Compellingness in Nash Implementation*

Shurojit Chatterji, Takashi Kunimoto, and Paulo Ramos†

First Version: July 2022
This Version: October 2023

Abstract

A social choice function (SCF) is said to be Nash implementable if there exists a mechanism in which every Nash equilibrium induces outcomes specified by the SCF. The main objective of this paper is to assess the impact of considering mixed strategy equilibria in Nash implementation. We call a mixed strategy equilibrium "uncompelling" if its outcome is strictly worse for any agent than that induced by the SCF. We show that if the finite environment and the SCF to be implemented jointly satisfy what we call *Condition COM*, we construct a finite mechanism which Nash implements the SCF in pure strategies and its any mixed strategy Nash equilibrium outcome is either uncompelling or consistent with the SCF. Our mechanism has several desirable features: transfers can be completely dispensable; only finite mechanisms are considered; integer games are not invoked; and agents' attitudes toward risk do not matter. These features make our result quite distinct from many other prior attempts to handle mixed strategy equilibria in the theory of implementation.

JEL Classification: C72, D78, D82.

Keywords: implementation, compelling equilibria, ordinality, mixed strategies, Nash equilibrium.

1 Introduction

The theory of implementation attempts to answer two questions. First, can one design a mechanism that successfully structures the interactions of agents in such a

*We thank Bhaskar Dutta, Matthew Jackson, Jiangtao Li, Peng Liu, Antonio Penta, Ching-Jen Sun, Jingyi Xue, Hamid Sabourian, Yves Sprumont, and Takuro Yamashita for helpful comments. This research is supported by the Ministry of Education, Singapore under MOE Academic Research Fund Tier 2 (MOE-T2EP402A20-0007). All remaining errors are our own.

†School of Economics, Singapore Management University, Singapore; shurojitc@smu.edu.sg, tkunimoto@smu.edu.sg, paulodsramos@smu.edu.sg

way that, in each state of the world, they always choose actions which result in the socially desirable outcomes for that state? Second, if agents possess information about the state and interact through a given mechanism, what properties do the resulting outcomes, viewed as a map from states to outcomes (and called social choice functions - henceforth, SCFs), possess? In answering these, the consequences of a given mechanism are predicted through the application of game theoretic solution concepts.[1]

In this paper we adopt Nash equilibrium as a solution concept, consider complete information environments, and ask if a given SCF is implementable, i.e., when we can design a mechanism in which "every" Nash equilibrium induces outcomes consistent with the SCF. Although the literature claims to care about all equilibria, it often ignores mixed strategy equilibria and only focuses on pure strategy equilibria. Jackson (1992) provides the most forceful argument for why the omission of mixed strategy equilibria brings about a serious consequence. In his Example 4, Jackson (1992) constructs a two-person environment and an SCF such that (i) there is a finite mechanism that pure Nash implements the SCF; and (ii) every finite pure Nash implementing mechanism always has a mixed strategy equilibrium that gives a lottery that is preferred by both agents to the outcome of the SCF. Thus, if we insist on using finite mechanisms, which is to be anticipated in an environment with finite number of alternatives and agents, we must question why agents would limit themselves to playing only pure strategies, particularly when there is a mixed strategy equilibrium that would be strictly preferred by both of them than any pure strategy equilibrium. In this paper, we revisit Jackson's example in Section 3.

To obtain the main result of the paper, we consider a finite environment with respect to an SCF on which we impose *Condition COM*, which delineates a set of conditions where it is always possible to construct a finite, pure Nash implementing mechanism such that every mixed strategy equilibrium outcome is either socially desirable or *uncompelling* in the sense that it is strictly Pareto dominated by the socially desirable outcome.[2] We call such a notion of implementation *compelling implementation.* Importantly, compelling implementation allows the implementing mechanism to admit mixed strategy equilibria that result in outcomes not consistent with the ones induced by the SCF, provided these mixed equilibria are uncom-

[1]See Jackson (2001), Maskin and Sjöström (2002), and Serrano (2004) for the survey of implementation theory.

