Revenue-capped efficient auctions*

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Abstract

We study an auction that maximizes the expected social surplus under an upper-bound constraint on the seller's expected revenue, which we call a *revenue cap*. Such a constrained-efficient auction may arise, for example, when: (i) the auction designer is "pro-buyer", that is, he maximizes the weighted sum of the buyers' and seller's auction payoffs, where the weight for the buyers is greater than that for the seller; (ii) the auction designer maximizes the (unweighted) total surplus in a multi-unit auction in which the number of units the seller owns is private information; or (iii) multiple sellers compete to attract buyers before the auction. We characterize the mechanisms for constrained-efficient auctions and identify their important properties. First, the seller sets no reserve price and sells the good for sure. Second, with a nontrivial

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revenue cap, "bunching" is necessary. Finally, with a sufficiently severe revenue cap, the constrained-efficient auction has a *bid cap*, so that bunching occurs at least "at the top," that is, "no distortion at the top" fails.

1 Introduction

We consider a single-object auction in which buyers have independent and private values. In the literature on the theory of auction design, much of the research focuses on auctions in which the designer's objective is a total surplus, revenue, or a convex combination of both. Under a standard "regularity" condition,[1] the designer's maximization problem reduces to unconstrained maximization of the expected virtual surplus and the solution is obtained by pointwise maximization. These objectives can be considered as weakly "pro-seller", because the designer essentially maximizes a weighted sum of the buyers' payoffs (their total values minus revenue) and the seller's payoffs (revenue), where the weight for the seller is weakly greater than that for the buyers.

In this paper, we consider the converse objective of the designer, which we believe is equally important. Namely, the auction designer is "pro-buyer" in the sense that his objective is a weighted sum of the buyers' and seller's payoffs where the weight for the seller is *smaller*: equivalently, his objective is a weighted sum of the total surplus and the revenue, where the weight for the revenue is *negative*.[2] Using the standard argument, we show that a similar mathematical problem also arises when the designer maximizes the total surplus under an upper-bound constraint for the seller's revenue, which we call a "revenue cap".[3]

The pro-buyer objective is reasonable, for example, if the auction designer is a social planner who tends to choose pro-consumer policies for political reasons, as discussed by Baron and Myerson (1982) and Baron (1988), in the context of the regulation of a monopolistic

[1]For example, the monotone virtual value condition in Myerson (1981).

[2]The extreme case is the maximization of the buyers' payoffs. McAfee and McMillan (1992) show that random allocation maximizes the buyers' payoffs in the presence of cartels.

[3]The same problem arises when the designer minimizes the revenue under a lower-bound constraint for the total surplus.

firm.[4] Even if the weights are potentially the same, the designer's problem may be represented in the same manner if there are some transaction costs for payments from buyers to the seller. For example, Caillaud et al. (1988) and Laffont and Tirole (1986) consider a situation in which the government, in regulating a monopolist, in principle aims to maximize the sum of the consumer surplus and the producer surplus, but the transaction costs distort the weights, which are higher for the consumers.[5] The designer also takes a pro-buyer objective when the marginal utility of money is greater for buyers than for the seller.[6] For example, a buyer may have more investment opportunities than the seller but face a liquidity problem, and hence the buyer's value of keeping money at hand is higher than the seller's.

Although closely related, our "revenue cap" formulation is relevant to the seller's incentive problem.[7] An important example is a multi-unit auction in which the number of units owned by the seller is private information. Suppose that the seller has two units of homogeneous objects and there are two buyers, each of whom demands at most one unit. An efficient mechanism allocates one unit for each buyer, which yields zero revenue with probability one. Thus, the seller is better off by throwing away one unit and pretending to have just one unit for sale. Given this incentive concern, the designer needs to modify the auction mechanism by decreasing the expected revenue in the single-unit auction, and/or increasing the expected revenue in the two-unit auction. More specifically, a constrained-efficient auction mechanism is characterized by $R \in \mathbb{R}_+$ such that: (i) it maximizes the expected surplus subject to revenue cap R with a single unit; and (ii) it maximizes the expected surplus subject to

[4]Besley and Coate (2003) report empirical evidence showing that regulators are more likely to choose pro-consumer policies if they are elected.

[5]Engel et al. (2013) discuss public–private partnerships in which such transaction costs exist, as well as another source of the smaller weight for the producer surplus; if the social planner is the government of a country, the planner will be in favor of consumers because producers are often firms from other countries.

[6]DeMarzo and Duffie (1999) analyze a security design problem in which the issuer's discount rate is higher than the market rate because of credit constraints.

[7]This formulation of the revenue cap is also related to the price cap in the context of the regulation of a monopolistic firm. Armstrong et al. (1995) analyze the expected-revenue regulation when non-linear pricing is allowed. Since they consider divisible goods mainly with a convex cost function, the firm increases the quantity of the production to fulfill the cap. In our model, the good is indivisible and such manipulation of the quantity of goods is impossible. See Armstrong and Sappington (2007) for a survey of the optimal regulation of a monopolist.

"revenue floor" R with two units. Since problem (ii) can be solved using the standard approach (such as in Myerson (1981)), we concentrate on problem (i) here. We formally discuss this multi-unit auction problem in Section 4.1.

Finally, even if an auction mechanism is designed by a revenue-maximizing seller, if buyers have outside options at the ex ante stage, and if those outside options are significant, then the seller's problem can be rewritten as a maximization of a weighted sum of the buyers' payoffs and revenue, where the revenue has a smaller weight. For example, such ex ante outside options may be relevant when there are multiple sellers competing to attract buyers.[8]

On the basis of the standard envelope conditions, the expected revenue can be represented as the expected virtual surplus. Therefore, our problem with a "revenue cap" reduces to the maximization of the expected surplus *minus* (a positive constant weight times) the expected virtual surplus subject to the standard monotonicity constraints and the revenue-cap constraint. This weight for the virtual surplus is uniquely determined by the given revenue cap. Constrained-efficient mechanisms given a negative weight for the virtual surplus have very different properties to those in the standard case with a positive weight. First, the above weighted sum is positive for all types, which implies that the seller always sells the object. This contrasts with the standard revenue-maximizing auction with a reserve price, in which no buyer obtains the object if every buyer's value is lower than the reserve price.

Second, with a nontrivial revenue cap, bunching is necessary. To provide some intuition, consider the designer's maximization of expected total surplus under a constraint whereby the seller's expected revenue does not exceed R, where R is strictly between 0 and the expected revenue in a second-price auction. Suppose, conversely, that the above weighted sum is nondecreasing in each buyer's value. Then, as in the standard case in the literature, the solution is obtained by pointwise maximization, and in the current case we obtain a second-price auction. However, this contradicts the revenue-cap constraint. This means that with a nontrivial revenue cap, the above weighted sum cannot be nondecreasing in each buyer's

[8]Peters and Severinov (1997, Section 3) study competition among sellers offering second-price auctions with reserve prices in which each buyer chooses a seller at the ex ante stage (i.e., before knowing his value).

value everywhere, and therefore monotonicity constraints must be binding somewhere.

Third, although the exact location for bunching depends on the shape of the value distribution, we show that if the revenue-cap constraint is sufficiently severe, then bunching must occur (at least) "at the top". As we see later, this essentially means that the constrained-efficient auction mechanism has a "bid cap", that is, the highest possible bid for each bidder is strictly less than his highest possible value. Obviously, such a bid cap lowers efficiency, but nevertheless it effectively constrains revenue.

The third property is of interest for the following reasons. First, it contrasts with the standard auction with a positive weight, which generally exhibits "no distortion at the top".[9] Our result shows that if the designer is sufficiently pro-buyer, then this well-known result no longer holds. Second, it has an important policy implication. Although a bid cap is often imposed in auctions in practice, such as in public procurement (Decarolis, 2009; Ishii, 2008; Eun, 2016) and in online auctions,[10] the standard approach (where "no distortion at the top" robustly holds) fails to provide a rationale for such a policy.[11] Our result provides a rationale by showing a logical connection between a revenue cap, which represents the degree to which the designer is pro-buyer, and a bid cap, which is a specific constraint on the auction rule. Finally, it suggests a potential reason why auctions with "Buy It Now" options have become popular in online auction environments such as eBay and Yahoo auction. For example, Anderson et al. (2008) and Reynolds and Wooders (2009) report that approximately 30–40% of eBay auctions in their data sets employ a "Buy It Now" option. Our result suggests that under harsh competition among multiple sellers to attract buyers, a seller may prefer to use "pro-buyer" auction formats that involve bid-capping.

The remainder of the paper is organized as follows. In Section 2, we introduce our

[9] For this terminology in a principal-agent model, see Fudenberg and Tirole (1991).

[10] For example, "Buy It Now" options in eBay, which provides services in many countries such as in the US, and "Buy Now" options in Yahoo auction, which provides services in several countries such as in Japan.

[11] Budish and Takeyama (2001) and Reynolds and Wooders (2009) show that adding a "Buy It Now" option can raise expected revenue when bidders are risk-averse. Mathews (2004) considers a dynamic environment, and shows that a "Buy It Now" option is useful for extracting rent from impatient buyers. Kirkegaard and Overgaard (2008) consider an alternative dynamic environment with multiple sellers, and show that an early seller has an incentive to use a "Buy It Now" option to avoid competition with late sellers.

model. In Section 3, we present our main results. We consider two applications in Section 4: a multiple-unit auction and ex ante competition among sellers. Section 5 concludes.

2 Model

We consider a single-unit auction with independent and private values, except in Section 4, where we consider a multi-unit auction (Section 4.1) and a simple competing auction environment (Section 4.2). A seller owns an indivisible object (valueless for the seller), and n (≥ 2) buyers potentially wish to buy it. Let $N = \{1, 2, \ldots, n\}$ denote the set of buyers.

For each buyer $i \in N$, his value v_i of the object is independently and identically distributed according to a cumulative distribution function F over $[0, 1]$.[12] We assume that this distribution is associated with a positive and continuously differentiable density function f on $[0, 1]$. Let $\psi(x) = x - \frac{1-F(x)}{f(x)}$ be the *virtual value* function. We denote a typical value profile by $v = (v_1, \ldots, v_n) \in [0, 1]^n$, and for each $i \in N$, let v_{-i} be the profile of the other buyers' values.

Each buyer $i \in N$ has a quasi-linear payoff function: if he obtains the object with probability $q_i \in [0, 1]$ and makes a monetary transfer $t_i \in \mathbb{R}$, then his payoff is $q_i v_i - t_i$. The seller's payoff is revenue $\sum_i t_i$.

A (direct) *auction mechanism* (or a *mechanism*) $(q, t) = (q_i, t_i)_{i \in N}$ comprises an assignment function $q : [0, 1]^n \to [0, 1]^n$ satisfying $\sum_{i \in N} q_i(v) \leq 1$ for all $v \in [0, 1]^n$,[13] and a transfer function $t : [0, 1]^n \to \mathbb{R}^n$. An assignment function is *symmetric* if for all permutations $\sigma : N \to N$, each $i \in N$ and each $v \in [0, 1]^n$, $q_i(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = q_{\sigma(i)}(v)$. For each $i \in N$, given that the other agents report truthfully, we denote the interim probability of winning the object by $Q_i(v_i) = E_{v_{-i}}\big[q_i(v_i, v_{-i})\big]$, and the interim transfer by $T_i(v_i) = E_{v_{-i}}\big[t_i(v_i, v_{-i})\big]$, where $E_{v_{-i}}[\cdot]$ denotes the expectation with respect to v_{-i}.

Therefore, the interim payoff for buyer $i \in N$ with value $v_i \in [0, 1]$ when he reports $\hat{v}_i$ is

[12] Our results can be generalized to the case with asymmetric distributions across agents.

[13] If $\sum_{i \in N} q_i(v) < 1$, the seller keeps the object with positive probability $1 - \sum_{i \in N} q_i(v)$.

given by

$$U_i(\hat{v}_i|\, v_i) = Q_i(\hat{v}_i)v_i - T_i(\hat{v}_i).$$

The mechanism (q, t) is (Bayesian) *incentive-compatible* if

$$U_i(v_i|\, v_i) \geq U_i(\hat{v}_i|\, v_i) \qquad \text{for each } i \in N \text{ and each } v_i, \hat{v}_i \in [0, 1]. \tag{IC}$$

The mechanism (q, t) is (interim) *individually rational* if

$$U_i(v_i|\, v_i) \geq 0 \qquad \text{for each } i \in N \text{ and each } v_i \in [0, 1]. \tag{IR}$$

We also assume the following constraint. The mechanism (q, t) satisfies *nonnegative payments* if

$$t_i(v) \geq 0 \qquad \text{for each } i \in N \text{ and each } v \in [0, 1]^n. \tag{NP}$$

We believe that this is a reasonable restriction on feasible mechanisms given that payment from the seller to the buyer is unrealistic in many auctions in practice. In the standard problem in the literature in which the designer assigns a positive weight for revenue, nonnegative payments are necessarily satisfied in the solution. However, in our problem, in which revenue has a negative weight, nonnegative payments play an important role. We further discuss this point in Remark 1 after we introduce our main optimization problem.

