Optimal Public Information Disclosure by Mechanism Designer*

Takuro Yamashita†

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Abstract

This paper considers mechanism design environments where a principal can disclose some information relevant to agents. As opposed to the standard "informed principal" approach with no commitment as to the principal's information disclosure, we consider fully committed (public) information disclosure by the principal, implying more tractability and hence wider applicability. In linear environments with no restriction on monetary transfers (e.g., auction of Myerson (1981)), we show that the principal finds it optimal to fully disclose his information. With a budget-balance restriction on monetary transfers (e.g., bilateral trading of Myerson and Satterthwaite (1983)), full disclosure may be suboptimal. In a bilateral-trading environment with uniformly distributed types, I characterize the second-best information disclosure policy, which is a simple *censoring* policy. The technique developed for this second-best characterization may be useful to tractably

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†Toulouse School of Economics, takuro.yamashita@tse-fr.eu

analyze general Bayesian persuasion problems with continuous state spaces.

1 Introduction

This paper considers mechanism design environments where a principal has some information relevant to agents. For example, a monopoly seller, who designs a nonlinear pricing contract, may be able to disclose some information about the quality of his good. As another example, a government, who regulates financial transactions, may be able to disclose some information about the future economic situations. In these cases, there are many important questions such as (i) what kind of information the principal would reveal to the agents, and (ii) how the optimal mechanism and optimal information disclosure interact with each other.

The key assumptions of the paper are (i) the principal commits to his disclosure rule at the ex ante stage, (ii) only public disclosure is allowed, and (iii) his mechanism (e.g., the monopoly price he charges) can be contingent on the disclosed information. One instance where these assumptions may be reasonable is when the principal himself does not have a technology to generate any hard evidence about his information, but before he observes his information, he can ask a third-party certifier to generate hard evidence.[1] Another possibility is that the principal himself cannot observe it, while he knows that a certifier can know it.

Our main questions include how much information the principal voluntarily discloses, and how the disclosed information and optimal mechanism interact with each other.

The basic economic intuitions that determine the optimal disclosure are as follows. First, more disclosure implies more flexibility in the mechanism. For example, the monopoly seller can charge different prices for different quality

[1]See Dranove and Jin (2010) for the literature review regarding (mainly third-party) economics of certification.

levels under full disclosure, while the price must be constant under no disclosure. This first effect makes the principal favor more disclosure. Second, on the other hand, more disclosure implies (at least weakly) more information rent to the agents. In its extremes, incentive compatibility must be satisfied given every possible realization of the quality parameters under full disclosure, while it must be satisfied only "in expectation" under no disclosure. Therefore, this second effect makes the principal favor less disclosure.[2]

Which effect dominates the other is not trivial, and depends on the environments. First, in linear environments with no restriction on monetary transfers (e.g., auction of Myerson (1981)), we show that the principal finds it optimal to fully disclose his information (Section 4.1). This is because the second "information-rent" effect is null in this environment. Because everyone is risk neutral, each agent's (ex ante) expected information rent (or as its mirror image, his expected virtual value) does not vary with the disclosure policy, for any fixed mechanism.

Second, in a bilateral-trading environment (Myerson and Satterthwaite (1983)) with a budget-balance restriction on monetary transfers, we show that full disclosure is suboptimal (Section 4.2). Even though everyone is risk neutral, and hence the agents' expected information rents do not vary with the disclosure policy, more disclosure means more dispersion in realization of information rents, compared to less or no disclosure case, which means that budget balance is most likely violated under full disclosure. Then, a natural next question is characterization of the second-best information disclosure policy. In Section 4.3, we characterize the second-best information disclosure policy with uniformly distributed types, which is a simple *censorship* policy (Kolotilin et al. (2015)).

The paper is related to the broad literature of information disclosure in mechanism design as follows. The classical "informed principal" literature assumes that the principal does not have a commitment power in terms of

[2]This second effect has been observed in Myerson (1983) and Skreta (2011) in mechanism design with an informed principal.

information disclosure. For example, see Myerson (1983), Maskin and Tirole (1990, 1992), and Mylovanov and Tröger (forthcoming). In this literature, as in standard signaling games, different types of the principal compete with each other through their design of mechanisms in a nontrivial manner. This is particularly challenging in mechanism design, where the sender's (principal's) action space is the set of mechanisms, highly multidimensional and complicated mathematical objects. This signaling issue makes the equilibrium characterization much less tractable. The commitment assumption in our paper makes the problems more tractable, and hence implies wider applicability. In reality, the principal's commitment ability may vary with contexts, ranging from full commitment (as in our paper) to no commitment at all (as in the informed-principal approach). In this sense, ex ante commitment approach can be complementary to the standard informed-principal approach.

Also, in the informed-principal approach, there are two elements that can potentially affect the principal's disclosure decision. One is the signaling effect, and the other is the allocation or information-rent effect. By assuming commitment, we can purely focus on the analysis of the allocation/rent effect. In this sense, even in situations where the principal's actual commitment ability is more limited, identifying the allocation/rent effect separately from the signaling effect would be useful.

Because of the commitment assumption, our methodology of analysing the information disclosure problem is based on the Bayesian persuasion literature. See Rayo and Segal (2010) and Kamenica and Gentzkow (2011), for example. Among those, as we discuss later, a recent paper by Kolotilin et al. (2015) would be most relevant to ours.

Some more recent papers, including some of the sequential-screening literature, consider committed information disclosure by a principal. For example, see Eső and Szentes (2007) and Bergemann and Pesendorfer (2007).[3]

[3] See also Bergemann and Wambach (2015), Li and Shi (2015), and Ganuza and Penalva (2014).

The key difference from those papers is that, in our setting, the mechanism can be contingent on the disclosed information (because it is public information disclosed by a third party), while in those papers, the principal does not observe disclosed information (only each agent observes a relevant information), and hence, the mechanism cannot be contingent on it. This means that our first "flexibility" effect does not exist in those papers, which (at least weakly) makes the principal favor less information disclosure. In fact, in an auction environment where we find full disclosure optimal, Bergemann and Pesendorfer (2007) find full disclosure suboptimal. On the other hand, Eső and Szentes (2007) find that, if the principal can charge a price for disclosure (as opposed to Bergemann and Pesendorfer (2007)), then the principal can essentially know such disclosed information for free, and therefore, we can treat the principal as if he knows such disclosed information as in our setting (and hence the "flexibility" effect revives). Indeed, our Theorem 1 in Section 4.1 can be seen as a generalization of their Proposition 1, showing optimality of full disclosure. One difference of our paper from those papers is that we consider general mechanism design problems, not only revenue maximization in auction. Moreover, we show that full disclosure may not be optimal in other environments than auction such as in bilateral trading of Myerson and Satterthwaite (1983), and characterize the optimal disclosure in a certain case. It would be an interesting question whether the same disclosure rule is optimal even when the mechanism cannot be contingent on the disclosed information.

The public nature of information disclosure also means that the principal cannot make the mechanism contingent on his information *without* disclosing it to the agents. In this respect, our paper is different from Skreta (2011), who allows for such a possibility, and in that case, the "flexibility" effect of information disclosure again disappears, because the mechanism can always be fully flexible, regardless of the disclosure level to the agents. Then Skreta (2011) shows that, in a quasi-linear auction setting, the principal is indifferent among any disclosure policy. Similarly, we do not allow for a possibility that different amount information disclosure are enjoyed by different agents or different types of the same agent. I believe that, given that we consider a third-party disclosure, it is a reasonable restriction to focus on public information disclosure. However, it would be interesting to study such "private" disclosure or "type-contingent" disclosure.[4]

Finally, our paper is related with the literature of more general information disclosure in game theory. (Committed) information disclosure problems have been studied in many games. For example, Ganuza and Penalva (2010, 2014) consider auction (with or without reserve price), and show that the seller does not want to reveal all the information relevant to the bidders. Morris and Shin (2002) and Angeletos and Pavan (2007) consider coordination games, and show that too much public information may hurt the players. Milgrom and Weber (1982) consider affiliated common-value auction, and show that disclosing affiliated signals increases the seller's expected revenue (called the "linkage principle").[5] Kamenica and Gentzkow (2011) study a general persuasion environment, and characterize the sender's optimal disclosure strategy. Often in those games, full information disclosure is suboptimal for the sender (i) because of misaligned preferences of the sender and receiver, or (ii) even if the sender is benevolent, because of some frictions in underlying games. Our results say that, if the principal can design the underlying game so that misalignment or friction is mitigated, then he sometimes finds it optimal to disclose all the relevant information (even though his preference and the agents' are not fully aligned), while in other cases, he still finds full disclosure suboptimal (even though he may be a benevolent mediator).

The paper is structured as follows. Section 2 introduces a mechanism design environment, and Section 3 introduces the principal's problem, mechanism design and information disclosure. Section 4 gathers the main results

[4]See Bergemann and Wambach (2015), Li and Shi (2015), Bergemann et al. (2015) and and Kolotilin et al. (2015).

[5]Information is correlated in some of these papers, while independent in our case. Therefore, direct comparison may be difficult.

of the paper. First, in linear environments with no restriction on monetary transfers (e.g., auction of Myerson (1981)), we show that the principal finds it optimal to fully disclose his information (Section 4.1). However, in a bilateral-trading environment (Myerson and Satterthwaite (1983)) with a budget-balance restriction on monetary transfers, full disclosure is suboptimal (Section 4.2). Then, a natural next question is characterization of the second-best information disclosure policy. In Section 4.3, we characterize the second-best information disclosure policy with uniformly distributed types, which is a simple *censorship* policy (Kolotilin et al. (2015)). Section 5 concludes the paper.

