Inclusive Collusion Neutrality on Networks

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Abstract

In the context of cooperative games with transferable utility, an inclusive collusion grants each colluding player access to resources of all colluding players and therefore transforms a given game. Inclusive collusion neutrality requires that no group of players can change their total payoff with an inclusive collusion. Assuming that collusion formation is governed by a network defined over players, we show that if the network is cyclic, no solution satisfies inclusive collusion neutrality, efficiency, and the null-player property. Tree (acyclic) networks allow us to escape the impossibility: affine combinations of the hierarchical solutions satisfy the three axioms. Further, we establish that the latter family of solutions are characterized by the three axioms and linearity.

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1 Introduction

Collusion is a common way of manipulating the system and promoting the interest of colluding agents. Collusion takes various forms. In industries, for example, firms in a cartel can limit output and raise prevailing prices (Stigler, 1964); alternatively, they may tacitly under-invest in product quality to increase their profits (Nocke, 2007). In auctions, bidders may form a ring and fix bids, so that all conspirators gain (Comanor and Schankerman, 1976; Graham and Marshall, 1987; McAfee and McMillan, 1992). In matching and social choice problems, some participants may jointly misrepresent their preferences to obtain a better outcome (Dasgupta et al., 1979; Green and Laffont, 1979; Dubins and Freedman, 1981). When bargaining, negotiators often resort to collective actions (Harsanyi, 1977).

The prevalence of collusion necessitates designing social institutions that are immune to collusive behavior. We search for solutions with the latter immunity in the context of cooperative games with transferable utility (henceforth, TU games). Specifically, we consider "inclusive collusions" (Haller, 1994)[1], namely those collusive agreements under which each colluding player has unfettered access to resources of all colluding players. An inclusive collusion transforms a given TU game into a new game by distorting worths of coalitions as follows: if a coalition contains any member of the collusion, then its worth equals the worth generated by the coalition and all colluding players; otherwise, the collusion plays no role. Inclusive collusion neutrality says that no group of players can change their total payoff with an inclusive collusion. In the presence of some other basic properties of solutions, inclusive collusion neutrality may turn out to be very demanding depending on the feasibility constraint on collusions and there may not exist solutions satisfying it. We provide a necessary and sufficient condition for the existence. Further, we characterize the family of solutions meeting the requirement when they exist.

The existence of solutions satisfying inclusive collusion neutrality depends crucially on feasibility of collusions, i.e., which group of players can enter into collusive agreements. In the spirit of Myerson (1977), we represent players' ability to form collusions by a network defined over players. A coalition of players can collude if and only if they

[1] Haller (1994) uses the term "association".

are all connected in the network. The case where any coalition can collude corresponds to a "complete" network. We show that under cyclic networks, no solution satisfies inclusive collusion neutrality, efficiency[2], and the null-player property[3] (Theorem 1).

If a (connected) network is acyclic, that is, it is a tree,[4] then the three axioms are compatible. We establish that the hierarchical solutions, due to Demange (2004), satisfy them and so do their affine combinations (Theorem 2). Once player i is chosen as the root of a tree network, the i-hierarchical solution is defined as follows. Given a game, player j's payoff is the worth created by j and all his successors (in the tree network with root i) *minus* the sum of worths, each created by a maximal subtree coalition of j's successors. Thus, the i-hierarchical solution assigns to each subtree coalition a total payoff that precisely equals the worth it can create separately. The hierarchical solutions are also studied in Herings et al. (2008), Béal et al. (2010), Mishra and Talman (2010), van den Brink (2012), and Park and Ju (2015).

Given that tree networks allow us to escape the impossibility, a natural question is whether we can characterize all solutions satisfying inclusive collusion neutrality, efficiency, and the null-player property. To answer, we first examine their behavior on the smaller domain of unanimity games. We find that if a solution satisfies the three axioms, it is an affine combination of the hierarchical solutions when restricted to the unanimity games (Theorem 3). The domain of unanimity games is a maximal domain on which the three axioms have such an implication (Remark 1). As far as non-unanimity games are concerned, a solution need not be an affine combination of the hierarchical solutions. However, such solutions are ruled out once linearity is imposed and we obtain a characterization: a solution satisfies inclusive collusion neutrality, efficiency, the null-player property, and linearity if and only if it is an affine combination of the hierarchical solutions (Theorem 4).

While the notion of inclusive collusion is old, it appears that no paper addresses the general existence issue on networks as we do. Haller (1994) restricts attention to the majority voting games and study when an inclusive collusion by two players increases,

[2]Efficiency requires that the worth of the grand coalition be fully allocated to all players.

[3]The null-player property requires that each null player (who contributes nothing to any coalition) should receive a zero payoff.

[4]For simplicity, we only consider connected networks for which acyclicity is equivalent to being a tree. However, our results easily extend to non-connected networks.

decreases, or does not change the sum of their Shapley values.[5] His findings translate to a sufficient and necessary condition under which the Shapley value satisfies (pairwise) inclusive collusion neutrality on the domain of majority voting games. Malawski (2002) characterize the Banzhaf value by inclusive collusion neutrality (or exclusive collusion neutrality, which is defined below), equal treatment of equals, the dummy-player property, and marginal contributions.[6] Segal (2003) also consider inclusive collusion neutrality but his focus is on the conditions on games that determine whether an inclusive collusion is profitable regardless of which random-order value, including the Shapley value, is applied.[7]

On the other hand, similar investigations to ours are available for a different type of colluding, which we call "exclusive collusion": van den Brink (2012) and Park and Ju (2015). An exclusive collusion gives each colluding player the right to exclude all colluding players from a coalition unless all of them are present (Malawski, 2002).[8] Both inclusive and exclusive collusions bind a group of players as one unit but they differ in how players attempt to gain leverage in their bargaining positions: in an inclusive collusion, each colluding player can force inclusion (participation) of the entire collusion group; in an exclusive collusion, he can force exclusion. Exclusive collusions, too, turn a game into a new game: if a coalition does not contain any of the colluding players, its worth is calculated as if the whole collusion group is absent from the coalition; otherwise, the collusion does not change the worth of a coalition. Exclusive collusion neutrality requires that no group of players can affect their total payoff by forming an exclusive collusion.

