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Two-person Pairwise Solvable Games

Takuya Iimura,[1] Toshimasa Maruta,[2] and Takahiro Watanabe[1]

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Abstract: A game is solvable if the set of Nash equilibria is nonempty and interchangeable. A pairwise solvable game is a two-person symmetric game in which any restricted game generated by a pair of strategies is solvable. We show that the set of equilibria in a pairwise solvable game is interchangeable. Under a quasiconcavity condition, we derive a complete order-theoretic characterization and some topological sufficient conditions for the existence of equilibria, and show that if the game is finite, then an iterated elimination of weakly dominated strategies leads precisely to the set of Nash equilibria, which means that such a game is both solvable and dominance solvable. All results are applicable to symmetric contests, such as the rent-seeking game and the rank-order tournament, which are shown to be pairwise solvable. Some applications to evolutionary equilibria are also given.

Keywords: Zero-sum games, quasiconcavity, interchangeability, iterated dominance, dominance solvability, Nash equilibrium, evolutionary equilibrium.

Journal of Economic Literature Classification Numbers: C70, C72.

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[1]Tokyo Metropolitan University, 1-1 Minami Ohsawa, Hachioji, Tokyo, 192-0397, Japan. Iimura and Watanabe gratefully acknowledge financial support from KAKENHI 25380233 and 16K03553.

[2]Corresponding Author. University Research Center and Population Research Institute, Nihon University, 12-5 Gobancho, Chiyoda, Tokyo, 102-8251, Japan. Maruta gratefully acknowledges financial support from KAKENHI 26380245. maruta.toshimasa@nihon-u.ac.jp

1 Introduction

Nash (1951) calls a strategic game *solvable* if the set of equilibria is nonempty and *interchangeable*, which means that it is precisely equal to the Cartesian product of sets of strategies, one for each player.[1] Moulin (1979), meanwhile, calls a game *dominance solvable* if for each player the strategies that survive the iterated elimination of weakly dominated strategies are all equivalent.[2] In this paper, we examine the consequences of requiring that "smaller components" of the game be solvable in the sense of Nash. Focusing on *two-person symmetric games*, this requirement leads to the notion of a *pairwise solvable game*. We show that pairwise solvable games not only have a number of notable properties, but also accommodate many games that frequently appear in applications. In particular, we establish a link between solvability and the dominance solvability. Under a quasiconcavity condition, any *finite* pairwise solvable game is shown to be dominance solvable.

A two-person symmetric game is pairwise solvable if any 2×2 game generated by a pair of strategies is solvable in the sense of Nash. Noting that any 2×2 symmetric game has a pure strategy Nash equilibrium, it follows that the restricted game is solvable if and only if the set of equilibria is interchangeable, which, in turn, is equivalent to the condition that either one strategy strictly dominates the other, or both are equivalent. Consequently, the whole game is pairwise solvable if and only if the trichotomy prevails for each pair of strategies.

It is straightforward to see that any two-person symmetric constant-sum game is pairwise solvable. It turns out that pairwise solvability extends much further. We consider *two-person symmetric contests*, in which each player is rewarded as either the winner or the loser, where the winning probability of a player is jointly and symmetrically determined, with or without externality, by a pair of costly actions by the players. This class of games includes the tournament game of Lazear and Rosen (1981), the rent-seeking game of Tullock (1980), and variations thereof. We show that two-person symmetric contests are pairwise solvable. Other examples include weakly unilaterally competitive games (Kats and Thisse 1992) and games with weak payoff externalities (Ania 2008). The class of pairwise solvable games is rich enough to merit special attention.

The formal results of the present paper are summarized as follows. Unless explicitly stated otherwise, we only consider equilibria in pure strategies.[3] We begin by showing that the set

[1]In fact, Nash (1951, p.290) simply defines solvability as the interchangeability. This is because he works on the mixed extension of a finite strategic game, which is shown by him to have at least one equilibrium. However, the formal definition of interchangeability is vacuously satisfied if there is no equilibrium. In more general settings, therefore, one may define solvability as the nonemptiness and the interchangeability of the set of all equilibria.

[2]The definition of dominance solvability varies. We adopt one in Moulin (1979, 1986), which implies neither uniqueness nor the payoff equivalence of equilibria. In Moulin (1984), dominance solvability implies uniqueness.

[3]Hence an equilibrium in mixed strategies should be regarded as a pure strategy equilibrium in the mixed

of all equilibria in any pairwise solvable game is interchangeable. The main implication is that questions on equilibria can be reduced to those on symmetric equilibria. In particular, the game is solvable if and only if there is a symmetric equilibrium, which does not always exist. When is a pairwise solvable game solvable? We offer two answers. First, we show that a strategy in a pairwise solvable game constitutes a symmetric equilibrium if and only if it is "maximal" in the binary relation generated by the trichotomy condition of pairwise solvability. This characterization is general, but not straightforward to apply. As the second approach, therefore, we focus on a special class of games. Let strategies be *linearly ordered*. We say that a two-person symmetric game with linearly ordered strategies is *quasiconcave at the diagonal* if the payoff function of a player is quasiconcave in her own-strategy at any symmetric strategy profile. This condition is weaker than the usual quasiconcavity on the real line. We derive a complete characterization of the existence of a Nash equilibrium in a pairwise solvable game that is quasiconcave at the diagonal. It is purely order-theoretic, and applies to any game with linearly ordered strategies. When they form a subset of the reals, it generates several topological sufficient conditions for existence, which accommodate discontinuous games such as the electoral competition model of Hotelling (1929).

Furthermore, for *finite* games, striking dominance relations emerge. We introduce a specific strategy elimination rule for a two-person finite symmetric game. Its critical feature is that it eliminates either the smallest strategy but nothing else, the largest strategy but nothing else, or no strategy at all. For a finite pairwise solvable game that is quasiconcave at the diagonal, we show that if the rule eliminates a strategy, then it must be a weakly dominated strategy. Conversely, if the rule fails to eliminate any strategy, then all strategies must be equivalent to each other, and hence, there is no weakly dominated strategy. The bottom line is that the successive application of the elimination rule generates an iterated elimination of weakly dominated strategies that leads *precisely* to the set of all Nash equilibria. Consequently, the game is dominance solvable in the sense of Moulin (1979).

In the last part of the paper, we apply the existence results to an investigation of the evolutionary equilibrium introduced by Schaffer (1989). It is known that a symmetric strategy profile in a two-person symmetric game is an evolutionary equilibrium if and only if it is a Nash equilibrium of the relative payoff game of the given game. By definition, the relative payoff game is a zero-sum game. Thus the game has an evolutionary equilibrium if the relative payoff game is quasiconcave at the diagonal. This line of reasoning leads to the notion of a *concave-convex* game. We show that if a two-person symmetric game is pairwise solvable, concave-convex, and sufficiently continuous, then it has simultaneously a symmetric Nash equilibrium and an evolutionary equilibrium. We also show that in any pairwise solvable game, any symmetric Nash equilibrium weakly Pareto-dominates any evolutionary equilibrium.

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extension.

The present paper compares with the previous literature as follows. Generalizing two-person strictly competitive games (Friedman 1983), Kats and Thisse (1992) introduce the notion of n-person weakly unilaterally competitive game. They show that the set of Nash equilibria in a two-person weakly unilaterally competitive game is interchangeable. A two-person symmetric weakly unilaterally competitive game is pairwise solvable but not vice versa. For two-person symmetric games, therefore, our interchangeability result generalizes theirs. In contrast to the present study, their analysis contains no results concerning existence nor dominance.[4] Duersch, Oechssler, and Schipper (2012a) show, among other things, that a two-person symmetric zero-sum game has a pure strategy Nash equilibrium if it is quasiconcave. Our existence result strengthens theirs and their study contains no discussion of dominance. As far as we know, no previous result demonstrates the dominance solvability in a class of games rich enough to include (finite formulations of) tournament games and rent-seeking games. Concerning concave-convex games, Duersch et al. (2012b) contains a result that is similar to ours, but the corresponding definitions and scopes are distinctively different. Hehenkamp, Leininger, and Possajennikov (2004) discover the Pareto-dominance relation between Nash and evolutionary equilibria in a class of symmetric n-person rent-seeking games with specific functional forms. In the class of two-person symmetric games, our result extends this to any pairwise solvable game.

Section 2 collects basic notations and definitions. In Section 3, we define pairwise solvability, derive the equivalent trichotomy condition, and introduce notions of skew-symmetry of a function, which prove useful in finding various examples of pairwise solvable games, including two-person symmetric contests. In Section 4, the interchangeability and the equilibrium characterization are shown for general pairwise solvable games. In Section 5, focusing on games with linearly ordered strategies that are quasiconcave at the diagonal, we derive characterizations and sufficient conditions for the existence of an equilibrium. In Section 6, we develop the analysis of the iterated elimination of weakly dominated strategies and the dominance solvability for finite games. In Section 7, we turn to evolutionary equilibria and their relation to Nash equilibria. We close the paper by giving some concluding remarks in Section 8.

2 Preliminaries

Let S be a nonempty set of strategies and $u : S \times S \to \mathbb{R}$ be a real valued function. The pair $G = \langle S, u \rangle$ defines a *two-person symmetric game*. In each strategy profile $(s,t) \in S \times S$, $u(s,t)$ is the payoff for the player who chooses $s \in S$ (the row player) and $u(t,s)$ is the payoff for the player who chooses $t \in S$ (the column player). A strategy profile (s,t) in G is a *Nash equilibrium*

[4] An equilibrium existence result for n-person symmetric weakly unilaterally competitive games is obtained in Iimura and Watanabe (2015).

