

[25] In what follows, we omit "strictly" since a CARA preference is either risk neutral ($\alpha = 0$) or strictly risk averse ($\alpha > 0$) or strictly risk loving ($\alpha < 0$).

3.2 Rational preferences

Absolute ambiguity attitudes describe how the decision maker's preferences over uncertain monetary alternatives vary as his wealth changes. This motivates the following behavioral definition, which adapts to our setting a standard notion for risk domains. We then proceed to characterize it for rational and for uncertainty averse preferences.

Definition 3 *A preference $\succsim$ on $\mathcal{F}$ is decreasing (increasing, constant) absolute ambiguity averse if, for any two levels w and w' of wealth, $w' > w$ implies that $\succsim^w$ is more (less, equally) ambiguity averse than $\succsim^{w'}$.*[26]

As this classification is not exhaustive, we say that a preference is (absolutely) *classifiable* in terms of absolute ambiguity aversion if it can be classified according to this definition, that is, if it is either decreasing or increasing or constant absolute ambiguity averse. The next result shows that being CARA is a necessary condition for a preference in order to be classifiable: in fact, in this way absolute risk attitudes do not intrude in wealth effects.

Proposition 3 *A rational preference $\succsim$ is classifiable only if it is CARA.*

We first characterize absolute ambiguity attitudes for rational preferences.

Theorem 2 *Let $\succsim$ be a rational preference on $\mathcal{F}$ with representation (u, I). The following statements are equivalent:*

(i) $\succsim$ is decreasing absolute ambiguity averse;

(ii) $\succsim$ is CARA and I is:

(a) concave (convex) at b provided $\succsim$ is risk averse (loving);

(b) constant superadditive provided $\succsim$ is risk neutral.

(iii) $\succsim$ is classifiable and I satisfies (a) or (b).

When $v_\alpha(c) = -\frac{1}{\alpha}e^{-\alpha c}$, and so $a = 1$ and $b = 0$, in point (a) concavity (convexity) at b reduces to positive superhomogeneity (subhomogeneity).[27]

Dual versions of this theorem are easily seen to hold for increasing and constant absolute ambiguity aversion (for this latter case see Corollary 1). In particular, by keeping the same premises, Theorem 2 takes a similar form with (i), (ii), and (iii) replaced by:

[26] Clearly, $\succsim^w$ is less ambiguity averse than $\succsim^{w'}$ if and only if $\succsim^{w'}$ is more ambiguity averse than $\succsim^w$. Similarly, equally ambiguity averse means that $\succsim^w$ is, at the same time, more and less ambiguity averse than $\succsim^{w'}$.

[27] See also Appendix A for the notions of concavity/convexity at b.

(i)' $\succsim$ is increasing absolute ambiguity averse;

(ii)' $\succsim$ is CARA and I is:

(a) convex (concave) at b provided $\succsim$ is risk averse (loving);

(b) constant subadditive provided $\succsim$ is risk neutral.

(iii)' $\succsim$ is classifiable and I satisfies (a) or (b).

The next result characterizes constant absolute ambiguity aversion for classifiable rational preferences. At the same time, the result still holds if instead of requiring $\succsim$ being classifiable we only require $\succsim$ to be CARA.[28]

Corollary 1 *Let $\succsim$ be a classifiable rational preference on $\mathcal{F}$ with representation (u, I). Then:*

(i) If $\succsim$ is risk neutral, it is constant absolute ambiguity averse if and only if I is constant additive.[29]

(ii) If $\succsim$ is not risk neutral, it is constant absolute ambiguity averse if and only if I is affine at b.[30]

When $v_\alpha(c) = -\frac{1}{\alpha}e^{-\alpha c}$, and so $a = 1$ and $b = 0$, in point (ii) the affinity at b reduces to positive homogeneity, that is, $I(\lambda\varphi) = \lambda I(\varphi)$ for all $\lambda > 0$. Risk neutrality and risk aversion of $\succsim$ may thus translate constant absolute ambiguity aversion in, respectively, constant additivity and positive homogeneity of I which are two mathematically and decision theoretically distinct properties.

Indeed, constant additivity and positive homogeneity can be obtained jointly by assuming C-Independence. The assumption of C-Independence could be equivalently rewritten as for each $f, g \in \mathcal{F}$, $x, y \in \Delta_0(\mathbb{R})$, and $\alpha, \beta \in (0, 1]$

$$\alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x \implies \beta f + (1-\beta)y \succsim \beta g + (1-\beta)y.$$

Thus, as argued in [27, p. 1454], C-Independence actually involves two types of independence: independence relative to mixing with constants and independence relative to the weights used in such mixing. The first type (Weak C-Independence) corresponds to I being constant additive while the second type, in the presence of a worst consequence, corresponds to I being positively homogeneous (see Chateauneuf and Faro [12]). Another class of preferences

[28] Recall that, by Proposition 3, classifiable preferences are CARA.

[29] Recall that $\operatorname{Im} u = \mathbb{R}$ in the risk neutral case.

[30] Recall that $b = \sup \operatorname{Im} u$ when $\succsim$ is risk averse and $b = \inf \operatorname{Im} u$ when $\succsim$ is risk loving.

that satisfy Weak C-Independence, but do not necessarily satisfy Convexity, is the class of vector expected utility preferences (see Siniscalchi [32]).[31]

Corollary 2 *A risk neutral rational preference is constant absolute ambiguity averse if and only if it satisfies Weak C-Independence.*

Along with Corollary 1, the next result shows that invariant biseparable preferences are a class of rational preferences that, when classifiable, are constant absolute ambiguity averse regardless of their risk attitudes.

Corollary 3 *Let $\succsim$ be an invariant biseparable preference $\succsim$ on $\mathcal{F}$. The following conditions are equivalent:*

(i) $\succsim$ *is classifiable;*

(ii) $\succsim$ *is constant absolute ambiguity averse;*

(iii) $\succsim$ *is CARA.*

As mentioned in the introduction, Corollary 2 (and Corollary 4 below) show that our analysis is consistent, under risk neutrality, with the approach of Grant and Polak [23].

Since we are dealing with acts yielding monetary lotteries, it is also possible to discuss *monetary* certainty equivalents. Given a canonical representation (u, I), we can define the functional $c : \mathcal{F} \to \mathbb{R}$ by the rule $c(f) = v^{-1}(I(u(f)))$. Note that, given $f \in \mathcal{F}$, the scalar $c(f)$ is the monetary amount that, received with certainty in each state of the world, makes the decision maker indifferent between f and the constant (risk free) act paying $c(f)$. We will say that c is wealth superadditive (resp., subadditive, additive) if and only if for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$c(f^w) \geq c(f) + w \qquad (\text{resp., } \leq, =).$$

Proposition 4 *Let $\succsim$ be a rational preference on $\mathcal{F}$ with representation (u, I). Then:*

(i) $\succsim$ *is decreasing absolute ambiguity averse if and only if c is wealth superadditive and* $\succsim$ *is CARA.*

(ii) $\succsim$ *is increasing absolute ambiguity averse if and only if c is wealth subadditive and* $\succsim$ *is CARA.*

(iii) $\succsim$ *is constant absolute ambiguity averse if and only if c is wealth additive and* $\succsim$ *is CARA.*

[31] Vector expected utility preferences, on top of being rational and satisfying Weak C-Independence, satisfy two other axioms of independence/invariance and an extra continuity axiom. In terms of framework, Σ is required to be countably generated.

3.3 Uncertainty averse preferences

Assume that $\succsim$ is an uncertainty averse preference. By definition, $\succsim$ is also rational. If (u, I) is a (rational) representation of $\succsim$, then there exists a unique minimal linearly continuous $G \in \mathcal{G}(\operatorname{Im} u \times \Delta)$ such that $I(\psi) = \inf_{p \in \Delta} G\left(\int \psi dp, p\right)$ for all $\psi \in B_0(\Sigma, \operatorname{Im} u)$. Uncertainty averse preferences are thus characterized by the pair (u, G). In particular, the function G is an index of ambiguity aversion.[32]

Now we characterize absolute ambiguity attitudes for uncertainty averse preferences in terms of the pair (u, G).

Theorem 3 *Let $\succsim$ be an uncertainty averse preference on $\mathcal{F}$ with representation (u, G). The following statements are equivalent:*

(i) $\succsim$ is decreasing absolute ambiguity averse;

(ii) $\succsim$ is CARA and G is such that:

(a) $G(\lambda t + (1-\lambda) b, p) \geq \lambda G(t, p) + (1-\lambda) b$ $(\leq)$ for all $(t, p) \in \operatorname{Im} u \times \Delta$ and for all $\lambda \in (0, 1)$ provided $\succsim$ is risk averse (loving);

(b) $G(t + k, p) \geq G(t, p) + k$ for all $(t, p) \in \operatorname{Im} u \times \Delta$ and for all $k \geq 0$ provided $\succsim$ is risk neutral.

(iii) $\succsim$ is classifiable and G satisfies (a) or (b).

As mentioned in the Introduction and above, our analysis is consistent, under risk neutrality, with the approach of Grant and Polak [23]. Indeed, the role of constant superadditivity in Theorem 3 shows that a similar consistency holds with the results of Xue [38] and [39] where decreasing absolute ambiguity aversion is modelled in terms of utility shifts.

In Theorem 3 as well, dual versions of this result hold in the increasing and constant absolute ambiguity averse case (with, respectively, opposite inequalities and equalities).

The next corollary shows that the behavioral characterization established in Corollary 2 leads to variational preferences when preferences are uncertainty averse.

Corollary 4 *A risk neutral uncertainty averse preference is constant absolute ambiguity averse if and only if it is a variational preference.*

The next result reports a noteworthy consequence of the previous theorem for uncertainty averse preferences which feature a concave G (or, equivalently, a concave I).

[32] These facts can be found in [9] (see also Appendix A). Because of the minimality of G, we have $G(t, p) = \sup_{f \in \mathcal{F}} \left\{u(x_f) : \int u(f)\, dp \leq t\right\}$ for all $(t, p) \in \operatorname{Im} u \times \Delta$. The function G is unique given u.

Corollary 5 *Let $\succsim$ be an uncertainty averse preference which is CARA and risk averse. If G is concave, then $\succsim$ is decreasing absolute ambiguity averse.*

This corollary can be sharpened for the class of variational preferences that are not maxmin, and so in particular are not invariant biseparable. This class features a concave G.

Corollary 6 *A variational preference, which is not maxmin and not risk neutral, satisfies:*

(i) decreasing absolute ambiguity aversion if and only if it is CARA and risk averse;

(ii) increasing absolute ambiguity aversion if and only if it is CARA and risk loving.

In order to characterize constant absolute ambiguity attitudes when the preference is not risk neutral, we need to consider homothetic preferences.

Corollary 7 *A risk nonneutral uncertainty averse preference is constant absolute ambiguity averse if and only if it is CARA and homothetic.*

To sum up, depending on risk attitudes, either homothetic or variational preferences characterize constant absolute ambiguity attitudes for uncertainty averse preferences.

3.4 Smooth ambiguity preferences

Let $\phi : \operatorname{Im} u \to \mathbb{R}$ be a strictly increasing and continuous function, and μ a Borel probability measure over Δ. The preferences represented by a pair (u, I), where

$$I(\varphi) = \phi^{-1}\left(\int \phi\left(\int \varphi dp\right) d\mu\right) \tag{5}$$

are called *smooth ambiguity preferences* (Klibanoff, Marinacci and, Mukerji [26]). They are uncertainty averse when ϕ is concave.

Proposition 5 *Let $\succsim$ be a CARA smooth ambiguity preference and $\phi(t) = -e^{-\gamma t}$ with $\gamma > 0$. Then,*

(i) If $\succsim$ is risk neutral, then it is constant absolute ambiguity averse.

(ii) If $\succsim$ is risk averse, then it is decreasing absolute ambiguity averse.

In our setup an exponential ϕ thus yields constant absolute ambiguity aversion, as argued in [26], as long as $\succsim$ is risk neutral. In the next result, using Theorem 2, we provide a full characterization of decreasing absolute ambiguity aversion within the smooth ambiguity model. Before doing so, we need to introduce some additional notions and terminology.

Given $\phi : \mathbb{R} \to \mathbb{R}$ and $w \in \mathbb{R}$, we define $\phi_w : \mathbb{R} \to \mathbb{R}$ to be such that $\phi_w(t) = \phi(t + w)$ for all $t \in \mathbb{R}$. Similarly, given $\phi : (-\infty, 0) \to \mathbb{R}$ (resp., $\phi : (0, \infty) \to \mathbb{R}$) and $\nu > 0$, we define $\phi_\nu(t) = \phi(\nu t)$ for all $t < 0$ (resp., $t > 0$).

Definition 4 *Let $\phi : \operatorname{Im} u \to \mathbb{R}$ be strictly increasing and continuous.*

(i) If $\operatorname{Im} u = \mathbb{R}$, we say that ϕ is DARA if for each $w', w \in \mathbb{R}$, with $w' > w$, there exists a strictly increasing and concave $f : \operatorname{Im} \phi \to \operatorname{Im} \phi$ such that $\phi_w = f \circ \phi_{w'}$.

(ii) If $\operatorname{Im} u = (-\infty, 0)$, we say that ϕ is IRRA if for each $\nu, \eta > 0$, with $\nu > \eta$, there exists a strictly increasing and concave $f : \operatorname{Im} \phi \to \operatorname{Im} \phi$ such that $\phi_\nu = f \circ \phi_\eta$.

(iii) If $\operatorname{Im} u = (0, \infty)$, we say that ϕ is DRRA if for each $\nu, \eta > 0$, with $\nu > \eta$, there exists a strictly increasing and concave $f : \operatorname{Im} \phi \to \operatorname{Im} \phi$ such that $\phi_\eta = f \circ \phi_\nu$.

Consider a function $\phi : \operatorname{Im} u \to \mathbb{R}$ which is twice continuously differentiable and such that $\phi' > 0$. Clearly, ϕ is DARA if and only if $-\phi''(t)/\phi'(t)$ is decreasing and similarly, ϕ is DRRA (resp., IRRA) if and only if $-t\phi''(t)/\phi'(t)$ is decreasing (resp., increasing).

Proposition 6 *Let $\succsim$ be a CARA smooth ambiguity preference with $b = 0$ in (4) and assume that Σ is nontrivial. Then,*

(i) If $\succsim$ is risk neutral, $\succsim$ is decreasing absolute ambiguity averse for all μ if and only if ϕ is DARA.

(ii) If $\succsim$ is risk averse, $\succsim$ is decreasing absolute ambiguity averse for all μ if and only if ϕ is IRRA.

(iii) If $\succsim$ is risk loving, $\succsim$ is decreasing absolute ambiguity averse for all μ if and only if ϕ is DRRA.

This result provides some behavioral guidance in the specification of the function ϕ, as the standard theory of absolute risk aversion of Arrow and Pratt provides guidance in the choice of the von Neumann-Morgenstern utility function.

Remark 1 Cherbonnier and Gollier [13, Proposition 2 and Corollary 1], in a different framework, characterize decreasing absolute *uncertainty* aversion (being the sum of risk and ambiguity) for the smooth ambiguity model (see also Section 4). Under the assumption that ϕ is concave, they show that a smooth ambiguity preference is decreasing absolute uncertainty averse if and only if v and $\phi \circ v$ are both DARA. The characterization in Proposition 6, where v is CARA, is consistent with their findings. At the same time, in our case, ϕ does not have to be concave.

Let $c_f(p) \in \mathbb{R}$ be the monetary certainty equivalent of act f under p, that is, $c_f(p) = v^{-1}\left(\int u(f)\, dp\right)$. By setting $w = \phi \circ v : \mathbb{R} \to \mathbb{R}$, the smooth ambiguity representation can be written as

$$V(f) = (v \circ w^{-1}) \left(\int w(c_f(p))\, d\mu \right)$$
$$= (v \circ w^{-1}) \left(\int (w \circ v^{-1}) \left(\int u(f)\, dp \right) d\mu \right).$$

The function w can be interpreted as aversion to epistemic uncertainty.[33] When v is the identity, we have $\phi = w$ and so point (i) of the Proposition 5 can be interpreted in terms of constant attitudes toward such uncertainty. When both $v(c) = -e^{-\alpha c}$ and $w(c) = -e^{-\beta c}$ are risk averse exponentials, with $\beta > \alpha > 0$, then $\phi(t) = -(-t)^{\frac{\beta}{\alpha}}$. The condition $\beta > \alpha$ can be interpreted as higher aversion to epistemic uncertainty than to risk (both being constant absolute averse). The next result shows that in this double exponential case the resulting absolute ambiguity aversion is decreasing.

Proposition 7 *Let $\succsim$ be a CARA smooth ambiguity preference, with $b \leq 0$ in (4), and suppose $\phi(t) = -(-t)^{\gamma}$ for all $t < 0$ with $\gamma > 1$. If $\succsim$ is risk averse, then it is decreasing absolute ambiguity averse.*

3.5 Portfolio problem

In this section we study how absolute ambiguity attitudes affect portfolio choices. To do so, we adapt to our setting the standard portfolio exercise of Arrow which originally was carried in a risk domain, as an illustration of the implications of absolute and relative risk attitudes (see Section 5.3 for the study of relative attitudes). Assume that $f : S \to \Delta_0(\mathbb{R})$ is a purely ambiguous asset, that is, for each state of the world f yields a deterministic consequence, interpreted as a return. Formally, as an Anscombe and Aumann act, we have that $f(s) = \delta_{r_s}$ for all $s \in S$ where $r_s > 0$ is the return in state s.[34] The risk free asset is instead modelled by the act g such that $g(s) = \delta_{r_f}$ for all $s \in S$ where $r_f > 0$ is the return on the risk free asset. The agent faces the following portfolio problem: he has wealth $w > 0$ which he has to allocate between the ambiguous asset and the risk free asset. We denote by β the amount of wealth invested in the ambiguous asset and by $w - \beta$ the amount invested in the risk free one. We assume that the agent cannot short any of the two securities and therefore $\beta \in [0, w]$. Note that the allocation $(\beta, w - \beta)$ generates an Anscombe and Aumann act that in each state of the world yields $\delta_{\beta r_s + (w-\beta) r_f}$ where $\beta r_s + (w - \beta) r_f = w r_f + \beta (r_s - r_f)$ is the final wealth level in state s. We denote the real-valued measurable random variable $s \mapsto r_s$ by r. Similarly, with a small abuse of notation, we denote by r_f the constant random variable that in each state s assumes value r_f.