The goal of the auction designer is to design a "pro-buyer" auction mechanism in a certain sense. Although there are several essentially equivalent formulations, we first introduce the surplus maximization problem under an upper-bound constraint for revenue, that is, a revenue cap. More specifically, let $R \in \mathbb{R}_+$ denote an exogenously given upper bound for expected revenue. Our problem is then expressed as follows:

$$\max_{(q,t)} E_v\big[\sum_{i\in N} q_i(v)v_i\big] \tag{$\mathcal{P}_0(R)$}$$

$$\text{subject to } E_v\big[\sum_{i\in N} t_i(v)\big] \leq R, \tag{RC}$$

$$\text{(IC), (IR), and (NP).}$$

Of course, the problem is trivial if $R \geq R^e$, where R^e is the expected revenue in the second-price auction without a revenue cap, because it achieves the highest surplus. The case with $R = 0$ is also trivial because the only possibility is to allocate the object to one buyer, possibly randomly but independent of the reported type profile. Therefore, in the following we assume that $R \in (0, R^e)$.

Remark 1. Without (NP), our problem is trivial. For example, consider the procedure whereby the seller first needs to pay an entrance fee R^e, which is distributed among the buyers, and then the seller runs a second-price auction. Then the expected revenue is always zero (and hence the revenue-cap constraint is always satisfied) and S is maximized. A problem with this procedure is that the seller incurs a loss if the buyers' realized values are low. Therefore, the seller's participation (although not explicitly modeled in this paper) may become problematic if his budget is constrained. Another potential problem is participation by "nonserious buyers".[14] Suppose that there are many "nonserious buyers" whose values are zero and who need to pay a fixed (socially wasteful) cost to participate in the auction (e.g., opportunity costs for more valuable economic activities). In an anonymous environment in which serious buyers cannot be distinguished from nonserious ones, nonserious buyers enter the auction to earn a part of the seller's entrance fee. The social cost of their entrance can almost countervail the surplus obtained in the second-price auction.[15]

[14] McAfee and McMillan (1992) discuss the effects of nonserious buyers in auctions with cartels.

[15] Formally, avoiding participation of nonserious buyers may be represented by $T_i(0) \geq 0$, $i \in N$, which is logically weaker than (ex post) nonnegative payments. However, in fact, we obtain the same results even if we replace the nonnegative-payment constraint by this weaker condition.

Another possibility may be to appoint one of the buyers randomly before their type reports (say buyer 1), run the second-price auction among all buyers except for buyer 1, and then give all the realized revenue to buyer 1. In this procedure, the seller's ex post revenue is zero, while the surplus is not maximized (because buyer 1 does not bid). Therefore, when n is small, this is not a good alternative. Conversely, when n is very large, the effect of excluding one buyer hardly affects the realized surplus; in fact, the expected surplus in this case converges to that in the (all-buyer) second-price auction as n goes to infinity. Instead, the problem again involves the possibility of nonserious buyers. As in the first alternative, many nonserious buyers may enter the game in an anonymous environment in which they may possibly be selected as the revenue absorber. Their socially wasteful entrance may again countervail the surplus obtained in the second-price auction.

Throughout the paper, we do not explicitly model the participation decisions of nonserious buyers. Instead, we simply impose the constraint (NP) to rule out such a possibility.

3 Constrained efficiency

In this section, we characterize the constrained-efficient auction mechanism and discuss its important properties.

The following revenue-equivalence result is standard in the literature, and hence we omit the proof.

Lemma 1. *(q, t) satisfies* (IC), (IR), *and* (NP), *if and only if, for each i, (i) $Q_i(v_i)$ is nondecreasing in v_i ("interim monotonicity"), and (ii) for each v_i,*

$$T_i(v_i) = Q_i(v_i)v_i - \int_0^{v_i} Q_i(\tilde{v}_i)d\tilde{v}_i,$$

where (ii) implies

$$E_v\big[\sum_{i\in N} t_i(v)\big] = E_v\big[\sum_{i\in N} q_i(v)\psi(v_i)\big].$$

Therefore, we can rewrite our original problem $\mathcal{P}_0(R)$ as follows, which we call *problem* $\mathcal{P}_1(R)$.

$$\max_q \; E_v\big[\sum_{i\in N} q_i(v)v_i\big] \tag{$\mathcal{P}_1(R)$}$$

$$\text{subject to } E_v\big[\sum_{i\in N} q_i(v)\psi(v_i)\big] \le R, \tag{RC$'$}$$

$$Q_i(v_i) \text{ is nondecreasing for all } i \in N. \tag{M}$$

Let $\mathcal{Q}_1^*(R)$ be the set of solutions to problem $\mathcal{P}_1(R)$.[16] We investigate this set $\mathcal{Q}_1^*(R)$ of solutions using a two-step procedure. First, we reduce problem $\mathcal{P}_1(R)$ to the maximization of a Lagrangian function shown to have a (weakly) larger set of solutions that may or may not satisfy the revenue-cap constraint. Second, among solutions in the reduced problem, we choose those that satisfy the revenue-cap constraint. Because the Lagrangian is the weighted sum of the expected surplus and the expected revenue, the maximization problem for a weighted sum of the buyers' and seller's payoffs is solved as a byproduct of solving the problem with a revenue cap. We note that, as emphasized in the Introduction, this Lagrangian maximization problem has economically significant implications.

Because the set of all assignment functions q that satisfy the constraints is (weak-*) compact and convex, for each $R \in (0, R^e)$, there exists a Lagrangian multiplier $\lambda > 0$ such that (q^*, λ) is a saddle point of the Lagrangian

$$\mathcal{L} = E_v\big[\sum_{i\in N} q_i(v)v_i\big] + \lambda\Big(R - E_v\big[\sum_{i\in N} q_i(v)\psi(v_i)\big]\Big).$$

[16] If $q \in \mathcal{Q}_1^*(R)$, then obviously any $\hat{q}$ that is interim monotonic and satisfies $q = \hat{q}$ almost everywhere is also in $\mathcal{Q}_1^*(R)$. Throughout the paper, we treat such q and $\hat{q}$ as equivalent and do not distinguish between them.

More formally, we have a reduced formulation of our problem as follows.

Proposition 1. *Let $R \in (0, R^e)$. There exists $\lambda > 0$ such that for any solution $q^* \in \mathcal{Q}_1^*(R)$, q^* is also a solution to the following problem.*

$$\max_q E_v\big[\sum_{i\in N} q_i(v)(v_i - \lambda\psi(v_i))\big] \qquad (\mathcal{P}_2(\lambda))$$

$$\text{subject to (M).}$$

Moreover, such $\lambda > 0$ is unique for each $R \in (0, R^e)$ and is nonincreasing in R.

The saddle point argument is standard, and hence we omit the proof.[17] Thus, the only subtlety here is the uniqueness of the Lagrangian multiplier λ. We provide a formal proof in Appendix A.1.

The objective of problem $\mathcal{P}_2(\lambda)$ can be interpreted as a weighted sum of the buyers' expected payoffs and the seller's expected revenue, where the weight for the seller is smaller than that for the buyers. In fact,

$$\begin{aligned} E_v\big[\sum_{i\in N} q_i(v)(v_i - \lambda\psi(v_i))\big] &= E_v\big[\sum_{i\in N} q_i(v)(v_i - \psi(v_i))\big] + (1-\lambda)E_v\big[\sum_{i\in N} q_i(v)\psi(v_i)\big] \\ &= E_v\big[(\text{Buyers' total payoffs}) + (1-\lambda)(\text{Seller's revenue})\big], \end{aligned}$$

where the seller's weight is $1 - \lambda < 1$.

Let $\mathcal{Q}_2^*(\lambda)$ be the set of all solutions to problem $\mathcal{P}_2(\lambda)$. We denote such a unique λ by $\lambda(R)$ when we want to be clear that λ is the Lagrangian multiplier corresponding to R. Then Proposition 1 implies that $\mathcal{Q}_1^*(R) \subseteq \mathcal{Q}_2^*(\lambda(R))$. Thus, for each $R \in (0, R^e)$, we can obtain the set of solutions $\mathcal{Q}_1^*(R)$ by solving problem $\mathcal{P}_2(\lambda)$ for each $\lambda > 0$, restricting the set of solutions in $\mathcal{Q}_2^*(\lambda)$ to those yielding expected revenue equal to R. Mathematically, problem $\mathcal{P}_2(\lambda)$ is viewed as a "subproblem" of $\mathcal{P}_1(R)$ in this sense. Thus, in what follows

[17] See Luenberger (1969, Chapter 8) for technical details of the Lagrangian method in an infinite-dimensional space.

we first solve problem $\mathcal{P}_2(\lambda)$.

In Section 3.3, we provide a sufficient condition for the shape of ψ with which $\mathcal{Q}_1^*(R) = \mathcal{Q}_2^*(\lambda(R))$. However, in general, it is possible that different $q, \hat{q} \in \mathcal{Q}_2^*(\lambda)$ yield different expected surplus and revenue pairs (even though, of course, the expected surplus minus λ times the expected revenue must be the same). Therefore, in solving $\mathcal{P}_2(\lambda)$ we are particularly interested in the extremal solutions (i.e., those that achieve the highest/lowest pairs of expected surplus and revenue).

Before the formal analysis, we provide some intuition regarding how to determine the winner of the auction in problem $\mathcal{P}_2(\lambda)$. Consider the second-price auction as a benchmark, which violates the revenue-cap constraint. We modify the mechanism from this benchmark to reduce the expected revenue or, equivalently, the expected virtual value,[18] without reducing the expected surplus by much. Imagine that at some value profile $v = (v_1, \ldots, v_n)$, the winner of an auction is changed from buyer i (who has the highest value by definition of the second-price auction) to buyer j ($\neq i$). Then the surplus decreases by $v_i - v_j$ and the virtual surplus decreases by $\psi(v_i) - \psi(v_j)$. Therefore, if we change the allocation at v, the best way is to give the object to buyer j who attains the highest ratio of $\frac{\psi(v_i)-\psi(v_j)}{v_i-v_j}$. In other words, the difference "rate" for the virtual value ψ, rather than the difference itself, matters in constructing the revenue-capped efficient auction. In fact, the form of the objective in problem $\mathcal{P}_2(\lambda)$ implies that buyer j wins the object at v if $v_j - \lambda\psi(v_j) > v_k - \lambda\psi(v_k)$ for all $k \neq j$ or, equivalently,

$$\lambda \frac{\psi(v_j) - \psi(v_k)}{v_j - v_k} > 1,$$

where λ may be interpreted as the shadow price for the revenue-cap constraint.

Although our basic idea is to modify the mechanism according to the argument above, such modification may well lead to a violation of the monotonicity constraint. Thus, to

[18] See Lemma 1 for the equivalence between the expected revenue and the expected virtual value.

fully characterize the solution to our problem, we use a "bunching" technique developed by Myerson (1981, Section 6) and Baron and Myerson (1982).[19]

Let $\varphi(x) = x - \lambda\psi(x)$ for each $x \in [0,1]$. We introduce a function $\bar{\varphi} : [0,1] \to \mathbb{R}$ as follows. Because the cumulative distribution function F has a positive density function, F is continuous and strictly increasing, and thus has a continuous inverse function F^{-1}. For each $p \in [0,1]$, let $h(p) = \varphi\big(F^{-1}(p)\big)$ and

$$H(p) = \int_0^p h(p')dp'.$$

Note that H is continuously differentiable because $h(p) = F^{-1}(p) - \lambda\psi(F^{-1}(p))$ is continuous. Let G be the "convex hull" of H, defined by

$$G(p) = \min\{\alpha H(p') + (1-\alpha)H(p'') \,|\, (p', p'', \alpha) \in [0,1]^3 \text{ s.t. } \alpha p' + (1-\alpha)p'' = p\}.$$

This function G is convex and continuously differentiable. Let g denote its derivative, and we define $\bar{\varphi}(x) = g(F(x))$. Since g is continuous, $\bar{\varphi}$ is also continuous. Intuitively, $\bar{\varphi}$ modifies φ by "flattening" the non-monotonic part of φ so that $\bar{\varphi}$ is monotonic. This modified version of the virtual value function, $\bar{\varphi}$, plays a central role in our characterization. We first show that $\bar{\varphi}(x)$ is always positive when $x > 0$.

Lemma 2. *For all $x > 0$ and all $\lambda > 0$, $\bar{\varphi}(x) > 0$.*

Proof. See Appendix A.2. □

This property is used repeatedly in our subsequent analysis. The following proposition provides a solution to $\mathcal{P}_2(\lambda)$.