With no restrictions on feasibility of collusions (i.e., complete networks), no solution satisfies exclusive collusion neutrality, efficiency, and the null-player property (van den Brink, 2012). In fact, the impossibility remains valid unless the network is the most trivial type: lines (Park and Ju, 2015). Yet a wider class of networks permits a possibility if we weaken the null-player property to the very-null-player property, which requires

[5]Haller (1994) considers another type of collusion, "proxy agreements", which is different from both inclusive and exclusive collusions.

[6]In Malawski (2002), the third characterization of the Banzhaf value involves "representation neutrality", which requires that a solution be neutral to proxy agreements considered by Haller (1994).

[7]Segal (2003) explores a similar question for exclusive collusion (to be introduced below).

[8]Malawski (2002) calls such collusion "distrust".

giving zero payoffs only to those null players who do not connect non-null players. On tree networks, the affine combinations of the hierarchical solutions are characterized by exclusive collusion neutrality, efficiency, the very-null-player property, and linearity (Park and Ju, 2015).

To compare our results with van den Brink (2012) and Park and Ju (2015), if inclusive collusion neutrality is imposed, a positive result obtains if and only if a network is a tree. In particular, for non-line tree networks, inclusive collusion neutrality, efficiency, and the null-player property are compatible, in contrast with Park and Ju (2015). Another difference is that we impose the null-player property in our characterization whereas Park and Ju (2015) impose the very-null-player property in theirs.[9]

The rest of the paper proceeds as follows. In Section 2, we set up the model and introduce our axiom. In Section 3, we present our impossibility and characterization results.

2 The Model and Axioms

Let $\boldsymbol{N} \equiv \{1, 2, \cdots, n\}$ be a finite set of players. A ***coalition*** is a non-empty subset of N. We sometimes call N the ***grand coalition***. A cooperative game with transferable utility, or a ***game***, is defined by a characteristic function $v : 2^N \to \mathbb{R}$, with $v(\emptyset) = 0$, which associates with each coalition $S \subseteq N$ the worth $v(S)$ generated by S. The unanimity games are examples. For each coalition $T \subseteq N$, in the ***$\boldsymbol{T}$-unanimity game***, denoted by $\boldsymbol{u^T}$, a coalition generates worth 1 if it contains T and 0 otherwise; i.e., for each coalition $S \subseteq N$, if $S \supseteq T$, $u^T(S) = 1$ and otherwise, $u^T(S) = 0$. Let $\boldsymbol{\mathcal{V}}$ be the set of all games. A ***payoff profile*** $x \equiv (x_i)_{i \in N} \in \mathbb{R}^N$ specifies, for each $i \in N$, player i's payoff x_i. A ***solution*** $f : \mathcal{V} \to \mathbb{R}^N$ associates with each game $v \in \mathcal{V}$ a payoff profile $f(v) \in \mathbb{R}^N$. For each $i \in N$, let $\boldsymbol{f_i(v)}$ be player i's payoff in $f(v)$. For each $S \subseteq N$, let $\boldsymbol{f_S(v)} \equiv \sum_{i \in S} f_i(v)$ and $\boldsymbol{x_S} \equiv \sum_{i \in S} x_i$.

We search for solutions that satisfy several desiderata—axioms. Our main axiom concerns collusive behavior of players. Players may enter into a collusive agreement that permits use of resources of all colluding players only in some pre-specified ways.

[9]Note that imposing the null-player property in Park and Ju (2015) leads to an impossibility unless the tree network is a line.

For instance, a collusion can be ***inclusive*** in that a colluding player can always use resources of all colluding players (Haller, 1994). The collusion then transforms a given game as follows: if a coalition contains any colluding player, its worth equals the worth generated by the coalition *and* all of the collusion members. Alternatively, a collusion can be ***exclusive*** in that a colluding player can use his resources only if all colluding players are present (Malawski, 2002). This type of collusion also transforms a game: each colluding player contributes nothing to a coalition unless all colluding players belong to the coalition. A natural question is which solutions are "neutral" to collusions (that is, for which solutions the total payoff for any collusion is unaffected). As mentioned in Section 1, several papers investigate solutions that are neutral to inclusive or exclusive collusions: Haller (1994), Malawski (2002), and Segal (2003); Malawski (2002), van den Brink (2012), and Park and Ju (2015). We ask whether there exists a solution that is neutral to inclusive collusions and characterize the family of such solutions when they exist.