(in pure strategies) if $u(s,t) \geq u(s',t)$ and $u(t,s) \geq u(t',s)$ for every $s',t' \in S$. In particular, (s,s) is a Nash equilibrium if and only if $u(s,s) \geq u(t,s)$ for every $t \in S$. Such an equilibrium is called a *symmetric Nash equilibrium*. In this paper, a Nash equilibrium always means a Nash equilibrium in pure strategies. This need not be a serious restriction, since we can always regard a mixed strategy equilibrium in a finite game as a pure strategy equilibrium in the mixed extension.[5] Let $\mathcal{E} = \mathcal{E}_G$ be the set of all Nash equilibria in G. Recall that if $S = \{s,t\}$, then $\mathcal{E} \neq \varnothing$. That is to say, any 2×2 symmetric game has a pure strategy equilibrium. Since the inequalities in the definition of an equilibrium are symmetric with respect to s and t, we have that

Lemma 2.1. *The set $\mathcal{E}$ of all Nash equilibria in a two-person symmetric game is symmetric: For every $(s,t) \in S \times S$, $(s,t) \in \mathcal{E}$ if and only if $(t,s) \in \mathcal{E}$.*

Strategy $s \in S$ *strictly dominates* strategy $s' \in S$ if $u(s,t) > u(s',t)$ for every $t \in S$. Strategy s *weakly dominates* strategy s' if $u(s,t) \geq u(s',t)$ for every $t \in S$ and there is $r \in S$ such that $u(s,r) > u(s',r)$. Let $T \subset S$ be a nonempty set of strategies. The *restricted game* generated by T is the two-person symmetric game with strategy set T and payoff function u that is restricted to $T \times T$. Strategies $s \in T$ and $s' \in T$ are *equivalent* in T if $u(s,t) = u(s',t)$ for every $t \in T$ (cf. Moulin, 1979, p.1339). Assume that s and s' are equivalent in T and pick any $t \in T$. Note that it *does not* follow that $u(t,s) = u(t,s')$. Among several equivalent strategies, it does not matter to the chooser which strategy she employs, but it may matter greatly to the other player.

3 Pairwise solvable games

3.1 Definition

Let $G = \langle S, u \rangle$ be a two-person symmetric game with strategy set S. Recall that $\mathcal{E}_G$ is the set of all Nash equilibria in G. $\mathcal{E} = \mathcal{E}_G$ is *interchangeable* if the following condition is satisfied: if $(s,t),(s',t') \in \mathcal{E}$, then $(s,t') \in \mathcal{E}$ (Nash 1951, p.290). Thanks to Lemma 2.1, we have an equivalent condition, which shall be useful in the sequel. We omit the straightforward proof.

Lemma 3.1. *Let $\mathcal{E}$ be the set of all Nash equilibria in a two-person symmetric game. A necessary and sufficient condition for $\mathcal{E}$ to be interchangeable is that*

(Int) $(s,t) \in \mathcal{E}$ *if and only if* $(s,s),(t,t) \in \mathcal{E}$.

For exposition, call a strategy that appears in an equilibrium an equilibrium strategy. Trivially, an equilibrium is a pair of equilibrium strategies. A pair of equilibrium strategies,

[5] Accordingly, we consider the mixed extension of a finite game only when we do so explicitly.

	s	t
s	$u(s,s), u(s,s)$	$u(s,t), u(t,s)$
t	$u(t,s), u(s,t)$	$u(t,t), u(t,t)$

Figure 1: The restricted game $g(s,t)$ generated by $\{s,t\}$.

however, is not an equilibrium in general, and this is where the interchangeability comes in: $\mathcal{E}$ is interchangeable if and only if any pair of equilibrium strategies is an equilibrium. Formally, $\mathcal{E}$ is interchangeable, or (Int) is true, if and only if $\mathcal{E} = E \times E$, where $E = \{\, s \in S \mid (s,s) \in \mathcal{E} \,\}$ is the set of *equilibrium strategies*. Following Nash (1951), let us call the game *solvable* if $\mathcal{E}$ is nonempty and interchangeable.

In this paper, we examine the consequences of requiring that "smaller components" of the game be solvable in the sense of Nash (1951). Let $G = \langle S, u \rangle$ be a two-person symmetric game. For each pair of distinct strategies $s, t \in S$, consider the restricted game generated by $\{s,t\}$, which is the symmetric 2×2 game depicted in Figure 1 as $g(s,t)$. Since $\mathcal{E}_{g(s,t)} \neq \varnothing$, $g(s,t)$ is solvable if and only if $\mathcal{E}_{g(s,t)}$ is interchangeable. The game G is *pairwise solvable* if every $g(s,t)$ is solvable, or equivalently, $\mathcal{E}_{g(s,t)}$ is interchangeable. Pairwise solvability admits a useful characterization in terms of dominance.

Lemma 3.2. *Consider the game $g(s,t)$ in Figure 1. $\mathcal{E}_{g(s,t)}$ is interchangeable if and only if either s strictly dominates t, t strictly dominates s, or s and t are equivalent.*

Proof. Assume that the game is interchangeable. We know that at least one of the four pure strategy profiles is in $\mathcal{E} = \mathcal{E}_{g(s,t)}$. Given Lemmas 2.1 and 3.1, $\mathcal{E}$ has to be equal to either $\{(s,s)\}$, $\{(t,t)\}$, or $\{s,t\} \times \{s,t\}$, which mean, respectively, that s strictly dominates t, t strictly dominates s, or s and t are equivalent. The converse is obvious. $\square$

Consequently, G is pairwise solvable if and only if the following condition is satisfied: For every distinct $s, t \in S$,

(PS) $u(s,s) > u(t,s)$ if and only if $u(s,t) > u(t,t)$.

In what follows, condition (PS) works as the definition of the pairwise solvability. Clearly, pairwise solvability is a purely ordinal concept. It is invariant under any order-preserving payoff transformation.

3.2 Skew-symmetries and pairwise solvability

Consider a real valued function $f : X \times X \to \mathbb{R}$. It is *skew-symmetric* (cf. von Neumann and Morgenstern 1944, p.166) if for every $x, y \in X$,

$$f(y,x) = -f(x,y).$$

For a real number $r \in \mathbb{R}$, let $\texttt{sgn}(r) = 1, -1, 0$ if $r > 0$, $r < 0$, $r = 0$, respectively. f is *skew-symmetric in sign* if for every $x, y \in X$

$$\texttt{sgn}(f(y,x)) = -\texttt{sgn}(f(x,y)).$$

If f is skew-symmetric in sign, then $f(x,x) = 0$ for every $x \in X$. If f is skew-symmetric, then it is skew-symmetric in sign. f is *skew-symmetric in deviation* if the function

$$\Delta_f(x,y) = f(x,y) - f(y,y)$$

is skew-symmetric.

Lemma 3.3. *Consider two functions $f, g : X \times X \to \mathbb{R}$.*

(1) f is skew-symmetric in deviation if and only if for every $x, y \in X$,

$$f(x,x) + f(y,y) = f(x,y) + f(y,x).$$

(2) If f is skew-symmetric, then it is skew-symmetric in deviation. If f is skew-symmetric in deviation, then Δ_f is skew-symmetric in sign.

(3) If f and g are skew-symmetric in deviation and $\alpha \in \mathbb{R}$, then $(f+g)(x,y) = f(x,y)+g(x,y)$ and $\alpha f(x,y)$ are skew-symmetric in deviation.

(4) A two-person symmetric game $G = \langle S, u \rangle$ is pairwise solvable if and only if Δ_u is skew-symmetric in sign.

Proof. By definition, f is skew-symmetric in deviation iff $f(y,x) - f(x,x) = -(f(x,y) - f(y,y)) = f(y,y) - f(x,y)$, or by rearranging, $f(x,x) + f(y,y) = f(x,y) + f(y,x)$. The former statement of (2) follows from $f(x,x) = f(y,y) = 0$ and (1). The latter statement of (2) is clear. (3) follows from (1). (4) is obvious from (PS). $\square$

A useful consequence of Lemma 3.3 is the following. Given a single variable function $g : X \to \mathbb{R}$, define $f : X \times X \to \mathbb{R}$ by setting $f(x,y) = g(x)$ for any $x, y \in X$. By Lemma 3.3.(1), f is skew-symmetric in deviation. Similarly for $h(x,y) = g(y)$. By Lemma 3.3.(3), if such a function is added to another function that is also skew-symmetric in deviation, then so is the resulting function.

3.3 Examples

The notion of skew-symmetry in deviation is particularly useful in finding pairwise solvable games. Throughout the following examples, let $G = \langle S, u \rangle$ be a two-person symmetric game.