[33] See Marinacci [28] for a discussion of this version of the smooth ambiguity model. The context should clarify that here w is a function and not a wealth level.

[34] As usual, $x = \delta_c$ is the degenerate lottery at c, that is, $x(d) = 1$ if $d = c$, and $x(d) = 0$ otherwise.

In terms of preferences, we assume that the agent has rational preferences $\succsim$ on $\mathcal{F}$ with canonical representation (u, I) and von Neumann-Morgenstern function $v : \mathbb{R} \to \mathbb{R}$. The portfolio problem amounts to

$$\max I\left(v\left(\beta r + (w - \beta) r_f\right)\right) \text{ subject to } \beta \in [0, w]. \tag{6}$$

In what follows, we assume that this problem always admits a unique solution for all $w > 0$, denoted by $\beta^*(w)$.

Proposition 8 *Let $\succsim$ be a rational preference on $\mathcal{F}$ with representation (u, I). If $\succsim$ is constant absolute ambiguity averse, then*

$$w' > w > 0 \implies \beta^*\left(w'\right) \geq \beta^*(w).$$

If, in addition, $\beta^*(w) \in (0, w)$ with $w > 0$ and $\succsim$ is risk averse and uncertainty averse, then

$$w' > w \implies \beta^*\left(w'\right) = \beta^*(w).$$

Before discussing the result, we comment on its generality. From a theoretical point of view, note that, differently from what happens under risk, the subclass of preferences which exhibit constant absolute attitudes is quite large. Under risk and the expected utility model, constant absolute attitudes coincide to a very specific form of v. In contrast, under ambiguity constant absolute attitudes encompass a family of preferences: α-maxmin, Choquet expected utility, variational under risk neutrality, vector expected utility under risk neutrality, homothetic under risk nonneutrality as well as the risk averse CARA smooth ambiguity preferences $\succsim$ of Proposition 7 when $b = 0$. That said, the relevance of this family to describe the behavior of decision makers is, in a final analysis, an empirical question.

We next discuss the second part of the statement. The result is indeed in line with intuition. If the decision maker is risk and uncertainty averse, then his preferences are convex in β, so the agent values diversification. It follows that if $\beta^*(w)$ is an interior solution, then an intermediate subjective optimal balance has been found between the certainty provided by the risk free asset and the potentially higher, yet uncertain, returns of the ambiguous asset. At the same time, if $w' > w$ and $\succsim$ is constant absolute ambiguity averse, then wealth does not impact the ambiguity attitudes of the decision maker. In other words, the increment in wealth $(w' - w) r_f$ is factored out and, as a consequence, the previously optimal balance between the risk free asset and the ambiguous one is unaffected, that is, $\beta^*(w') = \beta^*(w)$. In the first part of the statement, we only obtain a weak inequality since we impose no restriction on $\beta^*(w)$. This is easy to understand if, for example, we think of the case where $r_s > r_f$ for all $s \in S$. In such a case, the decision maker would always choose $\beta^*(w) = w$, no matter what and the inequality would trivially follow.

Note that the second part of the statement provides a testable implication for constant absolute ambiguity aversion which could be brought to portfolio composition data. Indeed, under the assumption the agent is risk averse and uncertainty averse as well as $\beta^*(w) > 0$, constant absolute ambiguity aversion yields that the share of wealth invested in the non risk free asset decreases with the person's wealth, that is,

$$w' > w > 0 \implies \frac{\beta^*(w)}{w} \geq \frac{\beta^*(w')}{w'}.$$

It eluded us to which extent a general portfolio result holds for decreasing absolute ambiguity aversion. We were able to prove such a result for two important classes of preferences: 1) risk neutral smooth ambiguity preferences and 2) CARA multiplier preferences. This is still somehow surprising in light of the negative result of Yaari [40, p. 322 and Figure 2]. In a nutshell, Yaari, in a mildly different framework, provides an example of convex preferences which are decreasing absolute *uncertainty* averse (see Section 4 and Proposition 11) for which the investment in the uncertain security, at least locally, decreases as wealth increases.

In the first case, by Proposition 6 and since $\succsim$ is risk neutral, by choosing v to be the identity, we know that decreasing absolute ambiguity aversion amounts to impose ϕ being DARA, provided Σ is nontrivial.

Proposition 9 *Let $\succsim$ be a CARA smooth ambiguity preference with ϕ twice continuously differentiable and such that $\phi' > 0$. If $\succsim$ is risk neutral, ϕ is concave and DARA, and $\beta^*(w) \in (0, w)$ with $w > 0$, then*

$$w' > w \implies \beta^*\left(w'\right) \geq \beta^*(w). \tag{7}$$

The second result instead deals with Hansen and Sargent [24] multiplier preferences. Recall that $\succsim$ is a multiplier preference if it admits a rational representation (u, I) where

$$I(\varphi) = -\frac{1}{\theta}\log\left(\int e^{-\theta\varphi} dq\right) = \min_{p\in\Delta}\left\{\int \varphi dp + \frac{1}{\theta} R(p||q)\right\}$$

where $\theta > 0$, q is a countably additive element of Δ, and $R(p||q)$ is the relative entropy of p with respect to q.[35] Multiplier preferences are variational. By Corollary 6, if $\succsim$ is risk averse then $\succsim$ is decreasing absolute ambiguity averse.

Proposition 10 *Let $\succsim$ be a CARA multiplier preference. If $\succsim$ is risk averse and $\beta^*(w) \in (0, w)$ with $w > 0$, then (7) holds.*

Remark 2 Cherbonnier and Gollier [13] carried out a portfolio analysis for decreasing absolute *uncertainty* averse smooth and α-maxmin preferences. It is, however, a different exercise than ours, based also on assumptions on returns. Combined with the differences in the frameworks, this makes their results not directly comparable with ours. In particular, in our setting also for the smooth ambiguity model we have a monotonicity result in wealth (cf. [13, Proposition 5]).

[35] See also Maccheroni, Marinacci, and Rustichini [27, Section 4.2.1] as well as Strzalecki [33].

4 General absolute risk attitudes

In an Anscombe and Aumann setting, under the usual interpretation there are two sources of uncertainty: one which is objective – risk, the lotteries in $\Delta_0(\mathbb{R})$ – and one which is subjective – ambiguity, the events in S. In studying how changes in wealth affect ambiguity attitudes, our analysis rested on the comparative notion of *being more ambiguity averse* contained in Definition 1. This notion has the desirable feature of equalizing risk attitudes (cf. Propositions 2 and 3). Thus, all the differences between $\succsim^w$ and $\succsim^{w'}$ can be rightfully attributed to attitudes toward ambiguity. Moreover, as we argued in the Introduction, this is in line with how comparative statics exercises are carried in Economics when a change in one variable generates effects through different channels.

At the same time, one could be interested in allowing for more general absolute risk attitudes. Clearly, if possible, any such analysis will encompass the CARA case (hence, this paper) as a particular, yet important, case. In general, allowing for more general absolute risk attitudes presents a difficulty. Namely, we are not able anymore to disentangle the effects of wealth on uncertainty attitudes coming from ambiguity and risk. We thus tackle the problem in a different way. We explore how much our characterization is portable to a non-CARA setting. In particular, outside the CARA framework, we identify absolute ambiguity attitudes with the functional properties found in the previous part of the paper and see how much this allows us to talk about absolute uncertainty attitudes.

Indeed, assume that one is interested in the overall *uncertainty attitudes* of the decision maker, that is, loosely speaking in the sum of risk and ambiguity attitudes. Formally, let T be equal to either $\mathbb{R}$ or $\mathbb{R}_{++}$ and $X = \Delta_0(T)$.[36]

Definition 5 *Given two preferences $\succsim_1$ and $\succsim_2$ on $\mathcal{F}$, we say that $\succsim_1$ is more uncertainty averse than $\succsim_2$ if, for each $f \in \mathcal{F}$ and $c \in T$, $f \succsim_1 \delta_c$ implies $f \succsim_2 \delta_c$.*[37]

Consider two rational preferences $\succsim_1$ and $\succsim_2$ with canonical representations (u_1, I_1) and (u_2, I_2). Let $v_1, v_2 : T \to \mathbb{R}$ be the respective von Neumann-Morgenstern utilities on monetary outcomes.[38] It follows that $\succsim_1$ is more uncertainty averse than $\succsim_2$ if and only if $c_1(f) \leq c_2(f)$ for all $f \in \mathcal{F}$ where, for $i \in \{1, 2\}$, $c_i : \mathcal{F} \to T$ is such that $c_i(f) = v_i^{-1}(I_i(u_i(f)))$ for all $f \in \mathcal{F}$. In other words, $\succsim_1$ is more uncertainty averse than $\succsim_2$ if and only if for each act f the monetary certainty equivalent of decision maker 1 is smaller than or equal to the monetary certainty equivalent of decision maker 2.

[36] For the case $\mathbb{R}_{++}$, we have to slightly modify the definition of x^w and $\succsim^w$. See Appendix B.3.1 for details.

[37] Recall that $x = \delta_c$ is the degenerate lottery at c, that is, $x(d) = 1$ if $d = c$, and $x(d) = 0$ otherwise. Recall also that lotteries are identified with constant acts.

[38] Recall that we always assume that v_1 and v_2 are strictly increasing and continuous.

Remark 3 Since lotteries are identified with constant acts, $\succsim_1$ is *more uncertainty averse* than $\succsim_2$ only if $\succsim_1$ is *more risk averse* than $\succsim_2$. In particular, v_2 is a strictly increasing and convex transformation of v_1.

By definition, if $\succsim_1$ is more *ambiguity* averse than $\succsim_2$, then $\succsim_1$ is more *uncertainty* averse than $\succsim_2$.[39] At the same time, there is something more to this mathematically trivial implication. Indeed, the economic reason why comparative ambiguity aversion implies comparative uncertainty aversion is that $\succsim_1$ coincides with $\succsim_2$ on $\Delta_0(T)$ so that all the differences come from ambiguity attitudes and those add up to the common behavior of $\succsim_1$ and $\succsim_2$ over risk.

Thus, in our attempt to combine our analysis on ambiguity attitudes with different absolute risk attitudes, we use Definition 5 to study how variations in wealth impact uncertainty attitudes:

Definition 6 *A preference $\succsim$ on $\mathcal{F}$ is decreasing (increasing, constant) absolute uncertainty averse if, for any two levels w and w' of wealth, $w' > w$ implies that $\succsim^w$ is more (less, equally) uncertainty averse than $\succsim^{w'}$.*

Remark 4 The standard notions of decreasing (increasing, constant) absolute *risk* aversion are defined using this definition, only restricted to lotteries. In what follows, we will refer to decreasing and increasing absolute risk aversion also as DARA and IARA.

As mentioned, intuitively, Definition 6 captures the combined effects of changes in wealth that come from two channels: risk and ambiguity. Hence, if we were to shut down one of the two, namely risk, all the effects should come from the other. The next simple result confirms this intuition.

Proposition 11 *Let $\succsim$ be a CARA rational preference on $\mathcal{F}$. The following statements are equivalent:*

(i) $\succsim$ is decreasing (increasing, constant) absolute uncertainty averse;

(ii) $\succsim$ is decreasing (increasing, constant) absolute ambiguity averse.

In words, once wealth's effects on risk are neutralized, the effects on uncertainty attitudes equate the effects on ambiguity attitudes.

The goal of this section is to allow for more general absolute attitudes toward risk and study how changes in wealth affect uncertainty attitudes where the latter are seen as the combination of ambiguity and risk attitudes.

[39] Degenerate lotteries are a subset of $\Delta_0(T)$.

On the one hand, since there are two channels where uncertainty kicks in, characterizations are very difficult and might be model dependent (see, for example, Cherbonnier and Gollier [13, Propositions 1 and 2]). Indeed, loosely speaking, one could think of the following scenario. Consider a decision maker who is:

1. decreasing absolute risk averse;
2. increasing absolute ambiguity averse (still to be defined outside a CARA setting).

A priori, we could still observe a decreasing absolute uncertainty averse behavior, provided absolute risk aversion quantitatively cancels the positive effects on uncertainty coming from the increasing ambiguity attitudes.

On the other hand, since our analysis is qualitative, we are only going to focus on the combined effects of risk and ambiguity when they both share the same nature. For example, we are going to study if decreasing absolute risk aversion and decreasing absolute ambiguity aversion yield indeed decreasing absolute uncertainty aversion.

By Remark 3, a decision maker who is decreasing absolute uncertainty averse must be DARA. Similar observations hold for increasing and constant absolute uncertainty attitudes. Therefore, in studying either of these three notions the attitudes on risk must necessarily match the ones on the overall uncertainty.

The conceptual issue in such an exercise is the exact meaning of decreasing absolute ambiguity aversion when behavior under risk is not confined to be CARA. Outside the CARA realm, we are going to identify decreasing absolute ambiguity aversion, DAAA, with the functional properties of I that characterize such a behavioral property in the CARA setting. By Theorem 2, recall that[40]

$$\text{Under risk aversion: DAAA} = \text{Concavity at } b \text{ of } I$$

and

$$\text{Under risk love: DAAA} = \text{Convexity at } b \text{ of } I.$$

So, in the next results we interpret concavity/convexity of I at b in terms of DAAA.

Proposition 12 *Let $\succsim$ be a rational preference with representation (u, I) that satisfies Weak C-Independence and Unboundedness. Then, $\succsim$ is decreasing absolute uncertainty averse if either of the following two conditions holds:*

(i) $\succsim$ is risk averse, DARA, and I is concave at b;

[40] As in the CARA case, we continue to set $b = \sup \operatorname{Im} u$ if $\operatorname{Im} u = (-\infty, b)$ and $b = \inf \operatorname{Im} u$ if $\operatorname{Im} u = (b, +\infty)$. At the same time, we set $b = 0$ if $\operatorname{Im} u = \mathbb{R}$. Note that if $\succsim$ is not CARA, then $\operatorname{Im} u$ can be the entire real line even if not risk neutral.

(ii) $\succsim$ *is risk loving, DARA, and* I *is convex at* b*.*

The previous proposition confirms that[41]

$$\text{DARA} + \text{DAAA} \implies \text{DAUA}.$$

This result is proved for a large class of preferences, namely the class of rational preferences that satisfy Weak C-Independence and therefore contains, inter alia, the class of invariant biseparable preferences [17], variational preferences [27], and vector expected utility preferences [32].

Under the same premises of Proposition 12, a dual version holds for increasing absolute uncertainty aversion: $\succsim$ is increasing absolute uncertainty averse if either of the following two conditions holds:

(i)' $\succsim$ is risk averse, IARA, and I is convex at b;

(ii)' $\succsim$ is risk loving, IARA, and I is concave at b.

The results that follow are aimed to further confirm the above intuitions. For example, the class of invariant biseparable preferences was identified in Corollary 3 as a large class of constant absolute ambiguity averse preferences (CAAA). Thus, one should expect that for this class, if attitudes over risk are assumed to be DARA, then the overall attitudes should be indeed DAUA, that is,

$$\text{DARA} + \text{CAAA} \implies \text{DAUA}.$$

This is the content of the next result.

Proposition 13 *Let* $\succsim$ *be an invariant biseparable preference on* $\mathcal{F}$*. The following conditions are equivalent:*

(i) $\succsim$ *is decreasing absolute uncertainty averse;*

(ii) $\succsim$ *is DARA.*

Remark 5 Cherbonnier and Gollier [13, Proposition 1] proved that an α-maxmin preference is decreasing absolute uncertainty averse if and only if it is DARA. The above result generalizes their result to the class of invariant biseparable preferences.[42] Compared to Proposition 12, we can dispense with the assumption of Unboundedness and $\succsim$ can be neither risk averse nor risk loving.

[41] Decreasing absolute uncertainty averse is abbreviated to DAUA.

[42] This class is much larger since it contains, inter alia, Choquet expected utility preferences which are not α-maxmin. At the same time, Cherbonnier and Gollier obtain their result in a different framework.

Proposition 14 *Let $\succsim$ be a rational preference with representation (u, I). If $\succsim$ is risk averse, DARA, and such that* $\operatorname{Im} u = (b, \infty)$ *and I is concave at b and constant superadditive, then $\succsim$ is decreasing absolute uncertainty averse.*

The previous result continues to hold if I is assumed to be concave, a much easier property to check.

Corollary 8 *Let $\succsim$ be a rational preference with representation (u, I). If $\succsim$ is risk averse, DARA, and such that* $\operatorname{Im} u = (b, \infty)$ *and I is concave, then $\succsim$ is decreasing absolute uncertainty averse.*

One more time, we obtain that decreasing absolute attitudes on risk and ambiguity compound and yield decreasing absolute attitudes on uncertainty (cf. Corollary 5).

Note that this corollary also provides an alternative way to experimentally test the validity of some models of choice under ambiguity, for example, variational preferences or some specification of the smooth ambiguity model. Indeed, variational preferences feature a concave functional I. For example, if an experimenter notices that the agent, whose choices she is observing, is risk averse and DARA, but not decreasing absolute uncertainty averse, then she can rule out that the agent can be represented by a concave functional I. Thus, in particular, the agent's preferences are neither variational nor smooth where I is as in the example below.[43] Note that a similar observation can be made using other results present in this section. For example, Proposition 13 could be used to test invariant biseparable preferences.[44]

Example 1 Let $v : \mathbb{R}_{++} \to \mathbb{R}$ be such that $v(c) = c^{\gamma}$ for all $c \in \mathbb{R}_{++}$ with $\gamma \in (0, 1)$. Clearly, we have that $\operatorname{Im} u = (b, \infty)$ with $b = 0$. Let $\phi : \operatorname{Im} u \to \mathbb{R}$ be the power function $\phi(t) = t^{\rho}$ with $\rho \in (0, 1)$. Let μ be a Borel probability measure over Δ. The rational preference represented by the pair (u, I), where

$$I(\varphi) = \phi^{-1}\left(\int \phi\left(\int \varphi dp\right) d\mu\right),$$

is a smooth ambiguity preference which satisfies all the hypotheses of Corollary 8. It follows that $\succsim$ is decreasing absolute uncertainty averse. ▲

[43] The experimenter should assume that choices under risk are explained by a von Neumann-Morgenstern whose image is such that $\operatorname{Im} u = (b, \infty)$. Clearly, there are several functional specifications with this property.