Formally, given a game v, an ***inclusive collusion of coalition*** $\boldsymbol{I} \subseteq N$ provides each member of I with the power of enforcing participation of all other members of I, so that game v turns into a new game $\boldsymbol{v_I}$ such that for each coalition $S \subseteq N$,

$$v_I(S) = \begin{cases} v(S \cup I) & \text{if } S \cap I \neq \emptyset; \\ v(S) & \text{otherwise.} \end{cases}$$

On the other hand, an ***exclusive collusion of coalition*** $\boldsymbol{I} \subseteq N$ gives each member of I the right to exclude participation of all other members of I. Thus, game v is transformed into a new game $\hat{v}_I$ such that for each coalition $S \subseteq N$,

$$\hat{v}_I(S) = \begin{cases} v(S \backslash I) & \text{if } S \not\supseteq I; \\ v(S) & \text{otherwise.} \end{cases}$$

Since our focus is on inclusive collusions, we sometimes write "collusion" for "inclusive collusion" when there is no ambiguity.

When there is no restriction on collusion formation, that is, any group of players can collude together, it turns out that no solution satisfying some other mild axioms is neutral to inclusive collusions (see Theorem 1 below). Thus, following Myerson (1977), we assume that a network structure (undirected graph) defined over players governs collusion formation. A ***network*** is a set of edges $\boldsymbol{L} \subseteq N \times N$. In reference to network L, we call elements of N ***nodes***. For all distinct $i, j \in N$, a ***path from i to j*** is a sequence of nodes $(i_1, \ldots, i_k)$ such that $i_1 = i$, $i_k = j$, and $(i_1, i_2), (i_2, i_3), \cdots, (i_{k-1}, i_k) \in L$. Network L is ***cyclic*** if there are distinct $i_1, \cdots, i_k \in N$, with $k \geq 3$, such that for each $j \in \{1, \cdots, k\}$, $(i_j, i_{j+1}) \in L$ (where $i_{k+1} = i_1$ by convention). Otherwise, L is ***acyclic***. It is a ***tree*** if for all distinct $i, j \in N$, there is a unique path from i to j. A coalition $S \subseteq N$ is ***connected*** (on L) if for all distinct $i, j \in S$, there is a path $(i_1, \cdots, i_k)$ from i to j such that $\{i_1, \cdots, i_k\} \subseteq S$. By definition, any singleton coalition is connected. Let $\boldsymbol{\mathcal{C}(L)}$ be the set of connected coalitions on L. Also, we say that L is a ***connected network*** if N is connected on L. Throughout the paper, we assume that L is a connected network and that only connected coalitions on L can collude.[10]

Now we state our main axiom: for no coalition connected on L should the inclusive collusion among the coalition members affect their total payoff.

Inclusive Collusion Neutrality. For each $v \in \mathcal{V}$ and each connected coalition $I \subseteq N$ on L, $\sum_{i \in I} f_i(v_I) = \sum_{i \in I} f_i(v)$.

A weaker neutrality property, called pairwise inclusive collusion neutrality, restricts attention to two-player collusions. This pairwise version is introduced by Haller (1994) and studied further by Malawski (2002) and Segal (2003), though without network considerations.

In the definition of inclusive collusion neutrality, players in coalition I enter into an inclusive collusion (as implied by v_I). If, instead, the collusion is exclusive, we can replace v_I in the definition by $\hat{v}_I$ and obtain another neutrality property: ***exclusive collusion neutrality***.[11] Malawski (2002) introduces this axiom and van den Brink (2012) and Ju and Park (2015) explore its implications on networks. Also, Malawski (2002) characterizes the Banzhaf value, imposing inclusive and exclusive collusion neutrality separately.

In addition, we consider the following three axioms, which are quite standard in the

[10]**XXX Do our results extend to the case of non-connected networks?**

[11]**XXX Any logical relation between exclusive collusion neutrality and inclusive collusion neutrality?**

cooperative game literature. First, we require that the worth of the grand coalition should be fully allocated to all players.

Efficiency. For each $v \in \mathcal{V}$, $\sum_{i \in N} f_i(v) = v(N)$.

Given $v \in \mathcal{V}$, player $i \in N$ is a ***null player*** for game v if his contribution to any coalition is zero; i.e., for all $S \subseteq N \backslash \{i\}$, $v(S \cup \{i\}) = v(S)$. Let $\boldsymbol{null(v)}$ be the set of null players for game v. Our next axiom requires that null players should receive a zero payoff.

Null-player Property. For each $v \in \mathcal{V}$, if $i \in N$ is a null player for v (i.e., $i \in null(v)$), then $f_i(v) = 0$.

Next is linearity of solutions considered by numerous authors in the cooperative game literature, first by Shapley (1953) and later by Lehrer (1988), Haller (1994), Malawski (2002), van den Brink (2009), and Mishra and Talman (2010).

Linearity. For all $v, w \in \mathcal{V}$ and each $c \in \mathbb{R}$, $f(v) + f(w) = f(v+w)$ and $f(cv) = cf(v)$.

3 Results

First, we examine whether there exists a solution satisfying inclusive collusion neutrality, efficiency, the null-player property under no restrictions on network L. It turns out that as long as L is cyclic, the three axioms are incompatible.

Theorem 1. *If L is cyclic, then no solution satisfies inclusive collusion neutrality, efficiency, and the null-player property.*

Proof. Suppose, by contradiction, that f satisfies the three axioms. Since L is cyclic, there are, say, $1, \cdots, m \in N$ (where $m \geq 3$) such that $(1,2), \cdots, (m-1, m), (m, 1) \in L$. Let $M \equiv \{1, \cdots, m\}$.