Proposition 2. *For each $\lambda > 0$, let $\mathcal{Q}_2^{**}(\lambda)$ be the set of interim monotonic assignment functions q such that*

[19] Toikka (2011) generalizes this technique. See Guensnerie and Laffont (1984) for a general methodology applying optimal control theory.

(i) for each $v \in [0,1]^n$, $\sum_{\{i|\bar{\varphi}(v_i)=\max_{j\in N}\bar{\varphi}(v_j)\}} q_i(v) = 1$, *and*

(ii) for each $i \in N$ *and each* $v_i \in [0,1]$, *if* $G(F(v_i)) < H(F(v_i))$, *then* Q_i *is differentiable at* v_i *and* $Q_i'(v_i) = 0$.

Then $\mathcal{Q}_2^{**}(\lambda) \subset \mathcal{Q}_2^{*}(\lambda)$, *and for any solution* $\tilde{q} \in \mathcal{Q}_2^{*}(\lambda)$ *there exists* $q \in \mathcal{Q}_2^{**}(\lambda)$ *such that* $q(v) = \tilde{q}(v)$ *with probability one.*

Proof. See Appendix A.3. □

Similar to the standard situation in the literature, the optimal assignment function maximizes (the modified version of) the virtual surplus, $E_v[\sum_i q_i(v)\bar{\varphi}(v_i)]$. Because Lemma 2 guarantees $\bar{\varphi}(v_i) > 0$ (with probability one), buyer i wins if $\bar{\varphi}(v_i)$ is the highest among all the buyers. However, owing to potential bunching, i does not necessarily win with probability one even if his value v_i is the highest among all the buyers, when there is another agent j who has $v_j < v_i$ but $\bar{\varphi}(v_i) = \bar{\varphi}(v_j)$. In fact, values in each interval $\{x \,|\, \bar{\varphi}(x) = \bar{\varphi}(v_i)\}$ may be bunched together with v_i in the sense that the interim winning probability for buyer i is constant in each of these intervals.

As suggested in the statement, there can be multiple solutions to problem $\mathcal{P}_2(\lambda)$. They differ in the tie-breaking rule, that is, who wins if there are multiple buyers who have the highest value of $\bar{\varphi}$. In the standard situation in which bunching does not occur, such differences are not essential because the probability of ties among virtual values is zero. However, in our environment, such events may occur with a strictly positive probability.

For each $\lambda > 0$, we introduce two symmetric assignment functions q^* and q^{**}, each of which turns out to be an "extreme" solution to problem $\mathcal{P}_2(\lambda)$ in the sense that they lead to the minimum or maximum expected revenue among all solutions.[20] Let q^* be an assignment

[20] The minimum and maximum coincide in some cases and differ in other cases. We discuss the multiplicity of expected revenues in Section 3.2.4.

function defined by, for each $i \in N$ and $v \in [0,1]^n$,

$$q_i^*(v) = \begin{cases} 1/\left|\arg\max_{j \in N} \bar{\varphi}(v_j)\right| & \text{if } i \in \arg\max_{j \in N} \bar{\varphi}(v_j), \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

This assignment function q^* assigns the object to one of the agents with the largest value of $\bar{\varphi}(v_i)$ with equal probability. Thus, bunching occurs on each interval on which $\bar{\varphi}$ is constant. That is, for each $i \in N$, $Q_i^*(v_i)$ is constant when $\bar{\varphi}(v_i)$ is constant.

The next assignment function q^{**} may not lead to bunching on the entire region within such an interval. For each $x \in [0,1]$, let

$$\underline{b}(x) = \max\{y \in [0,x] \,|\, H(F(y)) = G(F(y))\},$$
$$\overline{b}(x) = \min\{y \in [x,1] \,|\, H(F(y)) = G(F(y))\}.$$

Because $G(0) = H(0)$, $G(1) = H(1)$, and F, G, H are all continuous, the above maximum and minimum always exist. By definition, $\underline{b}(x) \leq \overline{b}(x)$ for all $x \in [0,1]$, and $\underline{b}(x) = \overline{b}(x)$ if and only if both are equal to x (or, equivalently, $H(F(x)) = G(F(x))$). For each $x \in [0,1]$, let $B(x)$ be either an open interval or a singleton defined by

$$B(x) = \begin{cases} \big(\underline{b}(x), \overline{b}(x)\big) & \text{if } \underline{b}(x) < \overline{b}(x), \\ \{x\} & \text{if } \underline{b}(x) = \overline{b}(x). \end{cases}$$

If $\underline{b}(x) < \overline{b}(x)$, then $B(x)$ is the maximum open interval containing x such that $H(F(y)) > G(F(y))$ for all y in the interval. By the definition of $\bar{\varphi}$, for each $x \in [0,1]$, $\bar{\varphi}$ must be constant on $B(x)$. It is possible, however, that $B(x)$ is not the maximal interval in which $\bar{\varphi}$ is constant, that is, $\bar{\varphi}$ may take the same value on $B(x)$ and $B(x')$ even when $B(x) \cap B(x') = \emptyset$. Let q^{**}

be defined by, for each $i \in N$ and $v \in [0,1]^n$,

$$q_i^{**}(v) = \begin{cases} 1/\left|\{j \in N \mid v_j \in B(\max_{k\in N} v_k)\}\right| & \text{if } v_i \in B(\max_{k\in N} v_k), \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

This assignment function q^{**} has the same property as q^* whereby the object is assigned to one of the agents with the largest value of $\bar{\varphi}(v_i)$, but is different from q^* in that even if $i \in \arg\max_{j\in N} \bar{\varphi}(v_j)$, it is possible that i wins with probability zero. This occurs if there exist $j \in N$ with $v_i < v_j$ and $y \in (v_i, v_j)$ such that $G(F(y)) = H(F(y))$. In this case, even if $\bar{\varphi}(v_j) = \bar{\varphi}(v_i)$, then $v_i \notin B(\max_{k\in N} v_k)$. Let $Q_i^{**}(v_i) = E_{v_{-i}}[q_i^{**}(v_i, v_{-i})]$ be the interim assignment probability.

Proposition 3. *For each $\lambda > 0$, both assignment functions q^* and q^{**} solve problem $\mathcal{P}_2(\lambda)$. Moreover, q^* yields the minimum expected revenue and the minimum expected surplus among $\mathcal{Q}_2^*(\lambda)$, and q^{**} yields the maximum expected revenue and the maximum expected surplus among $\mathcal{Q}_2^*(\lambda)$.*

Proof. See Appendix A.4. □

Proposition 3 shows how we can choose the solution among $\mathcal{Q}_2^*(\lambda)$ whose expected revenue equals the given revenue cap $R \in (0, R^e)$. Suppose that the expected revenues under q^* and q^{**} are R^* and R^{**}, respectively. Then Proposition 3 implies that $\mathcal{Q}_2^*(\lambda(R)) = \bigcup_{R' \in [R^*, R^{**}]} \mathcal{Q}_1^*(R')$. If $R^* = R^{**}$, then any solution to problem $\mathcal{P}_2(\lambda)$ yields the same level of expected revenue, and $\mathcal{Q}_1^*(R) = \mathcal{Q}_2^*(\lambda(R))$. If $R^* < R^{**}$ and $R \in [R^*, R^{**}]$, then $\mathcal{Q}_1^*(R) \subsetneq \mathcal{Q}_2^*(\lambda(R))$, and $\alpha q^* + (1-\alpha) q^{**}$ is a solution to problem $\mathcal{P}_1(R)$, where $\alpha = (R^{**} - R)/(R^{**} - R^*)$.

3.1 General properties of the constrained-efficient mechanism

Although the exact form of the constrained-efficient mechanism varies with parameters such as the distribution F, we can derive important properties in common.

The first property is that for almost all $v \in [0,1]^n$, the seller does not keep the object.

Proposition 4. *For any* $R \in (0, R^e)$ *and any solution* $q \in \mathcal{Q}_1^*(R)$*, we have* $\sum_{i \in N} q_i(v) = 1$ *with probability one.*

Proof. By Proposition 1, if $q \in \mathcal{Q}_1^*(R)$, then $q \in \mathcal{Q}_2^*(\lambda(R))$. The statement is a direct implication of (i) in Proposition 2. $\square$

This proposition says that in any constrained-efficient auction, a reserve price cannot be set. We note that this property is not as obvious as it seems. Since we do not assume that the virtual value function is monotonic, it is possible that the introduction of a reserve price *lowers* the expected revenue. Proposition 4 claims that bunching is a better tool to reduce the expected revenue even in such cases.

The second property of our constrained-efficient mechanism is that when $R \in (0, R^e)$, some bunching is necessary. With a revenue cap strictly less than R^e, the expected revenue (and the expected surplus) must be lowered from the (efficient) second-price auction. However, as in the previous result, the object is sold with probability one. This is the main intuitive reason why bunching is necessary to achieve revenue reduction.

Recall that Q_i^* is the interim probability of q_i^* defined in equality (1). According to the following proposition, values corresponding to some intervals are pooled (or bunched) together in the sense that Q_i^* is constant in this interval.

Proposition 5. *For each* $R \in (0, R^e)$ *and each* $i \in N$*, there exists an open interval* $(a, b) \subset [0, 1]$ *on which the interim probability* $Q_i^*(x)$ *is constant, where* q^* *is given by equality* (1) *with* $\lambda = \lambda(R)$*.*

Proof. Let $\lambda = \lambda(R)$. If $Q_i^*(v_i)$ is not constant in any open interval, then $Q_i^*(v_i)$ is strictly increasing by interim monotonicity. Then the definition of q^* implies that $\bar{\varphi}(x)$ is strictly increasing. In this case, the auction given by q^* is the same as the second-price auction that leads to efficiency. This contradicts $R < R^e$. Therefore $Q_i^*(v_i)$ is constant in some open interval. $\square$

We note that an analogous argument shows that $Q_i^{**}(v_i)$ is constant in some interval.[21]

In addition to Proposition 5, we can show that as the revenue cap R becomes more stringent so that $\lambda(R)$ becomes (weakly) higher, the bunching regions become larger in the set-inclusion order.

Proposition 6. *Suppose that for each $R \in (0, R^e)$ and each $\tilde{\lambda} > \lambda(R)$, q^* is the solution to problem $\mathcal{P}_2(\lambda(R))$ and $\tilde{q}^*$ is the solution to problem $\mathcal{P}_2(\tilde{\lambda})$ defined in equality* (1)*. Let Q_i^* and $\tilde{Q}_i^*$ be the interim probability for buyer i in q^* and $\tilde{q}^*$, respectively. For each $i \in N$, if Q_i^* is constant on (a, b), then $\tilde{Q}_i^*$ is constant (at least) on the same interval (a, b).*

Proof. See Appendix A.5. □

The exact regions where bunching occurs depend on the parameters. However, when the revenue-cap constraint is "severe enough" so that $\lambda(R) > 1/2$,[22] bunching must occur (at least) at the top, that is, the highest possible value of the object for a region close to 1. In the standard case with a positive weight on revenue, we generally have the property of "no distortion at the top", which implies that the buyer with $v_i = 1$ wins with probability one. However, this property is lost in our context.

Proposition 7. *For all $\lambda > 1/2$ and all $q \in \mathcal{Q}_2^*(\lambda)$, q exhibits distortion at the top for at least $n-1$ buyers, that is, there exists $j \in N$ such that $\lim_{v_i \nearrow 1} Q_i(v_i) < 1$ for each $i \neq j$. In particular, if q is symmetric, then $\lim_{v_i \nearrow 1} Q_i(v_i) < 1$ for each $i \in N$.*

Proof. First, observe that $\varphi'(1) = 1 - 2\lambda$. Thus, if $\lambda > 1/2$, then $\varphi'(1) < 0$. Because we assume the continuous differentiability of f, φ' is continuous. Thus, there exists $w < 1$ such that $\varphi'(x) < 0$ for all $x \in (w, 1)$. On the interval $(F(w), 1)$, we have $H''(p) = \varphi'(F^{-1}(p))\frac{dF^{-1}(p)}{dp} < 0$, and hence $H(p) - G(p) > 0$. By Proposition 2 (ii), $Q_i(x)$ is constant for all $x \in (w, 1)$ and all $i \in N$.

[21] If $Q_i^{**}(v_i)$ is strictly increasing, then $H(F(x)) = G(F(x))$ for almost all x by Proposition 2 (ii). This implies that $B(x) = \{x\}$ for almost all x, and thus q^{**} agrees with the efficient assignment function almost everywhere, contradicting $R < R^e$.

[22] As a caveat, we do not claim that R "severe enough" to induce $\lambda(R) > \frac{1}{2}$ always exists; for example, see the concave virtual-values case in Section 3.2.2.