Example 3.4. If G is a K-sum game, where $K \in \mathbb{R}$ is a constant, then $u(x, y) + u(y, x) = K$ for every $x, y \in S$. In particular, $2u(x, x) = K$ for every $x \in S$. Hence

$$u(x, x) + u(y, y) = u(x, y) + u(y, x) = K,$$

which shows, by Lemma 3.3.(1), that u is skew-symmetric in deviation. By (2) and (4) of the lemma, G is pairwise solvable. Let c and e be functions from S to $\mathbb{R}$. Then by (3) the game $\langle S, v\rangle$ defined by

$$v(x, y) = u(x, y) - c(x) + e(y)$$

is also pairwise solvable. The intended interpretations of c and e are, of course, the cost of one's own strategy and the externality generated by the other's strategy, respectively. $\square$

A *symmetric probability function* on $X \times X$ is a function $p : X \times X \to \mathbb{R}$ such that $p(x, y) \geq 0$ and $p(x, y) + p(y, x) = 1$ for every $x, y \in X$. In particular, $p(x, x) = 1/2$ for every $x \in S$. It is clear that any symmetric probability function is skew-symmetric in deviation. Let p be a symmetric probability function on $S \times S$ and assume that

$$u(x, y) = p(x, y)W + p(y, x)L - c(x) + e(y),$$

where W and L are constants, and c and e are the cost and the externality, respectively. By Lemma 3.3, the game is pairwise solvable. In this game, one of the two players is rewarded as the winner, the other is rewarded as the loser, and the probability of winning is determined by costly strategies of the players. Let us call this game *two-person symmetric contest*. Many games that are especially familiar in applications fall into this category.

Example 3.5. Let ϵ_1 and ϵ_2 be two random variables. For every $x, y \in S$, set

$$z_i(x) = x + \epsilon_i$$

and

$$p(x, y) = \text{Prob}(z_1(x) > z_2(y)).$$

If the probability that the difference $\epsilon_1 - \epsilon_2$ takes a particular value is zero, and each of the events $\epsilon_1 > \epsilon_2$ and $\epsilon_1 < \epsilon_2$ has probability 1/2, then $p(x, y)$ is a symmetric probability function. For example, if ϵ_1 and ϵ_2 are continuous IID random variables, as in the rank-order tournament of Lazear and Rosen (1981), then $p(x, y)$ is a symmetric probability function. $\square$

Example 3.6. Given $g : S \to \mathbb{R}$ such that $g(x) > 0$ for every $x \in S$, define

$$p(x, y) = \frac{g(x)}{g(x) + g(y)},$$

for every $x, y \in S$. Then p is a symmetric probability function. Setting $L = e(y) = 0$, the resulting contest is a rent-seeking game à la Tullock (1980), in which the constant W

	s_1	s_2
s_1	$5, 5$	$3, 4$
s_2	$4, 3$	$2, 2$

a

	s_1	s_2
s_1	$2, 2$	$3, 2$
s_2	$2, 3$	$3, 3$

b

Figure 2: Pairwise solvable games.

denotes the value of the rent. In the rent-seeking literature, Tullock (1980) and others (e.g., Hehenkamp et al. 2004) frequently consider the case in which $c(x) = x$, and $g(x) = x^r$, where $r > 0$. □

The existing literature contains more examples of pairwise solvable games.

Example 3.7. In an attempt to investigate classes of games that include strictly competitive games (Friedman 1983), Kats and Thisse (1992) introduced the notion of a weakly unilaterally competitive game. For a two-person symmetric game G, their definition simplifies as follows. The game is *weakly unilaterally competitive* if $u(t, s') \geq u(s, s')$ implies that $u(s', t) \leq u(s', s)$ for every $s, t, s' \in S$. In words, a game is weakly unilaterally competitive if and only if any *unilateral deviation* is a Pareto improvement only if the non-deviant's payoff is unchanged. Consider weakening this concept by restricting s' to be equal to either s or t. Namely, the game G is *pairwise competitive* if for each pair of distinct strategies $\{s, t\} \subset S$, the symmetric 2×2 game $g(s, t)$, which was depicted in Figure 1, is weakly unilaterally competitive. It is clear that a weakly unilaterally competitive game is pairwise competitive. One can verify that a pairwise competitive game is pairwise solvable. Note that the game in Figure 2a is not pairwise competitive. Hence the class of pairwise solvable games is strictly larger than that of weakly unilaterally competitive games. □

Example 3.8. Ania (2008) introduced the notion of a game with weak payoff externalities. For a two-person symmetric game G, her definition simplifies as follows. G has *weak payoff externalities* if $|u(t, s') - u(s, s')| > |u(s', t) - u(s', s)|$ for every $s, s', t \in S$. Let us consider a weaker condition. G has *pairwise weak payoff externalities* if $|u(t, s) - u(s, s)| > |u(s, t) - u(s, s)|$ for every $s, t \in S$. One can show that a two-person symmetric game with pairwise weak payoff externalities is pairwise solvable. Meanwhile, Figure 2a shows a pairwise solvable game without pairwise weak payoff externalities. □

4 Interchangeability and characterization of equilibria

Let $G = \langle S, u \rangle$ be a two-person symmetric game. In this section we derive two results that follow from pairwise solvability alone. First, the set of all equilibria in any pairwise solvable game is interchangeable.

Proposition 4.1. *Let $\mathcal{E}$ be the set of all Nash equilibria in a two-person symmetric game G. If G is pairwise solvable, then $\mathcal{E}$ is interchangeable.*

Proof. It suffices to show (Int) in Lemma 3.1. Assume first that $(s,t) \in \mathcal{E}$. For the row player, (a) $u(s,t) \geq u(r,t)$ for every $r \in S$. In particular, (b) $u(s,t) \geq u(t,t)$. By (PS), (c) $u(s,s) \geq u(t,s)$. For the column player, (d) $u(t,s) \geq u(r,s)$ for every $r \in S$. In particular, (e) $u(t,s) \geq u(s,s)$. By (c) and (e), (f) $u(t,s) = u(s,s)$. By (PS), (g) $u(s,t) = u(t,t)$. Putting (f) into (d), (h) $u(s,s) \geq u(r,s)$ for every $r \in S$, which shows that $(s,s) \in \mathcal{E}$. Putting (g) into (a), (i) $u(t,t) \geq u(r,t)$ for every $r \in S$, which shows that $(t,t) \in \mathcal{E}$.

Conversely, assume $(s,s),(t,t) \in \mathcal{E}$. Then (h) and (i) hold. In particular, $u(s,s) \geq u(t,s)$ and $u(t,t) \geq u(s,t)$. By (PS), $u(s,t) \geq u(t,t)$ and $u(t,s) \geq u(s,s)$. Hence (f) and (g) hold. Putting (g) into (i), we have (a). Putting (f) into (h), we have (d). By (a) and (d), $(s,t) \in \mathcal{E}$. $\square$

Corollary 4.2. *Assume that G is pairwise solvable. Then*

(1) $\mathcal{E} = E \times E$, where $E = \{\, s \in S \mid (s,s) \in \mathcal{E} \,\}$ is the set of equilibrium strategies.

(2) G is solvable in the sense of Nash (1951) if and only if $\mathcal{E} \neq \varnothing$.

(3) G has an equilibrium if and only if it has a symmetric equilibrium.

(4) All equilibrium strategies are equivalent in $\mathcal{E}$. Hence if there is a strict equilibrium, then it is a unique equilibrium.

Proof. For (4), recall the definition of equivalence of strategies, given at the end of Section 2. The rest are all clear. $\square$

Note that Proposition 4.1 does not imply that all equilibrium payoffs are identical. See the game in Figure 2b. Intuitively, thanks to the interchangeability, most questions on equilibria can be reduced to simpler ones on symmetric equilibria, which, in turn, boil down to questions on equilibrium strategies. An application of Proposition 4.1 is that if the mixed extension of a finite symmetric two-person game is pairwise solvable, then the mixed extension is solvable. Note that the mixed extension of a finite pairwise solvable game need not be pairwise solvable. For example, there is a finite symmetric two-person strictly competitive game whose mixed extension is not pairwise solvable, let alone strictly competitive.[6]

[6] In contrast, the mixed extension of a finite two-person zero-sum game is zero-sum.

Second, for pairwise solvable games, there is a simple characterization of Nash equilibria. Lemma 3.2, or equivalently, the condition (PS), naturally suggests the following binary relations on S: For all $x, y \in S$,

- $x \succ y$ if $u(x,x) > u(y,x)$ and $u(x,y) > u(y,y)$,
- $x \sim y$ if $u(x,x) = u(y,x)$ and $u(x,y) = u(y,y)$.

A two-person symmetric game is pairwise solvable if and only if exactly one of $x \succ y$, $y \succ x$, or $x \sim y$ holds for every $x, y \in S$. Note that the binary relations $\succ$ and $\sim$ need not be transitive. Still, the relation $\succ$ is useful for characterizing equilibria.

Proposition 4.3. *If G is pairwise solvable, then a symmetric strategy profile (x,x) is a Nash equilibrium if and only if there is no y such that $y \succ x$. Consequently, G possesses a Nash equilibrium if and only if there is $x \in S$ for which there is no $y \in S$ such that $y \succ x$.*

Proof. Assume that (x,x) is not a Nash equilibrium. Then there is y such that $u(y,x) > u(x,x)$. Since G is pairwise solvable, $u(y,y) > u(x,y)$. Hence $y \succ x$. Conversely, if $y \succ x$ then (x,x) cannot be an equilibrium. By the contrapositions, the first claim follows. By Corollary 4.2.(3), the second claim follows. $\square$

For exposition, let G be a pairwise solvable game in which $u(x,x) = 0$ for every $x \in S$.[7] Then $y \succ x$ if and only if $u(y,x) > u(x,x) = 0$. Hence G has no equilibrium if and only if for any strategy by the opponent, there is a strategy that pays off positive against it. This is exactly the characterization found by Duersch, Oechssler, Schipper (2012a) for (non)equilibrium in a symmetric two-person zero-sum game.[8] Proposition 4.3 generalizes theirs.