[44] In this case, the experimenter does not need to make any assumption or test about the risk attitudes (i.e., aversion/love) of the agent and the shape of $\operatorname{Im} u$.

5 Relative ambiguity aversion

5.1 Relative analysis

In this section we briefly explore relative ambiguity aversion. Due to the relevance of relative attitudes in applied work, we report the main definitions and characterizations.[45] For this reason, we focus on lotteries which yield only strictly positive numbers, interpreted as returns: $X = \Delta_0(\mathbb{R}_{++})$. As before, we consider a group of transformations on X, but this time, it is indexed by $\mathbb{R}_{++}$. In particular, given $\nu > 0$, we denote by ${}^{\nu} : \Delta_0(\mathbb{R}_{++}) \to \Delta_0(\mathbb{R}_{++})$ the affine and onto map such that $x^{\nu}(\nu c) = x(c)$ for all $c \in \mathbb{R}_{++}$ and for all $x \in \Delta_0(\mathbb{R}_{++})$. Given wealth $\nu > 0$, the lottery x^{ν} is interpreted as the distribution of final wealth if ν is invested in x. In this monetary setup, we have the following classic notion.

Definition 7 *A preference $\succsim$ on $\mathcal{F}$ is* constant relative risk averse *(CRRA) if, for any two strictly positive levels ν and η of wealth, the induced preferences $\succsim^{\nu}$ and $\succsim^{\eta}$ agree on $\Delta_0(\mathbb{R}_{++})$.*

This behavioral definition amounts to say that preferences over lotteries yielding returns are unaffected by changes in invested wealth. A routine argument shows that, if $\succsim$ is represented by an affine utility function $u : \Delta_0(\mathbb{R}_{++}) \to \mathbb{R}$,[46] then $\succsim$ is CRRA if and only if there exist $\gamma \in \mathbb{R}$, $a > 0$, and $b \in \mathbb{R}$ such that

$$v_{\gamma}(c) = \begin{cases} a\gamma c^{\gamma} + b & \text{if } \gamma \neq 0 \\ a \log c + b & \text{if } \gamma = 0 \end{cases}, \tag{8}$$

that is, if v_{γ} is either a power or the logarithm. Note that

$$\operatorname{Im} u = \begin{cases} (-\infty, b) & \text{if } \gamma < 0 \\ (b, +\infty) & \text{if } \gamma > 0 \\ (-\infty, +\infty) & \text{if } \gamma = 0 \end{cases}$$

and so $b = \sup \operatorname{Im} u$ when $\gamma < 0$ and $b = \inf \operatorname{Im} u$ when $\gamma > 0$. Again, this extremum role of b will play a key role momentarily (Theorem 4).

5.2 Relative ambiguity attitudes

Relative ambiguity attitudes describe how the decision maker's preferences over uncertain monetary *returns* vary as the wealth invested changes. This motivates the following behavioral definition, which adapts to our setting a standard notion for risk domains. We then proceed to characterize it for rational preferences.

[45] Proofs follow closely the ones carried out for the absolute case and are therefore omitted for brevity.

[46] Even in this section, we maintain the assumption that if $\succsim$ on $\Delta_0(\mathbb{R}_{++})$ is represented by an affine utility function, then its von Neumann-Morgenstern utility function is strictly increasing and continuous.

Definition 8 *A preference $\succsim$ on $\mathcal{F}$ is decreasing (increasing, constant) relative ambiguity averse if, for any two strictly positive levels ν and η of wealth, $\nu > \eta$ implies that $\succsim^{\eta}$ is more (less, equally) ambiguity averse than $\succsim^{\nu}$.*

Since also this classification of preferences is not exhaustive, we say that a preference is *relatively classifiable* (in terms of relative ambiguity aversion) if it can be classified according to this definition, that is, if it is either decreasing or increasing or constant relative ambiguity averse. The next result shows that being CRRA is a necessary condition for a preference to be relatively classifiable: indeed, in this way *relative risk attitudes* do not intrude in wealth's proportionality effects.

Proposition 15 *A rational preference $\succsim$ is relatively classifiable only if it is CRRA.*

We next characterize decreasing relative ambiguity attitudes for rational preferences.

Theorem 4 *Let $\succsim$ be a rational preference on $\mathcal{F}$ with representation (u, I). The following statements are equivalent:*

(i) $\succsim$ is decreasing relative ambiguity averse;

(ii) $\succsim$ is CRRA and I is:

  (a) concave (convex) at b provided $\gamma < 0$ ($\gamma > 0$);

  (b) constant superadditive provided $\gamma = 0$.

(iii) $\succsim$ is relatively classifiable and I satisfies (a) or (b).

Similar characterizations hold for increasing and constant relative ambiguity aversion.[47] We next provide a formal statement of a result mentioned in the Introduction which shows that our results provide behavioral guidance in the choice of the parameters of functional representations.

Proposition 16 *Let $\succsim$ be a CRRA smooth ambiguity preference with $b = 0$ in (8), $\gamma \in [0, 1)$, and assume that Σ is nontrivial. Then,*

(i) If $\gamma = 0$, $\succsim$ is constant relative ambiguity averse for all μ if and only if ϕ is CARA.[48]

[47] If we replace decreasing relative ambiguity aversion with increasing relative ambiguity aversion, then we must invert the role of concavity and convexity at b as well as change constant superadditivity in constant subadditivity. Similarly, if we replace decreasing relative ambiguity aversion with constant relative ambiguity aversion, then concavity and convexity at b (resp., constant superadditivity) will become affinity at b (resp., constant additivity).

[48] That is, $\phi : \mathbb{R} \to \mathbb{R}$ is a positive affine transformation of either $-\frac{1}{\beta}e^{-\beta t}$ where $\beta \neq 0$ or the identity.

(ii) If $\gamma \in (0,1)$, $\succsim$ is constant relative ambiguity averse for all μ if and only if ϕ is CRRA.[49]

Example 2 As common in Macroeconomics, let $v : \mathbb{R}_{++} \to \mathbb{R}$ be either $v(c) = c^{\gamma}$ with $\gamma \in (0,1)$ or $v(c) = \log c$. Clearly, we have that $\operatorname{Im} u = (b, \infty)$ with $b = 0$ in the former case and $\operatorname{Im} u = \mathbb{R}$ in the latter. Assume also that the agent has smooth ambiguity preferences, as common in applications, with $\phi : \operatorname{Im} u \to \mathbb{R}$ and μ a Borel probability measure over Δ. On the one hand, if v is a power function, then choosing ϕ to be $\phi(t) = t^{\rho}$ with $\rho \in (0,1)$ would yield constant relative ambiguity aversion no matter what is μ. On the other hand, things would be extremely different if v were to be chosen to be the logarithm. In this case, to obtain constant *relative ambiguity* aversion one should assume that ϕ is CARA. In the former case, Example 1 yields that the preference is also decreasing *absolute uncertainty* averse. In the latter case, by assuming that ϕ is concave too, Proposition 12 brings to the same conclusion. ▲

Also in this case, it is possible to introduce *monetary* certainty equivalents. Given a canonical representation (u, I), we can again define the functional $c : \mathcal{F} \to \mathbb{R}_{++}$ by the rule $c(f) = v^{-1}(I(u(f)))$. We will say that c is wealth superproportional (resp., subproportional, proportional) if and only if for each $f \in \mathcal{F}$ and for each $\nu \geq 1$

$$c(f^{\nu}) \geq \nu c(f) \qquad \text{(resp., } \leq, = \text{)}.$$

Proposition 17 *Let $\succsim$ be a rational preference on $\mathcal{F}$ with representation (u, I). Then:*

(i) $\succsim$ is decreasing relative ambiguity averse if and only if c is wealth superproportional and $\succsim$ is CRRA.

(ii) $\succsim$ is increasing relative ambiguity averse if and only if c is wealth subproportional and $\succsim$ is CRRA.

(iii) $\succsim$ is constant relative ambiguity averse if and only if c is wealth proportional and $\succsim$ is CRRA.

5.3 Portfolio problem and relative attitudes

We again consider the portfolio problem of Section 3.5. In a nutshell, we consider an agent with rational preferences (u, I) and von Neumann-Morgenstern function $v : \mathbb{R}_{++} \to \mathbb{R}$.[50] The decision maker is choosing an optimal portfolio which can consist of a mixture between a purely ambiguous asset, yielding returns $r_s > 0$ for all $s \in S$, and a risk free asset, yielding

[49] That is, $\phi : (0, \infty) \to \mathbb{R}$ is a positive affine transformation of either ρt^{ρ} where $\rho \neq 0$ or $\log t$.

[50] In Section 3.5, since we were studying absolute attitudes, v was defined over the entire real line. Nevertheless, given our assumptions on returns the set $(0, \infty)$ suffices.

a constant return $r_f > 0$. The agent has wealth $w > 0$ which has to be allocated between these two assets. The number β denotes the absolute amount of wealth invested in the ambiguous asset. The agent cannot short any of the two securities and therefore $\beta \in [0, w]$. Formally, the portfolio problem takes the form:

$$\max I\left(v\left(\beta r + (w - \beta) r_f\right)\right) \text{ subject to } \beta \in [0, w]. \tag{9}$$

Also here, we assume that this problem always admits a unique solution for all $w > 0$, denoted by $\beta^*(w)$.

Proposition 18 *Let $\succsim$ be a rational preference on $\mathcal{F}$ with representation (u, I). If $\succsim$ is constant relative ambiguity averse, then*

$$w' > w > 0 \implies \frac{\beta^*(w')}{w'} = \frac{\beta^*(w)}{w}.$$

In order to understand the previous result, we recall the standard result for constant relative risk attitudes under risk. In that case, if the decision maker is CRRA and expected utility, then the share of his wealth invested in the risky asset does not depend on the wealth level w. Our result is saying that if the risky asset is indeed perceived as ambiguous by the agent, then constant relative ambiguity attitudes would yield the same prediction. In other words, the share of wealth invested in the non risk free asset does not vary with the person's wealth. It is interesting to note that some papers in the literature exactly look at the share invested in the risky asset to test if CRRA preferences are consistent with the empirical evidence (see, e.g., Brunnermeier and Nagel [7] as well as Chiappori and Paiella [14]). Thus, the empirical evidence in favor of CRRA preferences might be indeed consistent with both CRRA and constant relative ambiguity attitudes. Recall that rational preferences, which are also constant relative ambiguity averse, are necessarily CRRA. So, examples of rational preferences that are constant relative ambiguity averse are: α-maxmin, Choquet expected utility, variational if v is the logarithm, vector expected utility if v is the logarithm, and the smooth ambiguity preferences of Example 2.

A Appendix: Mathematics

We denote by $B_0(\Sigma)$ the set of all real-valued Σ-measurable simple functions. If T is an interval of the real line, set $B_0(\Sigma, T) = \{\psi \in B_0(\Sigma) : \psi(s) \in T \text{ for all } s \in S\}$. We endow both $B_0(\Sigma)$ and $B_0(\Sigma, T)$ with the topology induced by the supnorm.

With a small abuse of notation, we denote by k both the real number and the constant function on S that takes value k. Let $\varphi, \psi \in B_0(\Sigma, T)$. A functional $I : B_0(\Sigma, T) \to \mathbb{R}$ is:

(i) *normalized* if $I(k) = k$ for all $k \in T$;

(ii) *monotone* if $\varphi \geq \psi$ implies $I(\varphi) \geq I(\psi)$;

(iii) *quasiconcave* if $I(\lambda\varphi + (1-\lambda)\psi) \geq \min\{I(\varphi), I(\psi)\}$ for all $\lambda \in (0,1)$;

(iv) *positively superhomogeneous* (*subhomogeneous*) if $I(\lambda\varphi) \geq (\leq) \lambda I(\varphi)$ for all $\lambda \in (0,1)$ such that $\lambda\varphi \in B_0(\Sigma, T)$;

(v) *positively homogeneous* if it is both: positively superhomogeneous and subhomogeneous;[51]

(vi) *concave* (*convex*) at $k \in \mathrm{cl}(T)$ if $I(\lambda\varphi + (1-\lambda)k) \geq (\leq) \lambda I(\varphi) + (1-\lambda)k$ for all $\lambda \in (0,1)$;

(vii) *affine* at $k \in \mathrm{cl}(T)$ if it is both concave and convex at k;

(viii) *constant superadditive* (*subadditive*) if $I(\varphi + k) \geq (\leq) I(\varphi) + k$ for all $k \geq 0$ such that $\varphi + k \in B_0(\Sigma, T)$.

(ix) *constant additive* if I is both constant superadditive and subadditive;[52]

(x) *constant linear* if $I(\lambda\varphi + k) = \lambda I(\varphi) + k$ for all $\lambda \in (0,1]$ and $k \in \mathbb{R}$ such that $\lambda\varphi + k \in B_0(\Sigma, T)$. If T is either $(-\infty, 0)$ or $(0, \infty)$ or $\mathbb{R}$, this amounts to impose that I is constant additive and positively homogeneous.

When $k = 0$, concavity (convexity) at k reduces to positive superhomogeneity (subhomogeneity).

As well known, the norm dual space of $B_0(\Sigma)$ can be identified with the set $ba(\Sigma)$ of all bounded finitely additive measures on (S, Σ). The set of probabilities in $ba(\Sigma)$ is denoted by Δ and is a (weak*) compact and convex subset of $ba(\Sigma)$. Elements of Δ are denoted by p or q. We endow Δ and any of its subsets with the weak* topology.

Functions of the form $G : T \times \Delta \to (-\infty, \infty]$, where T is an interval of the real line, play an important role in Section 3.3. We denote by $\mathcal{G}(T \times \Delta)$ the class of these functions such that:

(i) G is quasiconvex on $T \times \Delta$,

(ii) $G(\cdot, p)$ is increasing for all $p \in \Delta$,

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[51] When either $T = (-\infty, 0)$ or $T = (0, \infty)$ or $T = \mathbb{R}$, then I is positively homogeneous if and only if $I(\lambda\varphi) = \lambda I(\varphi)$ for all $\varphi \in B_0(\Sigma, T)$ and for all $\lambda > 0$. Often, in this paper, in talking about positive homogeneity properties of I, we will either say I is (sup/sub)homogeneous, dropping the qualifier positive, or equivalently say it is positive (sup/sub)homogeneous as well as positively (sup/sub)homogeneous.

[52] Note that I is constant additive if and only if $I(\varphi + k) = I(\varphi) + k$ for all $\varphi \in B_0(\Sigma, T)$ and for all $k \in \mathbb{R}$ such that $\varphi + k \in B_0(\Sigma, T)$. In other words, if $I(\varphi + k) = I(\varphi) + k$ holds for positive constants, then it also holds for $k < 0$, provided $\varphi, \varphi + k \in B_0(\Sigma, T)$.

(iii) $\inf_{p\in\Delta} G(t,p) = t$ for all $t \in T$.

A function $G : T\times\Delta \to (-\infty,\infty]$ is *linearly continuous* if the map

$$\psi \mapsto \inf_{p\in\Delta} G\left(\int \psi dp, p\right)$$

from $B_0(\Sigma, T)$ to $[-\infty,\infty]$ is extended-valued continuous. Finally, given a function, say $u : X \to \mathbb{R}$, we will denote its image, that is $u(X)$, by $\operatorname{Im} u$.

B Appendix: Proofs and related material

We begin with a preliminary result that will be used in the appendix.

Lemma 1 *Let $\succsim_1$ and $\succsim_2$ be two rational preferences on $\mathcal{F}$ with representations (u_1, I_1) and (u_2, I_2). The following statements are equivalent:*

(i) $\succsim_1$ is more ambiguity averse than $\succsim_2$;

(ii) There exist $a > 0$ and $b \in \mathbb{R}$ such that $u_1 = au_2 + b$ and $I_1 \leq I_2$ (provided $u_1 = u_2$).

B.1 Generic set of consequences

Proof of Proposition 1. Clearly, $\succsim^\circ$ is well defined. Moreover, we have

$$f \succ^\circ g \iff f \succsim^\circ g \text{ and } g \not\succsim^\circ f \iff f^\circ \succsim g^\circ \text{ and } g^\circ \not\succsim f^\circ \iff f^\circ \succ g^\circ.$$

(i). *Weak Order.* Since $\succsim$ satisfies Weak Order and Monotonicity, it follows that there exist $\bar{x}$ and $\bar{y}$ in X such that $\bar{x} \succ \bar{y}$. Since $^\circ$ is bijective, it follows that there exist $x, y \in X$ such that $\bar{x} = x^\circ$ and $\bar{y} = y^\circ$. By definition of $\succsim^\circ$, we have that

$$\bar{x} \succ \bar{y} \implies x^\circ \succ y^\circ \implies x \succ^\circ y,$$

proving that $\succsim^\circ$ is nontrivial. Consider $f, g \in \mathcal{F}$. Since $f^\circ, g^\circ \in \mathcal{F}$ and $\succsim$ satisfies Weak Order, we have that either $f^\circ \succsim g^\circ$ or $g^\circ \succsim f^\circ$. By definition of $\succsim^\circ$, this implies that either $f \succsim^\circ g$ or $g \succsim^\circ f$ or both, thus proving that $\succsim^\circ$ is complete. Next, consider $f, g, h \in \mathcal{F}$ and assume that $f \succsim^\circ g$ and $g \succsim^\circ h$. By definition of $\succsim^\circ$, we have that $f^\circ \succsim g^\circ$ and $g^\circ \succsim h^\circ$. Since $\succsim$ satisfies Weak Order, we can conclude that $f^\circ \succsim h^\circ$, that is, $f \succsim^\circ h$, proving that $\succsim^\circ$ is transitive. We can conclude that $\succsim^\circ$ satisfies Weak Order.

Monotonicity. Consider $f, g \in \mathcal{F}$ and assume that $f(s) \succsim^\circ g(s)$ for all $s \in S$. By definition of $\succsim^\circ$ and $^\circ$, it follows that $f^\circ(s) = f(s)^\circ \succsim g(s)^\circ = g^\circ(s)$ for all $s \in S$. Since $\succsim$ satisfies Monotonicity, we have that $f^\circ \succsim g^\circ$, that is, $f \succsim^\circ g$.