Suppose that there exists $j \in N$ such that $\lim_{v_j \nearrow 1} Q_j(v_j) = 1$. Then $Q_j(x) = 1$ for all $x \in (w, 1)$, that is, if $v_j \in (w, 1)$, then $q_j(v_j, v_{-j}) = 1$ for almost all $v_{-j} \in [0, 1]^{n-1}$. In particular, $q_i(v) = 0$ for all $i \neq j$ and almost all $v \in (w, 1)^n$. This implies that for all $i \neq j$ and almost all $v_i \in (w, 1)$, $Q_i(v_i) \leq F(w) < 1$, and thus $\lim_{v_i \nearrow 1} Q_i(v_i)$ is lower than 1.

If q is symmetric, for all $i \in N$ and all $v_i \in (w, 1)$, $Q_i(v_i) \leq (1 + (n-1)F(w))/n < 1$, and thus $\lim_{v_i \nearrow 1} Q_i(v_i)$ is lower than 1. $\square$

This result provides a logical connection between a revenue cap, which represents the degree to which the designer is pro-buyer, and a bid cap, which is a specific constraint on the auction rule.

So far, $\lambda(R)$ is not necessarily less than 1 for $R \in (0, R^e)$. This means that in the corresponding maximization problem $\mathcal{P}_2(\lambda(R))$ of a weighted sum of the buyers' and seller's payoffs, the weight for the seller may be negative. In such a case, we may consider it strange to interpret this problem as a social planner's weighted surplus maximization problem, because each person is usually assigned a nonnegative weight for such a problem. However, this "strange" situation never occurs in the regular cases in which F has a strictly increasing hazard rate. In fact, if $\lambda \geq 1$ and the hazard rate is strictly increasing, then $\varphi(v_i) = v_i - \lambda \cdot \left(v_i - \frac{1-F(v_i)}{f(v_i)}\right) = -(\lambda - 1)v_i + \lambda \frac{1-F(v_i)}{f(v_i)}$ is strictly decreasing in v_i. Therefore, given the interim monotonicity constraint for the assignment function, the solution to problem $\mathcal{P}_2(\lambda(R))$ is constant, which obviously implies that the seller's revenue is zero, contradicting the assumption $R > 0$.

We summarize the argument in the following proposition.

Proposition 8. *If F has a strictly increasing hazard rate, then $\lambda(R) \in (0, 1)$ for each $R \in (0, R^e)$.*

3.2 Convex/concave virtual values

Complete characterization of bunching regions is generally a daunting task, but can be straightforward in certain cases. In this section, we consider distributions F such that the virtual value function $\psi(x) = x - \frac{1-F(x)}{f(x)}$ is either convex or concave. As discussed before, the inverse of the difference rate of ψ is interpreted as a shadow price, which changes monotonically when ψ is either convex or concave. This monotonicity simplifies our analysis.

A variety of distributions satisfy such convexity or concavity. A class of special cases includes distributions with a constant or linear hazard rate, such as exponential and uniform distributions. Other examples follow.

Example 1. Let $F(x) = x^\alpha$, where α is a positive constant. Then the virtual value function is convex if $\alpha \in (0, 1]$, and concave if $\alpha \geq 1$.[23]

Example 2. Let $f(x) = \alpha(x - 1/2) + 1$, where $\alpha \in (-2, 2)$. Then the virtual value function is convex if $\alpha \in (-2, 0]$, and concave if $\alpha \in [0, 2)$.

3.2.1 Convex virtual values

We consider the case in which ψ is convex or, equivalently, φ is concave. In this case, bunching occurs only at the top.

Proposition 9. *Suppose that $\psi(x)$ is convex. For each $R \in (0, R^e)$, we have $\lambda(R) \geq 1/2$, and there exists a solution q^* to problem $\mathcal{P}_1(R)$ and $\bar{w} \in (0, 1)$ such that for each $i \in N$ and each $v \in [0, 1]^n$,*

$$q_i^*(v) = \begin{cases} 1/\left|\arg\max\{v_j \mid j \in N, v_j \geq \bar{w}\}\right| & \text{if } v_i \geq \bar{w}, \\ 1/\left|\arg\max\{v_j \mid j \in N\}\right| & \text{if } v_i = \max_{j \in N} v_j < \bar{w}, \\ 0 & \text{otherwise.} \end{cases}$$

[23]This example violates our regularity assumption that F has a positive and continuously differentiable density function on $[0, 1]$. However, we can show that our results so far hold true for these cases.

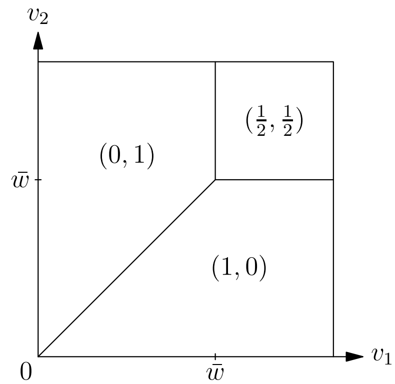

Figure 1: A constrained-efficient assignment function $q^*(v_1, v_2)$ when the virtual value function is convex.

Proof. We first show that $\lambda = \lambda(R) \geq 1/2$. Assume that $\lambda < 1/2$. Because $\varphi'(1) = 1 - 2\lambda > 0$, the concavity of φ implies that $\varphi'(x) > 0$ for all $x \in [0, 1]$, that is, φ is strictly increasing. Thus, $\bar{\varphi}$ is also strictly increasing. Proposition 2 implies that the expected revenue equals R^e. This contradicts the assumption $R < R^e$.

We now show that q^* is a solution for some $\bar{w} \in (0, 1)$. By the concavity of φ, there exists $w \in [0, 1]$ such that φ is nondecreasing on $[0, w]$ and nonincreasing on $[w, 1]$. Then $H(p) = \int_0^p \varphi(F^{-1}(p'))\, dp'$ is convex on $[0, F(w)]$ and concave on $[F(w), 1]$. This implies that there exists $\bar{w} \in [0, 1]$ such that $\bar{\varphi}$ is strictly increasing on $[0, \bar{w})$ and constant on $[\bar{w}, 1]$. We show that $\bar{w} \neq 0$ or 1. If $\bar{w} = 0$, the expected revenue equals R^e. If $\bar{w} = 1$, the expected revenue is zero. By Proposition 3, q^* takes the form in the statement. $\square$

The assignment function q^* in Proposition 9 is shown in Figure 1 for $n = 2$. This q^* can be achieved in a second-price auction with bid cap $\bar{w}$, where each buyer i bids the true value if $v_i \leq \bar{w}$, and $\bar{w}$ otherwise. By Lemma 1, a corresponding transfer function is the one in the second-price auction with bid cap $\bar{w}$.

Example 3. Let $n = 2$ and $F(x) = \sqrt{x}$ in $[0, 1]$. As noted in Example 1, this distribution is associated with a strictly convex virtual value function. Straightforward computation shows that for each $\bar{w} \in [0, 1]$, the auction in Proposition 9 yields expected revenue given by $\bar{w} - \frac{4}{3}\bar{w}^{\frac{3}{2}} + \frac{1}{2}\bar{w}^2$. This relationship between revenue cap R and $\bar{w}$ is described in Figure 2.

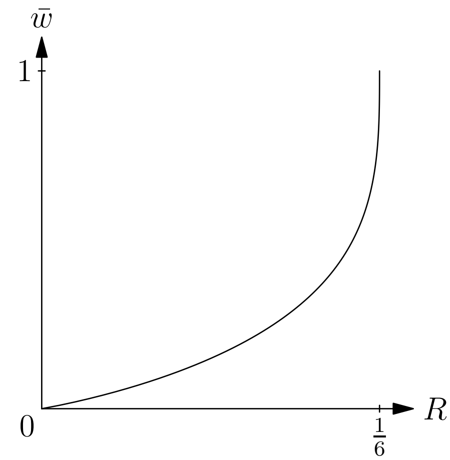

Figure 2: The threshold $\bar{w}$ when $n = 2$ and $F(x) = \sqrt{x}$.

In the efficient auction with $\bar{w} = 1$, the seller earns an expected revenue of 1/6. If R lies between 0 and 1/6, the mechanism in Proposition 10 is a constrained-efficient auction with bid cap $\bar{w}$ corresponding to R.

3.2.2 Concave virtual values

We consider the case in which ψ is concave or, equivalently, φ is convex. In this case, bunching occurs only at the bottom.

Proposition 10. *Suppose that $\psi(x)$ is concave. For each $R \in (0, R^e)$, $\lambda(R) \leq 1/2$ and there exist a solution q^* to problem $\mathcal{P}_1(R)$ and $\underline{w} \in (0,1)$ such that for each $i \in N$ and each $v \in [0,1]^n$,*

$$q_i^*(v) = \begin{cases} 1/\left|\arg\max_{j \in N} v_j\right| & \text{if } v_i = \max_{j \in N} v_j > \underline{w}, \\ 1/n & \text{if } \max_{j \in N} v_j \leq \underline{w}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We first show that $\lambda = \lambda(R) \leq 1/2$. Assume that $\lambda > 1/2$. Because $\varphi'(1) = 1 - 2\lambda < 0$, the convexity of φ implies that $\varphi'(x) < 0$ for all $x \in [0,1]$. Thus, $\bar{\varphi}$ is constant on $[0,1]$. Proposition 2 implies that the expected revenue equals 0. This contradicts the assumption

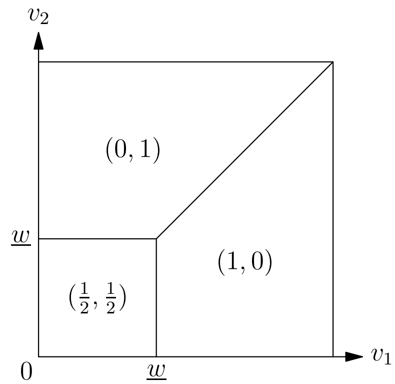

Figure 3: A constrained-efficient assignment function $q^*(v_1, v_2)$ when the virtual value function is concave.

$R > 0$.

We now show that q^* is a solution for some $\underline{w} \in (0,1)$. By the convexity of φ, there exists $w \in [0,1]$ such that $H(p) = \int_0^p \varphi(F^{-1}(p'))\, dp'$ is concave on $[0, F(w)]$, and convex on $[F(w), 1]$. This implies that there exists $\underline{w} \in [0,1]$ such that $\bar{\varphi}$ is constant on $[0, \underline{w}]$ and strictly increasing on $(\underline{w}, 1]$. By Proposition 3, q^* takes the form in the statement.

Finally, we show that $\underline{w} \neq 0$ or 1. If $\underline{w} = 0$, the expected revenue equals R^e. If $\underline{w} = 1$, the expected revenue is zero. Both contradict the assumption $R \in (0, R^e)$. $\square$

The assignment function q^* in Proposition 10 is shown in Figure 3 for $n = 2$. This q^* can be achieved using a modified form of the second-price auction with minimum bid $\underline{w}$, where each buyer i bids the true value if $v_i \geq \underline{w}$, and $\underline{w}$ otherwise. By Lemma 1, a corresponding transfer function is such that if the bid by i is the highest, then: (a) if this bid is $\underline{w}$, the winner's payment is zero; (b) if this bid exceeds $\underline{w}$ and the second highest bid is $\underline{w}$, then the payment is $\frac{n-1}{n}\underline{w}$; and (c) if both this bid and the second highest bid exceed $\underline{w}$, the payment equals the second highest bid.

Example 4. Let $n = 2$ and $F(x) = x^2$ in $[0,1]$. As noted in Example 1, this distribution is associated with a concave virtual value function. Straightforward computation shows that for each $\underline{w} \in [0,1]$, the auction in Proposition 9 yields expected revenue given by $\frac{8}{15} - \frac{1}{3}\underline{w}^3 - \frac{1}{5}\underline{w}^5$. This relationship between revenue cap R and $\underline{w}$ is illustrated in Figure 4.

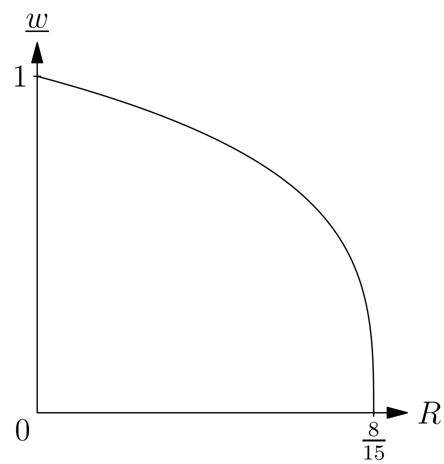

Figure 4: The threshold $\underline{w}$ when $n = 2$ and $F(x) = x^2$.

In the efficient auction with $\underline{w} = 0$, the seller earns an expected revenue of 8/15. If R lies between 0 and 8/15, the mechanism in Proposition 10 is a constrained-efficient auction with $\underline{w}$ corresponding to R.

3.2.3 Linear virtual values

If the virtual value $\psi(x)$ is an affine function,[24] $\psi(x)$ is both convex and concave, and thus both assignment functions defined in Propositions 9 and 10 are constrained-efficient. This observation suggests that multiple constrained-efficient auctions exist in this case. We pick the uniform distribution as a representative case and analyze this in more detail because the solutions can be computed directly.