Step 1: *For all distinct $i, j, k \in M$, $u_{M \backslash j}^{M \backslash i} = u_{M \backslash j}^{M \backslash k} = u_{M \backslash j}^{M}$.*

First, for each $j \in M$, $M \backslash j$ is connected on L, so that players in $M \backslash j$ can inclusively collude. Next, it is easy to verify that for each coalition $S \subseteq N$,

$$u_{M \backslash j}^{M \backslash i}(S) = u_{M \backslash j}^{M \backslash k}(S) = u_{M \backslash j}^{M}(S) = \begin{cases} 1 & \text{if } S \cap (M \backslash j) \neq \emptyset \text{ and } S \ni j; \\ 0 & \text{otherwise.} \end{cases}$$

Step 2: *Let $i \in M$. For each $j \in N$, $f_j(u^{M\setminus i}) = \frac{1}{m-1}$ if $j \in M\setminus i$; and 0 otherwise.*

Let $i \in M$. By the null-player property, it is enough to show that for each $j \in M\setminus i$, $f_j(u^{M\setminus i}) = \frac{1}{m-1}$. Let $j, k \in M\setminus i$ be distinct. Then

$$f_{M\setminus j}(u^{M\setminus i}) = f_{M\setminus j}(u^{M\setminus i}_{M\setminus j}) = f_{M\setminus j}(u^{M\setminus k}_{M\setminus j}) = f_{M\setminus j}(u^{M\setminus k}),$$

where the first and third equalities follow from inclusive collusion neutrality, and the second from Step 1. By the null-player property, $f_M(u^{M\setminus i}) = f_M(u^{M\setminus k}) = 1$. Combined with the above equation, this implies that $f_j(u^{M\setminus i}) = f_j(u^{M\setminus k})$. Denote the latter common value by α_j.

Now we pin down $(\alpha_j)_{j\in M}$. By the null-player property, for each $i \in M$, $\sum_{j\in M\setminus i} \alpha_j = f_{M\setminus i}(u^{M\setminus i}) = 1$. Solving m such equations simultaneously, we obtain that for each $j \in M$, $\alpha_j = \frac{1}{m-1}$.

Step 3: $f_M(u^M) = \frac{m(m-2)}{(m-1)^2} < 1$, *a contradiction.*

For each $j \in M$,

$$f_{M\setminus j}(u^M) = f_{M\setminus j}(u^M_{M\setminus j}) = f_{M\setminus j}(u^{M\setminus i}_{M\setminus j}) = f_{M\setminus j}(u^{M\setminus i}) = \frac{m-2}{m-1},$$

where the first three equalities follow from an argument similar to that in Step 2, and the last from Step 2. Summing up m equations of this type yields that $(m-1)f_M(u^M) = \frac{m(m-2)}{m-1}$. Thus, $f_M(u^M) = \frac{m(m-2)}{(m-1)^2} < 1$, violating the null-player property. □

If exclusive collusion neutrality, instead of inclusive collusion neutrality, is imposed in Theorem 1, a stronger impossibility obtains. Under the assumption of a "complete" network L (i.e., for all $i, j \in N$, $(i, j) \in L$), van den Brink (2012) proves that inclusive collusion neutrality is incompatible with efficiency and the null player property. According to Park and Ju (2015), this completeness assumption can be weakened substantially: unless network L is a line (i.e., each node is connected to at most two nodes), the impossibility holds.

Theorem 1 suggests that a solution can be neutral to inclusive collusion only if the network is acyclic, that is, it is a tree (recall that we restrict attention to connected networks). Indeed, there exists such a solution for tree networks. We now introduce them. Let L be a tree network. For all $i, j, h \in N$, a path $\{i_1, \cdots, i_k\}$ from i to j

contains h if $h \in \{i_1, \cdots, i_k\}$. Fix $i \in N$. Let $\boldsymbol{\overline{s}_i(j)} \equiv \{h \in N \backslash \{j\}$: the (unique) path from i to h contains $j\}$ be the set of ***successors*** of j from origin i. Let $\boldsymbol{s_i(j)} \equiv \{h \in \overline{s}_i(j) : (j,h) \in L\}$ be the set of ***immediate successors*** of j from origin i. Given a tree L, for each $i \in N$, the ***i-hierarchical solution,*** denoted $\boldsymbol{h^i}$, associates with each $v \in \mathcal{V}$ the payoff profile $h^i(v)$ such that for each $j \in N$,

$$h^i_j(v) = v\left(\overline{s}_i(j) \cup \{j\}\right) - \sum_{k \in s_i(j)} v\left(\overline{s}_i(k) \cup \{k\}\right).$$

Our next result shows that for tree networks, the hierarchical solutions—and their affine combinations—satisfy inclusive collusion neutrality, efficiency, and the null-player property.

Theorem 2. *Let L be a tree. For each $i \in N$, the i-hierarchical solution satisfies inclusive collusion neutrality, efficiency, and the null-player property. Further, so does each affine combination of the hierarchical solutions.*

Proof. We omit the simple proof that each affine combination of the hierarchical solutions satisfies efficiency and the null player property. To show that the hierarchical solutions satisfy inclusive collusion neutrality, let $i \in N$ and consider the i-hierarchical solution h^i. Let $v \in \mathcal{V}$. Let $I \subseteq N$ be a connected coalition on L. Note that $\{(j,k) : j,k \in I$ and $(j,k) \in L\}$ is a tree. By definition,

$$\begin{aligned} h^i_I(v) &= \sum_{j \in I}\left[v(\bar{s}_i(j) \cup \{j\}) - \sum_{k \in s_i(j)} v(\bar{s}_i(k) \cup \{k\})\right]; \text{ and} \\ h^i_I(v_I) &= \sum_{j \in I}\left[v_I(\bar{s}_i(j) \cup \{j\}) - \sum_{k \in s_i(j)} v_I(\bar{s}_i(k) \cup \{k\})\right]. \end{aligned}$$