5 Equilibria in games with linearly ordered strategies

Consider the rock-paper-scissors game. It is pairwise solvable, but we know and Proposition 4.3 shows that it lacks any (pure strategy) equilibrium. Rather than pursuing the existence problem in general, we focus on a special class of games. In symmetric contests and other applications, a strategy can be interpreted as an effort level. In those models, strategies are linearly ordered. In this section we consider pairwise solvable games in which the set of strategies are *linearly ordered*, and the payoff function satisfies a concavity condition, the *quasiconcavity at the diagonal*, which shall be defined shortly. These restrictions are enough to derive a complete and purely order-theoretic characterization of the existence of Nash equilibria.

[7] It is equivalent to assuming that G is *strictly competitive* (Friedman, 1983).

[8] Duersch et al. (2012a) call it the *generalized rock-paper-scissors* condition.

5.1 Linearly ordered sets

A binary relation $\geq$ on a nonempty set P is a *linear order* if it is complete ($a \geq b$ or $b \geq a$ for every $a, b \in P$), transitive, and anti-symmetric (if $a \geq b$ and $b \geq a$ then $a = b$). Write $a > b$ if $a \geq b$ and $a \neq b$. Assume that $\geq$ is a linear order on $P \neq \varnothing$. By abuse of notation, we write $b \leq a$, and so on. On the linearly ordered set $\langle P, \geq \rangle$, notions such as *upper bounds*, *lower bounds*, *maximum*, *minimum*, *supremum*, *infimum*, and adjectives such as *bounded from above* or *below* are defined, in exactly the same manner as in the real number system. For a subset $K \subset P$, denote the supremum of K by $\sup K$, and so forth.

A subset A of a linearly ordered set P is *closed downwards* if $a \in A$ and $b < a$ imply $b \in A$. A is *closed upwards* if $a \in A$ and $b > a$ imply $b \in A$. A subset I of P is an *interval* if whenever $a, b \in I$ and $a < c < b$, $c \in I$. It is clear that if A is closed downwards and B is closed upwards, then $A \cap B$ is an interval, which is possibly empty.

5.2 Quasiconcavity at the diagonal

Let $G = \langle S, u \rangle$ be a two-person symmetric game in which S is a linearly ordered set. For $x, y, z \in S$, we say that z is between x and y if $\min\{x, y\} \leq z \leq \max\{x, y\}$. G is *quasiconcave at the diagonal* (in own-strategy) if the following condition is satisfied:

(QCD) For any $x, y, z \in S$ such that z is between x and y, if $u(y, x) \geq u(x, x)$, then $u(z, x) \geq u(x, x)$.

The meaning of (QCD) is clear. Let

$$BT(x) = \{\, y \in S \mid u(y, x) \geq u(x, x) \,\},$$

the set of (weak) *better responses at* (x, x). One can verify that (QCD) is equivalent to the condition that $BT(x)$ is an interval for every $x \in S$.

Quasiconcavity at the diagonal is a weakening of quasiconcavity as single-peakedness. Duersch et al. (2012) define that a two-person symmetric game is *quasiconcave* (in own-strategy) if $u(z, w) \geq \min\{u(x, w), u(y, w)\}$ for every $x, y, z, w \in S$ such that z is between x and y. This condition requires that each "column" in the payoff matrix is single-peaked. If it is single-peaked, then the set of better responses is an interval. Hence quasiconcavity implies quasiconcavity at the diagonal. The converse does not hold, as (QCD) allows multiple peaks or valleys, and looks only at the better responses at symmetric profiles.[9]

[9] Quasiconcavity at the diagonal is an adaptation of the notion of *quasiconcavity at a point*, defined by Mangasarian (1969, Chapter 9). Specifically, regarding u as a function of single variable $u(\cdot, x)$, and applying his definition at the point (x, x), we obtain (QCD).

Example 5.1. Let S be an interval in $\mathbb{R}$ and F be a cumulative distribution function on S. Define $p(x,y)$ as follows: For $x, y \in S$,

$$p(x,y) = \begin{cases} F((x+y)/2), & \text{if } x < y, \\ 1 - F((x+y)/2), & \text{if } x > y, \\ 1/2, & \text{if } x = y. \end{cases}$$

In words, $p(x,y)$ is the probability that a random draw governed by F is closer to x than to y. The function p is a symmetric probability function. Let $x < z < y$ and assume that $p(y,x) = 1 - F((x+y)/2) \geq 1/2 = p(x,x)$. Then $p(z,x) = 1 - F((x+z)/2) \geq 1/2$ since $(x+z)/2 < (x+y)/2$ and F is nondecreasing. Similarly, $p(z,x) \geq 1/2$ if $x > z > y$ and $p(y,x) \geq 1/2$. Hence p is quasiconcave at the diagonal. Therefore the game defined by $u(x,y) = p(x,y)$ is a 1-sum game that is quasiconcave at the diagonal.[10] It is an electoral competition model à la Hotelling (1929) in which the strategy set is an interval in $\mathbb{R}$. $\square$

Quasiconcavity at the diagonal is a purely ordinal concept. It is invariant under any order-preserving payoff transformation.

5.3 Characterization of equilibria

Let S be a nonempty linearly ordered set and $G = \langle S, u \rangle$ be a two-person symmetric game that is *pairwise solvable and quasiconcave at the diagonal*. In this subsection, we derive a complete characterization of Nash equilibrium in G. At this point we make no assumption concerning the existence of special elements such as supremum, infimum, or upper and lower bounds.

Recall that $\mathcal{E}$ is the set of all Nash equilibria in G and $E = \{ x \in S \mid (x,x) \in \mathcal{E} \}$ is the set of equilibrium strategies. By Theorem 4.1, $\mathcal{E} = E \times E$. Now define

$$\begin{aligned} L_G &= \{ x \in S \mid u(x,x) \geq u(y,x) \text{ for every } y \leq x \}, \\ R_G &= \{ x \in S \mid u(x,x) \geq u(y,x) \text{ for every } y \geq x \}. \end{aligned}$$

Let us say that a strategy is *no better take lower* if, against itself, the strategy pays off better than any smaller strategy. L_G is the set of strategies that are no better take lower. R_G is the set of strategies that are *no better take higher*. Clearly, a strategy is a best response against itself if and only if it is both no better take lower and higher. Formally:

Lemma 5.2. *$E = L_G \cap R_G$. Hence $\mathcal{E} \neq \varnothing$ if and only if $L_G \cap R_G \neq \varnothing$.*

It turns out that L_G is closed downwards and R_G is closed upwards, from which a characterization of the *uniqueness* of a Nash equilibrium follows.

[10] Actually, one can verify that it is single-peaked.

Lemma 5.3. *For every $x \in S$, if there is $y < x$ such that $u(y, x) > u(x, x)$, then $u(y, x') > u(x', x')$ for every $x' > x$. If there is $y > x$ such that $u(y, x) > u(x, x)$, then $u(y, x') > u(x', x')$ for every $x' < x$.*

Proof. Assume that $y < x$ and $u(y, x) > u(x, x)$. By (PS), $u(y, y) > u(x, y)$. By (QCD), $u(y, y) > u(x', y)$ for every $x' > x$. By (PS), $u(y, x') > u(x', x')$. We have just shown the first claim. The second claim can be proved similarly. □

Lemma 5.4. *The set L_G is closed downwards. If $\sup L_G$ exists and $x < \sup L_G$, then $x \in L_G$. The set R_G is closed upwards. If $\inf R_G$ exists and $\inf R_G < x$, then $x \in R_G$.*

Proof. By Lemma 5.3, if $x' \notin L_G$ and $x > x'$, then $x \notin L_G$. By contraposition, if $x \in L_G$, then either $x \leq x'$ or $x' \in L_G$. Hence if $x \in L_G$ and $x' < x$, then $x' \in L_G$. Assume that $\sup L_G$ exists and $x < \sup L_G$. Then x is not an upper bound of L_G. Hence there is $x' \in L_G$ such that $x < x'$. By the result just shown, $x \in L_G$. Similar arguments apply for R_G. □

Corollary 5.5. *The set E of equilibrium strategies is an interval.*

Theorem 5.6. *Let $G = \langle S, u \rangle$ be a two-person symmetric game that is pairwise solvable and quasiconcave at the diagonal. Assume that $\sup L_G$ and $\inf R_G$ exist. Then strategy profile (x^*, x^*) is a unique Nash equilibrium if and only if $x^* = \max L_G = \min R_G$.*

Proof. If $x^* = \max L_G = \min R_G$, then $x^* \in L_G \cap R_G$. Thus by Lemma 5.2, (x^*, x^*) is a Nash equilibrium. It is unique, since if $y > x^*$ then $y \notin L_G$ and if $y < x^*$ then $y \notin R_G$. For the converse, assume that (x^*, x^*) is the unique Nash equilibrium. Then by Lemma 5.2, $\{x^*\} = L_G \cap R_G$. If there is $y \in L_G$ such that $x^* < y$, then since $x^* \in R_G$ and using Lemma 5.4, $y \in R_G$. Hence $y \in L_G \cap R_G$, which contradicts the assumption that $\{x^*\} = L_G \cap R_G$. Therefore $y \leq x^*$ for every $y \in L_G$. Since $x^* \in L_G$, $x^* = \max L_G$. Similarly, one can verify that $x^* = \min R_G$. □

Theorem 5.6 tells us when the intersection $L_G \cap R_G$ is a singleton. Let us proceed to characterize when it is nonempty.