Consider the uniform distribution on $[0, 1]$ under which the virtual value function is $\psi(x) = 2x - 1$. Let $\lambda = 1/2$. Then problem $\mathcal{P}_2(1/2)$ is rewritten as

$$\max_q E_v\Big[\frac{1}{2}\sum_{i\in N} q_i(v)\Big]$$
$$\text{subject to (M).}$$

Thus, any interim monotonic assignment function q is a solution to problem $\mathcal{P}_2(1/2)$ if q

[24]This holds if $F(x) = 1 - (1 - x)^\gamma$ for some $\gamma > 0$.

satisfies $E_v\left[\sum_{i\in N} q_i(v)\right] = 1$. For each $R \in (0, R^e)$, any solution to problem $\mathcal{P}_1(R)$ must satisfy $E_v\left[\sum_{i\in N} q_i(v)\right] = 1$ by Proposition 4. Hence, $\lambda(R) = 1/2$ for all $R \in (0, R^e)$.

Problem $\mathcal{P}_1(R)$ is rewritten as

$$\max_q E_v\Big[\sum_{i\in N} q_i(v)v_i\Big]$$
$$\text{subject to } E_v\Big[\sum_{i\in N} q_i(v)v_i\Big] \le \frac{1}{2}(R+1), \quad \text{and (M)},$$

where $E_v\left[\sum_{i\in N} q_i(v)\right] = 1$. Therefore, an interim monotonic assignment function q is a solution to problem $\mathcal{P}_1(R)$ if and only if q satisfies $E_v\left[\sum_{i\in N} q_i(v)\right] = 1$ and $E_v\left[\sum_{i\in N} q_i(v)v_i\right] = \frac{1}{2}(R+1)$. This implies that solutions in $\mathcal{Q}_2^*(1/2)$ yield multiple levels of the expected revenue and the expected surplus. Intuitively, we can obtain the constrained-efficient auction by bunching on any interval(s) if the resulting expected revenue equals R.

3.2.4 Relationship between $\mathcal{P}_1(R)$ and $\mathcal{P}_2(\lambda)$ in convex/concave cases

As informally discussed in the Introduction, problem $\mathcal{P}_1(R)$ involving surplus maximization subject to a revenue cap is related to the maximization of a weighted sum of the expected revenue and buyers' payoffs according to $\mathcal{P}_2(\lambda)$. We now discuss connections between these problems.

In Proposition 1, we proved that for each $R \in (0, R^e)$ there exists a unique $\lambda(R) > 0$ satisfying $\mathcal{Q}_1^*(R) \subseteq \mathcal{Q}_2^*(\lambda(R))$. In general, the converse is not true. In Section 3.2.3, we observed a special case in which problem $\mathcal{P}_2(\lambda)$ has multiple solutions that lead to multiple levels of the expected revenue. We now present another example in which $\mathcal{Q}_1^*(R) \subsetneq \mathcal{Q}_2^*(\lambda(R))$ holds true.

Example 5. Suppose that for each $x \in [0,1]$, $f(x) = e^{-x^2/2}/\Gamma$, where $\Gamma = \int_0^1 e^{-x^2/2}\,dx$. This is the density function for the truncated standard normal distribution defined on $[0,1]$. Numerical computation gives Figure 5, which shows the relationship between λ and R.

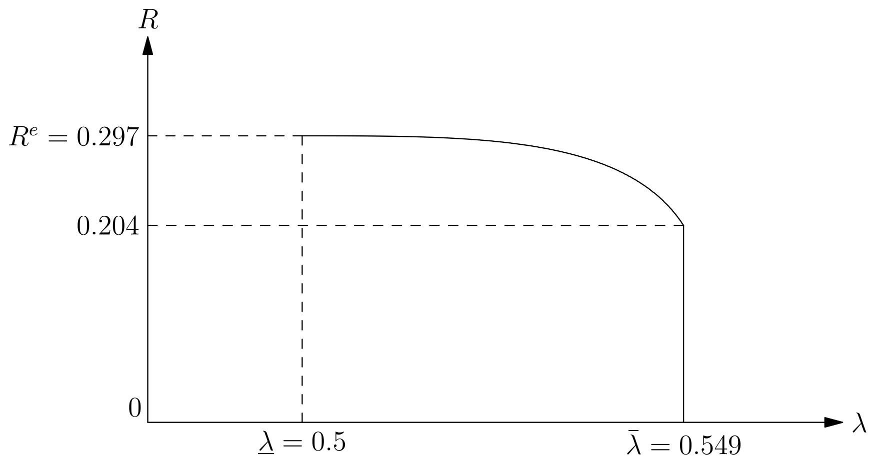

Figure 5: Relationship between λ and R under the standard normal distribution truncated on $[0, 1]$.

If $\lambda \leq 1/2$ $(=: \underline{\lambda})$, the essentially unique solution to problem $\mathcal{P}_2(\lambda)$ is the efficient assignment function, which leads to an expected revenue of $R^e = 0.297$. If $\underline{\lambda} < \lambda < \bar{\lambda} := 0.549$, the expected revenue in the solutions to problem $\mathcal{P}_2(\lambda)$ is unique and strictly decreasing in λ. If $\lambda = \bar{\lambda}$, problem $\mathcal{P}_2(\lambda)$ has solutions that lead to multiple levels of expected revenues: $R = 0.204$ under the assignment function q^{**} defined in (2), and $R = 0$ under the assignment function q^* defined in (1). Conversely, $\lambda(R)$ is constant on $(0, 0.204]$ and strictly decreasing on $(0.204, R^e)$.

We roughly explain this behavior of $\lambda(R)$. Under this distribution, there exists $\bar{x} \in (0, 1)$ such that the virtual value function ψ is strictly convex on $(0, \bar{x})$, and strictly concave on $(\bar{x}, 1)$. Since $\psi'(0) = \psi'(1) = 2$ in this specific case, $(\psi(y) - \psi(x))/(y - x) < 2$ for all $0 \leq x < y \leq 1$. By the "shadow price" argument, the solution of $\mathcal{P}_2(\lambda)$ with $\lambda \leq 1/2$ is the efficient assignment function. For $\lambda \in (1/2, \bar{\lambda})$, bunching occurs in the region with larger values of ψ'. Because $\psi'(x)$ is larger when x is close to 0 or close to 1, there exists $0 < \underline{w} < \overline{w} < 1$ such that the interim assignment probability $Q_i^*(v_i)$ is constant on $(0, \underline{w})$ and on $(\overline{w}, 1)$. As λ increases, $\underline{w}$ increases and $\overline{w}$ decreases. These changes make the bunching region larger and the expected revenue smaller. Eventually, $\underline{w}$ and $\overline{w}$ coincide when λ reaches some value $\bar{\lambda}$. In this limit as $\lambda \nearrow \bar{\lambda}$, the assignment function converges to q^{**} defined in

(2), under which $Q_i^{**}(v_i)$ is constant on the two intervals and discontinuous at $v_i = \underline{w} = \overline{w}$. Conversely, it can be shown that $(\psi(y) - \psi(x))/(y - x) > 1/\bar{\lambda}$ for almost all $0 \le x < y \le 1$. This implies that if $\lambda = \bar{\lambda}$, then $\bar{\varphi}(x)$ is constant on $[0, 1]$. Thus, q^* defined in (1) is constant on the entire domain, and leads to an expected revenue of zero.

The argument in this example suggests that multiplicity of the expected revenue may arise if there exists $0 \le a < b < c \le 1$ such that ψ is strictly convex on (a, b), and strictly concave on (b, c). We can show that revenue uniqueness holds in other special cases discussed in Sections 3.2.1 and 3.2.2.

Proposition 11. *Suppose that $R \in (0, R^e)$. If ψ is strictly convex, any solution $q \in \mathcal{Q}_2^*(\lambda(R))$ yields expected revenue equal to R. If ψ is strictly concave, any solution $q \in \mathcal{Q}_2^*(\lambda(R))$ yields expected revenue equal to R. Therefore, when ψ is strictly convex or strictly concave, $\mathcal{Q}_1^*(R) = \mathcal{Q}_2^*(\lambda(R))$ for all $R \in (0, R^e)$.*

Proof. Let $\lambda = \lambda(R)$. Suppose that ψ is strictly convex. By the strict concavity of φ, there exists $w \in [0, 1]$ such that φ is strictly increasing on $(0, w)$, and strictly decreasing on $(w, 1)$. Then $H(p) = \int_0^p \varphi(F^{-1}(p'))\, dp'$ is strictly convex on $(0, F(w))$, and strictly concave on $(F(w), 1)$. This implies that there exists unique $\bar{w} \in [0, w)$ such that $\bar{\varphi}$ is strictly increasing on $[0, \bar{w})$, and constant on $[\bar{w}, 1]$. In addition, we have $H(F(x)) > G(F(x))$ for all $x \in (\bar{w}, 1)$. Therefore, $q^* = q^{**}$ by Proposition 2. Hence, the expected revenue for solutions in $\mathcal{Q}_2^*(\lambda(R))$ is unique.

The case with strictly concave ψ can be shown similarly. □

Proposition 11 may be interpreted as a type of revenue-equivalence result, although there is an important difference. The standard revenue equivalence involves two mechanisms that have equivalent assignment functions, and claims that they yield the same expected revenue. The result here is different in that we consider two mechanisms that have different assignment functions. Nevertheless, when these maximize the same objective, that is, a pro-buyer-weighted expected surplus, then they yield the same expected revenue (and the same expected surplus) if ψ is either strictly convex or strictly concave.

4 Applications

4.1 Multi-unit auction

In this section, we consider auctions in which the seller possesses multiple units of a good with some positive probability.

For simplicity, assume that a seller possesses at most two units of homogeneous objects. The number of units owned by the seller is his private information. He has one unit with probability $\pi \in (0,1)$, and two units with probability $1-\pi$. If he has two units, he can falsely report that he has only one unit, and dispose of the remaining unit secretly. Conversely, it is impossible to misreport that he has two units when he has only one unit. There are two buyers, each of whom has a single-unit demand.[25]

The auction designer designs a direct mechanism $\left((q^k, t^k)\right)_{k=1}^2$ in which the seller and the buyers simultaneously report their private information; the seller reports the number of units k he possesses, and each buyer $i = 1, 2$ reports his value v_i. For each $k = 1, 2$, (q^k, t^k) is the auction mechanism when the seller provides k units. Feasibility requires $q_i^k \in [0,1]$ and $\sum_i q_i^k(v) \leq k$ for all i, k, v.

Let R^k be the expected revenue from auction (q^k, t^k) for each k. To rule out an incentive to misreport the number of units when the seller has two units, we need $R^1 \leq R^2$. Therefore, we consider the maximization problem

$$\max_{((q^1,t^1),(q^2,t^2))} \pi S^1 + (1-\pi) S^2 \tag{3}$$

$$\text{subject to (IC), (IR), (NP) for each } k = 1, 2, \text{ and}$$

$$R^1 \leq R^2, \tag{GRM}$$

[25]We consider this two-buyer, two-unit setting for simplicity, but we obtain a similar conclusion in more general settings.

where S^k is the expected social surplus in auction (q^k, t^k) for each $k = 1, 2$. We refer to constraint (GRM) as *goods revenue monotonicity*.[26]

In the efficient mechanism, if the seller has two units, then both buyers obtain the objects without payments, regardless of their values. This implies that $R^2 = 0$. Conversely, if the seller has only one unit, then the second-price auction yields a positive expected revenue $R^1 > 0$. Therefore, efficiency is incompatible with the constraint (GRM), and in the second-best problem (3), (GRM) must be binding. More specifically, let $\big((\bar{q}^1, \bar{t}^1), (\bar{q}^2, \bar{t}^2)\big)$ be a solution to problem (3). Then there exists a constant $\bar{R} > 0$ such that for each $k = 1, 2$, $(\bar{q}^k, \bar{t}^k)$ solves

$$\max_{(q^k, t^k)} S^k \tag{3.k}$$

$$\text{subject to} \quad \text{(IC), (IR), (NP), and } R^k = \bar{R}.$$

Since (GRM) is binding, $\bar{R}$ is less than or equal to the expected revenue from the efficient second-price auction with a single unit. Thus, problem (3.k) with $k = 1$ is the same as $\mathcal{P}_0(\bar{R})$ in Section 2. For $k = 2$, problem (3.k) can be analyzed in a standard manner, and the solution is a uniform-price auction with some reserve price. The level of $\bar{R}$ is determined to maximize the expected social surplus $\pi S^1 + (1 - \pi) S^2$.

4.2 Buyers' outside options at the ex ante stage

In some cases, buyers may have outside options before participating in the auction, and choose not to participate if the payoff from the outside option exceeds the ex ante expected payoff in the auction. For example, other sellers who sell competing goods may exist, and buyers choose the seller that gives the largest expected payoff.