To compare the terms in the latter two equations, recall that for each $S \subseteq N$, $v(S) \neq v_I(S)$ if and only if $S \cap I \neq \emptyset$ and $S \not\supseteq I$. Thus, if for some $k \in I$, $v(\bar{s}_i(k) \cup \{k\}) \neq v_I(\bar{s}_i(k) \cup \{k\})$, then there is $j \in I$ such that $k \in s_i(j)$. This means that (i) when $h^i_I(v)$ is calculated, $v(\bar{s}_i(k) \cup \{k\})$ cancels out in the summation; and (ii) when $h^i_I(v_I)$ is calculated, $v_I(\bar{s}_i(k) \cup \{k\})$ cancels out in the summation. For all other $k \in I$, $v(\bar{s}_i(k) \cup \{k\}) = v_I(\bar{s}_i(k) \cup \{k\})$. Thus, $h^i_I(v) = h^i_I(v_I)$.

Finally, we show that each affine combination of the hierarchical solutions satisfy inclusive collusion neutrality. Let $(\alpha_i)_{i\in N} \in \mathbb{R}^N$ be such that $\sum_{i\in N} \alpha_i = 1$ and consider solution $g \equiv \sum_{i\in N} \alpha_i h^i$. For each $v \in \mathcal{V}$ and each connected coalition $I \subseteq N$ on L, $g_I(v) = \sum_{i\in N} \alpha_i h^i_I(v) = \sum_{i\in N} \alpha_i h^i_I(v_I) = g_I(v_I)$, where the second equality follows from the fact that the hierarchical solutions satisfy inclusive collusion neutrality. $\square$

Theorem 2 stands in contrast with Park and Ju (2015). Exclusive collusion neutrality is compatible with efficiency and the null player property if and only if network L is a line. Inclusive collusion neutrality is compatible with the two axioms if and only if network L is a tree. Therefore, while there is no logical relation between exclusive and inclusive collusion neutrality, the former is more restrictive than the latter in the presence of efficiency and the null player property.

Given the possibility in Theorem 2, it is natural to ask if there exist solutions, other than the affine combinations of the hierarchical solutions, satisfying inclusive collusion neutrality, efficiency, and the null-player property. While the answer is yes, such a solution coincides with an affine combination of the hierarchical solutions when restricted to the unanimity games. In other words, the three axioms pin down the behavior of solutions on the domain of the unanimity games.

Theorem 3. *Let L be a tree. Assume that a solution on $\mathcal{V}$ satisfies inclusive collusion neutrality, efficiency, and the null-player property. When restricted to the unanimity games, the solution is an affine combination of the hierarchical solutions.*

Proof. Let f be a solution on $\mathcal{V}$ satisfying the three axiom. Let $i \in N$. Let $d(i)$ be the number of maximal connected components of $N\backslash\{i\}$. Let $D(i) = \{1, \cdots, d(i)\}$. Let $S_1, \cdots, S_{d(i)}$ be the maximal connected components of $N\backslash\{i\}$. Note that for each $j \in D(i)$, S_j and $N\backslash S_j$ are connected on L. Let $\alpha \equiv f(u^N)$ and for each coalition $S \subseteq N$, $\alpha_S \equiv \sum_{i\in S} \alpha_i$.We proceed in three steps.

Step 1: *For each $S \in \{S_1, \cdots, S_{d(i)}\}$ and each coalition $T \subseteq N$,* (i) *if $T \subseteq S$, then $f_S(u^T) = 1$;* (ii) *if $T \cap S = \emptyset$, then $f_S(u^T) = 0$; and* (iii) *if $T \cap S \neq \emptyset$ and $T\backslash S \neq \emptyset$, then $f_S(u^T) = \alpha_S$.*

First, (i) and (ii) follow from efficiency and the null player property. Before proving (iii), we first establish the following: for each $S \in \{S_1, \cdots, S_{d(i)}\}$ and each coalition

$T \subseteq N$, if $S \subsetneq T$, then $f_S(u^T) = \alpha_S$. Take such $S, T \subseteq N$. Since $N \backslash S$ is connected on L, players in $N \backslash S$ can inclusively collude. It is easy to verify that $u^T_{N\backslash S} = u^N_{N\backslash S}$.[12] Then

$$f_{N\backslash S}(u^T) = f_{N\backslash S}(u^T_{N\backslash S}) = f_{N\backslash S}(u^N_{N\backslash S}) = f_{N\backslash S}(u^N),$$

where the first and third equalities follow from inclusive collusion neutrality. Since, by efficiency, $f_N(u^T) = f_N(u^N) = 1$, it follows that $f_S(u^T) = f_S(u^N) = \alpha_S$.

Next, to prove (iii), let $S \in \{S_1, \cdots, S_{d(i)}\}$ and $T \subseteq N$ be such that $T \cap S \neq \emptyset$ and $T \backslash S \neq \emptyset$. Since S is connected on L, players in S can inclusively collude. It is easy to verify that $u^T_S = u^{T\cup S}_S$.[13] By an argument similar to that in the previous paragraph,

$$f_S(u^T) = f_S(u^T_S) = f_S(u^{T\cup S}_S) = f_S(u^{T\cup S}).$$

Further, since $T \cup S \supsetneq S$, the claim in the previous paragraph implies that $f_S(u^{T\cup S}) = \alpha_S$. Thus, $f_S(u^T) = \alpha_S$.