Continuity. Consider $f, g, h \in \mathcal{F}$ and a sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq [0,1]$ such that $\alpha_n \to \alpha$ and $\alpha_n f + (1-\alpha_n) g \succsim^\circ h$ for all $n \in \mathbb{N}$. By definition of $\succsim^\circ$ and since $^\circ$ is affine, we have $\alpha_n f^\circ + (1-\alpha_n) g^\circ = (\alpha_n f + (1-\alpha_n) g)^\circ \succsim h^\circ$ for all $n \in \mathbb{N}$. Since $\succsim$ satisfies Mixture Continuity, we have that $(\alpha f + (1-\alpha) g)^\circ = \alpha f^\circ + (1-\alpha) g^\circ \succsim h^\circ$. We can conclude that $\alpha f + (1-\alpha) g \succsim^\circ h$. Thus, the set $\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim^\circ h\}$ is closed. A symmetric argument yields the closure of $\{\alpha \in [0,1] : h \succsim^\circ \alpha f + (1-\alpha) g\}$.

Risk Independence. Consider $x, y, z \in X$, $\alpha \in (0,1)$, and assume that $x \sim^\circ y$. It follows that $x^\circ \sim y^\circ$. Since $\succsim$ satisfies Risk Independence and $^\circ$ is affine, we have that

$$(\alpha x + (1-\alpha) z)^\circ = \alpha x^\circ + (1-\alpha) z^\circ \sim \alpha y^\circ + (1-\alpha) z^\circ = (\alpha y + (1-\alpha) z)^\circ,$$

proving that $\alpha x + (1-\alpha) z \sim^\circ \alpha y + (1-\alpha) z$.

(ii). We only need to show that $\succsim^\circ$ also satisfies Convexity.

Convexity. Consider $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$ and assume that $f \sim^\circ g$. It follows that $f^\circ \sim g^\circ$. Since $\succsim$ satisfies Convexity and $^\circ$ is affine, we have that $(\alpha f + (1-\alpha) g)^\circ = \alpha f^\circ + (1-\alpha) g^\circ \succsim f^\circ$, that is, $\alpha f + (1-\alpha) g \succsim^\circ f$. ■

Proof of Proposition 2. By Proposition 1, both preferences $\succsim^\circ$ and $\succsim^\#$ are rational preferences. By Theorem 1, both preferences have a canonical representation: (u°, I°) and $(u^\#, I^\#)$. In particular, u° and $u^\#$ are nonconstant and affine. Since $\succsim^\circ$ is more ambiguity averse than $\succsim^\#$, we have that $y \succsim^\circ x$ implies $y \succsim^\# x$. Thus, we conclude that $u^\circ(y) \geq u^\circ(x)$ implies $u^\#(y) \geq u^\#(x)$. By [17, Corollary B.3], the statement follows. ■

The next result will be instrumental in proving Theorem 2, Propositions 12 and 14 as well as Corollary 8.

Proposition 19 *Let (u, I) and $(\bar{u}, \bar{I})$ be two canonical rational representations. The two representations (u, I) and $(\bar{u}, \bar{I})$ represent the same rational preference $\succsim$ if and only if there exist $a > 0$ and $b \in \mathbb{R}$ such that*

$$\bar{u} = au + b \text{ and } \bar{I}(\cdot) = aI\left(\frac{\cdot - b}{a}\right) + b.$$

Moreover,

(i) I is concave if and only if $\bar{I}$ is concave.

(ii) I is concave (convex, affine) at c if and only if $\bar{I}$ is concave (convex, affine) at $ac + b$.

(iii) I is constant superadditive (subadditive, additive) if and only if $\bar{I}$ is constant superadditive (subadditive, additive), provided $\operatorname{Im} u$ is unbounded from above.

Proof. The first part of the statement follows from [8, Proposition 1]. Define $f:\mathbb{R}\to\mathbb{R}$ as $f(t)=at+b$ for all $t\in\mathbb{R}$. Define $T:B_0(\Sigma,\operatorname{Im}\bar{u})\to B_0(\Sigma,\operatorname{Im}u)$ as $T(\varphi)=\frac{\varphi-b}{a}$ for all $\varphi\in B_0(\Sigma,\operatorname{Im}\bar{u})$. Note that both functions are bijective and $\bar{I}=f\circ I\circ T$ as well as $I=f^{-1}\circ\bar{I}\circ T^{-1}$.

(i). "Only if". Assume that I is concave. Since f and T are monotone and affine and $\bar{I}=f\circ I\circ T$, it follows that $\bar{I}$ is concave. "If". Note that $I=f^{-1}\circ\bar{I}\circ T^{-1}$. Assume that $\bar{I}$ is concave. Since f^{-1} and T^{-1} are monotone and affine, it follows that I is concave.

(ii). "Only if". Assume that I is concave (convex, affine) at $c\in\operatorname{cl}(\operatorname{Im}u)$. Note that $\bar{c}=ac+b\in\operatorname{cl}(\operatorname{Im}\bar{u})$. It follows that for each $\varphi\in B_0(\Sigma,\operatorname{Im}\bar{u})$ and for each $\lambda\in(0,1)$

$$
\begin{aligned}
\bar{I}(\lambda\varphi+(1-\lambda)\bar{c}) &= aI\left(\frac{\lambda\varphi+(1-\lambda)\bar{c}-b}{a}\right)+b=aI\left(\lambda\frac{\varphi-b}{a}+(1-\lambda)\frac{\bar{c}-b}{a}\right)+b\\
&= aI\left(\lambda\frac{\varphi-b}{a}+(1-\lambda)\frac{ac+b-b}{a}\right)+b\\
&= aI\left(\lambda\frac{\varphi-b}{a}+(1-\lambda)c\right)+b\\
&\geq(\leq,=)\,a\left(\lambda I\left(\frac{\varphi-b}{a}\right)+(1-\lambda)c\right)+b\\
&= \lambda\left(aI\left(\frac{\varphi-b}{a}\right)+b\right)+(1-\lambda)(ac+b)=\lambda\bar{I}(\varphi)+(1-\lambda)\bar{c},
\end{aligned}
$$

proving that $\bar{I}$ is concave (convex, affine) at $\bar{c}$. "If". Assume that $\bar{I}$ is concave (convex, affine) at $\bar{c}=ac+b$. It follows that for each $\varphi\in B_0(\Sigma,\operatorname{Im}u)$ and for each $\lambda\in(0,1)$

$$
\begin{aligned}
I(\lambda\varphi+(1-\lambda)c) &= \frac{1}{a}\bar{I}(a(\lambda\varphi+(1-\lambda)c)+b)-\frac{b}{a}\\
&= \frac{1}{a}\bar{I}(\lambda(a\varphi+b)+(1-\lambda)(ac+b))-\frac{b}{a}\\
&= \frac{1}{a}\bar{I}(\lambda(a\varphi+b)+(1-\lambda)\bar{c})-\frac{b}{a}\\
&\geq(\leq,=)\,\frac{1}{a}\left(\lambda\bar{I}(a\varphi+b)+(1-\lambda)\bar{c}\right)-\frac{b}{a}\\
&= \lambda\left(\frac{1}{a}\bar{I}(a\varphi+b)-\frac{b}{a}\right)+(1-\lambda)\left(\frac{\bar{c}}{a}-\frac{b}{a}\right)=\lambda I(\varphi)+(1-\lambda)c,
\end{aligned}
$$

proving that I is concave (convex, affine) at c.

(iii). "Only if". Assume that I is constant superadditive (subadditive, additive). It follows that for each $\varphi\in B_0(\Sigma,\operatorname{Im}\bar{u})$ and for each $k\geq 0$

$$
\begin{aligned}
\bar{I}(\varphi+k) &= aI\left(\frac{\varphi+k-b}{a}\right)+b=aI\left(\frac{\varphi-b}{a}+\frac{k}{a}\right)+b\\
&\geq(\leq,=)\,a\left(I\left(\frac{\varphi-b}{a}\right)+\frac{k}{a}\right)+b=aI\left(\frac{\varphi-b}{a}\right)+b+k=\bar{I}(\varphi)+k,
\end{aligned}
$$

proving that $\bar{I}$ is constant superadditive (subadditive, additive). "If". Assume that $\bar{I}$ is constant superadditive (subadditive, additive). It follows that for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and for each $k \geq 0$

$$I(\varphi + k) = \frac{1}{a}\bar{I}(a(\varphi + k) + b) - \frac{b}{a} = \frac{1}{a}\bar{I}((a\varphi + b) + ak) - \frac{b}{a}$$
$$\geq (\leq, =) \frac{1}{a}\left(\bar{I}(a\varphi + b) + ak\right) - \frac{b}{a} = \left(\frac{1}{a}\bar{I}(a\varphi + b) - \frac{b}{a}\right) + k = I(\varphi) + k,$$

proving that I is constant superadditive (subadditive, additive). ■

B.2 Monetary consequences

We next prove a couple of ancillary facts. Moreover, when $\succsim$ (on $\Delta_0(\mathbb{R})$) is represented by an affine u and is CARA, we first assume that v of u corresponds to (4) with $a = 1$ and $b = 0$, that is, we normalize the von Neumann-Morgenstern utility function v to be such that

$$\bar{v}_\alpha(c) = \begin{cases} -\frac{1}{\alpha}e^{-\alpha c} & \text{if } \alpha \neq 0 \\ c & \text{if } \alpha = 0 \end{cases}. \tag{10}$$

In this case, for each $w \in \mathbb{R}$ and for each lottery $x \in \Delta_0(\mathbb{R})$, either $u(x^w) = e^{-\alpha w}u(x)$ or $u(x^w) = u(x) + w$.

Lemma 2 *If $\succsim$ is a CARA rational preference with representation (u, I), then $\succsim^w$ is a rational preference with representation (u, I_w). Moreover, if we choose $v = \bar{v}_\alpha$ as in (10), then I_w is such that*

$$I_w(\varphi) = \begin{cases} I(\varphi + w) - w & \text{if } \succsim \text{ is risk neutral} \\ e^{\alpha w}I(e^{-\alpha w}\varphi) & \text{otherwise} \end{cases} \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u).$$

Proof. By Proposition 1, both preferences $\succsim^w$ and $\succsim$ are rational for all $w \in \mathbb{R}$. By assumption, $\succsim$ is CARA. Thus, $\succsim^w$ coincides with $\succsim$ on $\Delta_0(\mathbb{R})$ and it has a canonical representation (u_w, I_w) where v_w of u_w is either exponential or affine as in (4). Wlog, we can thus set $u = u_w$ and choose v as in (10). By [8, Proposition 1], we have that

$$I(\varphi) = u(x_g) \text{ where } x_g \sim g \text{ and } u(g) = \varphi$$

and

$$I_w(\varphi) = u(x_{f,w}) \text{ where } x_{f,w} \sim^w f \text{ and } u(f) = \varphi.$$

a) Assume that $v = \bar{v}_\alpha$ is exponential (risk nonneutral case), that is, $\bar{v}_\alpha(c) = -\frac{1}{\alpha}e^{-\alpha c}$ for all $c \in \mathbb{R}$. This implies that either $\operatorname{Im} u = (0, \infty)$ or $\operatorname{Im} u = (-\infty, 0)$, in particular, for each $w \in \mathbb{R}$ and $\varphi \in B_0(\Sigma, \operatorname{Im} u)$, we have that $e^{-\alpha w}\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Consider $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Then, there exists $f \in \mathcal{F}$ such that $u(f) = \varphi$. Call $x_{f,w}$ a certainty equivalent of f for the induced preference $\succsim^w$, that is, $x_{f,w} \sim^w f$. It follows that $I_w(\varphi) = u(x_{f,w})$. By definition of $\succsim^w$, we have that $f^w \sim x^w_{f,w}$. It follows that $u(f^w) = e^{-\alpha w} u(f) = e^{-\alpha w}\varphi$ and $u\left(x^w_{f,w}\right) = e^{-\alpha w} u(x_{f,w})$. If we define $g = f^w$, then we also have that x_g can be chosen to be $x^w_{f,w}$, that is,

$$I\left(e^{-\alpha w}\varphi\right) = I(u(g)) = u(x_g) = e^{-\alpha w} u(x_{f,w}) = e^{-\alpha w} I_w(\varphi),$$

and so $I_w(\varphi) = e^{\alpha w} I\left(e^{-\alpha w}\varphi\right)$.

b) Assume that $v = \bar{v}_\alpha$ is the identity (risk neutral case). This implies that $\operatorname{Im} u = \mathbb{R}$. Consider $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Then, there exists $f \in \mathcal{F}$ such that $u(f) = \varphi$. Call $x_{f,w}$ a certainty equivalent of f for the induced preference $\succsim^w$, that is, $x_{f,w} \sim^w f$. It follows that $I_w(\varphi) = u(x_{f,w})$. By definition of $\succsim^w$, we have that $f^w \sim x^w_{f,w}$. It follows that $u(f^w) = u(f) + w = \varphi + w$ and $u\left(x^w_{f,w}\right) = u(x_{f,w}) + w$. If we define $g = f^w$, then we also have that x_g can be chosen to be $x^w_{f,w}$, that is,

$$I(\varphi + w) = I(u(g)) = u(x_g) = u(x_{f,w}) + w = I_w(\varphi) + w,$$

and so $I_w(\varphi) = I(\varphi + w) - w$. ■

Proof of Proposition 3. Let $w, w' \in \mathbb{R}$ be such that $w \neq w'$ and $\circ = w$ and $\# = w'$. If $\succsim$ is decreasing or constant absolute ambiguity averse, wlog, we can assume that $w' > w$. If $\succsim$ is increasing absolute ambiguity averse, wlog, we can assume that $w > w'$. By Proposition 2 and since $\succsim$ is classifiable, we have that u_w is a positive affine transformation of $u_{w'}$ and this holds for all $w, w' \in \mathbb{R}$, proving that $\succsim$ is CARA. ■

Proof of Theorem 2. Let $\succsim$ be a rational preference with canonical representation (u, I) where u is such that $u(x) = \sum_{c\in\mathbb{R}} v(c)\, x(c)$ for every $x \in \Delta_0(\mathbb{R})$, with v strictly increasing and continuous. Before starting the proof, we add few extra points.

(iv) $\succsim$ is CARA and $I_w \leq I_{w'}$, provided $w' > w$ and $u_w = u_{w'} = u$ and $v = \bar{v}_\alpha$ is as in (10);

(v) $\succsim$ is CARA and, provided $v = \bar{v}_\alpha$ as in (10), for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and for each $w, w' \in \mathbb{R}$ such that $w' > w$, either

$$e^{\alpha w} I\left(e^{-\alpha w}\varphi\right) \leq e^{\alpha w'} I\left(e^{-\alpha w'}\varphi\right) \text{ if } \bar{v}_\alpha \text{ is exponential} \tag{11}$$

or

$$I(\varphi + w) - w \leq I\left(\varphi + w'\right) - w' \text{ if } \bar{v}_\alpha \text{ is the identity.} \tag{12}$$

(vi) $\succsim$ is CARA and, provided $v = \bar{v}_\alpha$ as in (10), I is:

(a) superhomogeneous (subhomogeneous) provided $\succsim$ is risk averse (loving);

(b) constant superadditive provided $\succsim$ is risk neutral.

(iii) implies (ii). By Proposition 3, we have that $\succsim$ is CARA. The implication trivially follows.

(ii) implies (vi). By assumption, $\succsim$ is CARA. We can thus choose a canonical representation $(\bar{u}, \bar{I})$ where $v = \bar{v}_\alpha$. In case $\succsim$ is risk averse (resp., loving) $\operatorname{Im} \bar{u} = (-\infty, 0)$ (resp., $\operatorname{Im} \bar{u} = (0, \infty)$). In both cases, we have that $\bar{b} = 0$. By Proposition 19, the implication follows.

(vi) implies (v). $\succsim$ is CARA and, provided $v = \bar{v}_\alpha$ is as in (10), we have three cases:

a. $\succsim$ is risk averse, that is, $\alpha > 0$. Consider $w' > w$. It follows that $\lambda = e^{\alpha(w-w')} \in (0,1)$. Next, consider $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Observe that $e^{-\alpha w}\varphi, e^{-\alpha w'}\varphi \in B_0(\Sigma, \operatorname{Im} u)$. We thus have that

$$I\left(e^{\alpha(w-w')}\left(e^{-\alpha w}\varphi\right)\right) \geq e^{\alpha(w-w')} I\left(e^{-\alpha w}\varphi\right) \implies e^{\alpha w'} I\left(e^{-\alpha w'}\varphi\right) \geq e^{\alpha w} I\left(e^{-\alpha w}\varphi\right),$$

since φ was arbitrarily chosen the statement follows.

b. $\succsim$ is risk loving, that is, $\alpha < 0$. Consider $w' > w$. It follows that $\lambda = e^{\alpha(w'-w)} \in (0,1)$. Next, consider $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Observe that $e^{-\alpha w}\varphi, e^{-\alpha w'}\varphi \in B_0(\Sigma, \operatorname{Im} u)$. We thus have that

$$I\left(e^{\alpha(w'-w)}\left(e^{-\alpha w'}\varphi\right)\right) \leq e^{\alpha(w'-w)} I\left(e^{-\alpha w'}\varphi\right) \implies e^{\alpha w} I\left(e^{-\alpha w}\varphi\right) \leq e^{\alpha w'} I\left(e^{-\alpha w'}\varphi\right),$$

since φ was arbitrarily chosen the statement follows.

c. $\succsim$ is risk neutral, that is, $\alpha = 0$ and $\bar{v}_\alpha$ is the identity. Consider $w' > w$. It follows that $k = (w' - w) > 0$. Next, consider $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Observe that $\varphi + w, \varphi + w' \in B_0(\Sigma, \operatorname{Im} u)$. We thus have that

$$I\left(\varphi + w + \left(w' - w\right)\right) \geq I(\varphi + w) + \left(w' - w\right) \implies I\left(\varphi + w'\right) - w' \geq I(\varphi + w) - w,$$

since φ was arbitrarily chosen the statement follows.