In this subsection, we assume that the hazard rate $\frac{f(x)}{1-F(x)}$ is strictly increasing. We focus on a fixed seller and suppose that n buyers participate in this seller's auction. We assume

[26] Muto and Shirata (2016) provide a restricted domain of valuation profiles in which efficient auctions satisfying goods revenue monotonicity exist among general combinatorial auctions.

symmetry between buyers, and denote by $u \geq 0$ the outside-option payoff for each buyer. To attract buyers, the seller has to guarantee to each buyer an ex-ante expected payoff greater than or equal to u. If the seller wants to maximize the expected revenue conditional on the given set of participating buyers, he solves

$$\max_{(q,t)} E_v \big[\sum_{i \in N} t_i(v)\big]$$
$$\text{subject to (IC), (IR), (NP), and (buyer's expected payoff)} \geq u.$$

We also assume that u is sufficiently high so that this constraint arising from the outside option is binding. Since symmetry between agents implies that

$$\begin{aligned}\text{(each buyer's ex-ante expected payoff)} &= \frac{1}{n}\Big(\text{(expected surplus)} - \text{(expected revenue)}\Big) \\ &= \frac{1}{n}\Big(E_v\big[\sum_{i \in N} q_i(v)v_i\big] - E_v\big[\sum_{i \in N} t_i(v)\big]\Big),\end{aligned}$$

the above problem is equivalent to

$$\max_{(q,t)} E_v \big[\sum_{i \in N} t_i(v)\big] \tag{4}$$
$$\text{subject to (IC), (IR), (NP), and } E_v\big[\sum_{i \in N} q_i(v)v_i\big] - E_v\big[\sum_{i \in N} t_i(v)\big] = nu.$$

We now revisit the original problem $\mathcal{P}_0(R)$.

$$\max_{(q,t)} E_v \big[\sum_{i \in N} q_i(v)v_i\big]$$
$$\text{subject to (IC), (IR), (NP), and } E_v\big[\sum_{i \in N} t_i(v)\big] \leq R.$$

Let S be the maximum social surplus in the above problem. Since we assumed strict monotonicity for the hazard rate, Proposition 8 implies that $\lambda(R) < 1$ for all $R \in (0, R^e)$. Since the differential coefficient for the Lagrangian at the optimal point is zero, the expected social surplus S in the constrained-efficient auction decreases at a speed slower than R does. Therefore, if S (and S') is the expected social surplus in the constrained-efficient auction with R (and R'), then $R > R'$ if and only if $S - R < S' - R'$. Thus, the revenue-cap constraint can be replaced by $E_v\left[\sum_{i\in N} q_i(v)v_i\right] - E_v\left[\sum_{i\in N} t_i(v)\right] \geq S - R$. Since we assumed that $R < R^e$, the revenue-cap constraint is binding. Therefore, our original problem $\mathcal{P}_0(R)$ is equivalent to

$$\max_{(q,t)} E_v\Big[\sum_{i\in N} q_i(v)v_i\Big]$$
$$\text{subject to (IC), (IR), (NP), and } E_v\Big[\sum_{i\in N} q_i(v)v_i\Big] - E_v\Big[\sum_{i\in N} t_i(v)\Big] = S - R.$$

This problem has the same form as (4) up to a constant. Hence, the problems with the outside option and our original problem have a common set of solutions when the hazard rate is strictly increasing. In conclusion, our analysis can be useful even in certain situations in which the seller's objective is revenue maximization.

5 Conclusion

We study a "pro-buyer" auction design problem involving maximization of the expected social surplus under a revenue-cap constraint. This problem is formally related to the maximization of a weighted sum of the expected social surplus and the expected revenue where the weight for the expected revenue is negative.

We characterize such a revenue-capped efficient auction and identify three general features that are in stark contrast to the standard case in the literature. First, the object is sold with probability one (i.e., no reserve price). Second, bunching must occur in some region of values (i.e., failure of pointwise maximization), and this bunching region stretches as the revenue-cap constraint becomes more stringent. Third, with a sufficiently severe revenue cap, bunching must occur at least at the top (i.e., failure of "no distortion at the top"). Thus, the introduction of a maximum bid is necessary in this case. The revenue-capped efficient auction takes a simple form if the virtual value function is either convex or concave. Finally, we identify two applications, a multi-unit auction and an ex ante competition among sellers, for which our analysis could be useful.

A Appendix

A.1 Proof of uniqueness of λ in problem $\mathcal{P}_1(R)$

Pick λ^1 and λ^2 satisfying $\lambda^1 < \lambda^2$. We fix a solution q^k to problem $\mathcal{P}_2(\lambda^k)$ for each $k = 1, 2$. Let S^k and R^k be the expected social surplus and expected revenue, respectively, under q^k for each $k = 1, 2$. We assume that $R^k \in (0, R^e)$ for each $k = 1, 2$ and prove that $R^1 > R^2$. This shows that $\lambda^1 \neq \lambda^2$ implies $R^1 \neq R^2$, and the uniqueness of λ follows for each $R \in (0, R^e)$.

We assume that $R^1 \leq R^2$, and derive a contradiction. Since the objective function of $\mathcal{P}_2(\lambda^k)$ equals $S^k - \lambda^k R^k$ for each k, the value of the objective at (q^2, λ^1) is $S^2 - \lambda^1 R^2$. Since q^1 is a solution to problem $\mathcal{P}_2(\lambda^1)$, we have $S^1 - \lambda^1 R^1 \geq S^2 - \lambda^1 R^2$. Since q^2 is a solution to problem $\mathcal{P}_2(\lambda^2)$, we have $S^2 - \lambda^2 R^2 \geq S^1 - \lambda^2 R^1$. Applying these two inequalities, we have $\lambda^1(R^2 - R^1) \geq S^2 - S^1 \geq \lambda^2(R^2 - R^1)$. Since we assumed that $R^1 \leq R^2$, then $(R^1, S^1) = (R^2, S^2)$. By this equality, problems $\mathcal{P}_2(\lambda^1)$ and $\mathcal{P}_2(\lambda^2)$ share the same set of solutions. Specifically, let q^* be the solution to problem $\mathcal{P}_2(\lambda^2)$ defined in equality (1) when $\lambda = \lambda^2$.[27] This must be a solution to problem $\mathcal{P}_2(\lambda^1)$. Since $R^2 < R^e$, there exists an open interval in which $\bar{\varphi}(x)$ is constant. (Otherwise, $\bar{\varphi}$ is strictly increasing, and q^* assigns the object efficiently. This contradicts the assumption $R < R^e$.)

Let (a, b) (where $0 \leq a < b \leq 1$) be the interval such that $\bar{\varphi}(x) < \bar{\varphi}(a)$ for all $x \in [0, a)$ and $\bar{\varphi}(x) > \bar{\varphi}(b)$ for all $x \in (b, 1]$. Since $\bar{\varphi}$ is continuous, $\bar{\varphi}(a) = \bar{\varphi}(b)$. Since $\bar{\varphi}(x) = \varphi(x)$ whenever $\bar{\varphi}(x)$ is strictly increasing in a small neighborhood of x, we have $\bar{\varphi}(a) = \varphi(a)$ and

[27] Proposition $\mathcal{P}_0(R)$ proves that q^* is the solution to problem $\mathcal{P}_2(\lambda^2)$. We note that Proposition $\mathcal{P}_0(R)$ does not rely on uniqueness of λ.

$\bar{\varphi}(b) = \varphi(b)$. Thus, $\varphi(a) = \varphi(b)$, that is, $a - \lambda^2\psi(a) = b - \lambda^2\psi(b)$. Because $b > a$, then $\psi(b) - \psi(a) = (b-a)/\lambda^2 > 0$. Let $\varphi_{\lambda^1}(x) := x - \lambda^1\psi(x)$ for each $x \in [0,1]$. Then we have $\varphi_{\lambda^1}(a) < \varphi_{\lambda^1}(b)$ because $\varphi_{\lambda^1}(b) - \varphi_{\lambda^1}(a) = (b-a) - \lambda^1(\psi(b) - \psi(a)) > (b-a) - \lambda^2(\psi(b) - \psi(a)) = \varphi(b) - \varphi(a) = 0$. Since φ_{λ^1} is continuous, there exist a' and b' such that $a < a' < b' < b$, $\max_{x\in[a,a']} |\varphi_{\lambda^1}(x) - \varphi_{\lambda^1}(a)| \leq (\varphi_{\lambda^1}(b) - \varphi_{\lambda^1}(a))/6$, and $\max_{x\in[b',b]} |\varphi_{\lambda^1}(x) - \varphi_{\lambda^1}(b)| \leq (\varphi_{\lambda^1}(b) - \varphi_{\lambda^1}(a))/6$. Let $\tilde{V} = \{v \in [0,1]^n \,|\, \exists j \in N \text{ such that } v_j \in [b',b] \text{ and } v_k \in [a,a'] \text{ for } \forall k \neq j\}$, and let $\tilde{q}$ be the following assignment function: for each $i \in N$ and each $v \in [0,1]$,

$$\tilde{q}_i(v) = \begin{cases} 1 & \text{if } v \in \tilde{V} \text{ and } v_i \in [b',b], \\ 0 & \text{if } v \in \tilde{V} \text{ and } v_i \in [a,a'], \\ q^*(v) & \text{if } v \notin \tilde{V}. \end{cases}$$

This assignment function satisfies monotonicity.

Let $\bar{a} \in \arg\max_{x\in[a,a']} \varphi_{\lambda^1}(x)$, $\bar{b} \in \arg\max_{x\in[b',b]} \varphi_{\lambda^1}(x)$, and $\underline{b} \in \arg\min_{x\in[b',b]} \varphi_{\lambda^1}(x)$. Since

$$E_v\Big[\sum_{i\in N} q_i^*(v)\varphi_{\lambda^1}(v_i)\Big| v \in \tilde{V}\Big] \leq \frac{1}{n}\varphi_{\lambda^1}(\bar{b}) + \frac{n-1}{n}\varphi_{\lambda^1}(\bar{a}) \text{ and}$$
$$E_v\Big[\sum_{i\in N} \tilde{q}_i(v)\varphi_{\lambda^1}(v_i)\Big| v \in \tilde{V}\Big] \geq \varphi_{\lambda^1}(\underline{b}),$$

we have

$$\begin{aligned}
&E_v\Big[\sum_{i\in N} \tilde{q}_i(v)\varphi_{\lambda^1}(v_i)\Big| v \in \tilde{V}\Big] - E_v\Big[\sum_{i\in N} q_i^*(v)\varphi_{\lambda^1}(v_i)\Big| v \in \tilde{V}\Big] \\
&\quad \geq \varphi_{\lambda^1}(\underline{b}) - \frac{1}{n}\varphi_{\lambda^1}(\bar{b}) - \frac{n-1}{n}\varphi_{\lambda^1}(\bar{a}) \\
&\quad = \frac{n-1}{n}\big(\varphi_{\lambda^1}(b) - \varphi_{\lambda^1}(a)\big) - \big(\varphi_{\lambda^1}(b) - \varphi_{\lambda^1}(\underline{b})\big) - \frac{1}{n}\big(\varphi_{\lambda^1}(\bar{b}) - \varphi_{\lambda^1}(b)\big) - \frac{n-1}{n}\big(\varphi_{\lambda^1}(\bar{a}) - \varphi_{\lambda^1}(a)\big) \\
&\quad \geq \frac{n-1}{n}\big(\varphi_{\lambda^1}(b) - \varphi_{\lambda^1}(a)\big) - \frac{1}{6}\big(\varphi_{\lambda^1}(b) - \varphi_{\lambda^1}(a)\big) - \frac{1}{6n}\big(\varphi_{\lambda^1}(b) - \varphi_{\lambda^1}(a)\big) - \frac{n-1}{6n}\big(\varphi_{\lambda^1}(b) - \varphi_{\lambda^1}(a)\big) \\
&\quad = \Big(\frac{n-1}{n} - \frac{1}{3}\Big)\big(\varphi_{\lambda^1}(b) - \varphi_{\lambda^1}(a)\big) > 0.
\end{aligned}$$

Therefore, $\tilde{q}$ improves the value of the objective in problem $\mathcal{P}_2(\lambda^1)$. This contradicts the assumption that q^* is a solution to problem $\mathcal{P}_2(\lambda^1)$.