Step 2: *For each coalition $T \subseteq N$,* (i) *if for some $j \in D(i)$, $T \subseteq S_j$, then $f_i(u^T) = 0$;* (ii) *otherwise, $f_i(u^T) = 1 - \sum_{j\in D(i), S_j\cap T\neq\emptyset} \alpha_{S_j}$.*

Let $T \subseteq N$ be a coalition. If for some $j \in D(i)$, $T \subseteq S_j$, then $i \notin T$, so that by the null player property, $f_i(u^T) = 0$. To prove (ii), assume that for any $j \in D(i)$, $T \nsubseteq S_j$. Then

$$\begin{aligned} f_i(u^T) &= 1 - \sum_{j\in D(i)} f_{S_j}(u^T) \\ &= 1 - \sum_{j\in D(i), S_j\cap T=\emptyset} f_{S_j}(u^T) - \sum_{j\in D(i), S_j\cap T\neq\emptyset} f_{S_j}(u^T) \\ &= 1 - \sum_{j\in D(i), S_j\cap T\neq\emptyset} \alpha_{S_j}, \end{aligned}$$

where the first equality follows from efficiency and the third from Step 1.

Step 3: *For each coalition $T \subseteq N$, $f_i(u^T) = \sum_{t\in N} \alpha_t h^t_i(u^T)$.*

Let $T \subseteq N$ be a coalition. First, we calculate $h^i_i(u^T)$. By definition, $h^i_i(u^T) =$

[12] For each coalition $A \subseteq N$, $u^T_{N\backslash S}(A) = u^N_{N\backslash S}(A) = 1$ if and only if $A \backslash S \neq \emptyset$ and $A \supseteq S$.

[13] For each coalition $A \subseteq N$, $u^T_S(A) = u^{T\cup S}_S(A) = 1$ if and only if $A \cap S \neq \emptyset$ and $A \supseteq T \backslash S$.

$u^T(N) - \sum_{j\in D(i)} u^T(S_j)$. If for some $j \in D(i)$, $T \subseteq S_j$, then $h_i^i(u^T) = 1 - 1 = 0$; otherwise, $h_i^i(u^T) = 1 - 0 = 1$. Next, let $t \in N\backslash\{i\}$ and we calculate $h_i^t(u^T)$. Let $j \in D(i)$ be such that $t \in S_j$. Again by definition,

$$h_i^t(u^T) = u^T(N\backslash S_j) - \sum_{k\in D(i)\backslash j} u^T(S_k) \tag{1}$$

Here, $u^T(N\backslash S_j) = 0$ if $T\cap S_j \neq \emptyset$; and $u^T(N\backslash S_j) = 1$ otherwise. Further, $\sum_{k\in D\backslash j} u^T(S_k) = 1$ if for some $k \in D(i)\backslash j$, $T \subseteq S_k$; and $\sum_{k\in D\backslash j} u^T(S_k) = 0$ otherwise. Substituting these into Eq. (1) yields that $h_i^t(u^T) = 1$ if $T \cap S_j = \emptyset$ and for any $k \in D(i)\backslash j$, $T \nsubseteq S_k$; $h_i^t(u^T) = 0$ otherwise.

Finally, we substitute into $\sum_{t\in N} \alpha_t h_i^t(u^T)$ the expressions for $h_i^i(u^T)$ and $h_i^t(u^T)$ for $t \in N\backslash\{i\}$ from the previous paragraph. Note that $\sum_{t\in N} \alpha_t h_i^t(u^T) = \alpha_i h_i^i(u^T) + \sum_{j\in D(i)} \sum_{t\in S_j} \alpha_t h_i^t(u^T)$. If for some $j \in D(i)$, $T \subseteq S_j$, then $\sum_{t\in N} \alpha_t h_i^t(u^T) = \alpha_i \cdot 0 + \sum_{j\in D(i)} \sum_{t\in S_j} \alpha_t \cdot 0 = 0$. If for any $j \in D(i)$, $T \nsubseteq S_j$,

$$\begin{aligned}
\sum_{t\in N} \alpha_t h_i^t(u^T) &= \alpha_i \cdot 1 + \sum_{j\in D(i), S_j\cap T=\emptyset} \sum_{t\in S_j} \alpha_t \cdot 1 \\
&= \alpha_i + \sum_{j\in D(i), S_j\cap T=\emptyset} \alpha_{S_j} \\
&= 1 - \sum_{j\in D(i), S_j\cap T\neq\emptyset} \alpha_{S_j}.
\end{aligned}$$

Thus, $f_i(u^T) = \sum_{t\in N} \alpha_t h_i^t(u^T)$. □

Remark 1. Say that a domain $\mathcal{D} \subseteq \mathcal{V}$ satisfies the ***affine-combination property*** if each solution satisfying inclusive collusion neutrality, efficiency, and the null-player property is an affine combination of the hierarchical solutions when restricted to $\mathcal{D}$. Theorem 3 shows that the set of all unanimity games, denoted $\mathcal{U}$, satisfies the affine-combination property. In fact, $\mathcal{U}$ is a maximal domain satisfying the property; i.e., for each $v \in \mathcal{V}\backslash\mathcal{U}$, $\mathcal{U} \cup \{v\}$ fails the affine-combination property. To see this, consider solution f defined as follows: (i) for each $v \in \mathcal{U}$, $f(v) = h^1(v)$; and (ii) for each $v \in \mathcal{V}\backslash\mathcal{U}$, $f(v) = h^2(v)$. Clearly, f satisfies inclusive collusion neutrality, efficiency, and the null-player property. However, there is no $v \in \mathcal{V}\backslash\mathcal{U}$ such that when restricted to $\mathcal{U} \cup \{v\}$,

f is an affine combination of the hierarchical solutions. That is, for each $v \in \mathcal{V} \backslash \mathcal{U}$, $\mathcal{U} \cup \{v\}$ violates the affine-combination property. $\triangle$