Proposition 5.7. *$S = L_G \cup R_G$. In particular, either $L_G \neq \varnothing$ or $R_G \neq \varnothing$.*

Proof. It suffices to show that if $x \notin L_G$ then $x \in R_G$. Pick $x \in S$ and assume that $x \notin L_G$. Then there is $z < x$ such that $u(z, x) > u(x, x)$. By Lemma 5.3, $u(z, y) > u(y, y)$ for every $y > x$. Fix such y. Since $y > x > z$, (QCD) implies that $u(x, y) \geq u(y, y)$. By (PS), $u(x, x) \geq u(y, x)$. Since this is true for every $y > x$, $x \in R_G$. □

Theorem 5.8. *Let $G = \langle S, u \rangle$ be a two-person symmetric game that is pairwise solvable and quasiconcave at the diagonal. Assume that $\sup L_G$ and $\inf R_G$ exist. Then $\inf R_G \leq \sup L_G$.*

Proof. For a contradiction, assume that $x_l = \sup L_G < \inf R_G = x_r$. By Proposition 5.7, observe that there is no $x \in S$ such that $x_l < x < x_r$. Now, since $x_r \notin L_G$, there is $x_1 < x_r$ such that $u(x_1, x_r) > u(x_r, x_r)$. Similarly, since $x_l \notin R_G$, there is $x_2 > x_l$ such that $u(x_2, x_l) > u(x_l, x_l)$. By the observation above, $x_1 \leq x_l$ and $x_r \leq x_2$. Hence there are $x_1 \leq x_l < x_r \leq x_2$ such that (1) $u(x_1, x_r) > u(x_r, x_r)$, and (2) $u(x_2, x_l) > u(x_l, x_l)$. By (1) and (PS), $u(x_1, x_1) > u(x_r, x_1)$. By (QCD), (3) $u(x_1, x_1) > u(x_2, x_1)$. Similarly, (2), (PS) and (QCD) imply that (4) $u(x_2, x_2) > u(x_1, x_2)$. But (3) and (4) violate (PS). □

Corollary 5.9. *If* $\max L_G$ *and* $\min R_G$ *exist, then* $\mathcal{E} \neq \varnothing$. *In particular, if* S *is finite, then* $\mathcal{E} \neq \varnothing$.

Theorem 5.10. *Let* $G = \langle S, u \rangle$ *be a two-person symmetric game that is pairwise solvable and quasiconcave at the diagonal. Assume that* $\sup L_G$ *and* $\inf R_G$ *exist. Then* $\mathcal{E} \neq \varnothing$ *if and only if either* $\inf R_G < \sup L_G$ *or both* $\max L_G$ *and* $\min R_G$ *exist and are equal.*

Proof. By Lemma 5.2, $\mathcal{E} \neq \varnothing$ if and only if $L_G \cap R_G \neq \varnothing$. Let $x \in L_G \cap R_G$ and assume that it is not true that $\inf R_G < \sup L_G$. Then by Theorem 5.8, there is $x^* \in S$ such that $x^* = \inf R_G = \sup L_G$. Since $x \in L_G$, $x \leq x^*$. Since $x \in R_G$, $x \geq x^*$. Thus $x = x^*$. Therefore $\sup L_G \in L_G$ and $\inf R_G \in R_G$, i.e., $x^* = \min R_G = \max L_G$. Conversely, assume that either $\inf R_G < \sup L_G$ or $\max L_G = \min R_G$ exist. If the latter is true, then $\mathcal{E} \neq \varnothing$ by Corollary 5.9 (or by Theorem 5.6). Thus assume that $x_r = \inf R_G < \sup L_G = x_l$. Then x_l is not a lower bound of R_G. Hence there is $x^* \in R_G$ such that $x^* < x_l$. Meanwhile, by $x^* < x_l$ and Lemma 5.4, $x^* \in L_G$. Consequently, $x^* \in L_G \cap R_G$. □

Assuming the existence of $\sup L_G$ and $\inf R_G$, the above results imply that there are multiple equilibria if and only if $\inf R_G < \sup L_G$, and that there is no equilibrium if and only if (L_G, R_G) constitutes a *Dedekind cut*.

5.4 Practical sufficient conditions

The results in the preceding subsection apply to any game that is pairwise solvable and quasiconcave at the diagonal. Let us see their implications in practical settings. If S is finite then there is an equilibrium by Corollary 5.9. Therefore, for the rest of this section, let S be a subset of $\mathbb{R}$, the real line equipped with the standard order, and let $G = \langle S, u \rangle$ be a two-person symmetric game that is pairwise solvable and quasiconcave at the diagonal.

The main results in the preceding subsection assume that $\sup L_G$ and $\inf R_G$ exist in S. We should begin with the question of when they exist.

Lemma 5.11. *Let* S *be a nonempty interval in* $\mathbb{R}$. *If* $L_G \neq S$ *and* $R_G \neq S$, *then neither* L_G *nor* R_G *is empty,* L_G *is bounded from above in* S, *and* R_G *is bounded from below in* S. *Consequently,* $\sup L_G$ *and* $\inf R_G$ *exist in* S.

Proof. Assume that neither L_G nor R_G is equal to S. By Proposition 5.7, $L_G = \varnothing$ implies $R_G = S$. By contraposition, $L_G \neq \varnothing$. Similarly, $R_G \neq \varnothing$. Pick $x \in S$ such that $x \notin L_G$. By Proposition 5.4, x is an upper bound of L_G. Hence L_G is bounded from above in $\mathbb{R}$. Hence $\sup L_G$ exists in $\mathbb{R}$. Pick any $y \in L_G$. Then it follows that $y \leq \sup L_G \leq x$. Since S is an interval, $\sup L_G \in S$. Similarly, $\inf R_G$ exists and is in S. □

Recall that $S \subset \mathbb{R}$ is an interval if and only if it is convex. It should be noted that $\sup L_G$ and $\inf R_G$ exist in S even if S is not bounded, e.g., $S = \mathbb{R}$, as long as $L_G \neq S$ and $R_G \neq S$.[11]

Example 5.12. Consider the electoral competition model in Example 5.1. For any cumulative distribution function F, there are $x, y \in S$ such that $y < x$ and $F((x+y)/2) > 1/2$. Thus $x \notin L_G$. Hence $L_G \neq S$. Similarly, $R_G \neq S$. By Lemma 5.11, $\sup L_G$ and $\inf R_G$ exist. ▯

By Corollary 5.9, a simple sufficient condition for the existence of a Nash equilibrium is that both $\max L_G$ and $\min R_G$ exist. The question is, then, when do they exist. An answer is given by (RC) and (LC).

(RC) If $u(y_n, x) \geq u(x, x)$, $y < y_n < x$, and $y_n \to y$, then $u(y, x) \geq u(x, x)$,

(LC) If $u(y_n, x) \geq u(x, x)$, $x < y_n < y$, and $y_n \to y$, then $u(y, x) \geq u(x, x)$.

In words, (RC) requires for every x that $u(\cdot, x)$ is upper semi-continuous *from the right* at every y such that $y < x$, and (LC) requires for every x that $u(\cdot, x)$ is upper semi-continuous *from the left* at every y such that $x < y$. Therefore the combination of (LC) and (RC) stipulates that for every x, $u(\cdot, x)$ is upper semi-continuous *outward* from (x, x). Note that the continuity with respect to own-strategy, as opposed to the full continuity in $S \times S$, is enough for the satisfaction of these conditions.

Proposition 5.13. *Let S be a nonempty interval in $\mathbb{R}$ and $G = \langle S, u \rangle$ be pairwise solvable and quasiconcave at the diagonal. Assume that $\sup L_G$ and $\inf R_G$ exist. If (LC) is satisfied, then $\max L_G$ exists. If (RC) is satisfied, then $\min R_G$ exists. Consequently, if (LC) and (RC) are satisfied, then $\mathcal{E} \neq \varnothing$.*

Proof. Let $x^* = \sup L_G$. We shall show that if (LC), then $x^* \in L_G$. Assume (LC). By the definition of sup, there is a sequence $\langle x_n \rangle$ such that $x_n \in L_G$ and $x_n \to x^*$. Pick $y < x^*$. Since $x_n \to x^*$, for all sufficiently large k, $y < x_k$. For all such k, $u(x_k, x_k) \geq u(y, x_k)$ since $x_k \in L_G$. By (PS), $u(x_k, y) \geq u(y, y)$. By (LC), $u(x^*, y) \geq u(y, y)$. By (PS), $u(x^*, x^*) \geq u(y, x^*)$. Hence $x^* \in L_G$. Similarly, if (RC) then $\inf R_G \in R_G$. By Corollary 5.9, there is an equilibrium. □

[11]What if the game at hand fails to satisfy the assumptions of Lemma 5.11? If $L_G = S$, say, then $\mathcal{E} = R_G \times R_G$. Such games should be considered on a case by case basis.