(v) is equivalent to (iv). By assumption, $\succsim$ is CARA. We consider two cases. For each $w, w' \in \mathbb{R}$:

a. $v = \bar{v}_\alpha$ is exponential. By Lemma 2, we have that

$$I_w \leq I_{w'} \iff e^{\alpha w} I\left(e^{-\alpha w}\varphi\right) \leq e^{\alpha w'} I\left(e^{-\alpha w'}\varphi\right) \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u).$$

b. $v = \bar{v}_\alpha$ is the identity. By Lemma 2, we have that

$$I_w \leq I_{w'} \iff I(\varphi + w) - w \leq I\left(\varphi + w'\right) - w' \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u).$$

Subpoints a. and b. prove the equivalence between (iv) and (v).

(iv) implies (i). Let $w' > w$. By Lemma 2 and since $\succsim$ is CARA, we have that both preferences, $\succsim^w$ and $\succsim^{w'}$, admit a representation (u_w, I_w) and $(u_{w'}, I_{w'})$. Since $\succsim$ is CARA, we can choose $u_w = u_{w'} = u$ with $v = \bar{v}_\alpha$ for all $w, w' \in \mathbb{R}$. By Lemma 1 and since $I_w \leq I_{w'}$, we can conclude that $\succsim^w$ is more ambiguity averse than $\succsim^{w'}$.

(i) implies (iv). By Proposition 3, since $\succsim$ is decreasing absolute ambiguity averse, $\succsim$ is CARA. By Lemma 2, we have that for each $w \in \mathbb{R}$ the preference $\succsim^w$ admits a canonical representation (u_w, I_w). Thus, we can choose $u_w = u$ for all $w \in \mathbb{R}$ with $v = \bar{v}_\alpha$. By Lemma 1 and since $u_w = u_{w'}$ for all $w, w' \in \mathbb{R}$, note that $\succsim^w$ is more ambiguity averse than $\succsim^{w'}$ only if $I_w \leq I_{w'}$.

(iv) implies (vi). By the previous part of the proof, we know that (iv) is equivalent to (v). We thus assume (v) and prove (vi). We have three cases.

a. $\succsim$ is risk averse, that is, $\alpha > 0$. In (11) set $w = 0$, so that

$$I(\varphi) \leq e^{\alpha w'} I\left(e^{-\alpha w'} \varphi\right) \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u), \forall w' > 0.$$

Since α is positive, it follows that $e^{\alpha w'} > 1$ and $\left\{e^{\alpha w'} : w' > 0\right\} = (1, \infty)$. This implies that $I(\varphi) \leq \gamma I(\varphi/\gamma)$ for all $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and for all $\gamma > 1$. In other words, $\lambda I(\varphi) \leq I(\lambda\varphi)$ for all $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and for all $\lambda \in (0,1)$, proving superhomogeneity.

b. $\succsim$ is risk loving, that is, $\alpha < 0$. In (11) set $w = 0$, so that

$$I(\varphi) \leq e^{\alpha w'} I\left(e^{-\alpha w'} \varphi\right) \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u), \forall w' > 0.$$

Since α is negative, it follows that $\left\{e^{\alpha w'} : w' > 0\right\} = (0, 1)$. This implies that $I(\varphi) \leq \gamma I(\varphi/\gamma)$ for all $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and for all $\gamma \in (0,1)$. If $\varphi \in B_0(\Sigma, \operatorname{Im} u)$, then $\lambda\varphi \in B_0(\Sigma, \operatorname{Im} u)$ for all $\lambda \in (0,1)$. We have that

$$I(\lambda\varphi) \leq \lambda I\left(\frac{1}{\lambda}(\lambda\varphi)\right) = \lambda I(\varphi) \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u), \forall \lambda \in (0,1),$$

proving subhomogeneity.

c. $\succsim$ is risk neutral, that is, $\bar{v}_\alpha$ is the identity. In (12) set $w = 0$ and $k = w'$, so that

$$I(\varphi) \leq I(\varphi + k) - k \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u), \forall k > 0.$$

In other words, $I(\varphi) + k \leq I(\varphi + k)$ for all $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and for all $k > 0$, proving superadditivity.

(vi) implies (ii). By assumption, $\succsim$ is CARA and represented by (u, I). We can thus choose a canonical representation $\left(\bar{u}, \bar{I}\right)$ where $v = \bar{v}_\alpha$. In case $\succsim$ is risk averse (resp., loving) $\operatorname{Im} \bar{u} = (-\infty, 0)$ (resp., $\operatorname{Im} \bar{u} = (0, \infty)$). In both cases, we have that $\bar{b} = 0$. By Proposition 19, the implication follows.

We thus proved that (iii) implies (ii) and (ii) is equivalent to (i), (iv), (v), and (vi). In particular, it follows that (ii) implies (i), thus $\succsim$ is classifiable, and I satisfies condition (a) or (b), that is, (ii) implies (iii). ■

B.3 Other proofs

Proof of Corollary 2. Call (u, I) the rational representation of $\succsim$ on $\mathcal{F}$. Since $\succsim$ is risk neutral, it follows that $\operatorname{Im} u = \mathbb{R}$ and $I : B_0(\Sigma) \to \mathbb{R}$.

"Only if." By point 1 of Corollary 1, it follows that $I(\varphi + k) = I(\varphi) + k$ for all $\varphi \in B_0(\Sigma)$ and for all $k \geq 0$. It is immediate to show that the equality holds for all $k \in \mathbb{R}$. By [27, Lemma 25], it follows that I is a normalized niveloid. By [27, Lemma 28], we can conclude that $\succsim$ satisfies Weak C-Independence.

"If." By [27, Lemma 28], it follows that I is a normalized niveloid. By [27, Lemma 25] and since $\operatorname{Im} u = \mathbb{R}$, it follows that $I(\varphi + k) = I(\varphi) + k$ for all $\varphi \in B_0(\Sigma)$ and for all $k \in \mathbb{R}$. By point 1 of Corollary 1 (recall that it holds by only assuming CARA in place of classifiable), the statement follows. ■

Proof of Corollary 3. Call (u, I) the rational representation of $\succsim$. Note that in all three points (i)–(iii), $\succsim$ is necessarily CARA. Thus, wlog, choose v to be such that $a = 1$ and $b = 0$. By [17], there also exists a normalized, monotone, and continuous functional $\hat{I} : B_0(\Sigma) \to \mathbb{R}$ such that for each $\varphi \in B_0(\Sigma)$

$$\hat{I}(\lambda\varphi + k) = \lambda \hat{I}(\varphi) + k \qquad \forall \lambda > 0, \forall k \in \mathbb{R}$$

and $f \succsim g$ if and only if $\hat{I}(u(f)) \geq \hat{I}(u(g))$. It follows that $\hat{I}$ and I coincide on $B_0(\Sigma, \operatorname{Im} u)$.

(i) implies (iii). By Proposition 3, the implication follows.

(iii) implies (ii). By Corollary 1 (recall that it holds by only assuming CARA in place of classifiable) and since $\hat{I}$ and I coincide on $B_0(\Sigma, \operatorname{Im} u)$, the implication follows.

(ii) implies (i). Trivially, $\succsim$ is classifiable. ■

Proof of Proposition 4. Let (u, I) be the canonical representation of $\succsim$. Wlog, if $\succsim$ is CARA, we choose v to be such that $a = 1$ and $b = 0$ (see equation (4)). In this case, by the definition of $c : \mathcal{F} \to \mathbb{R}$, we have that

$$c(f) = \begin{cases} -\frac{1}{\alpha} \log(-\alpha I(u(f))) & \alpha \neq 0 \\ I(u(f)) & \alpha = 0 \end{cases} \qquad \forall f \in \mathcal{F}.$$

Recall that for each $f \in \mathcal{F}$ and for each $w \in \mathbb{R}$

$$u(f^w) = \begin{cases} e^{-\alpha w} u(f) & \alpha \neq 0 \\ u(f) + w & \alpha = 0 \end{cases}.$$

(i). "Only if". By Proposition 3, $\succsim$ is CARA, we have three cases.

1. $\succsim$ is risk neutral, that is, $\alpha = 0$. It follows that $c(f^w) = I(u(f^w)) = I(u(f) + w)$ for all $f \in \mathcal{F}$ and for all $w \geq 0$. By Theorem 2, we have that for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$c(f^w) = I(u(f) + w) \geq I(u(f)) + w = c(f) + w,$$

proving that c is wealth superadditive.

2. $\succsim$ is risk averse, that is, $\alpha > 0$. It follows that $c(f^w) = v^{-1}(I(u(f^w))) = v^{-1}(I(e^{-\alpha w}u(f)))$ for all $f \in \mathcal{F}$ and for all $w \geq 0$. Note that if $w \geq 0$, then $e^{-\alpha w} \in (0, 1]$. By Theorem 2 and since $b = 0$, we have that for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$\begin{aligned} c(f^w) &= -\frac{1}{\alpha}\log\left(-\alpha I\left(e^{-\alpha w}u(f)\right)\right) \geq -\frac{1}{\alpha}\log\left(-\alpha e^{-\alpha w}I(u(f))\right) \\ &= -\frac{1}{\alpha}\log\left(e^{-\alpha w}(-\alpha I(u(f)))\right) = -\frac{1}{\alpha}\log\left(e^{-\alpha w}\right) + -\frac{1}{\alpha}\log(-\alpha I(u(f))) \\ &= c(f) + w, \end{aligned}$$

proving that c is wealth superadditive.

3. $\succsim$ is risk loving, that is, $\alpha < 0$. It follows that $c(f^w) = v^{-1}(I(u(f^w))) = v^{-1}(I(e^{-\alpha w}u(f)))$ for all $f \in \mathcal{F}$ and for all $w \geq 0$. Note that if $w \geq 0$, then $e^{\alpha w} \in (0, 1]$. By Theorem 2 and since $b = 0$, we have that for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$\begin{aligned} c(f) &= -\frac{1}{\alpha}\log(-\alpha I(u(f))) = -\frac{1}{\alpha}\log\left(-\alpha I\left(e^{\alpha w}\left(e^{-\alpha w}u(f)\right)\right)\right) \\ &\leq -\frac{1}{\alpha}\log\left(-\alpha e^{\alpha w}I\left(e^{-\alpha w}u(f)\right)\right) = -\frac{1}{\alpha}\log\left(e^{\alpha w}\right) + -\frac{1}{\alpha}\log\left(-\alpha I\left(e^{-\alpha w}u(f)\right)\right) \\ &= -w + c(f^w), \end{aligned}$$

proving that c is wealth superadditive.

"If". First, observe that

$$f \succsim g \iff I(u(f)) \geq I(u(g)) \iff v^{-1}(I(u(f))) \geq v^{-1}(I(u(g))) \iff c(f) \geq c(g).$$

Let $w' > w$ and $f \in \mathcal{F}$. Since $w' - w > 0$ and c is wealth superadditive, it follows that

$$c\left(f^{w'}\right) = c\left((f^w)^{w'-w}\right) \geq c(f^w) + w' - w,$$

that is, $c\left(f^{w'}\right) - w' \geq c(f^w) - w$. Next, let $x \in \Delta_0(\mathbb{R})$. Since $\succsim$ is CARA, we can conclude that

$$\begin{aligned} f \succsim^w x &\implies f^w \succsim x^w \implies c(f^w) \geq c(x^w) \implies c(f^w) \geq c(x) + w \\ &\implies c(f^w) - w \geq c(x) \implies c\left(f^{w'}\right) - w' \geq c(x) \implies c\left(f^{w'}\right) \geq c(x) + w' \\ &\implies c\left(f^{w'}\right) \geq c\left(x^{w'}\right) \implies f^{w'} \succsim x^{w'} \implies f \succsim^{w'} x. \end{aligned}$$

Since f, x, w, and w' were arbitrarily chosen, we have that $\succsim^w$ is more ambiguity averse than $\succsim^{w'}$, proving the statement.

(ii). "Only if". By Proposition 3, $\succsim$ is CARA, we have three cases.

1. $\succsim$ is risk neutral, that is, $\alpha = 0$. It follows that $c(f^w) = I(u(f^w)) = I(u(f) + w)$ for all $f \in \mathcal{F}$ and for all $w \geq 0$. By what follows right after Theorem 2, we have that for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$c(f^w) = I(u(f) + w) \leq I(u(f)) + w = c(f) + w,$$

proving that c is wealth subadditive.

2. $\succsim$ is risk averse, that is, $\alpha > 0$. It follows that $c(f^w) = v^{-1}(I(u(f^w))) = v^{-1}(I(e^{-\alpha w}u(f)))$ for all $f \in \mathcal{F}$ and for all $w \geq 0$. Note that if $w \geq 0$, then $e^{-\alpha w} \in (0, 1]$. By what follows right after Theorem 2 and since $b = 0$, we have that for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$\begin{aligned}
c(f^w) &= -\frac{1}{\alpha}\log\left(-\alpha I\left(e^{-\alpha w}u(f)\right)\right) \leq -\frac{1}{\alpha}\log\left(-\alpha e^{-\alpha w} I(u(f))\right) \\
&= -\frac{1}{\alpha}\log\left(e^{-\alpha w}(-\alpha I(u(f)))\right) = -\frac{1}{\alpha}\log\left(e^{-\alpha w}\right) + -\frac{1}{\alpha}\log(-\alpha I(u(f))) \\
&= c(f) + w,
\end{aligned}$$

proving that c is wealth subadditive.

3. $\succsim$ is risk loving, that is, $\alpha < 0$. It follows that $c(f^w) = v^{-1}(I(u(f^w))) = v^{-1}(I(e^{-\alpha w}u(f)))$ for all $f \in \mathcal{F}$ and for all $w \geq 0$. Note that if $w \geq 0$, then $e^{\alpha w} \in (0, 1]$. By what follows right after Theorem 2 and since $b = 0$, we have that for each $f \in \mathcal{F}$ and for each $w \geq 0$

$$\begin{aligned}
c(f) &= -\frac{1}{\alpha}\log(-\alpha I(u(f))) = -\frac{1}{\alpha}\log\left(-\alpha I\left(e^{\alpha w}\left(e^{-\alpha w}u(f)\right)\right)\right) \\
&\geq -\frac{1}{\alpha}\log\left(-\alpha e^{\alpha w} I\left(e^{-\alpha w}u(f)\right)\right) = -\frac{1}{\alpha}\log\left(e^{\alpha w}\right) + -\frac{1}{\alpha}\log\left(-\alpha I\left(e^{-\alpha w}u(f)\right)\right) \\
&= -w + c(f^w),
\end{aligned}$$

proving that c is wealth subadditive.

"If". First, recall that $f \succsim g$ if and only if $c(f) \geq c(g)$. Let $w' > w$ and $f \in \mathcal{F}$. Since $w' - w > 0$ and c is wealth subadditive, it follows that

$$c\left(f^{w'}\right) = c\left((f^w)^{w'-w}\right) \leq c(f^w) + w' - w,$$

that is, $c\left(f^{w'}\right) - w' \leq c(f^w) - w$. Next, let $x \in \Delta_0(\mathbb{R})$. Since $\succsim$ is CARA, we can conclude that

$$\begin{aligned}
f \succsim^{w'} x &\implies f^{w'} \succsim x^{w'} \implies c\left(f^{w'}\right) \geq c\left(x^{w'}\right) \implies c\left(f^{w'}\right) \geq c(x) + w' \\
&\implies c\left(f^{w'}\right) - w' \geq c(x) \implies c(f^w) - w \geq c(x) \implies c(f^w) \geq c(x) + w \\
&\implies c(f^w) \geq c(x^w) \implies f^w \succsim x^w \implies f \succsim^w x.
\end{aligned}$$

Since f, x, w, and w' were arbitrarily chosen, we have that $\succsim^{w'}$ is more ambiguity averse than $\succsim^w$, proving the statement.

(iii). It is an easy consequence of points (i) and (ii). ■

Proof of Theorem 3. Recall that an uncertainty averse preference is a rational preference. In particular, given a canonical representation (u, I), we have that

$$G(t,p) = \sup_{\varphi \in B_0(\Sigma, \operatorname{Im} u)} \left\{ I(\varphi) : \int \varphi dp \leq t \right\} \qquad \forall (t,p) \in \operatorname{Im} u \times \Delta.$$

(i) implies (ii). By Theorem 2, it follows that $\succsim$ is CARA and I is either concave at b, or convex at b, or constant superadditive, depending on $\succsim$ being, respectively, either risk averse, or risk loving, or risk neutral. We consider the three different cases separately:

- $\succsim$ is risk averse. Thus, $\operatorname{Im} u = (-\infty, b)$. Let $(t,p) \in \operatorname{Im} u \times \Delta$ and $\lambda \in (0,1)$. There exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq B_0(\Sigma, \operatorname{Im} u)$ such that $I(\varphi_n) \uparrow G(t,p)$ and $\int \varphi_n dp \leq t$ for all $n \in \mathbb{N}$. It follows that $\int (\lambda\varphi_n + (1-\lambda) b)\, dp \leq \lambda t + (1-\lambda) b \in \operatorname{Im} u$ for all $n \in \mathbb{N}$. Since I is concave at b, we have that for each $n \in \mathbb{N}$

$$G(\lambda t + (1-\lambda) b, p) \geq I(\lambda\varphi_n + (1-\lambda) b) \geq \lambda I(\varphi_n) + (1-\lambda) b.$$

By passing to the limit, it follows that $G(\lambda t + (1-\lambda) b, p) \geq \lambda G(t,p) + (1-\lambda) b$.

- $\succsim$ is risk loving. Thus, $\operatorname{Im} u = (b, \infty)$. Let $(t,p) \in \operatorname{Im} u \times \Delta$ and $\lambda \in (0,1)$. There exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq B_0(\Sigma, \operatorname{Im} u)$ such that $I(\varphi_n) \uparrow G(\lambda t + (1-\lambda) b, p)$ and $\int \varphi_n dp \leq \lambda t + (1-\lambda) b$ for all $n \in \mathbb{N}$. Define $\{\psi_n\}_{n\in\mathbb{N}}$ to be such that

$$\psi_n = \frac{\varphi_n - (1-\lambda) b}{\lambda} \qquad \forall n \in \mathbb{N}.$$

Note also that

$$\psi_n(s) > b \quad \forall s \in S, \quad \int \psi_n dp \leq t, \text{ and } \varphi_n = \lambda\psi_n + (1-\lambda) b \qquad \forall n \in \mathbb{N}.$$

Since I is convex at b, this implies that for each $n \in \mathbb{N}$

$$I(\varphi_n) = I(\lambda\psi_n + (1-\lambda) b) \leq \lambda I(\psi_n) + (1-\lambda) b \leq \lambda G(t,p) + (1-\lambda) b.$$

By passing to the limit, it follows that $G(\lambda t + (1-\lambda) b, p) \leq \lambda G(t,p) + (1-\lambda) b$.