Once the behavior of a solution on the domain of the unanimity games is known, we can characterize it on the full domain with the aid of linearity. Our next result shows that the affine combinations of the hierarchical solutions are the only solutions satisfying inclusive collusion neutrality, efficiency, the null-player property, and linearity.

Theorem 4. *Let L be a tree. A solution on $\mathcal{V}$ satisfies inclusive collusion neutrality, efficiency, and the null-player property, and linearity if and only if it is an affine combination of the hierarchical solutions.*

Proof. We only prove the "only if" part. Let f be a solution on $\mathcal{V}$ satisfying the four axioms. By Theorem 3, there is $(\alpha_i)_{i \in N}$ such that (i) $\sum_{i \in N} \alpha_i = 1$; and (ii) for each coalition $T \subseteq N$, $f(u^T) = \sum_{i \in N} \alpha_i h^i(u^T)$. Let $v \in \mathcal{V}$. Since $\left(u^T\right)_{T \in 2^N \backslash \{\emptyset\}}$ is a basis of $\mathcal{V}$, there is $(\lambda_T)_{T \in 2^N \backslash \{\emptyset\}} \in \mathbb{R}^{2^n - 1}$ such that $v = \sum_{T \in 2^N \backslash \{\emptyset\}} \lambda_T u^T$. By linearity,

$$\begin{aligned} f(v) &= \sum_{T \in 2^N \backslash \{\emptyset\}} \lambda_T f\left(u^T\right) = \sum_{T \in 2^N \backslash \{\emptyset\}} \sum_{i \in N} \lambda_T \alpha_i h^i(u^T) \\ &= \sum_{i \in N} \alpha_i \left[\sum_{T \in 2^N \backslash \{\emptyset\}} \lambda_T h^i(u^T) \right] = \sum_{i \in N} \alpha_i h^i \left(\sum_{T \in 2^N \backslash \{\emptyset\}} \lambda_T u^T \right) = \sum_{i \in N} \alpha_i h^i(v) \end{aligned}$$

$\square$

Remark 2. The axioms in Theorem 4 are independent. The Shapley value satisfies all but inclusive collusion neutrality (Haller, 1994). The negative of a hierarchical solution, i.e., $-h^i$ for some $i \in N$, satisfies all but efficiency. The equal division solution ED, defined as for each $v \in \mathcal{V}$ and each $i \in N$, $ED_i(v) = \frac{1}{n} v(N)$, satisfies all but the null player property. Finally, Theorem 3 indicates that one can easily construct a solution that satisfies all but linearity and yet is not an affine combination of the hierarchical solutions. A solution that applies different hierarchical solutions to $v \in \mathcal{V}$ depending on $v(N)$ is an example. Park and Ju (2015) consider such solutions and show that they satisfy exclusive collusion neutrality; clearly, they also satisfy inclusive collusion neutrality. $\triangle$

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4 Cho's Note

Game v is ***super-additive*** if for all disjoint coalitions $S, T \subseteq N$, $v(S)+v(T) \leq v(S \cup T)$.

Henceforth we only consider super-additive games. Let $\mathcal{V}_s$ be the set of all super-additive games.

Recall that super-additivity of v implies monotonicity of v.

Together with $v(\emptyset) = 0$, monotonicity of v implies non-negativity of v.

For each $S \subseteq N$, each connected $I \subseteq N$, $v_I(S) + \hat{v}_I(S^c) \leq v(N)$ and $v(S) + v(S^c) \leq v(N)$.

By definition, $\hat{v}_I \leq v \leq v_I$.

f is **non-negative** if for each $v \in \mathcal{V}_s$, $f(v) \geq 0$.

f is **additive** if for all $v, w \in \mathcal{V}_s$, $f(v + w) = f(v) + f(w)$.

f is **monotonic** if for all $v, w \in \mathcal{V}_s$ with $v \leq w$, $f(v) \leq f(w)$.

Example 1. *An inclusive collusion can turn a super-additive game into a non–super-additive game (i.e., $\mathcal{V}_s$ is not closed under inclusive collusion).* Let $N = \{1, 2, 3, 4\}$. Consider $v \in \mathcal{V}_s$ such that for each coalition $S \subseteq N$, $v(S) = \sum_{i \in S} i$. Clearly, v is super-additive (in fact, additive). Let $I = \{1, 2\}$ be a connected coalition on L and consider v_I. For disjoint coalitions $\{1, 3\}, \{2, 4\} \subseteq N$, $v_I(\{1, 3\}) + v_I(\{2, 4\}) = 6 + 7 \not\leq 10 = v_I(\{1, 3\} \cup \{2, 4\})$.