2 Mechanism Design with information disclosure

There is a set $I = \{1, \ldots, N\}$ of agents. A principal assigns an allocation $x \in X$. We can incorporate feasibility constraints on X. For example, in auction of Myerson (1981), an allocation comprises a probability of giving a good to each agent i, $q_i \in [0, 1]$, and his payment, $p_i \in \mathbb{R}$. Hence, $X = \{(q_i, p_i)_{i=1}^N \in ([0, 1] \times \mathbb{R})^N | \sum_{i=1}^n q_i \leq 1\}$. In bilateral trade of Myerson and Satterthwaite (1983), there are two agents ($N = 2$), a seller and a buyer, and an allocation comprises a trade probability between a buyer and a seller as well as the trading price. Hence, $X = \{(q_i, p_i)_{i=1}^2 \in ([0, 1] \times \mathbb{R})^2 | q_1 = q_2,\ p_1 + p_2 = 0\}$. In public-good provision of Mailath and Postlewaite (1990), an allocation comprises a probability that the public good is provided and each agent's payment, where the total payment must exceed the cost of the public good c. Hence, $X = \{(q_i, p_i)_{i=1}^N \in ([0, 1] \times \mathbb{R})^N | q_1 = \ldots = q_N,\ \sum_{i=1}^N p_i \geq c\}$.

Each agent $i \in I$ has private information, called his *type*, denoted by $v_i \in V_i \subseteq \mathbb{R}$. A type profile is denoted by $v = (v_i)_{i=1}^N$. The information the principal can control its disclosure level is denoted by $\theta \in \Theta \subseteq \mathbb{R}^d$. At the ex ante stage, the principal and agents share a common prior for the variables $(v, \theta) \in V \times \Theta$. We assume that $(v_1 \ldots, v_N, \theta)$ are mutually independently distributed,[6] denote by F_i the prior distribution of v_i, and by F_0 the prior distribution of θ. Let F_V denote the joint prior distribution of $v \in V$.

Given an allocation x and realization of random variables (v, θ), the utility of each agent i is denoted by $u_i(x, v, \theta)$, and the principal's utility is denoted by $u_0(x, v, \theta)$. Later, we impose more assumptions on their utility functions when we study optimal mechanism and disclosure policy.

The timing of the game is as follows. First, the principal sends a public message $m \in M$ to all the agents, which we call her *information disclosure*, where M denotes the principal's (exogenously given) message space. We assume that M is a sufficiently large[7] measurable space. The principal's *information disclosure strategy* is defined as a distributional strategy of Milgrom and Weber (1982), i.e., a joint distribution $\phi \in \Delta(\Theta \times M)$ such that the marginal of ϕ over Θ coincides with F_0.[8]

Second, after m is publicly observed, the principal designs a direct mechanism $x_m : V \to X$. The implicit assumption in this formulation is that the principal does not have more information about θ than the agents (i.e., than what the public message m reveals).[9]

Finally, each agent sends a message to the mechanism, and the allocation is realized.

[6]Note that we allow for dependence within θ, if $d \geq 2$.

[7]For example, any $M \supseteq \Delta(\Theta)$ will do.

[8]An alternative way is to define a disclosure strategy as a mapping $\xi : \Theta \to \Delta(M)$ (called a behavioral strategy). Obviously, together with $F_0 \in \Delta(\Theta)$, ξ induces a distributional strategy. See Milgrom and Weber (1982).

[9]As we discuss in the introduction, one instance where this assumption may be reasonable is when the principal himself does not have a technology to generate any hard information about θ, but there exists a third-party certifier that can generate hard information. It should be admitted that we make several simplifying assumptions. Of course, if a certifier charges a price (or perhaps different prices for different certification policies) then the analysis would be more complicated, which is left as an open question.

3 Principal's problem

Because the principal makes a sequential decision of information disclosure and mechanism design, we first consider the mechanism-design problem.

Because arbitrary posterior may be induced by some information disclosure, we define, for each posterior $\psi \in \Delta(\Theta)$, the optimal mechanism $\chi^*(\psi)$ and the principal's expected utility under the optimal mechanism, $S^*(\psi)$. Specifically, define

$$\begin{aligned} S^*(\psi) \ = \ \sup_x \quad & \int_\theta \int_v u_0(x(v), v, \theta) dF_V d\psi \\ \text{sub. to} \quad & \int_\theta \int_{v_{-i}} u_i(x(v), v, \theta) dF_{-i}(v_{-i}) d\psi(\theta) \\ & \quad \geq \int_\theta \int_{v_{-i}} u_i(x(v_i', v_{-i}), v, \theta) dF_{-i}(v_{-i}) d\psi(\theta), \ \forall i, v_i, v_i', v_{-i}, \\ & \int_\theta \int_{v_{-i}} u_i(x(v), v, \theta) dF_{-i}(v_{-i}) d\psi(\theta) \geq 0, \ \forall i, v_i, v_{-i}, \end{aligned}$$

and define $\chi^*(\psi)$ as a maximizer of this problem (if it exists).

The first constraint corresponds to each agent's Bayesian incentive compatibility condition, and the second constraint is each agent's interim individual rationality or participation condition. Because each agent observes a public message by the principal, the agent's expected utility is computed using his posterior ψ over θ.

Next, we consider the information disclosure problem. As is clear in the previous paragraphs, the posterior distribution induced by the information disclosure is crucial to determine the agents' incentives and the optimal mechanism. In this respect, the following well-known result in probability theory is useful in connecting the principal's distributional strategy and the system of posteriors.[10]

Proposition 1. ((Product-)regular-conditional-probability property) Each $\phi \in \Delta(\Theta \times M)$ induces a pair $(\mu, (\psi_m)_{m \in M})$ such that (i) $\mu \in \Delta(M)$, (ii) $\psi_m \in$

[10]For the proof, see Faden (1985), for example.

$\Delta(\Theta)$ for each m, and (iii) $\int_{m \in B} \psi_m(A) d\mu = \phi(A \times B)$ for each measurable $A \subseteq \Theta$ and $B \subseteq M$.

μ is the marginal distribution over the public message space M, and for each realization $m \in M$, ψ_m is the posterior over Θ. The last condition implies (by taking $B = M$) $\int_m \psi_m(A) d\mu = F_0(A)$ for each A, i.e., the system of posterior distributions $(\psi_m)_{m \in M}$ must satisfy a martingale property.[11]

Conversely, given a pair $(\mu, (\psi_m)_{m \in M}) \in \Delta(M) \times (\Delta(\Theta))^M$ such that $\int_m \psi_m(A) d\mu = F_0(A)$ for each measurable $A \subseteq \Theta$, there exists a distributional strategy ϕ that induces $(\mu, (\psi_m)_{m \in M})$.[12] Therefore, choosing ϕ and choosing $(\mu, (\psi_m)_{m \in M})$ are essentially equivalent. In what follows, we treat $(\mu, (\psi_m)_{m \in M})$ as the principal's information disclosure strategy because it is directly related to the system of the posteriors.

The optimal information disclosure problem can be written as follows.

$$\begin{aligned} \sup_{\mu, (\psi_m)_{m \in M}} \quad & \int_m S^*(\psi_m) d\mu(m) \\ \text{sub. to} \quad & \int_m \psi_m(\cdot) d\mu = F_0(\cdot). \end{aligned}$$

4 Optimal information disclosure in linear environments

Applying Jensen's inequality, optimality of full (no) disclosure is obtained when S^* is convex (concave) and continuous.[13] However, those characterization results based directly on the shape of S^* are not fully satisfactory

[11] This is called *Bayesian plausibility* by Kamenica and Gentzkow (2011).

[12] The proof idea is as follows. For each measurable $A \subseteq \Theta$ and $B \subseteq M$, define $\phi(A \times B) = \int_{m \in B} \psi_m(A) d\mu$. Then we can show that ϕ is a probability distribution on $\Theta \times M$, and moreover, its marginal on Θ is F_0. This means that ϕ is a distributional strategy.

[13] If $\Delta(\Theta)$ is finite-dimensional (e.g, Θ is finite), then convexity (concavity) of S^* implies its continuity. However, if $\Delta(\Theta)$ is infinite-dimensional, it is no longer the case, and for a convex but discontinuous S^*, Jensen's inequality may not hold. See, for example, Perlman (1974).

in our environment where S^* is an *endogenous* variable. To obtain fuller characterization of the optimal information disclosure strategy based on the conditions on the mechanism design environment, in this section, we consider the following simple class of environments.

An allocation is $x = (q_i, p_i)_{i \in N} \in X \subseteq \mathbb{R}^{2N}$, where $q_i \in \mathbb{R}$ is a decision variable that is payoff-relevant to agent i, and $p_i \in \mathbb{R}$ is a monetary transfer from agent i.

Each agent i's utility is $u_i(x, v, \theta) = q_i y_i(v_i, \theta) - p_i$, and the principal's utility is $u_0(q, v, \theta) = y_0(q, v, \theta) + \sum_{i=1}^N p_i$, where y_0, y_i are continuous in all their arguments, and $\frac{\partial y_i}{\partial v_i} > 0$. Hence, we assume (i) quasilinearity in monetary transfer for every party, (ii) linearity in the decision variable q_i for each agent i, (iii) private values for each i, and (iv) increasing difference in (q_i, v_i) for each i.