- $\succsim$ is risk neutral. Thus, $\operatorname{Im} u = \mathbb{R}$. Let $(t,p) \in \operatorname{Im} u \times \Delta$ and $k \geq 0$. There exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq B_0(\Sigma, \operatorname{Im} u)$ such that $I(\varphi_n) \uparrow G(t,p)$ and $\int \varphi_n dp \leq t$ for all $n \in \mathbb{N}$. It follows that $\int (\varphi_n + k)\, dp \leq t + k \in \operatorname{Im} u$ for all $n \in \mathbb{N}$. Since I is constant superadditive, we have that for each $n \in \mathbb{N}$

$$G(t+k, p) \geq I(\varphi_n + k) \geq I(\varphi_n) + k.$$

By passing to the limit, it follows that $G(t+k, p) \geq G(t,p) + k$.

(ii) implies (iii) and (i). Recall that

$$I(\psi) = \inf_{p\in\Delta} G\left(\int \psi dp, p\right) \qquad \forall \psi \in B_0(\Sigma, \operatorname{Im} u).$$

Observe also that $\succsim$ is CARA by assumption and G satisfies (a) or (b). As before, we consider three cases:

- $\succsim$ is risk averse. Let $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and $\lambda \in (0,1)$. We have that

$$\begin{aligned} I(\lambda\varphi + (1-\lambda)b) &= \inf_{p\in\Delta} G\left(\int (\lambda\varphi + (1-\lambda)b)\,dp, p\right) = \inf_{p\in\Delta} G\left(\lambda \int \varphi dp + (1-\lambda)b, p\right) \\ &\geq \inf_{p\in\Delta}\left(\lambda G\left(\int \varphi dp, p\right) + (1-\lambda)b\right) \\ &\geq \lambda \inf_{p\in\Delta} G\left(\int \varphi dp, p\right) + (1-\lambda)b = \lambda I(\varphi) + (1-\lambda)b, \end{aligned}$$

that is, I is concave at b.

- $\succsim$ is risk loving. Let $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and $\lambda \in (0,1)$. We have that

$$\begin{aligned} I(\lambda\varphi + (1-\lambda)b) &= \inf_{p\in\Delta} G\left(\int (\lambda\varphi + (1-\lambda)b)\,dp, p\right) = \inf_{p\in\Delta} G\left(\lambda \int \varphi dp + (1-\lambda)b, p\right) \\ &\leq \inf_{p\in\Delta}\left(\lambda G\left(\int \varphi dp, p\right) + (1-\lambda)b\right) \\ &= \lambda \inf_{p\in\Delta} G\left(\int \varphi dp, p\right) + (1-\lambda)b = \lambda I(\varphi) + (1-\lambda)b, \end{aligned}$$

that is, I is convex at b.

- $\succsim$ is risk neutral. Let $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and $k \geq 0$. We have that

$$\begin{aligned} I(\varphi + k) &= \inf_{p\in\Delta} G\left(\int (\varphi + k)\,dp, p\right) = \inf_{p\in\Delta} G\left(\int \varphi dp + k, p\right) \\ &\geq \inf_{p\in\Delta}\left(G\left(\int \varphi dp, p\right) + k\right) \geq \inf_{p\in\Delta} G\left(\int \varphi dp, p\right) + k = I(\varphi) + k, \end{aligned}$$

that is, I is constant superadditive.

It follows that $\succsim$ is CARA and I either satisfies (a) or (b) of point (ii) of Theorem 2. By Theorem 2, we can conclude that $\succsim$ is decreasing absolute ambiguity averse and is classifiable.

(iii) implies (ii). By Proposition 3 and since $\succsim$ is classifiable, we have that $\succsim$ is also CARA.

We thus have proved that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i), proving the statement. ■

Proof of Corollary 4. Recall that an uncertainty averse preference is a rational preference. By Corollary 2, we can conclude that a risk neutral uncertainty averse preference is constant absolute ambiguity averse if and only if it satisfies Weak C-Independence. At the same time, by definition, uncertainty averse preferences that satisfy Weak C-Independence are exactly variational preferences.

Proof of Corollary 5. Since $\succsim$ is CARA and risk averse, we have that $\operatorname{Im} u = (-\infty, b)$. Recall that $G(t,p) \geq t$ for all $(t,p) \in \operatorname{Im} u \times \Delta$. At the same time, note that for each $(t,p) \in \operatorname{Im} u \times \Delta$ and for each $\lambda \in (0,1)$

$$\begin{aligned} G(\lambda t + (1-\lambda) b, p) &\geq G(\lambda t + (1-\lambda) b_n, p) \geq \lambda G(t,p) + (1-\lambda) G(b_n, p) \\ &\geq \lambda G(t,p) + (1-\lambda) b_n \end{aligned}$$

where $b_n = b - \frac{1}{n}$ for all $n \in \mathbb{N}$. By passing to the limit and since (t,p) and λ were arbitrarily chosen, we have that $G(\lambda t + (1-\lambda) b, p) \geq \lambda G(t,p) + (1-\lambda) b$. By Theorem 3, the statement follows. ■

Proof of Corollary 6. Observe that a variational preference is a rational preference where the canonical representation (u, I) has the extra property of I being quasiconcave and constant additive. In particular, I is normalized and concave.

(i). By Theorem 2 and since $\succsim$ is not risk neutral, if $\succsim$ is either decreasing absolute ambiguity averse or CARA and risk averse, then v is a positive affine transformation of $-\frac{1}{\alpha}e^{-\alpha c}$ where $\alpha \neq 0$. Without loss of generality, we assume that either $\operatorname{Im} u = (-\infty, 0)$ or $\operatorname{Im} u = (0, \infty)$. The first case holds under risk aversion, the second one under risk love. In the first case, since I is normalized and concave, observe that for each $\lambda \in (0,1)$ and for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$, we have that $\lambda\varphi + (1-\lambda)\left(-\frac{1}{n}\right) \in B_0(\Sigma, \operatorname{Im} u)$ and

$$\begin{aligned} I\left(\lambda\varphi + (1-\lambda)\left(-\frac{1}{n}\right)\right) &\geq \lambda I(\varphi) + (1-\lambda) I\left(-\frac{1}{n}\right) \\ &\geq \lambda I(\varphi) - (1-\lambda)\frac{1}{n} \qquad \forall n \in \mathbb{N}. \end{aligned}$$

By passing to the limit, it follows that I is concave at 0, that is, I is superhomogeneous. In the second case, since I is normalized and concave, observe that for each $\lambda \in (0,1)$ and for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$, we have that $\lambda\varphi + (1-\lambda)\frac{1}{n} \in B_0(\Sigma, \operatorname{Im} u)$ and

$$\begin{aligned} I\left(\lambda\varphi + (1-\lambda)\frac{1}{n}\right) &\geq \lambda I(\varphi) + (1-\lambda) I\left(\frac{1}{n}\right) \\ &\geq \lambda I(\varphi) + (1-\lambda)\frac{1}{n} \qquad \forall n \in \mathbb{N}. \end{aligned}$$

By passing to the limit, it follows that I is concave at 0, that is, I is again superhomogeneous.

"If". By Theorem 2 and since I is concave at 0, if $\succsim$ is CARA and risk averse, it follows that $\succsim$ is decreasing absolute ambiguity averse. "Only if". By Theorem 2, if $\succsim$ is decreasing absolute ambiguity averse, then $\succsim$ is CARA. Since $\succsim$ cannot be risk neutral, it can either be risk averse or risk loving. By contradiction, assume it is risk loving. By Theorem 2, it follows that I is convex at 0, that is, I is subhomogeneous. From the previous part of the proof, we can conclude that I is homogeneous. To sum up, we would have that I is normalized, monotone, continuous, concave, constant additive, and homogeneous, that is, $\succsim$ is maxmin, a contradiction.

(ii). It follows from analogous arguments. ■

Proof of Corollary 7. "If". Since $\succsim$ is risk nonneutral, if $\succsim$ is CARA, then either $\succsim$ is risk averse or it is risk loving. If $\succsim$ is homothetic uncertainty averse, then, in both cases, I is positively homogeneous, proving the statement.

"Only if". By Proposition 3 and since $\succsim$ is constant absolute ambiguity averse and uncertainty averse, we have that $\succsim$ is CARA. Since $\succsim$ is uncertainty averse and risk non-neutral, we can consider a canonical representation (u, I) such that either $\operatorname{Im} u = (-\infty, 0)$ or $\operatorname{Im} u = (0, \infty)$. Since $\succsim$ is constant absolute ambiguity averse, we also have that I is positively homogeneous. Define $\bar{I} : B_0(\Sigma) \to [-\infty, \infty)$ by

$$\bar{I}(\varphi) = \sup \{I(\psi) : B_0(\Sigma, \operatorname{Im} u) \ni \psi \leq \varphi\} \qquad \forall \varphi \in B_0(\Sigma).$$

By [9, Theorem 36], it follows that $\bar{I}$ is monotone, lower semicontinuous, quasiconcave, and such that $\bar{I}_{|B_0(\Sigma, \operatorname{Im} u)} = I$. We next show that also $\bar{I}$ is positively homogeneous. Consider $\varphi \in B_0(\Sigma)$ and $\lambda > 0$. We have two cases:

1. $\{I(\psi) : B_0(\Sigma, \operatorname{Im} u) \ni \psi \leq \varphi\} = \emptyset$. Since $B_0(\Sigma, \operatorname{Im} u)$ is a cone, $\{I(\psi) : B_0(\Sigma, \operatorname{Im} u) \ni \psi \leq \lambda\varphi\} = \emptyset$, which yields that $\bar{I}(\lambda\varphi) = -\infty = \bar{I}(\varphi) = \lambda\bar{I}(\varphi)$.

2. $\{I(\psi) : B_0(\Sigma, \operatorname{Im} u) \ni \psi \leq \varphi\} \neq \emptyset$. Let $\{\psi_n\}_{n \in \mathbb{N}} \subseteq B_0(\Sigma, \operatorname{Im} u)$ be such that $\psi_n \leq \varphi$ for all $n \in \mathbb{N}$ and $I(\psi_n) \uparrow \bar{I}(\varphi)$. Let now $\lambda > 0$. Since $B_0(\Sigma, \operatorname{Im} u)$ is a cone, it follows that $\{\lambda\psi_n\}_{n \in \mathbb{N}} \subseteq B_0(\Sigma, \operatorname{Im} u)$ and it is such that $\lambda\psi_n \leq \lambda\varphi$ for all $n \in \mathbb{N}$. In particular, by the definition of $\bar{I}$, we have that $\bar{I}(\lambda\varphi) \geq I(\lambda\psi_n) = \lambda I(\psi_n) \to \lambda\bar{I}(\varphi)$. We just proved that $\bar{I}(\lambda\varphi) \geq \lambda\bar{I}(\varphi)$ for all $\varphi \in B_0(\Sigma)$ and for all $\lambda > 0$. By choosing $1/\lambda$ with $\lambda > 0$, it follows that

$$\bar{I}(\varphi) = \bar{I}\left(\frac{1}{\lambda}(\lambda\varphi)\right) \geq \frac{1}{\lambda}\bar{I}(\lambda\varphi),$$

that is, $\lambda\bar{I}(\varphi) \geq \bar{I}(\lambda\varphi)$, proving positive homogeneity.

Consider $G : \mathbb{R} \times \Delta \to [-\infty, \infty]$ defined by

$$G(t, p) = \sup \left\{\bar{I}(\varphi) : \int \varphi dp \leq t\right\} \qquad \forall (t, p) \in \mathbb{R} \times \Delta.$$

By [10], we have that G is lower semicontinuous, quasiconvex, and such that

$$\bar{I}(\varphi) = \min_{p\in\Delta} G\left(\int \varphi dp, p\right) \qquad \forall \varphi \in B_0(\Sigma) \tag{13}$$

and $G(\lambda t, p) = \lambda G(t,p)$ for all $\lambda > 0$, for all $t \in \mathbb{R}$, and for all $p \in \Delta$. Define $c_1, c_2 : \Delta \to [0,\infty]$ to be such that

$$c_1(p) = \frac{1}{G(1,p)} \text{ and } c_2(p) = -G(-1,p) \qquad \forall p \in \Delta.$$

We now consider two cases:

Risk averse case. $\operatorname{Im} u = (-\infty, 0)$. Since $\bar{I} \leq 0$ and $\bar{I}(-1) = I(-1) = -1$, observe that $G(-1,p) \leq 0$ and $G(-1,p) \geq -1$, that is, $c_2(p) \geq 0$ and $c_2(p) \leq 1$ for all $p \in \Delta$. Next, we have that for each $\alpha \in \mathbb{R}$

$$\{p \in \Delta : c_2(p) \geq \alpha\} = \{p \in \Delta : -G(-1,p) \geq \alpha\} = \{p \in \Delta : G(-1,p) \leq -\alpha\}.$$

Since G is quasiconvex and lower semicontinuous, the set is convex and closed, proving that c_2 is quasiconcave and upper semicontinuous. By (13), we can conclude that for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$

$$I(\varphi) = \bar{I}(\varphi) = \min_{p\in\Delta} G\left(\int \varphi dp, p\right) = \min_{p\in\Delta}\left(-\int \varphi dp\right) G(-1,p) = \min_{p\in\Delta} c_2(p) \int \varphi dp.$$

Since $-1 = \bar{I}(-1) = \min_{p\in\Delta} -c_2(p)$, we have that c_2 is normalized. The statement follows by setting $c = c_2$.

Risk loving case. $\operatorname{Im} u = (0,\infty)$. Since $\bar{I}(1) = I(1) = 1$, observe that $G(1,p) \geq 1$, that is, $0 \leq c_1(p) \leq 1$. Next, we have that for each $\alpha \in (0,\infty)$

$$\{p \in \Delta : c_1(p) \geq \alpha\} = \left\{p \in \Delta : \frac{1}{G(1,p)} \geq \alpha\right\} = \left\{p \in \Delta : G(1,p) \leq \frac{1}{\alpha}\right\}.$$

Since G is quasiconvex and lower semicontinuous, for each $\alpha \in (0,\infty)$ the set is convex and closed. Since $\{p \in \Delta : c_1(p) \geq \alpha\} = \Delta$ for all $\alpha \leq 0$, it follows that c_1 is quasiconcave and upper semicontinuous. By (13), we can conclude for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$

$$I(\varphi) = \bar{I}(\varphi) = \min_{p\in\Delta}\left(\int \varphi dp\right) G(1,p) = \min_{p\in\Delta} \frac{\int \varphi dp}{c_1(p)}.$$

Since $1 = \bar{I}(1) = \min_{p\in\Delta} \frac{1}{c_1(p)}$, we have that c_1 is normalized. The statement follows by setting $c = c_1$. ■

Proof of Proposition 5. Since there exists $\gamma > 0$ such that $\phi(t) = -e^{-\gamma t}$ for all $t \in \mathbb{R}$, we have that I, defined as in (5), can be defined over the entire space $B_0(\Sigma)$. Moreover, by [9, Proposition 54], I is normalized, concave and constant additive. In particular, it is concave at b, in case $\succsim$ is either risk averse or risk loving.

(i). By Corollary 1 (recall that it holds by only assuming CARA in place of classifiable) and since I is constant additive, if $\succsim$ is risk neutral, then $\succsim$ is constant absolute ambiguity averse.

(ii). By Corollary 5 and since $\succsim$ is CARA, if $\succsim$ is risk averse, then $\succsim$ is decreasing absolute ambiguity averse. ■

Proof of Proposition 6. We only prove point (ii). Point (iii) follows from a completely specular argument. Point (i) instead follows from similar techniques (see also Marinacci and Montrucchio [29, Theorem 12]).