Recall the set $BT(x) = \{ y \in S \mid u(y,x) \geq u(x,x) \}$ of better responses at (x,x). It is straightforward to show that (LC) and (RC) hold if and only if $BT(x)$ is closed for every $x \in S$. Meanwhile, quasiconcavity at the diagonal is equivalent to the convexity of $BT(x)$ when S is an interval in $\mathbb{R}$. Hence:

Proposition 5.14. *Let S be a nonempty interval in $\mathbb{R}$ and $G = \langle S, u\rangle$ be pairwise solvable. Assume that $\sup L_G$ and $\inf R_G$ exist. If $BT(x)$ is closed and convex for every $x \in S$, then there is a Nash equilibrium.*

Finally, let us verify that our results accommodate the electoral competition model in Example 5.1.

Lemma 5.15. *In the game in Example 5.1, $\max L_G$ exists and (RC) is satisfied.*

Proof. By the observation in Example 5.12, $\sup L_G$ and $\inf R_G$ exist. Since F is right-continuous, the payoff function satisfies (RC). Details are left to the reader. For the existence of $\max L_G$, let $x^* = \sup L_G$. By definition of sup there is a sequence $\langle x_n\rangle$ such that $x_n \in L_G$ and $x_n \leq x^*$ for every n and $x_n \to x^*$. Since $x_n \in L_G$, $u(x_n - 2/n, x_n) = F(x_n - 1/n) \leq u(x_n, x_n) = 1/2$. Now set $z_n = x_n - 1/n$ and consider the sequence $\langle z_n\rangle$. Then $F(z_n) \leq 1/2$ for every n and $z_n \to x^*$ from the left. Therefore, $F(z_n) \to F(x^* - 0) \leq 1/2$, where $F(x^* - 0)$ is the left-hand limit of F at x^*, which exists since F is a cumulative distribution function. Since F is nondecreasing, $F(z) \leq 1/2$ for every $z < x^*$. Take $y < x^*$. Then $(x^* + y)/2 < x^*$. Hence $u(y, x^*) = F((x^* + y)/2) \leq 1/2 = u(x^*, x^*)$, which shows that $x^* \in L_G$, or $x^* = \max L_G$. □

6 Dominance solvability

A pairwise solvable game is solvable in the sense of Nash (1951) if it is quasiconcave at the diagonal and satisfies one of the conditions found in the previous section. If the game is *finite*, we can say more. We shall show that in a finite pairwise solvable game, quasiconcavity at the diagonal generates a special type of dominance relation among the strategies. Throughout this section, let $G = \langle S, u\rangle$ be a two-person symmetric game in which $S = \{s_1, \ldots, s_n\}$. We call such a game an *$n \times n$ game*. We shall assume that strategies are linearly ordered according to the respective subscripts. Hence $s_1 = \min S$ and $s_n = \max S$. For an $n \times n$ game, we write u_{ij} for $u(s_i, s_j)$ and so forth.

We start by introducing the notion of a symmetric elimination of strategies. Let T be a nonempty *proper* subset of S. Imagine removing, simultaneously, all the strategies not in T from the payoff matrix of G. This results in a game $G' = \langle T, u\rangle$, the restricted game generated by T. We call this procedure a *symmetric elimination of strategies*. A crucial feature of symmetric elimination of strategies is that any diagonal entry remains a diagonal entry if the corresponding strategy survives the elimination at all. Likewise the non-diagonal entries.

Lemma 6.1. *Let G be an $n \times n$ game, and consider a symmetric elimination of strategies, which results in an $m \times m$ game G', where $1 \leq m < n$. If G is pairwise solvable and quasiconcave at the diagonal, then so is G'.*

It is clear that G' is pairwise solvable. Quasiconcavity is preserved since the strategy s_k is between s_i and s_j in G' only if they stand in that relation in G. We omit the details.

A symmetric elimination of *weakly dominated strategies* is a symmetric elimination of strategies such that every strategy in $S \setminus T$ is weakly dominated. We allow the possibility that T contains weakly dominated strategies.

Lemma 6.2. *Let G be an $n \times n$ game, and consider a symmetric elimination of weakly dominated strategies, which results in an $m \times m$ game G', where $1 \leq m < n$. Then any Nash equilibrium in G' is a Nash equilibrium in G.*

Proof. As a binary relation on S, weak domination is transitive, and no strategy weakly dominates itself. It follows that for each weakly dominated strategy in a *finite* game, there is an undominated strategy that weakly dominates it. The rest is left to the reader. $\square$

By Lemma 6.1, both pairwise solvability and quasiconcavity at the diagonal will be preserved throughout an *iterated* symmetric elimination of strategies. In what follows, we extensively discuss eliminations of strategies and iterations thereof. In so doing, we *rename* or *renumber* the strategies, and we do so *implicitly*.

6.1 Iterated elimination of weakly dominated strategies

Given an $n \times n$ game G, consider the following strategy elimination rule, which we call (E):

(E1) If $u_{11} < u_{n1}$, then eliminate s_1. If $u_{11} > u_{n1}$, then eliminate s_n. If $u_{11} = u_{n1}$, then go to (E2).

(E2) If there is j, $1 < j < n$, such that $u_{11} < u_{j1}$, then eliminate s_1. If there is no such row, then go to (E3).

(E3) If there is j, $1 < j < n$, such that $u_{jn} > u_{nn}$, then eliminate s_n. If there is no such row, then eliminate no strategy.

When (E) eliminates a strategy, it is to be understood that the corresponding row and the column are simultaneously removed from the payoff matrix of G. Hence an application of (E) to G results in either G itself or one-strategy-smaller $(n-1) \times (n-1)$ game, depending on whether or not (E) eliminates a strategy. When it does, (E) generates a symmetric elimination of strategy, and it eliminates either the *smallest* strategy or the *largest* strategy. It never eliminates one in the middle.

Theorem 6.3. *Let G be a pairwise solvable $n \times n$ game that is quasiconcave at the diagonal. If rule* (E) *eliminates a strategy in G, then it is weakly dominated.*

Proof. Assume that s_1 is eliminated by rule (E). Then $u_{11} \leq u_{n1}$ and there is $k > 1$ such that $u_{11} < u_{k1}$. Since $u_{11} \leq u_{n1}$, (QCD) implies that $u_{11} \leq u_{j1}$ for every j. By (PS),

$$u_{1j} \leq u_{jj} \text{ for every } j. \tag{1}$$

On the other hand, by Corollary 5.9 there is s_{j^*} such that (s_{j^*}, s_{j^*}) is a Nash equilibrium. Hence $u_{j^*j^*} \geq u_{jj^*}$ for every j. By (PS),

$$u_{j^*j} \geq u_{jj} \text{ for every } j. \tag{2}$$

By (1) and (2), $u_{j^*j} \geq u_{1j}$ for every j. Recall that there is $k > 1$ such that $u_{11} < u_{k1}$. For this k, $u_{1k} < u_{kk}$ by (PS). Hence $u_{j^*k} \geq u_{kk} > u_{1k}$, which means that s_{j^*} weakly dominates s_1. An entirely analogous argument shows that if s_n is eliminated by rule (E), then s_{j^*}, where $j^* < n$ (since (s_n, s_n) cannot be a Nash equilibrium), weakly dominates s_n. □

A sort of converse of Theorem 6.3 is also true.

Theorem 6.4. *Let G be a pairwise solvable $n \times n$ game that is quasiconcave at the diagonal. If rule* (E) *fails to eliminate any strategy, then all strategies in G are equivalent. In particular, there is no weakly dominated strategy.*

Proof. Assume that (E1), (E2), and (E3) fail to eliminate any strategy. Since (E1) failed, $u_{11} = u_{n1}$. By (QCD), $u_{j1} \geq u_{11}$ for every $j = 1, \ldots, n$. Since (E2) failed, $u_{11} = u_{21} = \cdots = u_{n1}$. By (PS), $u_{1k} = u_{kk}$ for every $k = 1, \ldots, n$. By (QCD), $u_{jk} \geq u_{kk}$ for every k and every $j \leq k$. If $u_{jk} > u_{kk}$ for some $j < k$, then $u_{jj} > u_{kj}$ by (PS). By (QCD), $u_{jj} > u_{nj}$. By (PS), $u_{jn} > u_{nn}$, but this would have allowed (E3) to eliminate strategy s_n. Hence $u_{1k} = u_{2k} = \cdots = u_{kk}$ for every k. In particular, $u_{1n} = u_{2n} = \cdots = u_{nn}$. From this, one can similarly show that $u_{kk} = u_{(k+1)k} = \cdots = u_{nk}$. Consequently, $u_{1k} = u_{2k} = \cdots = u_{nk}$ for every k. □

Apply rule (E) to the $n \times n$ game G iteratively. The iteration terminates in at most $(n-1)$ steps and generates the set of surviving strategies $S^* \subset S$. Consider the restricted game $G^* = \langle S^*, u \rangle$ generated by S^*. It is a game in which rule (E) fails to eliminate any strategy.

Corollary 6.5. *Consider $G^* = \langle S^*, u \rangle$. Then $S^* \times S^*$ is the final outcome of the iterated elimination of weakly dominated strategies in G, and G^* contains no weakly dominated strategies.*

Proof. By Lemma 6.1, any game that appears during the iterative procedure is pairwise solvable and quasiconcave at the diagonal. By Theorem 6.3, any eliminated strategy is a weakly dominated strategy. By Theorem 6.4, there is no weakly dominated strategy in G^*. □

6.2 The set of Nash equilibria

We are now ready to consider the relationship between the set $\mathcal{E}$ of Nash equilibria in G and the final outcome $S^* \times S^*$ of the iterated elimination.