We believe that many economically important environments are in this class. For example, the auction, bilateral trade, and public good environments briefly discussed in the previous section are all in this class, by restricting X appropriately. Later, we also discuss to what extent these assumptions could be weakened.

4.1 Optimality of full disclosure

Throughout this section, we assume that monetary transfers are not restricted, i.e., the feasible allocation set satisfies

$$X = \{(q_i, p_i)_{i=1}^N | q = (q_i)_{i=1}^N \in Q, \ p_i \in \mathbb{R}, \ \forall i\},$$

for some $Q \subseteq [0, 1]^N$. For example, auction environments as in Myerson (1981) satisfy this condition, but a balanced-budget bilateral trade as in Myerson and Satterthwaite (1983) or a public-good provision as in Mailath and Postlewaite (1990) do not.

We further assume that each F_j, $j = 0, 1, \ldots, N$ has a continuous and full-support density f_j.

We show that full disclosure of θ is optimal for the principal.

Definition 1. An information-disclosure strategy $(\mu, (\psi_m)_{m\in M})$ exhibits *full disclosure* if, for each $m \in M$, ψ_m is degenerated, i.e., $\psi_m(\{\theta\}) = 1$ for some $\theta \in \Theta$.

Theorem 1. An optimal information-disclosure strategy exhibits full disclosure.

Proof. It suffices to show that $S^*(\psi)$ is convex and continuous in ψ, because then, given any information disclosure strategy $(\mu, (\psi_m)_{m\in M})$, Jensen's inequality implies $S^*(\psi_m) \le \int_\theta S^*(\delta_\theta)d\psi_m$ where δ_θ is a Dirac distribution on θ (i.e., $\delta_\theta(\{\theta\}) = 1$), which further implies

$$\begin{aligned}\int_m S^*(\psi_m)d\mu(m) &\le \int_m \int_\theta S^*(\delta_\theta)d\psi_m d\mu(m)\\ &= \int_\theta S^*(\delta_\theta)dF_0.\end{aligned}$$

We first show S^* is convex. By the standard procedure in mechanism design (e.g., see Myerson (1981)), the maximization problem for $S^*(\psi)$ can be expressed as a virtual-surplus maximization problem subject to certain monotonicity constraints.

Lemma 1.

$$\begin{aligned} S^*(\psi) \;=\; \sup_q \quad & \int_\theta \int_v \sum_{j=0}^N \gamma_j(q(v), v, \theta) dF_V d\psi \\ \text{sub. to} \quad & E_{v_{-i}}[q_i(v_i, v_{-i})] \le E_{v_{-i}}[q_i(v_i', v_{-i})],\ \forall i, v_{-i}, v_i < v_i', \end{aligned}$$

where $\gamma_0(q, v, \theta) = y_0(q, v, \theta)$, $\gamma_i(q, v, \theta) = q_i[y_i(v_i, \theta) - \frac{\partial y_i}{\partial v_i}(v_i, \theta)\frac{1-F_i(v_i)}{f_i(v_i)}]$ for each $i = 1, \dots, N$, are the principal's and each agent's virtual value, respectively.

We omit the proof of this standard result.

For each $q : [0,1]^N \to Q$ that satisfies the monotonicity constraints above, let

$$S(q, \psi) = \int_\theta \int_v \sum_{j=0}^N \gamma_j(q(v), v, \theta) dF_V d\psi$$

denote the objective of the problem. Recall that $S^*(\psi) = S(\chi^*(\psi), \psi)$.

Fix arbitrary $\psi, \psi' \in \Delta(\Theta)$ and $\alpha \in (0,1)$. Let $\psi'' = \alpha\psi + (1-\alpha)\psi'$. Because $\chi^*(\psi'')$ satisfies the monotonicity constraints above, we have

$$\begin{aligned} \alpha S(\chi^*(\psi), \psi) &\geq \alpha S(\chi^*(\psi''), \psi), \\ (1-\alpha)S(\chi^*(\psi'), \psi') &\geq (1-\alpha)S(\chi^*(\psi''), \psi'), \end{aligned}$$

and therefore,

$$\begin{aligned} \alpha S^*(\psi) + (1-\alpha)S^*(\psi') &\geq \alpha S(\chi^*(\psi''), \psi) + (1-\alpha)S(\chi^*(\psi''), \psi') \\ &= S(\chi^*(\psi''), \psi'') \\ &= S^*(\psi''), \end{aligned}$$

where the first equality is because of linearity of $S(q, \psi)$ in ψ. Therefore, $S^*(\psi)$ is convex in ψ.

Next, we show S^* is continuous. Let Ψ denote the set of all finite signed measures on Θ. Endowed with a total variation norm and its induced topology, Ψ is a topological vector space. Let $\Psi_1 \subseteq \Psi$ denote the set of all signed measures with norm 1. Then, $\Delta(\Theta) \subseteq \Psi_1$.

For each $\psi \in \Psi$, define $S(q, \psi)$ the same way as above. For each given q, $S(q, \psi)$ is homogeneous of degree one with respect to ψ. Hence, for $\psi \neq 0$, we have $S(q, \psi) = \|\psi\| S(q, \frac{\psi}{\|\psi\|})$, which implies $\chi^*(\psi) = \chi^*(\frac{\psi}{\|\psi\|})$ (if it exists) and $S^*(\psi) = \|\psi\| S^*(\frac{\psi}{\|\psi\|})$.

Obviously, $S^*(\frac{\psi}{\|\psi\|})$ is finite, and therefore, $S^*(\psi)$ is finite too. Also, $S^*(\psi)$ is convex, as explained in the previous paragraph. Those imply that $S^*(\psi)$ is continuous on Ψ (and hence on $\Delta(\Theta)$).[14] $\square$

The proof exploits the simple property that a value function ($S^*(\psi)$) is convex in a parameter (ψ) if the feasible set (the set of all monotonic q) does not vary with this parameter.

To provide some economic intuition for the result, observe that there are two channels where information disclosure affects implementable allocation

[14] See Aliprantis and Border (2006) (p.188, Theorem 5.42).

rules. First, more disclosure implies more flexibility in the mechanism. In its extreme, q can vary with θ under full disclosure, while q must be constant in θ under no disclosure. This first effect makes the principal favor more disclosure. Second, more disclosure implies (weakly) tighter incentive constraints. Again in its extreme, truth-telling must be optimal given every possible realization θ under full disclosure, while truth-telling needs to be optimal only "in expectation" under no disclosure. Because of this, either expected information rent may be higher or implementable allocation rules may be smaller under full disclosure (than under less disclosure). Hence, this second effect makes the principal favor less disclosure.

However, in the current environment, the second effect is null. Because every player (in particular, the principal) has a quasilinear payoff, the expected information rent is linear in ψ, and in this sense, more disclosure does not imply strictly tighter incentive constraints. Therefore, only the first effect exists, leading to optimality of full disclosure.

Remark 1. The assumptions of linearity and private values can be weakened to some extent (though not completely dispensable). We briefly see that the same result is obtained if either (i) $u_i(x, v, \theta) = x_i(q, v_{-i})y_i(v_i, \theta)$, (ii) $u_i(x, v, \theta) = x_i(q, v_{-i}, \theta)y_i(v_i)$, or (iii) $u_i(x, v, \theta) = x_i(q, v)y_i(\theta)$. Case (i) is a direct generalization of the one used in this section, allowing for externalities in q and interdependence, as long as those are multiplicably separated from $y_i(v_i, \theta)$. Either v_i or θ (but not both) can be moved into x_i instead of y_i, leading to Case (ii) or (iii).

For (i), agent i's incentive compatibility is equivalent to combination of (a) (interim) monotonocity:

$$E_{v_{-i}}[x_i(q(v_i, v_{-i}), v_{-i})] \text{ is nondecreasing in } v_i,$$

and (b) envelope formula:

$$\begin{aligned}&E_{v_{-i},\theta}[x_i(q(v),v_{-i})y_i(v_i,\theta)-p_i(v)]\\&=E_{v_{-i},\theta}[x_i(q(0,v_{-i}),v_{-i})y_i(v_i,\theta)-p_i(0,v_{-i})]\\&+\int_0^{v_i}E_{v_{-i}}[x_i(q(0,v_{-i}),v_{-i})\frac{\partial y_i}{\partial v_i}(\tilde{v}_i,\theta)]d\tilde{v}_i,\end{aligned}$$

where expectation with respect to θ is based on a posterior ψ.

Therefore, the set of implementable q does not vary with ψ and the information rent is linear in ψ, which implies optimality of full disclosure. We omit the other cases (ii) and (iii).

Remark 2. Although this paper only considers Bayesian incentive compatibility, the same sort of exercise is possible for other solution concepts, and the optimal disclosure strategy naturally varies with the underlying solution concept. For example, we may consider ex post incentive compatibility with respect to the agents' types, i.e., each agent i finds truth-telling optimal regardless of the other agents' type realization, v_{-i}. On the other hand, we still keep the assumption that the principal has a full control over disclosure of θ and can design the agents' posterior about θ.

To provide a more concrete idea, consider an interdependent-value auction environment where each i's utility is $q_i y_i(v,\theta)-p_i$ with $\frac{\partial y_i}{\partial v_i}>0$. Then, ex post incentive compatibility (with respect to v) means, for each i, v_i, v_i', v_{-i},

$$q_i(v_i,v_{-i})E_\theta[y_i(v_i,v_{-i},\theta)]-p_i(v_i,v_{-i})\geq q_i(v_i',v_{-i})E_\theta[y_i(v_i,v_{-i},\theta)]-p_i(v_i',v_{-i}),$$

where expectation with respect to θ is based on a posterior ψ.