(ii). Fix μ. By Theorem 2 and since $\succsim$ is risk averse and $b = 0$, we have that $\succsim$ is decreasing absolute ambiguity averse if and only if I is positive superhomogeneous. Thus, to prove point (ii), we only need to show that I is positive superhomogeneous for all μ if and only if ϕ is IRRA. Since $\succsim$ is risk averse and $b = 0$, we also have that $\operatorname{Im} u = (-\infty, 0)$ and $\phi : (-\infty, 0) \to \mathbb{R}$. For each $\nu > 0$, define $\phi_\nu : (-\infty, 0) \to \mathbb{R}$ to be such that $\phi_\nu(t) = \phi(\nu t)$ for all $t \in (-\infty, 0)$. Note that $\phi_1 = \phi$. Finally, we have that $\operatorname{Im}\phi = \operatorname{Im}\phi_\nu$ for all $\nu > 0$. "If" Let μ be generic. Consider $\nu > \eta > 0$. It follows that, $\phi_\nu = f \circ \phi_\eta$ where $f : \operatorname{Im}\phi \to \operatorname{Im}\phi$ is strictly increasing and concave. By the Jensen's inequality, it follows that if $\nu > \eta > 0$, then

$$\phi_\nu^{-1}\left(\int \phi_\nu\left(\int \varphi dp\right) d\mu\right) \le \phi_\eta^{-1}\left(\int \phi_\eta\left(\int \varphi dp\right) d\mu\right) \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u).$$

If we let $\eta \in (0,1)$ and $\nu = 1$, we have that for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$

$$\begin{aligned} \phi\left(\eta\phi^{-1}\left(\int \phi\left(\int \varphi dp\right) d\mu\right)\right) &= \phi_\eta\left(\phi^{-1}\left(\int \phi\left(\int \varphi dp\right) d\mu\right)\right) \\ &\le \int \phi\left(\eta \int \varphi dp\right) d\mu. \end{aligned}$$

We can conclude that for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$

$$I(\eta\varphi) = \phi^{-1}\left(\int \phi\left(\eta\int \varphi dp\right) d\mu\right) \ge \eta\phi^{-1}\left(\int \phi\left(\int \varphi dp\right) d\mu\right) = \eta I(\varphi),$$

proving that I is positive superhomogeneous. "Only if" Let $\nu > \eta > 0$. Consider $\frac{\eta}{\nu} \in (0,1)$. Fix μ. Since I is positive superhomogeneous, it follows that for each $\varphi \in B_0(\Sigma, \operatorname{Im} u)$

$$\begin{aligned} \phi^{-1}\left(\int \phi_\eta\left(\int \varphi dp\right) d\mu\right) = \phi^{-1}\left(\int \phi\left(\eta\int \varphi dp\right) d\mu\right) &= \phi^{-1}\left(\int \phi\left(\int \frac{\eta}{\nu}\nu\varphi dp\right) d\mu\right) \\ &\ge \frac{\eta}{\nu}\phi^{-1}\left(\int \phi\left(\int \nu\varphi dp\right) d\mu\right) \\ &= \frac{\eta}{\nu}\phi^{-1}\left(\int \phi_\nu\left(\int \varphi dp\right) d\mu\right), \end{aligned}$$

yielding that

$$\begin{aligned}\phi_\eta^{-1}\left(\int \phi_\eta\left(\int \varphi dp\right) d\mu\right) &= \frac{1}{\eta}\phi^{-1}\left(\int \phi_\eta\left(\int \varphi dp\right) d\mu\right)\\ &\geq \frac{1}{\nu}\phi^{-1}\left(\int \phi_\nu\left(\int \varphi dp\right) d\mu\right)\\ &= \phi_\nu^{-1}\left(\int \phi_\nu\left(\int \varphi dp\right) d\mu\right) \qquad \forall \varphi \in B_0\left(\Sigma, \operatorname{Im} u\right).\end{aligned}$$

Since both ϕ_ν and ϕ_η are both strictly increasing and continuous, there exists a strictly increasing function $h : \operatorname{Im}\phi \to \operatorname{Im}\phi$ such that $\phi_\nu = h \circ \phi_\eta$. It follows that

$$h\left(\int \phi_\eta\left(\int \varphi dp\right) d\mu\right) \geq \int h\left(\phi_\eta\left(\int \varphi dp\right)\right) d\mu \qquad \forall \varphi \in B_0\left(\Sigma, \operatorname{Im} u\right). \tag{14}$$

Since μ was arbitrarily chosen, (14) holds for all μ. Since Σ is nontrivial, there exists $E \in \Sigma$ such that $E \neq \emptyset, S$. Consider $s_1, s_2 \in S$ such that $s_1 \in E$ and $s_2 \in E^c$. Note that $\delta_{s_i} \in \Delta$ and $\{\delta_{s_i}\} \in \mathcal{B}$ for $i \in \{1,2\}$.[53] Let $\mu = \lambda\delta_{\delta_{s_1}} + (1-\lambda)\,\delta_{\delta_{s_2}}$ with $\lambda \in (0,1)$. Consider also $k_1, k_2 \in \operatorname{Im}\phi$. It follows that there exist $t_1, t_2 \in (-\infty, 0)$ such that $\phi_\eta(t_i) = k_i$ for $i \in \{1,2\}$. Define $\varphi = t_1 1_E + t_2 1_{E^c} \in B_0(\Sigma, \operatorname{Im} u)$. By (14), we have that

$$\begin{aligned}h\left(\lambda k_1 + (1-\lambda) k_2\right) &= h\left(\lambda\phi_\eta(t_1) + (1-\lambda)\phi_\eta(t_2)\right)\\ &= h\left(\lambda\phi_\eta\left(\int \varphi d\delta_{s_1}\right) + (1-\lambda)\phi_\eta\left(\int \varphi d\delta_{s_2}\right)\right)\\ &= h\left(\int \phi_\eta\left(\int \varphi dp\right) d\mu\right) \geq \int h\left(\phi_\eta\left(\int \varphi dp\right)\right) d\mu\\ &= \lambda h\left(\phi_\eta\left(\int \varphi d\delta_{s_1}\right)\right) + (1-\lambda) h\left(\phi_\eta\left(\int \varphi d\delta_{s_2}\right)\right)\\ &= \lambda h\left(\phi_\eta(t_1)\right) + (1-\lambda) h\left(\phi_\eta(t_2)\right) = \lambda h(k_1) + (1-\lambda) h(k_2),\end{aligned}$$

proving that h is concave and ϕ is IRRA. ■

Proof of Proposition 7. Since $\succsim$ is a smooth ambiguity preference, it admits a canonical representation (u, I) where I is as in (5). Since $\succsim$ is CARA and risk averse and $b \leq 0$, we also have that I is defined over $B_0(\Sigma, (-\infty, 0)) \supseteq B_0(\Sigma, \operatorname{Im} u)$. The functional $\hat{I}$: $B_0(\Sigma, (0,\infty)) \to (0,\infty)$ defined by

$$\hat{I}(\varphi) = \left(\int\left(\int \varphi dp\right)^\gamma d\mu\right)^{\frac{1}{\gamma}} \qquad \forall \varphi \in B_0\left(\Sigma, (0,\infty)\right).$$

is normalized, monotone, continuous, positively homogeneous, and quasiconvex. It follows that $I : B_0(\Sigma, (-\infty, 0)) \to \mathbb{R}$, which is such that $I(\varphi) = -\hat{I}(-\varphi)$, is normalized, monotone,

[53] If $s \in S$, then we denote by δ_s the Dirac at s. We denote by $\mathcal{B}$ the Borel σ-algebra over Δ.

continuous, positively homogeneous, and quasiconcave. In particular, by [10, Proposition 7 and its proof, WP version, Carlo Alberto Notebook 80], it is concave. By Corollary 5 and since $\succsim$ is CARA and risk averse, the statement easily follows. ■

Proof of Proposition 8. Since $\succsim$ is constant absolute ambiguity averse, then $\succsim$ is CARA and I is either constant additive or affine at b, depending on $\succsim$ being risk neutral or not. As usual, without loss of generality we can normalize $a = 1$ and $b = 0$ (see equation (4)). In both cases, it follows that

$$\begin{aligned} \beta \mapsto v^{-1}\left(I\left(v\left(\beta r+(w-\beta) r_f\right)\right)\right) &= v^{-1}\left(I\left(v\left(w r_f+\beta\left(r-r_f\right)\right)\right)\right) \\ &= v^{-1}\left(I\left(v\left(\beta\left(r-r_f\right)\right)\right)\right)+w r_f . \end{aligned}$$

Thus, for each $w \in (0,\infty)$, maximizing $\beta \mapsto I\left(v\left(\beta r+(w-\beta) r_f\right)\right)$ subject to $\beta \in [0,w]$ is equivalent to maximize $\beta \mapsto v^{-1}\left(I\left(v\left(\beta\left(r-r_f\right)\right)\right)\right)$ subject to $\beta \in [0,w]$. Define $f : [0,\infty) \to \mathbb{R}$ by $f(\beta) = v^{-1}\left(I\left(v\left(\beta\left(r-r_f\right)\right)\right)\right)$ for all $\beta \geq 0$. Let $w' > w$. We have two cases:

1. $\beta^*(w') \geq w$. This implies that $\beta^*(w') \geq w \geq \beta^*(w)$.
2. $\beta^*(w') < w$. Since $\beta^*(w')$ maximizes f on $[0,w']$ and $0 \leq \beta^*(w) \leq w \leq w'$, we have that $f(\beta^*(w')) \geq f(\beta^*(w))$. Since $\beta^*(w)$ maximizes f on $[0,w]$ and $0 \leq \beta^*(w') < w$, we have that $f(\beta^*(w)) \geq f(\beta^*(w'))$. This implies that $\beta^*(w')$ is a maximizer of f on $[0,w]$. Since the solution of (6) is unique for all $w > 0$, we can conclude that $\beta^*(w') = \beta^*(w)$.

Points 1 and 2 yield the main statement.

Note that if $\succsim$ is risk averse and uncertainty averse, it follows that $f(\beta) = v^{-1}\left(I\left(v\left(\beta\left(r-r_f\right)\right)\right)\right)$ is quasiconcave on $[0,\infty)$. Let $w' > w$. By contradiction, assume that $\beta^*(w') \neq \beta^*(w)$. From the previous part of the proof, it follows that $\beta^*(w') > \beta^*(w)$. Consider $\hat{\beta} \in (\beta^*(w), \min\{w, \beta^*(w')\}) \subseteq (0,w) \subseteq (0,w')$. Since $\beta^*(w), \hat{\beta} \in (0,w)$ and the former is the unique maximizer of f on $[0,w]$, it follows that $f(\beta^*(w)) > f\left(\hat{\beta}\right)$. Similarly, since $\beta^*(w'), \hat{\beta} \in [0,w']$ and the former is the unique maximizer of f on $[0,w']$, it follows that $f(\beta^*(w')) > f\left(\hat{\beta}\right)$. On the one hand, we can conclude that $\min\{f(\beta^*(w)), f(\beta^*(w'))\} > f\left(\hat{\beta}\right)$. On the other hand, by construction of $\hat{\beta}$, we also have that there exists $\lambda \in (0,1)$ such that

$$\hat{\beta} = \lambda\beta^*(w) + (1-\lambda)\beta^*\left(w'\right).$$

Since f is quasiconcave, this implies that $f\left(\hat{\beta}\right) \geq \min\{f(\beta^*(w)), f(\beta^*(w'))\}$, a contradiction. ■

Proof of Proposition 9. Since $\succsim$ is risk neutral, without loss of generality, let v be the identity. Note that

$$\beta \mapsto I\left(v\left(\beta r+(w-\beta) r_f\right)\right) = \phi^{-1}\left(\int \phi\left(\beta \int r dp+(w-\beta) r_f\right) d\mu\right).$$

Define $\hat{r} : \Delta \to \mathbb{R}$ by $\hat{r}(p) = \int r dp$ for all $p \in \Delta$. We have that $\hat{r} \geq 0$ is a bounded, real-valued, Borel measurable function. It follows that the problem in (6) is equivalent to solve

$$\max \left(\int \phi \left(\beta \hat{r} + (w - \beta) r_f \right) d\mu \right) \text{ subject to } \beta \in [0, w],$$

which is mathematically equivalent to the usual expected utility portfolio choice problem. Since ϕ is concave and DARA, it is twice continuously differentiable and such that $\phi' > 0$, and $\beta^* (w) \in (0, w)$ with $w > 0$, we have that (7) holds. ■

Proof of Proposition 10. Since $\succsim$ is a risk averse multiplier preference, note that

$$\beta \mapsto I \left(v \left(\beta r + (w - \beta) r_f \right) \right) = \phi^{-1} \left(\int \phi \left(v \left(\beta r + (w - \beta) r_f \right) \right) dq \right)$$

where $v(c) = -a \frac{1}{\alpha} e^{-\alpha c} + b$ for all $c \in \mathbb{R}$, with $\alpha, a > 0$ and $b \in \mathbb{R}$, and $\phi(t) = -e^{-\theta t}$ for all $t \in \mathbb{R}$, with $\theta > 0$. Define $\hat{v} = \phi \circ v : \mathbb{R} \to \mathbb{R}$. It follows that the problem in (6) is equivalent to solve

$$\max \left(\int \hat{v} \left(\beta r + (w - \beta) r_f \right) dq \right) \text{ subject to } \beta \in [0, w],$$

which is mathematically equivalent to the usual expected utility portfolio choice problem. Since $\hat{v}$ is concave and DARA, it is twice continuously differentiable and such that $\hat{v}' > 0$, and $\beta^* (w) \in (0, w)$ with $w > 0$, we have that (7) holds. ■

B.3.1 Non-CARA preferences

Let T be either $\mathbb{R}$ or $\mathbb{R}_{++}$. Consider a rational preference $\succsim$ with canonical representation (u, I), where u has von Neumann-Morgenstern utility $v : T \to \mathbb{R}$ with v strictly increasing and continuous. Fix $w \in T$. For this section, define $v_w : T \to \mathbb{R}$ to be such that $v_w (c) = v (c + w)$ for all $c \in T$ and $u_w : \Delta_0 (T) \to \mathbb{R}$ to be the associated expected utility. Note that $\operatorname{Im} u_w = \operatorname{Im} v_w \subseteq \operatorname{Im} v = \operatorname{Im} u$ for all $w \in T$. Note that if $T = \mathbb{R}_{++}$, then we have to slightly modify the definition of x^w. Indeed, using the current definition, x^w is defined to be such that

$$x^w (c) = x (c - w) \qquad \forall c \in \mathbb{R}_{++}.$$

If $T = \mathbb{R}_{++}$, in this way, we can only define $x^w (c)$ for all values $c > w$. To overcome such an issue, we set $x^w (c) = 0$ for all the values $c \in (0, w]$. Clearly, this convention is in line with our interpretation of x^w. Since if the decision maker has wealth $w > 0$ and outcomes are strictly positive, he cannot get a final wealth level which is smaller than or equal to w. In light of this, note that we always have

$$u (x^w) = \sum_{c \in T} v (c) x^w (c) = \sum_{c \in T} v (c + w) x (c) = \sum_{c \in T} v_w (c) x (c) = u_w (x) \quad \forall x \in \Delta_0 (T).$$

It follows that

$$I\left(u\left(f^{w}\right)\right)=I\left(u_{w}\left(f\right)\right)\qquad\forall f\in\mathcal{F}.$$

Moreover, as in the previous part of the paper, for any $w\in T$, we define $\succsim^{w}$ on $\mathcal{F}$ by

$$f\succsim^{w}g\overset{def}{\iff}f^{w}\succsim g^{w}.$$

Define $c_{w}:\mathcal{F}\to T$ to be such that $c_{w}\left(f\right)=v_{w}^{-1}\left(I\left(u_{w}\left(f\right)\right)\right)$ for all $f\in\mathcal{F}$. We can conclude that

$$\begin{aligned}f\succsim^{w}g&\iff f^{w}\succsim g^{w}\iff I\left(u\left(f^{w}\right)\right)\geq I\left(u\left(g^{w}\right)\right)\\&\iff I\left(u_{w}\left(f\right)\right)\geq I\left(u_{w}\left(g\right)\right)\iff c_{w}\left(f\right)\geq c_{w}\left(g\right).\end{aligned}$$

Proof of Proposition 11. Let $w',w\in T$ be such that $w'>w$. Recall that $\succsim$ is CARA. Hence, $\succsim^{w}$ and $\succsim^{w'}$ agree on $\Delta_{0}\left(T\right)$.

(i) implies (ii). Since $\succsim$ is decreasing (resp., increasing) absolute uncertainty averse, we have that $c_{w}\left(f\right)\leq c_{w'}\left(f\right)$ (resp., $c_{w}\left(f\right)\geq c_{w'}\left(f\right)$) for all $f\in\mathcal{F}$. Consider $f\in\mathcal{F}$ and $x\in\Delta_{0}\left(T\right)$. We need to show that

$$f\succsim^{w}x\implies f\succsim^{w'}x\text{ (resp., }f\succsim^{w'}x\implies f\succsim^{w}x).$$

Assume that $f\succsim^{w}x$ (resp., $f\succsim^{w'}x$). It follows that $c_{w'}\left(f\right)\geq c_{w}\left(f\right)\geq c_{w}\left(x\right)$ ($c_{w}\left(f\right)\geq c_{w'}\left(f\right)\geq c_{w'}\left(x\right)$), that is, $\delta_{c_{w'}(f)}\succsim^{w}x$ (resp., $\delta_{c_{w}(f)}\succsim^{w'}x$). Since $\succsim^{w}$ coincides with $\succsim^{w'}$ on $\Delta_{0}\left(T\right)$, this implies that $\delta_{c_{w'}(f)}\succsim^{w'}x$ (resp., $\delta_{c_{w}(f)}\succsim^{w}x$), that is, $c_{w'}\left(f\right)\geq c_{w'}\left(x\right)$ (resp., $c_{w}\left(f\right)\geq c_{w}\left(x\right)$) and $f\succsim^{w'}x$ (resp., $f\succsim^{w}x$). This proves the implication for the two distinct cases of decreasing and increasing attitudes. Since having constant attitudes means having both of the above features, the full implication follows.

(ii) implies (i). It is trivial since the notion of more ambiguity averse implies the notion of more uncertainty averse. ■

Lemma 3 *Let $v:T\to\mathbb{R}$ be strictly increasing, continuous, and concave (resp., convex) and $w',w\in T$ be such that $w'>w$. If v is DARA, then there exists a strictly increasing and convex function $\phi:\operatorname{Im}v_{w}\to\operatorname{Im}v_{w'}$ such that*

$$v\left(c+w'\right)=\phi\left(v\left(c+w\right)\right)\qquad\forall c\in T$$

where $0\leq\phi'_{+}\left(t\right)\leq1$ (resp., $\phi'_{+}\left(t\right)\geq1$) for all $t\in\operatorname{Im}v_{w}$. Moreover, $\phi=v_{w'}\circ v_{w}^{-1}$.