[2]Note that Moore and Repullo (1990) identify Condition μ as a necessary and sufficient condition for pure strategy Nash implementation when there are at least three agents. In addition, Dutta and Sen (1991) and Moore and Repullo (1990) identify Condition β and Condition $\mu 2$, respectively, as a necessary and sufficient condition for pure Nash implementation when there are only two agents.

pelling.[3] Hence, compelling implementation is considered a compromise between pure Nash implementation where only pure strategies are considered and mixed Nash implementation where all mixed strategy equilibria are fully considered.

To locate our contribution in a broader context, we first acknowledge that every prior work cited in the table below exploits some combination of the following five ingredients to handle mixed strategy equilibria in complete information environments: (i) infinite mechanisms; (ii) rationalizability as a stronger requirement than Nash equilibrium;[4] (iii) refinements of Nash equilibrium, such as subgame perfect equilibrium and undominated Nash equilibrium; (iv) environments with transfers or ones similar to separable environments of Jackson, Palfrey, and Srivstava (1994); and (v) cardinal utilities.[5]

Combination of ingredients used	Previous works which handle mixed strategy equilibria in complete information environments
(i)	Kartik and Tercieux (2012), Maskin (1999), Maskin and Sjöström (2002), Mezzetti and Renou (2012a)
(i) × (v)	Kunimoto (2019), Serrano and Vohra (2010)
(i) × (ii) × (v)	Bergemann, Morris, and Tercieux (2011), Jain (2021), Kunimoto and Serrano (2019), Xiong (2022)
(ii) × (iv) × (v)	Abreu and Matsushima (1992), Chen, Kunimoto, Sun, and Xiong (2021)
(iii) × (iv)	Goltsman (2011), Jackson, Palfrey, and Srivastava (1994), Moore and Repullo (1988), Sjöström (1994)
(iii) × (iv) × (v)	Abreu and Matsushima (1994)
(iv)	Mezzetti and Renou (2012b)
(iv) × (v)	Chen, Kunimoto, Sun, and Xiong (2022)

Table 1: The list of prior works handling mixed strategy equilibria in complete information environments.

We next emphasize that we obtain the main result of the paper without using any of the five ingredients used in the previous works. Of course, there is the cost associated with this result, as our implementing mechanism might admit mixed strategy equilibria which are uncompelling. In addition to the information about the agents' ordinal strict preferences, what is required is the information regarding the smallest difference in cardinal utilities between any two distinct alternatives. We can think of such information as the smallest unit in which the agents' utilities

[3]Our compelling implementation is similar to the notion of repeated implementation adopted by Lee and Sabourian (2015). They design a sequence of simple, finite mechanisms such that every pure strategy subgame perfect equilibrium "repeatedly" implements the efficient social choice function, while every mixed strategy subgame perfect equilibrium is strictly Pareto dominated by the pure equilibrium.

[4]Rationalizability is a more demanding requirement than Nash equilibrium because every action played with positive probability in a mixed strategy Nash equilibrium is rationalizable.

[5]This table, by no means, exhausts all related papers.

are measured. As long as that unit of measure is positive, we can construct a mechanism that compellingly implements the SCF. In this sense, while compelling implementation is not completely ordinal, it can be made as ordinal as it can possibly be. We consider Nash implementation as the right notion of implementation if we insist on the robustness to information perturbations. This is so because Aghion, Fudenberg, Holden, Kunimoto, and Tercieux (2012) and Chung and Ely (2003) both show that Maskin monotonicity, a necessary condition for Nash implementation, is also necessary if we want implementation using refinements of Nash equilibria to be robust to information perturbations.[6] Our mechanism is finite so that it does not use the *integer games* which are often considered a questionable device in the literature.[7] The use of transfers can be dispensed with completely, which allows us to apply our result to an important class of environments including the models of voting and matching in which monetary transfers are simply unavailable.