Again, this is equivalent to combination of (a) ex post monotonicity:

$$q_i(v_i,v_{-i})\text{ is nondecreasing in }v_i,$$

and (b) envelope formula:

$$\begin{aligned}&q_i(v)E_\theta[y_i(v,\theta)]-p_i(v)\\&=q_i(0,v_{-i})E_\theta[y_i(0,v_{-i},\theta)]-p_i(0,v_{-i})\\&+\int_0^{v_i}q_i(\tilde{v}_i,v_{-i})E_\theta[\frac{\partial y_i(\tilde{v}_i,v_{-i},\theta)}{\partial v_i}]d\tilde{v}_i.\end{aligned}$$

Therefore, the set of implementable q does not vary with ψ and the information rent is linear in ψ, which implies optimality of full disclosure.

4.2 Suboptimality of full disclosure

As discussed in the last section, the property that the set of implementable q does not vary with ψ is crucial for the convexity of S^* (and hence for optimaility of full disclosure).

In this section, we observe that the feasible set varies with ψ in balanced-budget bilateral trading. This is because the budget balance condition constrains the agents' total expected information rents, an expression that varies with ψ. In this environment, we show that full disclosure is suboptimal. More specifically, there exists a subset of Θ where the principal prefers not to reveal its realization.

Following Myerson and Satterthwaite (1983), we assume that there are two agents, a seller ($i = 1$) and a buyer ($i = 2$). For each $i = 1, 2$, F_i has a density f_i such that $f_i(v_i) \in [d_1, d_2]$ for some $0 < d_1 \leq d_2 < \infty$. For θ, we assume that F_0 has a full-support density f_0 on $\Theta = (a, b) \subseteq \mathbb{R}$ with $a < 1 < b$. The seller's payoff is $-v_1 q_1 - p_1$ and the buyer's payoff is $(v_2 + \theta)q_2 - p_2$, where $q_1 = q_2 \in [0, 1]$ is the trade probability and $p_i \in \mathbb{R}$ is the monetary transfer from i. Hence, higher θ means that trade between the seller and buyer is more socially desirable.[15] The budget balance condition requires that $p_1 + p_2 \geq 0$.[16] Therefore, the feasible allocation set is given by

$$X = \{(q_i, p_i)_{i=1}^2 \in [0, 1]^2 \times \mathbb{R}^2 | q_1 = q_2,\ p_1 + p_2 \geq 0\}.$$

The principal's objective is the trade surplus, $(v_2 + \theta - v_1)q$, as in Myerson and Satterthwaite (1983). Then, the value function for the principal given

[15]Although θ only enters into the buyer's payoff here, we obtain the same result as long as the total surplus is $(v_2 + \theta - v_1)q$, i.e., it does not matter whose payoff function θ appears in.

[16]This is also called a "no-deficit" condition. In the optimal mechanism, the constraint binds with equality anyway.

any posterior ψ is $S^*(\psi)$, where

$$
\begin{aligned}
S^*(\psi) \;=\; & \sup_{(q,p_1,p_2):[0,1]^2\to X} && \int_\theta \int_v (v_2+\theta-v_1)q(v)dF_V d\psi \\
& \text{sub. to} && \int_\theta \int_{v_2} -v_1 q(v) - p_1(v) dF_2(v_2) d\psi(\theta) \\
& && \quad \geq \max\left\{0, \int_\theta \int_{v_2} -v_1 q(v_1', v_2) - p_1(v_1', v_2) dF_2(v_2) d\psi(\theta)\right\},\ \forall v_1, v_1', v_2, \\
& && \int_\theta \int_{v_1} v_2 q(v) - p_2(v) dF_1(v_1) d\psi(\theta) \\
& && \quad \geq \max\left\{0, \int_\theta \int_{v_1} v_2 q(v_1, v_2') - p_2(v_1, v_2') dF_1(v_1) d\psi(\theta)\right\},\ \forall v_1, v_2, v_2', \\
& && p_1(v) + p_2(v) \geq 0,\ \forall v.
\end{aligned}
$$

Again, with the standard machinery based on an envelope theorem and integration by parts, we obtain

$$
\begin{aligned}
S^*(\psi) \;\leq\; & \sup_{q:[0,1]^2\to[0,1]} \int_v (v_2 + E_\psi\theta - v_1) q(v) dF_V \\
& \text{sub. to} \int_v (v_2 + E_\psi(\theta) - v_1 - \frac{1-F_2(v_2)}{f_2(v_2)} - \frac{F_1(v_1)}{f_1(v_1)}) q(v) dF_V \geq 0,
\end{aligned}
$$

where $E_\psi(\theta) = \int_\theta \theta d\psi$. We denote the value of the right hand side by $S^{**}(E_\psi(\theta))$, to make it explicit that the value varies with $E_\psi(\theta)$ (but not with the other moments of ψ). The inequality is because (i) the ex post budget balance condition is replaced by the interim one, and (ii) the monotonicity condition is ignored. However, as in the standard argument, this holds with equality as long as each F_i has a monotone hazard rate: throughout this section, we assume that $\frac{1-F_2(v_2)}{f_2(v_2)}$ is decreasing in v_2, and $\frac{F_1(v_1)}{f_1(v_1)}$ is increasing in v_1.

The key difference of this problem from the one in the previous subsection is that the constraint now varies with ψ. Therefore, the proof in the previous section does not apply. Indeed, we show that the principal's value function $S^*(\psi)$ is not convex, implying that the principal rather prefers to hide some realization of θ.

Theorem 2. Full disclosure is suboptimal.

Proof. First, consider any ψ such that $E_\psi(\theta) \geq 1$. In this case, regardless of the agents' values v, making them trade is always efficient (with $p_2(v) = 1 = -p_1(v)$ so that both participation and budget-balance constraints are satisfied). Therefore, we obtain

$$S^{**}(E_\psi(\theta)) = S^*(\psi) = E(v_2 - v_1) + E_\psi(\theta).$$

Similarly, for any ψ such that $E_\psi(\theta) < -1$, regardless of the agents' values v, making them trade is always inefficient. Therefore, we obtain

$$S^{**}(E_\psi(\theta)) = 0.$$

Next, fix arbitrary $x \in (-1, 1)$. The following lemma is crucial, whose proof is in the appendix.

Lemma 2. There exist $\overline{\varepsilon} > 0$ and a continuous function $z : (0, \overline{\varepsilon}) \to \mathbb{R}_{++}$ such that, for $\varepsilon \in (0, \overline{\varepsilon})$, $S^{**}(1-\varepsilon) \leq E(v_2 + 1 - \varepsilon - v_1) - z(1-\varepsilon)$.

Because F_0 has a support around $\theta = 1$, there exists $\tilde{\varepsilon} > 0$ such that $E_{F_0}(\theta | \theta \in (1 - \overline{\varepsilon}, 1 + \tilde{\varepsilon})) = 1$. Consider the following disclosure strategy: the principal announces $m = \theta$ if $\theta \notin (1 - \varepsilon_2, 1 + \varepsilon_3)$, and announces $m = 1$ if $\theta \in (1 - \varepsilon_2, 1 + \varepsilon_3)$. Conditional on $\theta \in (1 - \varepsilon_2, 1 + \varepsilon_3)$ (which occurs with a positive probability since F_0 has a full-support), this disclosure strategy achieves expected surplus $E(v_2 + 1 - v_1)$, which is strictly higher than

$$\begin{aligned} E_{F_0}(S^{**}(\theta) | \theta \in (1 - \varepsilon_2, 1 + \varepsilon_3)) &= \int_{1-\overline{\varepsilon}}^{1+\tilde{\varepsilon}} \frac{E(v_2 + 1 - v_1) - z(\theta)}{F_0(\theta \in (1 - \overline{\varepsilon}, 1 + \tilde{\varepsilon})} dF_0 \\ &= E(v_2 + 1 - v_1) - \int_{1-\overline{\varepsilon}}^{1+\tilde{\varepsilon}} \frac{z(\theta)}{F_0(\theta \in (1 - \overline{\varepsilon}, 1 + \tilde{\varepsilon})} dF_0 \\ &< E(v_2 + 1 - v_1). \end{aligned}$$

Therefore, full disclosure is suboptimal. $\square$

4.3 Second-best disclosure strategy

Given the previous result, a natural next step is characterization of the second-best disclosure strategy. It is out of the scope of the paper to provide such characterization in a general setting. Instead, in this section, we provide the optimal disclosure strategy when each F_i is a uniform distributions on $[0,1]$. As in the previous section, we assume $\Theta = (a,b) \subseteq \mathbb{R}$ with $a < 1 < b$, and $F_0 \in \Delta(\Theta)$ admits a full-support density f_0. Because the support is around $\theta = 1$, as we show in the previous section, full disclosure is suboptimal. We first examine the shape of S^{**} more fully.

Lemma 3. $\frac{dS^{**}}{dx}$ exists and continuous for all x. $\frac{d^2S^{**}}{dx^2}$ exists and continuous for all $x \neq 1$. More specifically, there exists $\hat{x} \in (\frac{1}{3}, 1)$ such that

$$\frac{d^2S^{**}}{dx^2}\begin{cases} \geq 0 & \text{if} \quad x < \hat{x}, \\ < 0 & \text{if} \quad x \in (\hat{x}, 1), \\ = 0 & \text{if} \quad x > 1, \end{cases}$$

The last part of the lemma says that S^{**} is convex on $(-\infty, \hat{x})$ and concave on $(\hat{x}, \infty)$. The proof is in the appendix.