A.2 Proof of Lemma 2

Note that, for all $x \in [0,1]$,

$$\begin{aligned}\int_0^x \psi(y)\,dF(y) &= \int_0^x \big(yf(y) - (1-F(y))\big)dy \\ &= -x(1-F(x)).\end{aligned}$$

Set $\lambda > 0$ arbitrarily. For all $x \in [0,1]$,

$$\begin{aligned}H(F(x)) &= \int_0^x \big(y - \lambda\psi(y)\big)dF(y) \\ &= \int_0^x y\,dF(y) + \lambda x(1-F(x)).\end{aligned}$$

Thus, $H(0) = 0$ and $H(p) > 0$ for all $p \in (0,1]$, and by the property of a convex hull, we have $G(0) = 0$ and $G(p) > 0$ for all $p \in (0,1]$. This implies that $g(0) \geq 0$, and by the convexity of G, $g(p) \geq 0$ for all $p \in [0,1]$. If $g(p) = 0$ for some $p \in (0,1]$, then by the convexity of G, $g(p') = 0$ for all $p' \in [0,p]$, which implies that $G(p) = 0$. Because this is a contradiction, we have $g(p) > 0$ for all $p \in (0,1]$. Hence, $\bar{\varphi}(x) > 0$ for all $x > 0$ and all $\lambda > 0$.

A.3 Proof of Proposition 2

The proof is based on Myerson (1981, Section 6). The objective in problem $\mathcal{P}_2(\lambda)$ is rewritten as follows:

$$\begin{aligned}&\sum_{i\in N} E_v\big[q_i(v)\big(v_i - \lambda\psi(v_i)\big)\big] = \sum_{i\in N} E_{v_i}\big[Q_i(v_i)\varphi(v_i)\big] \\ &= \sum_{i\in N} E_{v_i}\Big[Q_i(v_i)\big(g(F(v_i)) + h(F(v_i)) - g(F(v_i))\big)\Big]\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\in N} E_{v_i}\big[Q_i(v_i)\bar{\varphi}(v_i)\big] + \sum_{i\in N} E_{v_i}\big[Q_i(v_i)\big(h(F(v_i)) - g(F(v_i))\big)\big] \\
&= \sum_{i\in N} E_{v_i}\big[Q_i(v_i)\bar{\varphi}(v_i)\big] \\
&\quad + \sum_{i\in N}\big[Q_i(v_i)\big(H(F(v_i)) - G(F(v_i))\big)\big]_0^1 - \sum_{i\in N}\int_0^1 \big(H(F(v_i)) - G(F(v_i))\big)\, dQ_i(v_i).
\end{aligned}
$$

Because $G(0) = H(0)$ and $G(1) = H(1)$ by the definition of G, then $\sum_{i\in N}\big[Q_i(v_i)\big(H(F(v_i)) - G(F(v_i))\big)\big]_0^1 = 0$. Therefore,

$$
\sum_{i\in N} E_{v_i}\big[Q_i(v_i)\varphi(v_i)\big] = \sum_{i\in N} E_{v_i}\big[Q_i(v_i)\bar{\varphi}(v_i)\big] + \sum_{i\in N}\int_0^1 \big(G(F(v_i)) - H(F(v_i))\big)\, dQ_i(v_i). \quad (5)
$$

We first show that $\mathcal{Q}_2^{**}(\lambda) \subset \mathcal{Q}_2^{*}(\lambda)$. It suffices to show that if $q \in \mathcal{Q}_2^{**}(\lambda)$, then q maximizes both terms on the right-hand side of (5) separately. First, q maximizes the first term $\sum_{i\in N} E_{v_i}\big[Q_i(v_i)\bar{\varphi}(v_i)\big]$ by Lemma 2 and (i) in the statement. Second, q maximizes the second term on the right-hand side of (5) because of the following. Since $H(p) \geq G(p)$ for all $p \in [0,1]$ by the definition of G, the second term is nonpositive. We show that if q satisfies (ii) in the statement, then the second term is zero. Suppose that there exists $x \in (0,1)$ such that $G(F(x)) < H(F(x))$. Because G and H are continuous, there exists $v_i, v_i' \in (0,1)$ such that $v_i < x < v_i'$ and $G(F(y)) < H(F(y))$ for all $y \in (v_i, v_i')$. By (ii), $Q_i(x)$ is constant in (v_i, v_i'). This implies that $\sum_{i\in N}\int_{v_i}^{v_i'}\big(G(F(x)) - H(F(x))\big)\, dQ_i(x) = 0$. Because $x \in (0,1)$ was arbitrary, the second term is zero. This shows that q is a solution to problem $\mathcal{P}_2(\lambda)$. It is easy to see that $q^* \in \mathcal{Q}_2^{**}(\lambda)$, where q^* is defined in equality (1), and thus $\mathcal{Q}_2^{**}(\lambda) \neq \emptyset$.

We now show that if $\tilde{q}$ is a solution to problem $\mathcal{P}_2(\lambda)$, then there exists $q \in \mathcal{Q}_2^{**}(\lambda)$ such that $\text{Prob}\{v \mid \tilde{q}(v) = q(v)\} = 1$. Note that the argument in the above paragraph shows that q^* maximizes both terms on the right-hand side of (5). Thus, if $\tilde{q} \in \mathcal{Q}_2^{*}(\lambda)$, then $\tilde{q}$ also maximizes both terms on the right-hand side of (5). Because $\tilde{q}$ maximizes the first term, $\tilde{q}$ satisfies (i) in the statement with probability one. Because $\tilde{q}$ maximizes the second term (i.e., the value of the second term is zero) and $\tilde{Q}_i$ is nondecreasing, for each $v_i \in (0,1)$, if $H(F(v_i)) > G(F(v_i))$, then there must be a small neighborhood of v_i in which $\tilde{Q}_i(x)$ is constant. Thus, $\tilde{Q}_i$ is differentiable at v_i and $\tilde{Q}_i'(v_i) = 0$. Therefore, $\tilde{q}$ satisfies (ii) in the statement with probability one.

A.4 Proof of Proposition 3

It is obvious that q^* and q^{**} satisfy conditions (i) and (ii) in Proposition 2, and thus $q^*, q^{**} \in \mathcal{Q}_2^*(\lambda)$.

To clarify the dependence on λ, we denote $\varphi_\lambda(x) = x - \lambda\psi(x)$ and $H_\lambda(p) = \int_0^p \varphi_\lambda(F^{-1}(p'))\,dp'$ for each $\lambda > 0$. Let $G_\lambda(p)$ be the convex hull of $H_\lambda(p)$, and let $\bar{\varphi}_\lambda(x) = G_\lambda'(F^{-1}(x))$. We now prove the following lemma.

Lemma 3. *For each $x, x' \in [0,1]$ with $x < x'$ and each $\bar{\lambda} > 0$: (i) if $\bar{\varphi}_{\bar{\lambda}}(x) = \bar{\varphi}_{\bar{\lambda}}(x')$, then $\bar{\varphi}_\lambda(x) = \bar{\varphi}_\lambda(x')$ for all $\lambda \geq \bar{\lambda}$; and (ii) if $\bar{\varphi}_{\bar{\lambda}}(x) < \bar{\varphi}_{\bar{\lambda}}(x')$, then there exists $\hat{\lambda} > \bar{\lambda}$ such that $\bar{\varphi}_\lambda(x) < \bar{\varphi}_\lambda(x')$ for all $\lambda \in [\bar{\lambda}, \hat{\lambda}]$.*

Proof of Lemma 3. Let $x, x' \in [0,1]$ with $x < x'$ and $\bar{\lambda} > 0$. First we prove (i). Suppose that $\bar{\varphi}_\lambda(x) = \bar{\varphi}_{\bar{\lambda}}(x')$. Equivalently, there exists $a, b \in [0,1]$ with $a \leq x < x' \leq b$ such that for all $y \in (a,b)$,

$$H_{\bar{\lambda}}(F(y)) \geq H_{\bar{\lambda}}(F(a)) + (F(y) - F(a))\frac{H_{\bar{\lambda}}(F(b)) - H_{\bar{\lambda}}(F(a))}{F(b) - F(a)}.$$

Observe that for any $\lambda \geq \bar{\lambda}$,

$$\begin{aligned}
&\frac{1}{\lambda}\Big(H_\lambda(F(y)) - H_\lambda(F(a)) - (F(y) - F(a))\frac{H_\lambda(F(b)) - H_\lambda(F(a))}{F(b) - F(a)}\Big)\\
&- \frac{1}{\bar{\lambda}}\Big(H_{\bar{\lambda}}(F(y)) - H_{\bar{\lambda}}(F(a)) - (F(y) - F(a))\frac{H_{\bar{\lambda}}(F(b)) - H_{\bar{\lambda}}(F(a))}{F(b) - F(a)}\Big)\\
&\quad = (\frac{1}{\bar{\lambda}} - \frac{1}{\lambda})\Big((F(y) - F(a))\frac{\int_a^b w dF(w)}{F(b) - F(a)} - \int_a^y w\, dF(w)\Big)\\
&\quad = (\frac{1}{\bar{\lambda}} - \frac{1}{\lambda})\Big(\frac{(F(y) - F(a))\int_y^b w dF(w)}{F(b) - F(a)} - \frac{(F(b) - F(y))\int_a^y w dF(w)}{F(b) - F(a)}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\geq \Big(\frac{1}{\bar{\lambda}} - \frac{1}{\lambda}\Big)\Big(\frac{(F(y)-F(a))(F(b)-F(y))y}{F(b)-F(a)} - \frac{(F(b)-F(y))(F(y)-F(a))y}{F(b)-F(a)}\Big) \\
&= 0.
\end{aligned}
$$

Therefore, for all $y \in (a, b)$ and all $\lambda \geq \bar{\lambda}$ we obtain

$$
H_\lambda(F(y)) \geq H_\lambda(F(a)) + (F(y) - F(a))\frac{H_\lambda(F(b)) - H_\lambda(F(a))}{F(b) - F(a)},
$$

which implies that $\bar{\varphi}_\lambda(x) = \bar{\varphi}_\lambda(x')$.

We now prove (ii). Suppose that $\bar{\varphi}_{\bar{\lambda}}(x) < \bar{\varphi}_{\bar{\lambda}}(x')$. Equivalently, for any $a, b \in [0, 1]$ with $a \leq x < x' \leq b$ there exists $y \in (a, b)$ such that

$$
H_{\bar{\lambda}}(F(y)) < H_{\bar{\lambda}}(F(a)) + (F(y) - F(a))\frac{H_{\bar{\lambda}}(F(b)) - H_{\bar{\lambda}}(F(a))}{F(b) - F(a)}.
$$

By the continuity of $H_\lambda(F(a)), H_\lambda(F(b))$ and $H_\lambda(F(y))$ in λ, there exists $\hat{\lambda} > \bar{\lambda}$ such that, for all $\lambda \in [\bar{\lambda}, \hat{\lambda}]$,

$$
H_\lambda(F(y)) < H_\lambda(F(a)) + (F(y) - F(a))\frac{H_\lambda(F(b)) - H_\lambda(F(a))}{F(b) - F(a)}.
$$

Therefore, $\bar{\varphi}_\lambda(x) < \bar{\varphi}_\lambda(x')$ for all $\lambda \in [\bar{\lambda}, \hat{\lambda}]$. □

Let $q^*(\cdot; \lambda)$ be the assignment function defined in (1). We show that $q^*(\cdot; \lambda)$ achieves the minimum expected revenue for each $\lambda > 0$.

Let $v \in [0, 1]^n$. Applying Lemma 3 for each pair of $(x, x') \in \{v_1, \ldots, v_n\}^2$ with $x < x'$, we obtain $\hat{\lambda} > \bar{\lambda}$ such that for each $i, j \in N$ and each $\lambda \in [\bar{\lambda}, \hat{\lambda}]$,

$$
\bar{\varphi}_{\bar{\lambda}}(v_i) \lesseqqgtr \bar{\varphi}_{\bar{\lambda}}(v_j) \iff \bar{\varphi}_\lambda(v_i) \lesseqqgtr \bar{\varphi}_\lambda(v_j).
$$

Hence, $q^*(v; \lambda)$ gives the same assignment for all $\lambda \in [\bar{\lambda}, \hat{\lambda}]$.

Applying this for each $v \in [0, 1]^n$, we obtain pointwise convergence of $\sum_i q_i^*(v; \lambda)\psi(v_i)f(v)$

to $\sum_i q_i^*(v;\bar\lambda)\psi(v_i)f(v)$ as $\lambda \searrow \bar\lambda$. Because $\sum_i q_i^*(v;\lambda)\psi(v_i)f(v)$ is bounded uniformly across all v, by the dominated convergence theorem we have

$$\begin{aligned} E_v\big[\sum_{i\in N} q_i^*(v;\lambda)\psi(v_i)\big] &= \int_{v\in[0,1]^n} \sum_{i\in N} q_i^*(v;\lambda)\psi(v_i)f(v)dv \\ &\to \int_{v\in[0,1]^n} \sum_{i\in N} q_i^*(v;\bar\lambda)\psi(v_i)f(v)dv \qquad (\text{as } \lambda \searrow \bar\lambda) \\ &= E_v\big[\sum_{i\in N} q_i^*(v;\bar\lambda)\psi(v_i)\big], \end{aligned}$$

that is, the expected revenue under assignment function $q^*(\cdot;\lambda)$ converges to the expected revenue under assignment function $q^*(\cdot;\bar\lambda)$ as $\lambda \searrow \bar\lambda$.