Theorem 6.6. *Let $S^* \times S^*$ be the final outcome of the iterated application of rule* (E) *to an $n \times n$ game G, which is pairwise solvable and quasiconcave at the diagonal. Then $\mathcal{E} = S^* \times S^*$.*

Proof. By Corollary 4.2, $\mathcal{E} = E \times E$, where $E = \{ s \in S \mid (s, s) \in \mathcal{E} \}$. It suffices to show that $S^* = E$. Rule (E) eliminates a strategy s_i only if there is a row $j \neq i$ such that $u_{ji} > u_{ii}$, which implies that (s_i, s_i) is not a Nash equilibrium in G. Hence if $(s_i, s_i) \in \mathcal{E}$, then $(s_i, s_i) \in S^* \times S^*$. That is, $E \subset S^*$. Conversely, pick $s_i \in S^*$. By Theorem 6.4, (s_i, s_i) is a Nash equilibrium in G^*. By Corollary 6.5, $S^* \times S^*$ is the final outcome of an iterated symmetric elimination of weakly dominated strategies. Since the game is finite, the number of iterations is finite. By applying Lemma 6.2 the appropriate number of times, we know that (s_i, s_i) is in $\mathcal{E}$. That is, $s_i \in E$. Therefore $S^* = E$. □

In words, there is a procedure of iterated elimination of weakly dominated strategies that leads precisely to the solution, in the sense of Nash. In other words, a strategy survives the iterated elimination if and only if it is an equilibrium strategy.[12] If all the weak dominations are in fact strict, as they would be in a game with no payoff ties, then there is a unique equilibrium, and the equilibrium strategy is a unique rationalizable strategy (Bernheim 1984, Pearce 1984).[13]

Example 6.7. Consider discretizing the electoral competition model in Example 5.1. In several textbook treatments (e.g., Dixit and Skeath 2004, Watson 2013), equilibrium has been found as the final outcome of the iterated elimination of dominated strategies. Theorem 6.6 ensures that this is always the case. ▯

In some games, the iterated elimination governed by rule (E) becomes trivial in that an equilibrium strategy emerges as the dominant strategy after just one or two steps. In other games, the iterated elimination maintains itself non-trivial until the penultimate step.

Example 6.8. Set $S = \{0, 1/5, 2/5, 3/5, 4/5, 1\}$ and let $p(s, t) = s^{1/2}/(s^{1/2} + t^{1/2})$ for every $s, t \in S \times S$ such that $(s, t) \neq (0, 0)$, and let $p(0, 0) = 1/2$. Consider the 6×6 game defined by

$$u(s, t) = 4p(s, t) - s.$$

[12] By Corollary 5.5, therefore, S^* is an interval in S.

[13] This follows from the fact that any rationalizable strategy survives the iterated eliminations of strictly dominated strategies (Pearce 1984, p.1035). Moreover, the unique equilibrium in this case is *robust* in the sense of Kajii and Morris (1997).

	0	1/5	2/5	3/5	4/5	1
0	2.000	0.000	0.000	0.000	0.000	0.000
1/5	3.800	1.800	1.457	1.264	1.133	1.036
2/5	3.600	1.943	1.600	1.398	1.257	1.150
3/5	3.400	1.936	1.602	1.400	1.256	1.146
4/5	3.200	1.867	1.543	1.344	1.200	1.089
1	3.000	1.764	1.450	1.254	1.111	1.000

a

	s_1	s_2	s_3
s_1	$1, 1$	$0, 1$	$-1, 1$
s_2	$1, 0$	$0, 0$	$0, -1$
s_3	$1, -1$	$-1, 0$	$-1, -1$

b

Figure 3: Pairwise solvable games.

This is a finite symmetric contest which is quasiconcave at the diagonal. The payoff matrix is given by Figure 3a, in which the payoffs are rounded off to three decimal place, and only those of the row player are shown. Rule (E) eliminates strategy 0 first, and subsequently, 1, 1/5, 4/5, and finally, 2/5. The surviving strategy profile $(3/5, 3/5)$ is a unique Nash equilibrium. Note that the equilibrium strategy 3/5 emerges as a dominant strategy only at the final step. $\square$

It is well known that the final outcome of the iterated elimination of weakly dominated strategies is sensitive to the details as to whether several strategies may be eliminated at a time, whether all the dominated strategies are eliminated at a time, and what is the order of elimination when there are many. In the elimination governed by rule (E), only a single strategy may be eliminated at a time even when there are multiple weakly dominated strategies. Under other elimination rules, the final outcome need not coincide with $\mathcal{E}$.

Example 6.9. Consider the game in Figure 3b. It is pairwise solvable and quasiconcave at the diagonal. The outcome under rule (E) is $\{s_1, s_2\} \times \{s_1, s_2\}$, which is the set of all Nash equilibria. Note that s_2 weakly dominates s_1. Hence the outcome under the rule that eliminates all the weakly dominated strategies at a time is the singleton $\{(s_2, s_2)\}$.[14] $\square$

The preceding example shows that the final outcome of the iterated elimination of weakly dominated strategies does depend on the details of the elimination rule. Nonetheless, it follows from Lemma 6.1 and Theorem 6.4 that under any elimination rule that generates a *symmetric* elimination of weakly dominated strategies, the strategies that survive the iterated elimination are equivalent. Therefore G is *dominance solvable* in the sense of Moulin (1979, 1986), who

[14]In this game, the set of Nash equilibria, $\{s_1, s_2\} \times \{s_1, s_2\}$, is a solution, but not a strong solution in the sense of Nash (1951). In addition, all strategies are rationalizable.

stipulates that the elimination procedure remove *all* weakly dominated strategies at *each step*, and then requires that all surviving strategies be equivalent.[15]

7 Evolutionary equilibria

Let $G = \langle S, u\rangle$ be a two-person symmetric game. A symmetric profile $(x, x) \in S \times S$ in G is called a *(symmetric) evolutionary equilibrium* (Schaffer 1989) if $u(y, x) \leq u(x, y)$ for every $y \in S$. Let us say that a strategy y is a *beating deviation* from (x, x) if $u(y, x) > u(x, y)$. The profile (x, x) is an evolutionary equilibrium if there is no beating deviation from there. The strategy in an evolutionary equilibrium is a version of the *finite-population evolutionarily stable strategy*, which is introduced by Schaffer (1988) to capture the spiteful behavior within a finite population in an evolutionary setting.

Define $r(x, y) = u(x, y) - u(y, x)$ for $x, y \in S$, and consider $r(G) = \langle S, r\rangle$. It is called the *relative payoff game* of G. It is pointed out by Schaffer (1989) that (x, x) is an evolutionary equilibrium in G if and only if it is a Nash equilibrium in $r(G)$. Clearly, $r(G)$ is a zero-sum game. Hence it is pairwise solvable. The results in Section 5 then imply that G has an evolutionary equilibrium whenever $r(G)$ is quasiconcave at the diagonal and $r(x, y)$ is sufficiently continuous.

Let $S \subset \mathbb{R}$. We say that u is *concave-convex* if

$$\frac{u(z, w) - u(x, w)}{z - x} \leq \frac{u(y, w) - u(x, w)}{y - x} \quad \text{and} \quad \frac{u(w, z) - u(w, x)}{z - x} \geq \frac{u(w, y) - u(w, x)}{y - x}$$

for every $x, y, z, w \in S$ such that $x < y < z$. In practice, S is either a finite set or an interval. In the next result, we focus on the case that S is an interval in $\mathbb{R}$, but it is straightforward to translate the result to the case of finite games. Neither do we search for a minimal continuity condition. Given that S is an interval, we simply assume that u is *coordinatewise continuous*: $u(\cdot, y)$ is continuous for any fixed $y \in S$ and $u(x, \cdot)$ is continuous for any fixed $x \in S$.[16] Recall from subsection 5.3 that L_G is the set of strategies that are no better take lower, and R_G is the set of strategies that are no better take higher.

Proposition 7.1. *Let $G = \langle S, u\rangle$ be a two-person symmetric game in which S is an interval in $\mathbb{R}$ such that $L_G \neq S$ and $R_G \neq S$. If u is concave-convex and coordinatewise continuous, then G possesses an evolutionary equilibrium.*

Proof. It is clear that if u is coordinatewise continuous, then $r(x, y)$ is continuous with respect to own-strategy. By Lemma 5.11 and Proposition 5.13, therefore, it suffices to show that $r(G)$ is

[15] The notion of dominance solvability does not imply payoff equivalence between equilibria. Moulin (1986, p.73) explicitly allows a game like the one in Figure 2b to be dominance solvable.