With this simple cutoff structure, we obtain the following simple optimal disclosure policy.

Proposition 2. There exists $x^* \in [a, \hat{x}]$ such that the following disclosure strategy is optimal: fully disclose the realized θ if $\theta \leq x^*$, and not (at all) disclose it if $\theta > x^*$.

This strategy is sometimes called (upper) *censoring* (Kolotilin et al. (2015)). Given this disclosure strategy, if $\theta \leq x^*$, then the conditional expected value of θ is θ itself, while if $\theta > x^*$, then it is always $x^{**} \equiv \frac{\int_{x^*}^b \theta dF_0}{1-F_0(x^*)}$. Hence, letting

$$G^*(x) = \begin{cases} 0 & \text{if} \quad x < a, \\ F_0(x) & \text{if} \quad x \in [a, x^*), \\ F_0(x^*) & \text{if} \quad x \in [x^*, \frac{\int_{x^*}^b \theta dF_0}{1-F_0(x^*)}), \\ 1 & \text{if} \quad x \geq \frac{\int_{x^*}^b \theta dF_0}{1-F_0(x^*)}, \end{cases}$$

be the cdf of the agents' posterior mean of θ, the (ex ante) expected surplus is $\int_x S^{**}(x)dG^*(x)$.

Within this class of upper-censoring disclosure strategies, it is straightforward to find the optimal one because each policy in this class is identified by a single parameter x^*. We now show that, in fact, the upper-censoring disclosure strategy with optimally chosen x^* is optimal among all (not necessarily upper-censoring) disclosure strategies.

In what follows, we provide the main idea of the proof. The formal proof is in the appendix. Fix an arbitrary disclosure strategy $(\mu, (\psi_m)_{m\in M})$. Given each message m, θ's posterior mean is given by $x = E(\theta|m) = \int_\theta \theta d\psi_m$. Let G denote the (ex ante) cdf of this x. Observing m with $E(\theta|m) = x$, it is possible that realized θ is different from x, but it must be (conditionally on x) distributed with mean x. In other words, F_0, the (ex ante) distribution of θ, is second-order stochastically dominated by G, while they have the same (ex ante) expected value.

Thus, we consider the following problem to identify the second-best policy.

$$\begin{aligned}
\sup_G \quad & V = \int_a^b S^{**}(x)dG(x) \\
\text{sub. to} \quad & G \text{ is non-decreasing, right-continuous,} \\
& G(a) = 0,\ G(b) = 1, \\
& \int_a^b xdG(x) = \int_a^b xdF_0(x)\ (\Leftrightarrow \int_a^b G(y)dy \le \int_a^b F_0(y)dy), \\
& \int_a^x G(y)dy \le \int_a^x F_0(y)dy,\ \forall x,
\end{aligned}$$

where the constraints in the first line say that G is a cdf on (a, b), the constraint in the second line says that G and F_0 have the same (ex ante) expected value, and the last constraint says that G second-order stochastically dominates F_0.

Let $H(x) = \int_a^x G(y)dy$ and $H_0(x) = \int_a^x F_0(y)dy$. Then H is convex and continuously differentiable, and H_0 is strictly convex and continuously differentiable. By integration by parts,[17]

$$V = S^{**}(b) - \int_a^b \frac{dS^{**}(x)}{dx} dH(x),$$

and hence, the same problem can be rewritten as follows.

$$\begin{aligned}
\sup_H \quad & V = S^{**}(b) - \int_a^b \frac{dS^{**}(x)}{dx} dH(x) \\
\text{sub. to} \quad & H \text{ is convex, continuously differentiable,} \\
& H'(a) = 0,\ H'(b) = 1, \\
& H(a) = 0,\ H(b) = H_0(b), \\
& H(x) \leq H_0(x),\ \forall x.
\end{aligned}$$

In what follows, we assume that S^{**} is twice continuously differentiable so that we can further rewrite

$$V = S^{**}(b) - \frac{dS^{**}(b)}{dx} H_0(b) + \int_a^b \frac{d^2 S^{**}(x)}{dx^2} H(x) dx.$$

In fact, in the current case, $\frac{dS^{**}(b)}{dx}$ is not continuously differentiable at $x = 1$, and hence the following argument is not valid. However, the main idea can still be delivered. See the appendix for the formal proof.

Fix $\hat{h} \in \mathbb{R}$, and we characterize H such that $H(\hat{x}) = \hat{h}$ and maximize $\frac{d^2 S^{**}(x)}{dx^2} H(x) dx$. By the constraints on feasible H, we must have (i) $\hat{h} \leq H_0(\hat{x})$ because $H(\hat{x}) \leq H_0(\hat{x})$, (ii) $\hat{h} \geq 0$ because $H(\hat{x}) \geq H(a) + H'(a)(\hat{x} - a) = 0$ by convexity, and (iii) $\hat{h} \geq H_0(b) - b + \hat{x}$ because $H(x) \geq H(b) - H'(b)(b - \hat{x})$ again by convexity. Thus, we fix an arbitrary $\hat{h} \in [\max\{H_0(0, H_0(b) - b + \hat{x}\}, H_0(\hat{x})]$.[18]

We now introduce x^* and x^{**}.

[17] Integration by parts can be applied here because $S^{**}(x)$ and $G(x)$ are non-decreasing and bounded.

[18] The interval is nonempty because $H_0(\hat{x}) \geq 0$ and $H_0(\hat{x}) - (H_0(b) - b + \hat{x}) = (b - \hat{x}) \left(1 - \frac{H_0(b) - H_0(\hat{x})}{b - \hat{x}}\right) \geq 0$ by convexity of H_0.

Let $x^* \in [a, \hat{x}]$ be the point such that the line that takes $\hat{h}$ at $x = \hat{x}$ and takes $H_0(x^*)$ at $x = x^*$ supports H_0 from below. Formally, define x^* so that, if $\hat{h} = H_0(\hat{x})$, then $x^* = \hat{x}$; otherwise, $x^* = \arg\max_{x\in[a,\hat{x})} \frac{\hat{h}-H_0(x)}{\hat{x}-x}$.[19]

The slope of this tangent line is $H_0'(x^*) < 1$, and thus, it crosses with another tangent line that takes $H_0(b)$ at $x = b$ and has slope one. Let $x^{**} \in [\hat{x}, b)$ be this cross point. Formally, define x^{**} as the solution x of the following equation:

$$\hat{h} + (x - \hat{x})H_0'(x^*) = H_0(b) + (x - b) \cdot 1,$$

or equivalently,

$$x^{**} = \frac{\hat{h} - H_0(b) + b - \hat{x}H_0'(x^*)}{1 - H_0'(x^*)} = \frac{\int_{x^*}^b y dF_0(y)}{1 - F_0(x^*)},$$

where the second equality is because $\hat{h} = H_0(x^*) + (\hat{x} - x^*)H_0'(x^*)$.

Consider maximization of $\int_a^b \frac{d^2S^{**}(x)}{dx^2} H(x)dx$ subject to (i) $H(x) \le H_0(x)$ for $x \in [a, x^*]$, (ii) $H(x) \le \hat{h} + (x - \hat{x})H_0'(x^*)$ for $x \in (x^*, \hat{x})$, (iii) $H(x) \ge \hat{h}+(x-\hat{x})H_0'(x^*)$ for $x \in (\hat{x}, x^{**}]$, and (iv) $H(x) \ge H_0(b)+x-b$ for $x \in (x^{**}, b]$. This is a relaxed problem of the original one in the sense that every feasible H must satisfy all the constraints (i)-(iv).[20]

Because $\frac{d^2S^{**}(x)}{dx^2} \ge 0$ for $x \in [a, \hat{x})$ and $\frac{d^2S^{**}(x)}{dx^2} \le 0$ for $x \in (\hat{x}, b]$, we can maximize $\int_a^b \frac{d^2S^{**}(x)}{dx^2} H(x)dx$ by making all the constraints (i)-(iv) binding. Therefore, the solution H^* satisfies

$$H^*(x) = \begin{cases} H_0(x) & \text{if} \quad x \in [a, x^*], \\ \hat{h} + (x - \hat{x})H_0'(x^*) & \text{if} \quad x \in (x^*, x^{**}), \\ H_0(b) + x - b & \text{if} \quad x \in (x^{**}, b]. \end{cases}$$

By differentiating H^*, we obtain the cdf of x, $G^*(x)$, as defined right after the statement of the proposition.

[19] If $\hat{h} < H_0(\hat{x})$, then $\frac{\hat{h}-H_0(x)}{\hat{x}-x} \to -\infty$ as $x \uparrow \hat{x}$, and thus, $\max_{x\in[a,hatx)} \frac{\hat{h}-H_0(x)}{\hat{x}-x}$ is well-defined. The uniqueness of the maximizer is because H_0 is strictly convex.

[20] (i) is trivial. (ii) and (iii) are implied by convexity of H and $H(\hat{x}) = \hat{h}$. (iv) is implied by convexity of H, $H(b) = H_0(b)$ and $H'(b) = 1$.

Remark 3. Although our formal result focuses on the case where S^{**} has a unique inflection point $\hat{x}$, as suggested by the argument above, it is conceptually straightforward to extend our result to arbitrary S^{**} which are twice continuously differentiable and have finitely many inflection points (e.g., when S^{**} is a polynomial function). In such a case, we first fix $H(\hat{x}_k)$ for each inflection point $\hat{x}_k$, apply the argument above to reduce the original infinite-dimensional problem to a finite-dimensional one, and then apply standard techniques such as first-order conditions.