Because $\lambda(R)$ is a nonincreasing function, the limit of the expected revenue under $q^*(\cdot;\lambda)$ as $\lambda \searrow \bar\lambda$ is weakly less than the expected revenue under any $q \in \mathcal{Q}_2^*(\lambda)$. This means that the expected revenue under q^* is minimum among all $q \in \mathcal{Q}_2^*(\lambda)$.

We now prove the following lemma.

Lemma 4. *For each $\bar\lambda > 0$: (i) for each $x \in [0,1]$, if $G_{\bar\lambda}(F(x)) = H_{\bar\lambda}(F(x))$, then $G_\lambda(F(x)) = H_\lambda(F(x))$ for all $\lambda \in (0,\bar\lambda]$; and (ii) for each $x, x' \in [0,1]$ with $x < x'$, if $G_{\bar\lambda}(F(y)) > H_{\bar\lambda}(F(y))$ for all $y \in [x,x']$, then there exists $\hat\lambda \in (0,\bar\lambda)$ such that $G_\lambda(F(y)) > H_\lambda(F(y))$ for all $\lambda \in [\hat\lambda,\bar\lambda]$ and all $y \in [x,x']$.*

Proof of Lemma 4. Let $x \in [0,1]$ and $\bar\lambda > 0$. We first prove (i). Suppose that $G_{\bar\lambda}(F(x)) = H_{\bar\lambda}(F(x))$. Equivalently, for any $a, b \in [0,1]$ with $a < x < b$,

$$H_{\bar\lambda}(F(x)) \le H_{\bar\lambda}(F(a)) + (F(x)-F(a))\frac{H_{\bar\lambda}(F(b)) - H_{\bar\lambda}(F(a))}{F(b)-F(a)}.$$

Observe that for any $\lambda \in (0,\bar\lambda]$,

$$\begin{aligned} &\frac{1}{\lambda}\Big(-H_\lambda(F(x)) + H_\lambda(F(a)) + (F(x)-F(a))\frac{H_\lambda(F(b)) - H_\lambda(F(a))}{F(b)-F(a)}\Big) \\ &- \frac{1}{\bar\lambda}\Big(-H_{\bar\lambda}(F(x)) + H_{\bar\lambda}(F(a)) + (F(x)-F(a))\frac{H_{\bar\lambda}(F(b)) - H_{\bar\lambda}(F(a))}{F(b)-F(a)}\Big) \end{aligned}$$

$$
\begin{aligned}
&= \Big(\frac{1}{\lambda} - \frac{1}{\bar{\lambda}}\Big)\Big((F(x) - F(a))\frac{\int_a^b w dF(w)}{F(b) - F(a)} - \int_a^x w\, dF(w)\Big) \\
&= \Big(\frac{1}{\lambda} - \frac{1}{\bar{\lambda}}\Big)\Big(\frac{(F(x) - F(a)) \int_x^b w dF(w)}{F(b) - F(a)} - \frac{(F(b) - F(x)) \int_a^x w dF(w)}{F(b) - F(a)}\Big) \\
&\geq \Big(\frac{1}{\lambda} - \frac{1}{\bar{\lambda}}\Big)\Big(\frac{(F(x) - F(a))(F(b) - F(x))x}{F(b) - F(a)} - \frac{(F(b) - F(x))(F(x) - F(a))x}{F(b) - F(a)}\Big) \\
&= 0.
\end{aligned}
$$

Therefore, for all $x \in (a, b)$ and all $\lambda \in (0, \bar{\lambda}]$, we obtain

$$
H_\lambda(F(y)) \leq H_\lambda(F(a)) + (F(y) - F(a))\frac{H_\lambda(F(b)) - H_\lambda(F(a))}{F(b) - F(a)},
$$

which implies that $G_\lambda(F(x)) = H_\lambda(F(x))$.

Now we prove (ii). Suppose that $G_{\bar{\lambda}}(F(y)) > H_{\bar{\lambda}}(F(y))$ for all $y \in [x, x']$. Equivalently, there exists $a, b \in [0, 1]$ such that $a \leq x < x' \leq b$, and for all $y \in [x, x']$,

$$
H_{\bar{\lambda}}(F(y)) > H_{\bar{\lambda}}(F(a)) + (F(y) - F(a))\frac{H_{\bar{\lambda}}(F(b)) - H_{\bar{\lambda}}(F(a))}{F(b) - F(a)}.
$$

By the continuity of $H_\lambda(F(a)), H_\lambda(F(b))$ and $H_\lambda(F(x))$ in λ, there exists $\hat{\lambda} \in (0, \bar{\lambda}]$ such that, for all $y \in [x, x']$ and all $\lambda \in [\hat{\lambda}, \bar{\lambda}]$,

$$
H_\lambda(F(y)) > H_\lambda(F(a)) + (F(y) - F(a))\frac{H_\lambda(F(b)) - H_\lambda(F(a))}{F(b) - F(a)}.
$$

Therefore, $G_\lambda(F(y)) > H_\lambda(F(y))$ for all $y \in [x, x']$ and all $\lambda \in [\hat{\lambda}, \bar{\lambda}]$. □

Let $q^{**}(\cdot; \lambda)$ be the assignment function defined in (2). We show that $q^{**}(\cdot; \lambda)$ achieves the maximum expected revenue for each $\lambda > 0$.

Applying Lemma 4 for each $x, x' \in [0, 1]$ with $x < x'$, there exists $\hat{\lambda} \in (0, \bar{\lambda})$ such that for each $\lambda \in [\hat{\lambda}, \bar{\lambda}]$,

$$G_{\bar{\lambda}}(y) > H_{\bar{\lambda}}(y) \text{ for all } y \in [x, x'] \iff G_{\lambda}(y) > H_{\lambda}(y) \text{ for all } y \in [x, x'].$$

Thus, for each $v \in [0,1]^n$ there exists $\hat{\lambda} \in (0, \bar{\lambda})$ such that for each $i, j \in N$ and each $\lambda \in [\hat{\lambda}, \bar{\lambda}]$,

$$B_{\bar{\lambda}}(v_i) = B_{\bar{\lambda}}(v_j) \iff B_{\lambda}(v_i) = B_{\lambda}(v_j),$$

where for each λ, $B_{\lambda}(x)$ is the set $B(x)$ introduced before the definition of q^{**}. Hence, for each $v \in [0,1]^n$, $q^{**}(v; \lambda)$ gives the same assignment for all $\lambda \in [\hat{\lambda}, \bar{\lambda}]$.

Applying this, for each $v \in [0, 1]$ we obtain pointwise convergence of $\sum_i q_i^{**}(v; \lambda)\psi(v_i)f(v)$ to $\sum_i q_i^{**}(v; \bar{\lambda})\psi(v_i)f(v)$ as $\lambda \nearrow \bar{\lambda}$. Because $\sum_i q_i^{**}(v; \lambda)\psi(v_i)f(v)$ is bounded uniformly across all v, by the dominated convergence theorem,

$$\begin{aligned}
E_v\big[\sum_{i \in N} q_i^{**}(v; \lambda)\psi(v_i)\big] &= \int_{v \in [0,1]^n} \sum_{i \in N} q_i^{**}(v; \lambda)\psi(v_i)f(v)dv \\
&\to \int_{v \in [0,1]^n} \sum_{i \in N} q_i^{**}(v; \bar{\lambda})\psi(v_i)f(v)dv \qquad (\text{as } \lambda \nearrow \bar{\lambda}) \\
&= E_v\big[\sum_{i \in N} q_i^{**}(v; \bar{\lambda})\psi(v_i)\big],
\end{aligned}$$

that is, the expected revenue under assignment function $q^{**}(\cdot; \lambda)$ converges to the expected revenue under assignment function $q^{**}(\cdot; \bar{\lambda})$ as $\lambda \nearrow \bar{\lambda}$.

Because $\lambda(R)$ is a nonincreasing function, the limit of the expected revenues under $q^{**}(\cdot; \lambda)$ as $\lambda \nearrow \bar{\lambda}$ is weakly greater than the expected revenue under any $q \in \mathcal{Q}_2^*(\lambda)$. This means that the expected revenue under q^{**} is maximum among all $q \in \mathcal{Q}_2^*(\lambda)$.

A.5 Proof of Proposition 6

To clarify the dependence on λ, we denote $\varphi_\lambda(x) = x - \lambda\psi(x)$ and $H_\lambda(p) = \int_0^p \varphi_\lambda(F^{-1}(p'))\, dp'$ for each $\lambda > 0$. Let $G_\lambda(p)$ be the convex hull of $H_\lambda(p)$, and let $\bar{\varphi}_\lambda(x) = G'_\lambda(F^{-1}(x))$.

By Proposition 1, $q \in \mathcal{Q}_2^*(\lambda(R))$. By Lemma 3 (i) in Appendix A.4, for each $x, x' \in [0,1]$ and each $\tilde{\lambda} \geq \lambda(R)$, $\bar{\varphi}_{\lambda(R)}(x) = \bar{\varphi}_{\lambda(R)}(x')$ implies that $\bar{\varphi}_{\tilde{\lambda}}(x) = \bar{\varphi}_{\tilde{\lambda}}(x')$. Therefore, for each $i \in N$, if $Q_i^*(v_i)$ is constant on (a,b), then $\tilde{Q}_i^*(v_i)$ is also constant on (a,b).

References

Anderson, S., Friedman, D., Milam, G., and Singh, N. (2008). Buy it now: A hybrid internet market institution. *Journal of Electronic Commerce Research*, 9(2):137–153.

Armstrong, M., Cowan, S., and Vickers, J. (1995). Nonlinear pricing and price cap regulation. *Journal of Public Economics*, 58(1):33–55.

Armstrong, M. and Sappington, D. E. M. (2007). Recent developments in the theory of regulation. In *Handbook of Industrial Organization*, volume 3, pages 1557–1699. North Holland.

Baron, D. P. (1988). Regulation and legislative choice. *The RAND Journal of Economics*, 19(3):467–477.

Baron, D. P. and Myerson, R. B. (1982). Regulating a monopolist with unknown costs. *Econometrica*, 50(4):911–930.

Besley, T. and Coate, S. (2003). Elected versus appointed regulators: Theory and evidence. *Journal of the European Economic Association*, 1(5):1176–1206.

Budish, E. B. and Takeyama, L. N. (2001). Buy prices in online auctions: irrationality on the internet? *Economics Letters*, 72(3):325–333.

Caillaud, B., Guesnerie, R., Rey, P., and Tirole, J. (1988). Government intervention in production and incentives theory: A review of recent contributions. *The RAND Journal of Economics*, 19(1):1–26.

Decarolis, F. (2009). When the highest bidder loses the auction: Theory and evidence from public procurement. Bank of Italy Temi di Discussione (Working Paper) No. 717.

DeMarzo, P. and Duffie, D. (1999). A liquidity-based model of security design. *Econometrica*, 67(1):65–99.

Engel, E., Fischer, R., and Galetovic, A. (2013). The basic public finance of public private partnerships. *Journal of the European Economic Association*, 11(1):83–111.

Eun, D. J. (2016). Reining in ex-post adjustment: a new procurement mechanism from Korea. Working paper.

Fudenberg, D. and Tirole, J. (1991). *Game Theory*. MIT Press, Cambridge.

Guensnerie, R. and Laffont, J.-J. (1984). A complete solution to a class of principal-agent problems with an application to the control of a self-managed firm. *Journal of Public Economics*, 25(3):329–369.

Ishii, R. (2008). Collusion in repeated procurement auction: A study of a paving market in Japan. ISER Discussion Paper No. 710.

Kirkegaard, R. and Overgaard, P. B. (2008). Buy-out prices in auctions: seller competition and multi-unit demands. *RAND Journal of Economics*, 39(3):770–789.

Laffont, J.-J. and Tirole, J. (1986). Using cost observation to regulate firms. *Journal of Political Economy*, 94(3):614–641.

Luenberger, D. G. (1969). *Optimization by Vector Space Methods*. Wiley-Interscience, New York.

Mathews, T. (2004). The impact of discounting on an auction with a buyout option: a theoretical analysis motivated by eBay's Buy-It-Now feature. *Journal of Economics*, 81(1):25–52.

McAfee, R. P. and McMillan, J. (1992). Bidding rings. *The American Economic Review*, 82(3):579–599.

Muto, N. and Shirata, Y. (2016). Manipulation via endowments in auctions with multiple goods. Mimeo.

Myerson, R. B. (1981). Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73.

Peters, M. and Severinov, S. (1997). Competition among sellers who offer auctions instead of prices. *Journal of Economic Theory*, 75(1):141–179.

Reynolds, S. S. and Wooders, J. (2009). Auctions with a buy price. *Economic Theory*, 38(1):9–39.

Toikka, J. (2011). Ironing without control. *Journal of Economic Theory*, 146(6):2510–2526.