Proof. Consider $w,w'\in T$ such that $w'>w$. Since v is DARA, it follows that $c\mapsto v_{w}\left(c\right)=v\left(c+w\right)$ is more risk averse than $c\mapsto v_{w'}\left(c\right)=v\left(c+w'\right)$. Thus, there exists a strictly increasing and convex function $\phi:\operatorname{Im}v_{w}\to\operatorname{Im}v_{w'}$ such that

$$v\left(c+w'\right)=\phi\left(v\left(c+w\right)\right)\qquad\forall c\in T.$$

Let $c' > c > 0$. Since v is strictly increasing, define $h = v(c' + w) - v(c + w) > 0$. It follows that

$$v\left(c' + w'\right) - v\left(c + w'\right) = \phi\left(v\left(c' + w\right)\right) - \phi\left(v\left(c + w\right)\right) = \phi\left(v\left(c + w\right) + h\right) - \phi\left(v\left(c + w\right)\right),$$

that is,

$$\frac{v(c' + w') - v(c + w')}{v(c' + w) - v(c + w)} = \frac{v(c' + w') - v(c + w')}{h} = \frac{\phi(v(c + w) + h) - \phi(v(c + w))}{h}.$$

Since v is concave (resp., convex) and v and ϕ are strictly increasing, it follows that

$$(\text{resp., } 1 \leq\ )\ 1 \geq \frac{v(c' + w') - v(c + w')}{v(c' + w) - v(c + w)} = \frac{\phi(v(c + w) + h) - \phi(v(c + w))}{h} \geq 0. \qquad (15)$$

If we define $c'_n = c + \frac{1}{n}$ for all $n \in \mathbb{N}$, we have that $0 < h_n = v(c'_n + w) - v(c + w) \to 0$. By (15) and since ϕ is convex and c and c' were arbitrarily chosen, this implies that $0 \leq \phi'_+(v(c + w)) \leq 1$ (resp., $\phi'_+(v(c + w)) \geq 1$) for all $c \in T$. Consider $t \in \operatorname{Im} v_w$. It follows that there exists $\hat{c} \in T$ such that $v(\hat{c} + w) = v_w(\hat{c}) = t$. It follows that $\phi'_+(t) = \phi'_+(v_w(\hat{c})) = \phi'_+(v(\hat{c} + w)) \in [0, 1]$ (resp., ≥ 1), proving the statement. ■

Lemma 4 *Let $v : T \to \mathbb{R}$ be strictly increasing, continuous, and concave (resp., convex) and $w', w \in T$ be such that $w' > w$. If v is DARA and $\phi = v_{w'} \circ v_w^{-1}$, then there exist $\{a_\alpha\}_{\alpha \in A} \subseteq (0, 1]$ (resp., $\{a_\alpha\}_{\alpha \in A} \subseteq [1, +\infty)$) and $\{b_\alpha\}_{\alpha \in A} \subseteq \mathbb{R}$ such that*

$$\phi(t) = \sup_{\alpha \in A} \{a_\alpha t + b_\alpha\} \qquad \forall t \in \operatorname{Im} v_w$$

Moreover, if $\operatorname{Im} v = (0, \infty)$ and v is concave, then $b_\alpha > 0$ for all $\alpha \in A$.

Proof. Recall that $\phi : \operatorname{Im} v_w \to \operatorname{Im} v_{w'}$ is strictly increasing and convex. By [6, Corollary 2.1.3] and since ϕ is convex, we have that for each $t^* \in \operatorname{Im} v_w$

$$\phi(t) \geq \phi(t^*) + \phi'_+(t^*)(t - t^*) \qquad \forall t \in \operatorname{Im} v_w.$$

This implies that

$$\phi(t) = \sup_{t^* \in \operatorname{Im} v_w} \left\{\phi'_+(t^*)\, t + \left(\phi(t^*) - \phi'_+(t^*)\, t^*\right)\right\} \qquad \forall t \in \operatorname{Im} v_w.$$

By Lemma 3, if we define $A = \operatorname{Im} v_w$ and $a_{t^*} = \phi'_+(t^*)$ as well as $b_{t^*} = \phi(t^*) - \phi'_+(t^*)\, t^*$, the main statement follows.[54] Next, assume that $\operatorname{Im} v = (0, \infty)$ and v is concave. Note that

[54] For the case in which v is concave, Lemma 3 only says that $\phi'_+(t^*) \geq 0$ for all $t^* \in \operatorname{Im} v_w$. At the same time, since $\operatorname{Im} v_w$ is an open interval and ϕ is convex and strictly increasing, we cannot have $\phi'_+(t^*) = 0$ for any $t^* \in \operatorname{Im} v_w$. Otherwise, $0 \leq \phi'_+(\hat{t}) \leq \phi'_+(t^*) = 0$ for all $\hat{t} \in \operatorname{Im} v_w$ such that $\hat{t} < t^*$, yielding that ϕ is constant on $\operatorname{Im} v_w \cap (-\infty, t^*) \neq \emptyset$, which is a contradiction with ϕ being strictly increasing.

for each $t^* \in \operatorname{Im} v_w$ we have that $v(c+w) = t^*$ for some $c \in T$. Since v is strictly increasing, this implies that

$$\phi(t^*) = \phi(v(c+w)) = v\left(c+w'\right) > v(c+w) = t^* \quad \forall t^* \in \operatorname{Im} v_w \subseteq (0,\infty). \tag{16}$$

By (16) and since v is concave, we have that $b_{t^*} = \phi(t^*) - \phi'_+(t^*)\, t^* > t^*\left(1 - \phi'_+(t^*)\right) \geq 0$ for all $t^* \in \operatorname{Im} v_w$. ■

Lemma 5 *Let $\succsim$ be a rational preference with representation (u, I). If $\succsim$ is DARA and I is such that for each $w, w' \in T$ with $w' > w$*

$$I(\varphi) \leq \phi^{-1}(I(\phi(\varphi))) \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u_w) \subseteq B_0(\Sigma, \operatorname{Im} u), \tag{17}$$

where $\phi = v_{w'} \circ v_w^{-1} : \operatorname{Im} v_w \to \operatorname{Im} v_{w'}$, then $\succsim$ is decreasing absolute uncertainty averse.

Proof. Let $w', w \in T$ be such that $w' > w$. Consider $f \in \mathcal{F}$. Since $\succsim$ is DARA, note that $c_w(f(s)) \leq c_{w'}(f(s))$ for all $s \in S$. Define $\psi \in B_0(\Sigma, T)$ to be such that $\psi = v_w^{-1}(u_w(f))$. Define also $\psi' \in B_0(\Sigma, T)$ to be such that $\psi' = v_{w'}^{-1}(u_{w'}(f))$. It follows that $\psi \leq \psi'$. Define $\varphi = u_w(f)$ and $\varphi' = u_{w'}(f)$. We have that

$$u_{w'}(f) = v_{w'}\left(\psi'\right) \text{ and } \psi = v_w^{-1}(\varphi)$$

as well as

$$\phi(\varphi) = v_{w'}\left(v_w^{-1}(\varphi)\right) = v_{w'}(\psi) \in B_0(\Sigma, \operatorname{Im} u_{w'}) \subseteq B_0(\Sigma, \operatorname{Im} u)$$

By (17), it follows that

$$\begin{aligned} I(u_w(f)) &= I(\varphi) \leq \phi^{-1}(I(\phi(\varphi))) = \phi^{-1}(I(v_{w'}(\psi))) \leq \phi^{-1}\left(I\left(v_{w'}\left(\psi'\right)\right)\right) \\ &= \phi^{-1}(I(u_{w'}(f))), \end{aligned}$$

that is, $I(u_w(f)) \leq \phi^{-1}(I(u_{w'}(f))) = v_w\left(v_{w'}^{-1}(I(u_{w'}(f)))\right)$ which yields

$$c_w(f) = v_w^{-1}(I(u_w(f))) \leq v_{w'}^{-1}(I(u_{w'}(f))) = c_{w'}(f).$$

Since f was arbitrarily chosen, the statement follows. ■

Proof of Proposition 12. Since $\succsim$ is a rational preference that satisfies Weak C-Independence, it follows that $\succsim$ admits a canonical representation (u, I) where $I : B_0(\Sigma, \operatorname{Im} u) \to \mathbb{R}$ is also normalized, monotone, and constant additive. Since $\succsim$ satisfies Unboundedness, $\operatorname{Im} u$ is unbounded. Since T is open and v is strictly increasing and continuous, $\operatorname{Im} u = \operatorname{Im} v$ is an open set. We thus have three cases: either $\operatorname{Im} u = (-\infty, b)$ or $\operatorname{Im} u = (b, +\infty)$ or $\operatorname{Im} u = (-\infty, \infty)$. In the first two cases, without loss of generality, we can assume that $b = 0$. In this way, by Proposition 19, concavity at b (resp., convexity at b) becomes superhomogeneity (resp., subhomogeneity) of I. In the third case, by assumption, concavity at b (resp., convexity at b) is superhomogeneity (resp., subhomogeneity) of I. By [11] and since $\operatorname{Im} u$ is unbounded, I admits a (unique) extension to $B_0(\Sigma)$ which is normalized, monotone, and constant additive. We will denote the extension by $\bar{I}$.

(i). Given Lemma 5, we only need to prove that I satisfies

$$\phi(I(\varphi)) \leq I(\phi(\varphi)) \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u_w) \subseteq B_0(\Sigma, \operatorname{Im} u)$$

where $\phi = v_{w'} \circ v_w^{-1}$ when $w, w' \in T$ with $w' > w$. By Lemma 4 and since $\succsim$ is risk averse, it follows that $\phi(t) = \sup_{\alpha \in A} \{a_\alpha t + b_\alpha\}$ where $\{a_\alpha\}_{\alpha \in A} \subseteq (0, 1]$ and $\{b_\alpha\}_{\alpha \in A} \subseteq \mathbb{R}$ for all $t \in \operatorname{Im} u_w$. Let $\varphi \in B_0(\Sigma, \operatorname{Im} u_w) \subseteq B_0(\Sigma, \operatorname{Im} u)$. We have that $\phi(\varphi) \in B_0(\Sigma, \operatorname{Im} u_{w'}) \subseteq B_0(\Sigma, \operatorname{Im} u)$ and $a_\alpha \varphi \in B_0(\Sigma, \operatorname{Im} u)$ as well as $\phi(\varphi) \geq a_\alpha \varphi + b_\alpha \in B_0(\Sigma)$ for all $\alpha \in A$. We can conclude that for each $\alpha \in A$

$$I(\phi(\varphi)) = \bar{I}(\phi(\varphi)) \geq \bar{I}(a_\alpha \varphi + b_\alpha) = \bar{I}(a_\alpha \varphi) + b_\alpha = I(a_\alpha \varphi) + b_\alpha \geq a_\alpha I(\varphi) + b_\alpha.$$

We can conclude that

$$I(\phi(\varphi)) \geq \sup_{\alpha \in A} \{a_\alpha I(\varphi) + b_\alpha\} = \phi(I(\varphi)),$$

proving the statement.

(ii). Given Lemma 5, we only need to prove that I satisfies

$$\phi(I(\varphi)) \leq I(\phi(\varphi)) \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u_w) \subseteq B_0(\Sigma, \operatorname{Im} u)$$

where $\phi = v_{w'} \circ v_w^{-1}$ when $w, w' \in T$ with $w' > w$. By Lemma 4 and since $\succsim$ is risk loving, it follows that $\phi(t) = \sup_{\alpha \in A} \{a_\alpha t + b_\alpha\}$ where $\{a_\alpha\}_{\alpha \in A} \subseteq [1, \infty)$ and $\{b_\alpha\}_{\alpha \in A} \subseteq \mathbb{R}$ for all $t \in \operatorname{Im} u_w$. Let $\varphi \in B_0(\Sigma, \operatorname{Im} u_w) \subseteq B_0(\Sigma, \operatorname{Im} u)$. We have that $\phi(\varphi) \in B_0(\Sigma, \operatorname{Im} u_{w'}) \subseteq B_0(\Sigma, \operatorname{Im} u)$ and $a_\alpha \varphi \in B_0(\Sigma, \operatorname{Im} u)$ as well as $\phi(\varphi) \geq a_\alpha \varphi + b_\alpha \in B_0(\Sigma)$ for all $\alpha \in A$. We can conclude that for each $\alpha \in A$

$$I(\phi(\varphi)) = \bar{I}(\phi(\varphi)) \geq \bar{I}(a_\alpha \varphi + b_\alpha) = \bar{I}(a_\alpha \varphi) + b_\alpha = I(a_\alpha \varphi) + b_\alpha \geq a_\alpha I(\varphi) + b_\alpha$$

where the last inequality follows from the fact that subhomogeneity implies that $I(\lambda \varphi) \geq \lambda I(\varphi)$ for all $\lambda \in [1, \infty)$.[55] We can conclude that

$$I(\phi(\varphi)) \geq \sup_{\alpha \in A} \{a_\alpha I(\varphi) + b_\alpha\} = \phi(I(\varphi)),$$

[55] For $\lambda = 1$, the inequality is obvious. Let $\lambda > 1$ and $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. It follows that $\frac{1}{\lambda} \in (0, 1)$. We have that

$$I(\varphi) = I\left(\frac{1}{\lambda}(\lambda \varphi)\right) \leq \frac{1}{\lambda} I(\lambda \varphi) \implies \lambda I(\varphi) \leq I(\lambda \varphi).$$

proving the statement. ■

Proof of Proposition 13. (i) implies (ii). Let $w, w' \in T$ such that $w' > w$. By assumption, $\succsim^w$ is more uncertainty averse than $\succsim^{w'}$. By Remark 3, $\succsim^w$ is more risk averse than $\succsim^{w'}$, proving that $\succsim$ is DARA.

(ii) implies (i). Call (u, I) the representation of $\succsim$. By [17], there also exists a normalized, monotone, and continuous functional $\hat{I} : B_0(\Sigma) \to \mathbb{R}$ such that $\hat{I}(\lambda\varphi + k) = \lambda\hat{I}(\varphi) + k$ for all $\lambda > 0$, for all $k \in \mathbb{R}$, and for all $\varphi \in B_0(\Sigma)$ and such that $f \succsim g$ if and only if $\hat{I}(u(f)) \geq \hat{I}(u(g))$. It follows that $\hat{I}$ and I coincide on $B_0(\Sigma, \operatorname{Im} u)$. Given Lemma 5, we only need to prove that I satisfies

$$\phi(I(\varphi)) \leq I(\phi(\varphi)) \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u_w) \subseteq B_0(\Sigma, \operatorname{Im} u)$$

where $\phi = v_{w'} \circ v_w^{-1}$ when $w, w' \in T$ with $w' > w$. Since $\succsim$ is DARA, ϕ is strictly increasing and convex. It follows that $\phi(t) = \sup_{\alpha \in A}\{a_\alpha t + b_\alpha\}$ where $\{a_\alpha\}_{\alpha \in A} \subseteq \mathbb{R}_+$ and $\{b_\alpha\}_{\alpha \in A} \subseteq \mathbb{R}$. Let $\varphi \in B_0(\Sigma, \operatorname{Im} u_w) \subseteq B_0(\Sigma, \operatorname{Im} u)$. We have that $\phi(\varphi) \in B_0(\Sigma, \operatorname{Im} u_{w'}) \subseteq B_0(\Sigma, \operatorname{Im} u)$ and $\phi(\varphi) \geq a_\alpha \varphi + b_\alpha \in B_0(\Sigma)$ for all $\alpha \in A$. We can conclude that for each $\alpha \in A$

$$I(\phi(\varphi)) = \hat{I}(\phi(\varphi)) \geq \hat{I}(a_\alpha \varphi + b_\alpha) = a_\alpha \hat{I}(\varphi) + b_\alpha = a_\alpha I(\varphi) + b_\alpha.$$

We can conclude that

$$I(\phi(\varphi)) \geq \sup_{\alpha \in A}\{a_\alpha I(\varphi) + b_\alpha\} = \phi(I(\varphi)),$$

proving the statement. ■

Proof of Proposition 14. Call (u, I) the representation of $\succsim$. Without loss of generality, by Proposition 19, we can assume that $b = 0$. Given Lemma 5, we only need to prove that I satisfies

$$\phi(I(\varphi)) \leq I(\phi(\varphi)) \qquad \forall \varphi \in B_0(\Sigma, \operatorname{Im} u_w) \subseteq B_0(\Sigma, \operatorname{Im} u)$$

where $\phi = v_{w'} \circ v_w^{-1}$ when $w, w' \in T$ with $w' > w$. Since I is superhomogeneous and constant superadditive, we have that

$$I\left(a\varphi + \hat{b}\right) \geq I(a\varphi) + \hat{b} \geq aI(\varphi) + \hat{b} \qquad \forall a \in (0,1], \forall \hat{b} \geq 0, \forall \varphi \in B_0(\Sigma, \operatorname{Im} u). \tag{18}$$

By Lemma 4 and since $\succsim$ is DARA and risk averse, it follows that $\phi(t) = \sup_{\alpha \in A}\{a_\alpha t + b_\alpha\}$ where $\{a_\alpha\}_{\alpha \in A} \subseteq (0,1]$ and $\{b_\alpha\}_{\alpha \in A} \subseteq \mathbb{R}_{++}$. Let $\varphi \in B_0(\Sigma, \operatorname{Im} u_w) \subseteq B_0(\Sigma, \operatorname{Im} u)$. By (18), we have that

$$I(\phi(\varphi)) \geq I(a_\alpha \varphi + b_\alpha) \geq a_\alpha I(\varphi) + b_\alpha \qquad \forall \alpha \in A.$$

We can conclude that

$$I(\phi(\varphi)) \geq \sup_{\alpha \in A}\{a_\alpha I(\varphi) + b_\alpha\} = \phi(I(\varphi)),$$

proving the statement. ■

Proof of Corollary 8. Call (u, I) the representation of $\succsim$. Without loss of generality, by Proposition 19, we can assume that $b = 0$. In light of Proposition 14, we only need to show concavity at b (that is, superhomogeneity) and constant superadditivity. Consider $\lambda \in (0,1)$ and $k \geq 0$ as well as $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Note that there exists a sequence $\{k_n\}_{n\in\mathbb{N}} \subseteq (0,\infty)$ such that $k_n \to k$. Moreover, we have that $\lambda\varphi, \lambda\varphi + k \in B_0(\Sigma, \operatorname{Im} u)$ as well as $\lambda\varphi + k_n, \frac{k_n}{1-\lambda} \in B_0(\Sigma, \operatorname{Im} u)$ for all $n \in \mathbb{N}$. Since I is normalized, continuous, and concave, this implies that

$$\begin{aligned} I(\lambda\varphi + k) &= \lim_n I(\lambda\varphi + k_n) = \lim_n I\left(\lambda\varphi + (1-\lambda)\frac{k_n}{1-\lambda}\right) \\ &\geq \lim_n \left[\lambda I(\varphi) + (1-\lambda) I\left(\frac{k_n}{1-\lambda}\right)\right] = \lim_n \left[\lambda I(\varphi) + (1-\lambda)\frac{k_n}{1-\lambda}\right] \\ &= \lambda I(\varphi) + \lim_n k_n = \lambda I(\varphi) + k. \end{aligned}$$

Since λ, k, and φ were arbitrarily chosen and I is continuous, I is superhomogeneous and constant superadditive and the statement follows. ■

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