5 Conclusion

This paper considers mechanism design environments where a principal has some information relevant to agents. As opposed to the standard "informed principal" approach with no commitment as to the principal's information disclosure, we consider fully committed public information disclosure by the principal, implying more tractability and hence wider applicability. In linear environments with no restriction on monetary transfers (e.g., auction of Myerson (1981)), we show that the principal finds it optimal to fully disclose his information. With a budget-balance restriction on monetary transfers (e.g., bilateral trading of Myerson and Satterthwaite (1983)), full disclosure may be suboptimal. In a bilateral-trading environment with uniformly distributed types, I characterize the second-best information disclosure policy, which is a simple censoring policy. The technique developed for this second-best characterization may be useful to tractably analyze general Bayesian persuasion problems with continuous state spaces.

A Proof of Lemma 2

The Lagrangian of the problem of $S^{**}(x)$ is

$$\int_v (v_2 + x - v_1 + \lambda[v_2 + x - v_1 - \frac{1 - F_2(v_2)}{f_2(v_2)} - \frac{F_1(v_1)}{f_1(v_1)}])q(v)dF_V.$$

Given any $\lambda \geq 0$, the pointwise maximization of the Lagrangian yields $q(v) = 1$ if

$$(v_2 + x - v_1)(1 + \lambda) \geq (\frac{1 - F_2(v_2)}{f_2(v_2)} + \frac{F_1(v_1)}{f_1(v_1)})\lambda,$$

and $q(v) = 0$ otherwise.

Note that such q satisfies the monotonicity (i.e., nonincreasing in v_1 and nondecreasing in v_2) because of the monotone hazard rate assumption. Therefore, the optimal q is determined by identifying the smallest λ such that the constraint is satisfied, which we denote by $\lambda(x)$. As shown by Myerson and Satterthwaite (1983), $\lambda(x) > 0$ for $x \in (-1, 1)$. Obviously, λ is decreasing in x.

Lemma 4. $\lim_{x \uparrow 1} \lambda(x) = 0$.

Proof. (of the lemma) Suppose not, and let $\lambda^* = \lim_{x \uparrow 1} \lambda(x) > 0$. Then $\lim_{x \uparrow 1} S^{**}(x) < S^{**}(1)$.

Consider a (monotone) trading rule such that $q(v) = 1$ if and only if

$$(v_2 + x - v_1)(1 + \lambda) \geq (\frac{1 - F_2(v_2)}{f_2(v_2)} + \frac{F_1(v_1)}{f_1(v_1)})\lambda,$$

for $\lambda = \frac{\lambda^*}{2}$. By continuity of the budget surplus, $v_2 + x - v_1 - \frac{1-F_2(v_2)}{f_2(v_2)} - \frac{F_1(v_1)}{f_1(v_1)}$, there exists $\overline{x} < 1$ such that this trading rule (with $x = \overline{x}$) does not violate the budget-balance constraint. Because $\lambda(\overline{x}) \geq \lambda^* > \frac{\lambda^*}{2}$, $S^{**}(\overline{x})$ is strictly less than the expected surplus given this trading rule (with $x = \overline{x}$), which is a contradiction. $\square$

Let $\delta(x)$ be the supremum value of v_1 such that $q(v_1, 0) = 1$ given $\lambda(x)$, and $\beta(x)$ be the supremum value of v_1 such that $q(v_1, 1) = 1$ given $\lambda(x)$. The lemma implies that there exists $\varepsilon_1 > 0$ such that $\delta(x) > 0$ and $\beta(x) = 1$ for any $x \in (1 - \varepsilon_1, 1)$. In the following, x is always taken in this range.

For each v_1, let $\alpha(v_1, x)$ be the infimum value of v_2 such that $q(v_1, v_2) = 1$. By setting

$$\begin{aligned}\underline{\alpha}(v_1, x) &= v_1 - x + \frac{\lambda(x)}{1+\lambda(x)} \frac{F_1(\delta(1-\varepsilon_1))}{d_2}, \\ \overline{\alpha}(v_1, x) &= v_1 - x + \frac{\lambda(x)}{1+\lambda(x)} \frac{2}{d_1},\end{aligned}$$

we have

$$\begin{aligned}\alpha(v_1, x) &= v_1 - x + \frac{\lambda(x)}{1+\lambda(x)} \left(\frac{F_1(v_1)}{f_1(v_1)} + \frac{1 - F_2(\alpha(v_1, x))}{f_2(\alpha(v_1, x))}\right) \\ &\in [\underline{\alpha}(v_1, x), \overline{\alpha}(v_1, x)].\end{aligned}$$

Similarly, by setting

$$\begin{aligned}\underline{\delta}(x) &= x - \frac{\lambda(x)}{1+\lambda(x)} \frac{2}{d_1}, \\ \overline{\delta}(x) &= x - \frac{\lambda(x)}{1+\lambda(x)} \frac{F_1(\delta(1-\varepsilon_1))}{d_2},\end{aligned}$$

we have

$$\begin{aligned}\delta(x) &= x - \frac{\lambda(x)}{1+\lambda(x)} \left(\frac{F_1(\delta(x))}{f_1(\delta(x))} + \frac{1}{f_2(0)}\right) \\ &\in [\underline{\delta}(x), \overline{\delta}(x)].\end{aligned}$$

For the constraint, let $B(v, x) = v_2 + x - v_1 - \frac{F_1(v_1)}{f_1(v_1)} - \frac{1-F_2(v_2)}{f_2(v_2)}$. Let x be sufficiently small so that $B(v, x) < 0$ at $v = (\underline{\delta}(x), \overline{\alpha}(1, x))$. This is possible because B is continuous in (v,x) and $B((0,1),1) < 0$. Then,

$$\begin{aligned}
0 &\leq \int_{\delta(x)}^{1}\int_{\alpha(v_1,x)}^{1} B(v,x)dF + \int_{0}^{\delta(x)}\int_{0}^{1} B(v,x)dF \\
&= x - 1 + \int_{\delta(x)}^{1}\int_{0}^{\alpha(v_1,x)} (-B(v,x))dF \\
&\leq x - 1 + \int_{\delta(x)}^{1}\int_{0}^{\alpha(v_1,x)} (-B((1,0),x))dF \\
&\leq x - 1 + \int_{\underline{\delta}(x)}^{1}\int_{0}^{\overline{\alpha}(v_1,x)} (-x + 1 + \frac{2}{d_1}))d_2^2 dv \\
&\leq x - 1 + (1 - x + \frac{\lambda(x)}{1+\lambda(x)}\frac{2}{d_1})^2(1 - x + \frac{2}{d_1}))d_2^2,
\end{aligned}$$

where the equality is because $\int_0^1\int_0^1 B(v,1)dF = 0$, the second inequality is because $-B$ is increasing in $(v_1, -v_2)$, and the last inequality is because B is negative for all $v \geq (\underline{\delta}(x), \overline{\alpha}(1,x))$.

Let $y = \sqrt{1-x} \in (0, \sqrt{\varepsilon_1})$. The inequality above implies that

$$\frac{\lambda(x)}{1+\lambda(x)} \geq \frac{d_1 y}{2d_2}\sqrt{\frac{d_1}{2+d_1y^2}} - \frac{d_1y^2}{2},$$

which further implies that

$$\begin{aligned}
\overline{\delta} &\leq 1 - y^2 - \frac{F_1(\delta(1-\varepsilon_1))}{d_2}(\frac{d_1 y}{2d_2}\sqrt{\frac{d_1}{2+d_1y^2}} - \frac{d_1y^2}{2}) \\
&\leq 1 - \frac{F_1(\delta(1-\varepsilon_1))d_1}{2d_2^2}\sqrt{\frac{d_1}{2+d_1\varepsilon_1}}y \\
&= 1 - \phi y, \\
\underline{\alpha}(v_1,x) &\geq v_1 - 1 + \phi y
\end{aligned}$$

where $\phi \equiv \frac{F_1(\delta(1-\varepsilon_1))d_1}{2d_2^2}\sqrt{\frac{d_1}{2+d_1\varepsilon_1}}$. In the following, y is taken so that $y \leq \phi$ (which is satisfied if x is taken sufficiently close to one), or equivalently, $1 - \phi y \leq 1 - y^2 = x$.

Now,

$$\begin{aligned} E(v_2+x-v_1)-S^{**}(x) &\geq \int_{\overline{\delta}(x)}^{1}\int_{0}^{\underline{\alpha}(v_1,x)}(v_2+1-y^2-v_1)dF \\ &\geq \int_{1-\phi y}^{1}\int_{0}^{v_1-1+\phi y}(v_2+1-y^2-v_1)dF, \end{aligned}$$

where we denote the last term (on RHS) by $h(y)$. It is strictly positive if y is sufficiently close to but greater than 0 (or equivalently, if x is sufficiently close to but smaller than 1), because by Taylor expansion,

$$\begin{aligned} h(y) &= h(0)+h'(0)y+\frac{1}{2}h''(0)y^2+\frac{1}{6}h'''(0)y^3+o(y^3) \\ &= \frac{1}{6}\phi^3 f_1(1)f_2(0)y^3+o(y^3). \end{aligned}$$

By continuity of h, there exists $\overline{y}>0$ such that, for any $y\in(0,\overline{y})$, $h(y)>0$. Therefore, we complete the proof by letting $\overline{\varepsilon}=\overline{y}^2$ and $z(x)=\frac{h(1-y^2)}{2}$.