We finally take up Korpela (2016) which is perhaps the closest to our paper.[8] Korpela (2016) uses a weaker notion of implementation than our compelling implementation in the sense that his notion of implementation ignores all uncompelling Nash equilibria, "regardless of whether they are pure or mixed." Therefore, Korpela's (2016) notion of implementation does not necessarily imply pure strategy Nash implementation, whereas our compelling implementation does.[9]

We organize the rest of the paper as follows: Section 2 presents the environment, notation, mechanisms and solution concepts, as well as a small discussion on Maskin Monotonicity. Section 3 revisits Example 4 of Jackson (1992), which motivates our inquiry. Section 4 slightly modifies Example 4 of Jackson (1992) and presents an illustration of this paper's main result. Section 5 contains the main result of the paper when there are at least three agents and proposes a canonical mechanism that can achieve compelling implementation under Condition COM. Section 6 argues that Condition COM is indispensable for compelling implementation in the sense that our mechanism fails to achieve compelling implementation when at least one property of Condition COM is violated. Section 7 extends the main result of Section 5 to the case of two agents. In Section 8, we compare Properties 1 and 2 of Condition COM with Condition μ of Moore and Repullo (1990)

[6]Aghion, Fudenberg, Holden, Kunimoto, and Tercieux (2012) and Chung and Ely (2003) adopt subgame perfect equilibrium and undominated Nash equilibrium as a solution concept, respectively.

[7]In the integer game, each agent announces some integer and the person who announces the highest integer gets to name his favorite outcome.

[8]This paper has been developed independently of Korpela (2016) and we only became aware of it after we completed the first draft of the paper.

[9]In the rest of the paper, we will further make the connection to Korpela (2016) wherever necessary.

for pure Nash implementation. Section 9 concludes the paper and the Appendix contains the proofs omitted from the main body of the paper.

2 Preliminaries

There is a finite set of agents, denoted by $I = \{1, 2, \dots, n\}$. Let Θ be the finite set of states. It is assumed that the underlying state $\theta \in \Theta$ is commonly certain among the agents. This is the complete information assumption. Let A denote the set of social alternatives, which are assumed to be independent of the information state. We shall assume that A is finite, and denote by $\Delta(A)$ the set of probability distributions over A. Associated with each state θ is a preference profile $\succeq^{\theta} = (\succeq_i^{\theta})_{i\in I}$ where $\succeq_i^{\theta}$ is agent i's preference relation over A at θ. We write $a \succeq_i^{\theta} a'$ when agent i weakly prefers a to a' in state θ. We also write $a \succ_i^{\theta} a'$ if agent i strictly prefers a to a' in state θ and $a \sim_i^{\theta} a'$ if agent i is indifferent between a and a' in state θ. We can now define an *environment* as $\mathcal{E} = \left(I, A, \Theta, (\succeq_i^{\theta})_{i\in I, \theta\in\Theta}\right)$, which is implicitly understood to be commonly certain among the agents.

We assume that any preference relation $\succeq_i^{\theta}$ is representable by a von Neumann-Morgenstern utility function $u_i(\cdot, \theta) : \Delta(A) \to \mathbb{R}$. We say that $u_i(\cdot, \theta)$ is consistent with $\succeq_i^{\theta}$ if, for any $a, a' \in A$, $u_i(a, \theta) \geq u_i(a', \theta) \Leftrightarrow a \succeq_i^{\theta} a'$. We denote by $\mathcal{U}_i^{\theta}$ the set of all possible cardinal representations $u_i(\cdot, \theta)$ that are consistent with $\succeq_i^{\theta}$. We formally define $\mathcal{U}_i^{\theta}$ as follows:

$$\mathcal{U}_i^{\theta} = \left\{ u_i(\cdot, \theta) \in [0,1]^{|A|} \;\middle|\; \begin{array}{l} u_i(\cdot, \theta) \text{ is consistent with } \succeq_i^{\theta}; \min_{a\in A} u_i(a, \theta) = 0; \\ \text{and } \max_{a\in A} u_i(a, \theta) = 1 \end{array} \right\},$$