By contrast, $\mathcal{V}_s$ is closed under exclusive collusion. To see this, let $v \in \mathcal{V}_s$ and $I \subseteq N$. Let $S, T \subseteq N$ be disjoint coalitions. If $S \cup T \supseteq I$, then $\hat{v}_I(S) + \hat{v}_I(T) \leq v(S) + v(T) \leq v(S \cup T) = \hat{v}_I(S \cup T)$. On the other hand, if $S \cup T \not\supseteq I$, then since $S \not\supseteq I$ and $T \not\supseteq I$, it follows that $\hat{v}_I(S) + \hat{v}_I(T) = v(S \backslash I) + v(T \backslash I) \leq v\left((S \backslash I) \cup (T \backslash I)\right) = v\left((S \cup T) \backslash I\right) = \hat{v}_I(S \cup T)$.

Game v is **monotonic** if for all coalitions $S, T \subseteq N$ with $S \subseteq T$, $v(S) \leq v(T)$.

Let $\boldsymbol{\mathcal{V}_m}$ be the set of all monotonic games. Note that $\mathcal{V}_s \subseteq \mathcal{V}_m$.

$\mathcal{V}_m$ is closed under both inclusive and exclusive collusion.

A **collusion method *c*** associates with each $v \in \mathcal{V}_m$ and each coalition $I \subseteq N$ connected on L a game $v_I^c \in \mathcal{V}_m$ such that $v_I^c(N) = v(N)$. The property $v_I^c(N) = v(N)$ requires that as far as the woth of the grand coalition N is concerned, the collusion by any coalition makes no difference. Collusion method c is **uniform** if for each $v \in \mathcal{V}_m$

and each coalition $I \subseteq N$ connected on L, the collusion by I via method c transforms game v in one direction, that is, $v_I^c \leq v$ or $v_I^c \geq v$.[14] Clearly, inclusive and exclusive collusion are uniform.

***c*-Collusion Neutrality.** For each $v \in \mathcal{V}$ and each connected coalition $I \subseteq N$ on L, $\sum_{i \in I} f_i(v_I^c) = \sum_{i \in I} f_i(v)$.

Strong *c*-Collusion Neutrality. For each $v \in \mathcal{V}$ and each connected coalition $I \subseteq N$ on L, $f(v_I^c) = f(v)$.

Proposition 1. *Let c be a uniform collusion method. On $\mathcal{V}_m$, monotonicity and efficiency together imply (strong) c-collusion neutrality.*

Proof. Let f be a solution satisfying monotonicity and efficiency. We first show that for all $v, w \in \mathcal{V}_m$, if $v \leq w$ and $v(N) = w(N)$, then $f(v) = f(w)$. Let $v, w \in \mathcal{V}_m$ be such that $v \leq w$ and $v(N) = w(N)$. By monotonicity, $f(v) \leq f(w)$. By efficiency, $f_N(v) = v(N) = w(N) = f_N(w)$. If there is $i \in N$ with $f_i(v) < f_i(w)$, then $f_N(v) = f_N(w)$ implies that there is $j \in N \backslash \{i\}$ with $f_j(v) > f_j(w)$, contradicting $f(v) \leq f(w)$. Thus, $f(v) = f(w)$.

Now let c be a uniform collusion method. Let $v \in V$ and let $I \subseteq N$ be a connected coalition on L. Note that $v_I^c(N) = v(N)$; and $v_I^c \leq v$ or $v_I^c \geq v$. Thus, by the argument in the previous paragraph, $f(v_I^c) = f(v)$, so that f satisfies (strong) c-collusion neutrality. □

[14]One may consider a stronger propery, which requires that either (i) for each $v \in \mathcal{V}_m$ and each coalition $I \subseteq N$ connected on L, $v_I^c \leq v$; or (ii) for each $v \in \mathcal{V}_m$ and each coalition $I \subseteq N$ connected on L, $v_I^c \geq v$. For our results, the weaker requirement of uniformity is sufficient.

- Haller (1994)
  - Two types of collusion: proxy agreement and association agreement (the latter is, in my language, pairwise inclusive collusion)
  - Considers pairwise inclusive collusion neutrality and shows that the Shapley value violates it.
- Malawski (2002)
  - Three types of collusion: proxy (representation), association, and distrust ("exclusive collusion"), all in pairwise versions
  - Consider three neutrality properties, including inclusive collusion neutrality
  - Characterize the Banzhaf value by each of the three neutrality properties and other axioms

$v_I(S) = v(S \cup I)$ if $I \cap S \neq \emptyset$ and $I \cap S^c \neq \emptyset$; and $v_I(S) = v(S)$ otherwise.

$\hat{v}_I(S) = v(S \backslash I)$ if $I \cap S \neq \emptyset$ and $I \cap S^c \neq \emptyset$; and $\hat{v}_I(S) = v(S)$ otherwise.

Summary of van den Brink, Park and Ju, and Cho and Ju:

	Exclusive Collusion Neutrality	
L: complete	$\nexists\varphi$: (Pairwise) ECN, E, NPP (van den Brink)	
L: cyclic	$\nexists\varphi$: ECN, E, NPP (Park and Ju)	
L: tree and not line	$\nexists\varphi$: ECN, E, NPP (Park and Ju) (but **VNPP** gives a possibility)	
L: tree and line	$\exists\varphi$: ECN, E, NPP (Park and Ju)	
L: tree	Hierarchicals characterized by ECN, E, **VNPP**, linearity	Hierarch

Summary of Malawski (2002):

Thm 1–3: Equal treatment of equals, dummy-player property, equal treatment of games with equal marginal contributions, and (***exclusive collusion neutrality*** **or** ***inclusive collusion neutrality*** **or** representation collusion neutrality) $\Longrightarrow$ Banzhaf value