B Proof of Lemma 3

Given that v follows an independent uniform distribution on $[0,1]$, the second-best trading rule satisfies $q(v)=1$ if

$$\begin{aligned} & (v_2+x-v_1)(1+\lambda(x)) \geq (1-v_2+v_1)\lambda(x) \\ \Leftrightarrow\quad & v_2+x-v_1 \geq \frac{(1+x)\lambda(x)}{1+2\lambda(x)} \equiv \eta(x), \end{aligned}$$

where $\lambda(x)$ is the Lagrange multiplier of the problem of $S^{**}(x)$, and $q(v)=0$ otherwise. The first-best efficiency is achieved if and only if $\eta(x)=0$, and the size of $\eta(x)$ represents inefficiency of the second-best trading.

We first characterize $\eta(x)$ for each $x\in\mathbb{R}$. By Myerson and Satterthwaite (1983), $\eta(x)=0$ if and only if $x\leq-1$ or $x\geq 1$ (called the “gap” cases), where the first-best efficiency is achieved by a simple posted-price mechanism, implying $S^{**}(x)=0$ for $x\leq-1$ and $S^{**}(x)=E[v_2+x-v_1]=x$ for $x\geq 1$.

Thus, in the following, we consider the other case with $x \in (-1, 1)$, where $\eta(x)$ must be strictly positive.

Recall that $\lambda(x)$ (and hence $\eta(x)$) is determined to satisfy the budget-balance constraint with equality. Given arbitrarily given $\eta > 0$, consider a trading rule such that $q(v) = 1$ if $v_2 + x - v_1 \geq \eta$, and $q(v) = 0$ otherwise. The budget surplus is then

$$\begin{aligned} B(x, \eta) &= \int_v (2v_2 - 2v_1 + x - 1) q(v) dF(v) \\ &= \begin{cases} 0 & \text{if } \eta \geq 1 + x, \\ (1 + x - \eta)^2 (\frac{1+x-\eta}{3} - \frac{1+x-2\eta}{2}) & \text{if } \eta \in [x, 1 + x), \\ x - 1 + \frac{(1-x+\eta)^2(5+x-4\eta)}{6} & \text{if } \eta \in (0, x). \end{cases} \end{aligned}$$

With respect to η, B is continuously differentiable and single-crossing. We have

$$\eta(x) \gtreqless x \Leftrightarrow B(x, x) \lesseqgtr 0 \Leftrightarrow x \lesseqgtr \frac{1}{3}.$$

Case (I): $x \in (-1, \frac{1}{3})$.

In this case, we have

$$(1 + x - \eta(x))^2 (\frac{1 + x - \eta(x)}{3} - \frac{1 + x - 2\eta(x)}{2}) = 0,$$

and hence, $\eta(x) = \frac{1+x}{4}$.

We obtain $S^{**}(x) = \frac{9(1+x)^3}{64}$, and therefore, $S^{**}(x)$ is convex in this region. $\frac{dS^{**}(x)}{dx}$ and $\frac{d^2 S^{**}(x)}{dx^2}$ exist and continuous in this region, and moreover, $\lim_{x \downarrow -1} \frac{dS^{**}(x)}{dx} = \lim_{x \downarrow -1} \frac{d^2 S^{**}(x)}{dx^2} = 0$, and hence, $\frac{dS^{**}(x)}{dx}$ and $\frac{d^2 S^{**}(x)}{dx^2}$ are continuous at $x = 0$ too. At the other extreme point, $\lim_{x \uparrow \frac{1}{3}} \frac{dS^{**}(x)}{dx} = \frac{3}{4}$ and $\lim_{x \uparrow \frac{1}{3}} \frac{d^2 S^{**}(x)}{dx^2} = \frac{9}{8}$.

Case (II): $x \in (\frac{1}{3}, 1)$.

In this case, we have

$$x - 1 + \frac{(1 - x + \eta(x))^2 (5 + x - 4\eta(x))}{6} = 0,$$

and hence, we do not have a closed-form expression for $\eta(x)$.

To simplify the expression, let $z(x) = 1 - x + \eta(x) \in (0, 1)$. Then the budget-balance condition becomes

$$x = \frac{9z(x)^2 - 4z(x)^3 - 6}{3z(x)^2 - 6}.$$

By the implicit function theorem, we have

$$\begin{aligned}
z'(x) &= \frac{-3(2 - z(x)^2)^2}{4(z(x)^4 - 6z(x)^2 + 6z(x))} < 0, \\
z''(x) &= \frac{-9(2 - z(x)^2)^3}{8(z(x)^4 - 6z(x)^2 + 6z(x))^3}(-2z(x)^3 + 9z(x)^2 - 12z(x) + 6) < 0, \\
z'''(x) &= \frac{-27(2 - z(x)^2)^4}{32(z(x)^4 - 6z(x)^2 + 6z(x))^5} 6(1 - z(x))^2 (z(x)^4 (2 - z(x))^2 + 24(1 - z(x))^2 + 12) < 0.
\end{aligned}$$

The expected social surplus is

$$\begin{aligned}
S^{**}(x) &= \int_{v_1=0}^{1} \int_{v_2=0}^{1} v_2 + x - v_1 dv - \int_{v_1=1-z(x)}^{1} \int_{v_2=0}^{v_1-(1-z(x))} v_2 + x - v_1 dv \\
&= x - \frac{z(x)^2(z(x) + x - 1)}{2} + \frac{z(x)^3}{6},
\end{aligned}$$

and thus, under the budget-balance condition,

$$\begin{aligned}
S^{**}(x) &= \frac{9z(x)^2 - 4z(x)^3 - 6}{3z(x)^2 - 6} - \frac{z(x)^2(z(x) + x - 1)}{2} + \frac{z(x)^3}{6} \\
&= \frac{z(x)^3}{6} - z(x)^2 + 1.
\end{aligned}$$

To examine the shape of S^{**}, let $T(z) = \frac{z^3}{6} - z^2 + 1$ for $z \in (0, 1)$. Then, $T'(z) = z^2 - 2z$, $T''(z) = 2z - 2$, $T'''(z) = 2$, and

$$\begin{aligned}
\frac{dS^{**}(x)}{dx} &= T'(z(x))z'(x), \\
\frac{d^2 S^{**}(x)}{dx^2} &= T''(z(x))(z'(x))^2 + T'(z(x))z''(x), \\
\frac{d^3 S^{**}(x)}{dx^3} &= T'''(z(x))(z'(x))^3 + 3T''(z(x))z'(x)z''(x) + T'(z(x))z'''(x).
\end{aligned}$$

Hence, $\lim_{x \downarrow \frac{1}{3}} \frac{dS^{**}(x)}{dx} = \frac{3}{4}$, $\lim_{x \uparrow 1} \frac{dS^{**}(x)}{dx} = 1$, $\lim_{x \downarrow \frac{1}{3}} \frac{d^2S^{**}(x)}{dx^2} = \frac{9}{8}$, and $\lim_{x \uparrow 1} \frac{d^2S^{**}(x)}{dx^2} = -\infty$. Therefore, $\frac{dS^{**}(x)}{dx}$ exists and continuous for $x \in [\frac{1}{3}, 1]$. $\frac{dS^{**}(x)}{dx}$ exists and continuous for $x \in [\frac{1}{3}, 1)$, but $\frac{d^2S^{**}(1-0)}{dx^2} = -\infty \neq 0 = \frac{d^2S^{**}(1+0)}{dx^2}$.

Finally, observe that

$$
\begin{aligned}
\frac{d^3S^{**}(x)}{dx^3} &= T'''(z(x))(z'(x))^3 + 3T''(z(x))z'(x)z''(x) + T'(z(x))z'''(x) \\
&= \frac{-27(2 - z(x)^2)^4 z(x)^2}{32(z(x)^4 - 6z(x)^2 + 6z(x))^5}[1 + \\
&\quad (1 - z(x))(71(1 - 2z(x))^2 + 67z(x) + 579z(x)^2(1 - z(x))^2 \\
&\quad + z(x)^3(401 - 217z(x)^2 - 9z(x)^3 - z(x)^5 - z(x)^6) + 44z(x)^4 + 21z(x)^7)] \\
&< 0,
\end{aligned}
$$

which means that there exists a unique $\hat{x} \in (\frac{1}{3}, 1)$ such that

$$\frac{d^2S^{**}(x)}{dx^2} \lesseqgtr 0 \Leftrightarrow x \gtreqless \hat{x}.$$

C Proof of Proposition 2

As discussed before, if $\frac{dS^{**}(x)}{dx}$ is continuously differentiable, then we can apply integration by parts one more time and then prove the proposition. However, $\frac{dS^{**}(x)}{dx}$ is not continuously differentiable at $x = 1$. Therefore, we first consider a function that is twice continuously differentiable and approximate S^{**}.