where $|A|$ denotes the cardinality of A. Let $\mathcal{U}^{\theta} \equiv \times_{i\in I} \mathcal{U}_i^{\theta}$ and $\mathcal{U} \equiv \times_{\theta\in\Theta} \mathcal{U}^{\theta}$. We denote any subset of $\mathcal{U}_i^{\theta}$ by $\hat{\mathcal{U}}_i^{\theta}$, any subset of $\mathcal{U}^{\theta}$ by $\hat{\mathcal{U}}^{\theta}$, any subset of $\mathcal{U}$ by $\hat{\mathcal{U}}$, respectively.

The planner's objective is specified by a *social choice function (henceforth, SCF)* $f : \Theta \to \Delta(A)$, which takes only ordinal information about the state θ as input, but is allowed to have lotteries as outputs. Although many papers deal with multi-valued social choice correspondences in the literature of Nash implementation, we focus only on single-valued SCFs.

2.1 Compelling Implementation

Let $\Gamma = ((M_i)_{i\in I}, g)$ be a finite mechanism where M_i is a nonempty *finite* set of messages available to agent i; $g : M \to \Delta(A)$ (where $M \equiv \times_{i\in I} M_i$) is the *outcome function*. At each state $\theta \in \Theta$ and profile of representations $u \in \mathcal{U}$, the environment and the mechanism together constitute a *game with complete information* which we denote by $\Gamma(\theta, u)$. By $\Gamma(\theta)$ we mean the game in which the preference profile

$(\succeq_i^\theta)_{i\in N}$ is commonly certain among the agents so that any representation $u \in \mathcal{U}$ is admissible. Note that the restriction of M_i to a finite set rules out the use of integer games (See, for example, Maskin (1999)).

Let $\sigma_i \in \Delta(M_i)$ be a mixed *strategy* of agent i in the game $\Gamma(\theta, u)$. A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n) \in \times_{i\in I}\Delta(M_i)$ is said to be a mixed-strategy *Nash equilibrium* of the game $\Gamma(\theta, u)$ if, for all agents $i \in I$ and all messages $m_i \in \text{supp}\,(\sigma_i)$ and $m_i' \in M_i$, we have

$$\sum_{m_{-i}\in M_{-i}} \prod_{j\neq i} \sigma_j(m_j) u_i(g(m_i, m_{-i}), \theta) \geq \sum_{m_{-i}\in M_{-i}} \prod_{j\neq i} \sigma_j(m_j) u_i(g(m_i', m_{-i}), \theta).$$

A *pure-strategy* Nash equilibrium is a mixed-strategy Nash equilibrium σ such that each agent i's mixed-strategy σ_i assigns probability one to some $m_i \in M_i$. Let $NE(\Gamma(\theta, u))$ denote the set of mixed-strategy Nash equilibria of the game $\Gamma(\theta, u)$ and $pureNE(\Gamma(\theta))$ denote the set of pure strategy Nash equilibria of the game $\Gamma(\theta)$. As far as we are only concerned with pure strategy equilibria, we only need ordinal preferences so that we can write $pureNE(\Gamma(\theta))$. We also define

$$NE(\Gamma(\theta)) = \bigcup_{u\in\mathcal{U}^\theta} NE(\Gamma(\theta, u))$$

as the set of all Nash equilibria of the class of games $\Gamma(\theta, u)$ across all possible representation $u \in \mathcal{U}^\theta$. Since it does not depend upon cardinal utilities, $NE(\Gamma(\theta))$ is defined only in terms of ordinal preferences. We say that an SCF f is *pure Nash implementable* if there exists a mechanism $\Gamma = (M, g)$ such that for any state θ, the following two conditions hold: (i) $pureNE(\Gamma(\theta)) \neq \emptyset$; and (ii) $m \in pureNash(\Gamma(\theta)) \Rightarrow g(m) = f(\theta)$.