[16] If u is concave-convex, the coordinatewise continuity reduces to the requirement that the coordinatewise payoff functions are continuous at the boundaries of S, if there are any.

quasiconcave at the diagonal. By definition of $r(G)$, it suffices to show that if $u(y,x) \geq u(x,y)$ then $u(z,x) \geq u(x,z)$ for any $z \in (\min\{x,y\}, \max\{x,y\})$. Thus assume that $u(y,x) \geq u(x,y)$, where $x \neq y$. Consider the case that $x < y$. Then

$$\frac{u(y,x)-u(x,x)}{y-x} \geq \frac{u(x,y)-u(x,x)}{y-x}.$$

Choose z such that $x < z < y$. Since u is concave with respect to the first argument, $(u(z,x) - u(x,x))/(z-x) \geq (u(y,x)-u(x,x))/(y-x)$. Since u is convex with respect to the second argument, $(u(x,y)-u(x,x))/(y-x) \geq (u(x,z)-u(x,x))/(z-x)$. The three inequalities imply that

$$\frac{u(z,x)-u(x,x)}{z-x} \geq \frac{u(x,z)-u(x,x)}{z-x}.$$

Since $z > x$, it follows that $u(z,x) \geq u(x,z)$. The analogous argument works if $y < x$. □

Corollary 7.2. *Assume that G satisfies the assumptions in Proposition 7.1. If G is pairwise solvable, then G possesses a symmetric Nash equilibrium and an evolutionary equilibrium.*

Proof. Assume that u is concave-convex and coordinatewise continuous. Then it is clear that u is quasiconcave at the diagonal and own-strategy continuous. Hence G has a Nash equilibrium by Proposition 5.13. □

In contrast to pairwise solvability and quasiconcavity at the diagonal, the property of being concave-convex is not an ordinal concept. An order-preserving transformation of a concave-convex payoff function need not be concave-convex. One can verify, however, that the set of Nash (respectively, evolutionary) equilibria is invariant under any order-preserving transformation of the payoff function. Namely:

Corollary 7.3. *Let $G = \langle S, u\rangle$ and $G' = \langle S, v\rangle$ be two-person symmetric games. Assume that G satisfies the assumptions in Corollary 7.2 and that v is an order-preserving transformation of u. Then G' possesses a symmetric Nash equilibrium and an evolutionary equilibrium.*

Example 7.4. Consider a two-person symmetric contest

$$u(x,y) = p(x,y)W + p(y,x)L - c(x),$$

where $p(x,y)$ is a symmetric probability function. In this game, $u(x,y)$ is concave-convex if p is concave with respect to the first argument and $c(x)$ is convex. For example, recall the rent-seeking game in Example 3.6, in which

$$p(x,y) = \frac{g(x)}{g(x)+g(y)},$$

for every $x, y \in S$. If g is concave and the cost function c is convex, then this game is concave-convex, and it has a Nash equilibrium and an evolutionary equilibrium. □

In a two-person symmetric game, all symmetric strategy profiles are (weakly) Pareto-ranked. It is natural to ask which equilibrium, Nash or evolutionary, dominates the other. To compare, let (x^N, x^N) be a Nash equilibrium and (x^E, x^E) be an evolutionary equilibrium. From the definitions of these equilibria, it follows that (n) $u(x^N, x^N) \geq u(x^E, x^N)$ and (e) $u(x^E, x^N) \geq u(x^N, x^E)$. In general, $u(x^E, x^E)$ may be larger than $u(x^N, x^N)$, smaller than $u(x^N, x^E)$, or somewhere in between. But if the game is pairwise solvable, then (n) implies (ps) $u(x^N, x^E) \geq u(x^E, x^E)$. By (n), (e), and (ps), $u(x^N, x^N) \geq u(x^E, x^E)$. Consider the row player in (x^E, x^E). By deviating to x^N, she never earns less, by (ps). But she may refrain from doing so since that would allow the other to beat her, by (e). In this way, spiteful behavior keeps players trapped in an inefficient outcome.

Proposition 7.5. *Let G be a pairwise solvable game. If (x^N, x^N) is a Nash equilibrium and (x^E, x^E) is an evolutionary equilibrium, then $u(x^N, x^N) \geq u(x^E, x^E)$.*

The next example illustrates an extreme case.

Example 7.6. Let v be a real valued function on a nonempty set S and consider $G = \langle S, u\rangle$, where $u(x, y) = v(y)$. By Lemma 3.3, G is pairwise solvable. In this game, all strategy profiles are Nash equilibria. Consider the relative payoff game $r(G) = \langle S, r\rangle$, where $r(x, y) = v(y) - v(x)$. A symmetric profile (x, x) is an evolutionary equilibrium if and only if $0 = r(x, x) \geq r(y, x) = v(x) - v(y)$ for every $y \in S$, or equivalently, $u(x, x) = \min_{y \in S} u(y, y)$. □

8 Concluding remarks

In a pairwise solvable game, each restricted game generated by a pair of strategies has a pure strategy equilibrium, satisfies the interchangeability condition, and is, a fortiori, dominance solvable. Does any of these extend to the whole game? For the case of interchangeability, the problem is answered in the affirmative. For the others, we have found some additional conditions. A classical result along a similar line of reasoning is Shapley (1964, p.6, Theorem 2.1): "If A is the matrix of a zero-sum two-person game, and if every 2-by-2 submatrix of A has a saddle point, then A has a saddle point."

Moulin (1979, 1984, 1986) gives several sufficient conditions for dominance solvability. Each of them applies to either the strategic form of an extensive game with perfect information, or a strategic game with differentiable payoff functions. In contrast, our dominance solvability result concerns finite pairwise solvable games, which are totally independent of any sequential procedure in the background. Thus our result seems new, and it applies to many games that appear frequently in applications, including symmetric contests.[17]

[17] Milgrom and Roberts (1990) show that in a *supermodular game*, the iterated elimination of strictly dominated strategies leads to a minimum and a maximum Nash equilibria. Subsequently, Milgrom and Shannon (1994,

	s_1	s_2	s_3	s_4
s_1	$0,0$	$0,1$	$1,0$	$0,0$
s_2	$1,0$	$1,1$	$1,1$	$0,1$
s_3	$0,1$	$1,1$	$1,1$	$1,0$
s_4	$0,0$	$1,0$	$0,1$	$0,0$

a

	s_1	s_2	s_3
s_1	$3,3$	$1,4$	$2,1$
s_2	$4,1$	$2,2$	$1,3$
s_3	$1,2$	$3,1$	$1,1$

b

Figure 4: Symmetric games.

Friedman (1983) shows that the set of all equilibria in a two-person strictly competitive game is interchangeable. This is generalized by Kats and Thisse (1992) to weakly unilaterally competitive games. Unfortunately, however, these papers contain no concrete economic applications. For symmetric games, we not only generalize their results but exhibit a number of familiar games possessing interchangeability.[18]

One can make use of the results in Section 5 to establish the existence of an equilibrium in an *n-person* symmetric game. For details, see Iimura, Maruta, and Watanabe (2016), which investigates a class of n-person symmetric games that includes weakly unilaterally competitive games and games with weak payoff externalities (Ania 2008).

A pairwise solvable game need not have potential functions. Consider the game in Figure 4a, which is pairwise solvable and quasiconcave at the diagonal. One can verify that it admits neither a generalized ordinal potential (Monderer and Shapley 1996) nor a best-response potential (Voorneveld 2000).[19]

Several questions remain to be investigated. One is to see to what extent the current analysis can be extended to asymmetric games. A moment's reflection reveals that a payoff transformation argument would extend some of the results to a class of asymmetric games that are essentially symmetric.[20] Another is to explore whether dominance solvability generalizes

p.175) generalize this result to the class of games with ordinal strategic complementarities. These results need not apply to pairwise solvable games. Consider the game in Figure 3a. In this game, the difference $u(3/5, y) - u(2/5, y)$ alternates its sign twice as y increases from 0 to 1, which means that the game lacks the single crossing property (Milgrom and Shannon 1994, p.160), or that the game is not one with ordinal strategic complementarities.

[18]Yasuda (2016) shows that a version of interchangeability must hold in any two-person supermodular game.

[19]Note that the game in Figure 4a has a generalized potential in the sense of Morris and Ui (2005), which is maximized at the set of Nash equilibria, $\{s_2, s_3\} \times \{s_2, s_3\}$.

[20]Let $G_u = \langle S, u\rangle$ and $G_v = \langle S, v\rangle$ be pairwise solvable games. Assume that u and v are *order-equivalent at the diagonal*, that is:

For every $x, y \in S$, $u(x,x) \geq u(y,x)$ iff $v(x,x) \geq v(y,x)$, and $u(y,x) \geq u(x,x)$ iff $v(y,x) \geq v(x,x)$.

Now consider a two-person "asymmetric" strategic game $G_a = \langle S_1, S_2, u_1, u_2\rangle$ in which $S_1 = S_2 = S$, $u_1(x,y) =$

to games with infinitely many strategies. Finally, we note that no results seem to be retained under a weaker definition of pairwise solvability. Consider the game in Figure 4b. It is a quasiconcave game that is "weakly" pairwise solvable, in that in every 2×2 restricted game, one strategy weakly dominates the other. The game has no equilibrium in pure strategies.

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$u(x, y)$, and $u_2(x, y) = v(y, x)$ for every $x, y \in S$. We claim that these three games are essentially the same. It is straightforward to verify that G_u and G_v have precisely the same set of pure strategy Nash equilibria and the same set of equilibrium strategies, and that if S is finite and the games are quasiconcave at the diagonal, the rule (E) eliminates a strategy in G_u precisely when it eliminates the same strategy in G_v. Consequently, G_a too has the same set of interchangeable equilibria and is dominance solvable. Note that order-equivalence at the diagonal is considerably weaker than the standard order-preserving payoff transformation. As a consequence, the former need not preserve the set of evolutionary equilibria.

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