Fix arbitrary $\varepsilon \in (0, 1 - \hat{x})$. By definition of $\hat{x}$, $\frac{d^2S^{**}(x)}{dx^2} < 0$ for $x \in (\hat{x}, 1)$ and hence $\frac{dS^{**}(x)}{dx}$ is decreasing for $x \in (\hat{x}, 1)$. Let $v_\varepsilon : (a, b) \to \mathbb{R}$ be a twice-continuously-differentiable function such that its second derivative $v''_\varepsilon(x)$ is continuous and satisfies

$$v''_\varepsilon \begin{cases} = \frac{d^2S^{**}(x)}{dx^2} & \text{if} \quad x \notin (1 - \varepsilon, 1), \\ \in (\frac{d^2S^{**}(x)}{dx^2}, 0] & \text{if} \quad x \in (1 - \varepsilon, 1), \end{cases}$$

which is possible by Tietze extension theorem. The first derivative satisfies

$$\begin{aligned} v'_\varepsilon(x) &= \frac{dS^{**}(1-\varepsilon)}{dx} + \int_{1-\varepsilon}^{x} v''_\varepsilon(y)dy \\ &\leq \frac{dS^{**}(1-\varepsilon)}{dx}, \end{aligned}$$

and therefore, $v'_\varepsilon(x) - \frac{dS^{**}(x)}{dx} \in [0, \frac{dS^{**}(1-\varepsilon)}{dx} - \frac{dS^{**}(0)}{dx}]$ and this difference is nondecreasing.

Then we have[21]

$$\begin{aligned} V &= S^{**}(b) + \int_a^b (v'_\varepsilon(x) - \frac{dS^{**}(x)}{dx})dH(x) - \int_a^b v'_\varepsilon(x)dH(x) \\ &= S^{**}(b) + \int_a^b (v'_\varepsilon(x) - \frac{dS^{**}(x)}{dx})dH(x) - v'_\varepsilon(b)H(b) + \int_a^b v''_\varepsilon(x)H(x)dx. \end{aligned}$$

Although H appears both in the second and fourth terms in the last expression, we first consider H that maximizes the fourth term, $\int_a^b v''_\varepsilon(x)H(x)dx$, and then verify later that the same H maximizes the second term.

Fix $\hat{h} \in \mathbb{R}$, and we characterize the optimal H such that $H(\hat{x}) = \hat{h}$. By the constraints on feasible H, we must have (i) $\hat{h} \leq H_0(\hat{x})$ because $H(\hat{x}) \leq H_0(\hat{x})$, (ii) $\hat{h} \geq 0$ because $H(\hat{x}) \geq H(a) + H'(a)(\hat{x} - a) = 0$ by convexity, and (iii) $\hat{h} \geq H_0(b) - b + \hat{x}$ because $H(x) \geq H(b) - H'(b)(b - \hat{x})$ again by convexity. Thus, we fix an arbitrary $\hat{h} \in [\max\{H_0(0, H_0(b) - b + \hat{x}\}, H_0(\hat{x})]$.[22]

We now introduce x^* and x^{**}.

Let $x^* \in [a, \hat{x}]$ be the point such that the line that takes $\hat{h}$ at $x = \hat{x}$ and takes $H_0(x^*)$ at $x = x^*$ supports H_0 from below. Formally, define x^* so that, if $\hat{h} = H_0(\hat{x})$, then $x^* = \hat{x}$; otherwise, $x^* = \arg\max_{x \in [a,\hat{x})} \frac{\hat{h} - H_0(x)}{\hat{x} - x}$.[23]

The slope of this tangent line is $H'_0(x^*) < 1$, and thus, it crosses with another tangent line that takes $H_0(b)$ at $x = b$ and has slope one. Let

[21]Integration by parts can be applied here because v'_ε and H are continuously differentiable.

[22]The interval is nonempty because $H_0(\hat{x}) \geq 0$ and $H_0(\hat{x}) - (H_0(b) - b + \hat{x}) = (b - \hat{x})\left(1 - \frac{H_0(b) - H_0(\hat{x})}{b - \hat{x}}\right) \geq 0$ by convexity of H_0.

[23]If $\hat{h} < H_0(\hat{x})$, then $\frac{\hat{h} - H_0(x)}{\hat{x} - x} \to -\infty$ as $x \uparrow \hat{x}$, and thus, $\max_{x \in [a, hatx)} \frac{\hat{h} - H_0(x)}{\hat{x} - x}$ is well-defined. The uniqueness of the maximizer is because H_0 is strictly convex.

$x^{**} \in [\hat{x}, b)$ be this cross point. Formally, define x^{**} as the solution x of the following equation:

$$\hat{h} + (x - \hat{x})H_0'(x^*) = H_0(b) + (x - b) \cdot 1,$$

or equivalently,

$$x^{**} = \frac{\hat{h} - H_0(b) + b - \hat{x}H_0'(x^*)}{1 - H_0'(x^*)} = \frac{\int_{x^*}^{b} y dF_0(y)}{1 - F_0(x^*)},$$

where the second equality is because $\hat{h} = H_0(x^*) + (\hat{x} - x^*)H_0'(x^*)$.

Consider the "relaxed" problem where we maximize the fourth term of V, $\int_a^b v_\varepsilon''(x)H(x)dx$, subject to (i) $H(x) \leq H_0(x)$ for $x \in [a, x^*]$, (ii) $H(x) \leq \hat{h} + (x - \hat{x})H_0'(x^*)$ for $x \in (x^*, \hat{x})$, (iii) $H(x) \geq \hat{h} + (x - \hat{x})H_0'(x^*)$ for $x \in (\hat{x}, x^{**}]$, and (iv) $H(x) \geq H_0(b) + x - b$ for $x \in (x^{**}, b]$. This is a relaxed problem in the sense that every feasible H must satisfy all the constraints (i)-(iv). (i) is trivial. If (ii) is violated (with $x^* < \hat{x}$), then, letting $\alpha \in (0, 1)$ be such that $x = \alpha x^* + (1 - \alpha)\hat{x}$,

$$\begin{aligned} H(x) &> \hat{h} + (x - \hat{x})\frac{\hat{h} - H_0(x^*)}{\hat{x} - x^*} \\ &= (1 - \alpha)\hat{h} + \alpha H_0(x^*) \\ &\geq (1 - \alpha)\hat{h} + \alpha H(x^*), \end{aligned}$$

and hence H is not convex. If (iii) is violated, then, in case $x^* < \hat{x}$,

$$\begin{aligned} H(x) &< \hat{h} + (x - \hat{x})\frac{\hat{h} - H_0(x^*)}{\hat{x} - x^*} \\ \Leftrightarrow \frac{H(x) - \hat{h}}{x - \hat{x}} &< \frac{\hat{h} - H_0(x^*)}{\hat{x} - x^*}, \end{aligned}$$

which violates convexity of H, and in case $x^* = \hat{x}$ (which implies $\hat{h} = H_0(\hat{x})$ and $H'(\hat{x}) = H_0'(\hat{x})$),

$$\begin{aligned} H(x) &< \hat{h} + (x - \hat{x})H_0'(\hat{x}) \\ \Leftrightarrow \frac{H(x) - \hat{h}}{x - \hat{x}} &< H'(\hat{x}), \end{aligned}$$

which again violates convexity of H. If (iv) is violated,

$$\begin{aligned} H(x) &< H_0(b) + x - b \\ \Leftrightarrow \frac{H(b) - H(x)}{b - x} &> 1 = H'(b), \end{aligned}$$

which violates convexity of H.

Because $v''_\varepsilon(x) \geq 0$ for $x \in [a, \hat{x})$ and $v''_\varepsilon(x) \leq 0$ for $x \in (\hat{x}, b]$, we can maximize $\int_a^b v''_\varepsilon(x) H(x) dx$ by making all the constraints (i)-(iv) binding:

$$H^*(x) = \begin{cases} H_0(x) & \text{if} \quad x \in [a, x^*], \\ \hat{h} + (x - \hat{x}) H'_0(x^*) & \text{if} \quad x \in (x^*, x^{**}), \\ H_0(b) + x - b & \text{if} \quad x \in (x^{**}, b]. \end{cases}$$

By differentiating H^*, we obtain the cdf of x, $G^*(x)$, as in the statement of the proposition (recall $x^{**} = \frac{\int_{x^*}^b \theta dF_0}{1 - F_0(x^*)}$):

$$G^*(x) = \begin{cases} 0 & \text{if} \quad x < a, \\ F_0(x) & \text{if} \quad x \in [a, x^*), \\ F_0(x^*) & \text{if} \quad x \in [x^*, x^{**}), \\ 1 & \text{if} \quad x \geq x^{**}. \end{cases}$$

We complete the proof of the proposition by showing that H^* further maximizes the second term of V, $\int_a^b (v'_\varepsilon(x) - \frac{dS^{**}(x)}{dx}) dH(x)$, among all feasible H with $H(\hat{x}) = \hat{h}$. Recall that $v'_\varepsilon(x) - \frac{dS^{**}(x)}{dx}$ takes a strictly positive value only for $x \in (1 - \varepsilon, 1) \subseteq (\hat{x}, 1)$, and is nondecreasing. Therefore, it suffices to show that, for any feasible H with $H(\hat{x}) = \hat{h}$, (i) $H^*(x) \leq H(x)$ for any $x \in (\hat{x}, 1)$, and (ii) $\int_{\hat{x}}^b dH^*(x) = \int_{\hat{x}}^b dH(x)$, i.e., H^* "first-order stochastically dominates" H (treating H^* and H as finite measures of x over the interval $(\hat{x}, b)$). Indeed, (ii) is always satisfied because $\int_{\hat{x}}^b dH^*(x) = H_0(b) - \hat{h} = \int_{\hat{x}}^b dH(x)$. For (i), recall that any feasible H must satisfy

$$H(x) \geq \begin{cases} \hat{h} + (x - \hat{x}) H'_0(x^*) & \text{if} \quad x \in (\hat{x}, x^{**}), \\ H_0(b) + x - b & \text{if} \quad x \in (x^{**}, b], \end{cases}$$

but all these inequalities are binding under H^*. Therefore, $H^*(x) \leq H(x)$ for any $x \in (\hat{x}, 1)$.

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