We strengthen the notion of pure Nash implementation by requiring that any mixed equilibrium outcome, if exists, be either socially desirable or "uncompelling" in the sense that it is strictly Pareto dominated by the socially desirable outcome. For every mixed strategy profile $\sigma \in \times_{i\in I}\Delta(M_i)$, we write

$$g(\sigma) \equiv \sum_{m\in M} \sigma(m) g(m).$$

Our notion of implementation can then be formally defined as follows:

Definition 1 *Let $\hat{\mathcal{U}} \subseteq \mathcal{U}$. An SCF f is* ***compellingly implementable (C-implementable) with respect to $\hat{\mathcal{U}}$*** *if there exists a finite mechanism $\Gamma = (M, g)$ such that for every state $\theta \in \Theta$, (i) $pureNE(\Gamma(\theta)) \neq \emptyset$; (ii) $m \in pureNE(\Gamma(\theta)) \Rightarrow g(m) = f(\theta)$; and (iii) for any $u \in \hat{\mathcal{U}}^\theta$ and $\sigma \in NE(\Gamma(\theta, u))$, $g(\sigma) \neq f(\theta) \Rightarrow u_i(f(\theta), \theta) > u_i(g(\sigma), \theta)$ for all $i \in I$.*

Remark 1 *The implementing mechanism may have two types of mixed strategy equilibria. We call the first type of it a "good" mixed strategy equilibrium in the sense that its outcome is socially desirable and call the second type of it a "bad" mixed strategy equilibrium in the sense that its outcome is strictly worse for all agents than the socially desirable outcome. Our notion of compelling implementation says that the planner should ignore bad mixed strategy equilibria in the mechanism.*

3 The Relevance of Mixed Strategy Equilibria in Nash Implementation

In this section, we articulate a compelling reason why we need to be worried about mixed strategy equilibria in Nash implementation. To do so, we revisit Example 4 of Jackson (1992), which shows that the omission of mixed strategy equilibria brings about a serious blow to Nash implementation.

We revisit Example 4 of Jackson (1992). Suppose that there are two agents $I = \{1, 2\}$; four alternatives $A = \{a, b, c, d\}$; and two states $\Theta = \{\theta, \theta'\}$. Suppose that agent 1 has the state-independent preference $a \succ_1 b \succ_1 c \sim_1 d$ and agent 2 has the preference $a \succ_2^\theta b \succ_2^\theta d \succ_2^\theta c$ at state θ and preference $b \succ_2^{\theta'} a \succ_2^{\theta'} c \sim_2^{\theta'} d$ at state θ'. Consider the SCF f such that $f(\theta) = a$ and $f(\theta') = c$.

First, Jackson (1992) constructs a finite mechanism $\Gamma = (M, g)$ (described in Table 2) that implements the SCF f in pure-strategy Nash equilibria:

$g(m)$		Agent 2		
		m_2^1	m_2^2	m_2^3
Agent 1	m_1^1	c	d	d
	m_1^2	d	a	b
	m_1^3	d	b	a

Table 2: The mechanism introduced in Example 4 of Jackson (1992).

There are two pure strategy Nash equilibria, (m_1^2, m_2^2) and (m_1^3, m_2^3), in the game $\Gamma(\theta)$, both of which result in outcome a. In the game $\Gamma(\theta^{'})$, the unique pure-strategy Nash equilibrium is (m_1^1, m_2^1), which results in outcome c. Thus, the SCF f is implementable by the above finite mechanism in pure-strategy Nash equilibria. Due to the necessity of Maskin monotonicity for Nash implementation, we know that the SCF f satisfies Maskin monotonicity. However, in the game $\Gamma(\theta^{'})$, there is a mixed-strategy Nash equilibrium, where each agent i plays m_i^2 and m_i^3 with equal probability, which results in outcomes a and b, each with probability 1/2. Both agents strictly prefer any outcome of the mixed-strategy equilibrium to the outcome of the pure-strategy equilibrium. Thus, according to our terminology, this mixed strategy Nash equilibrium is compelling. Note that there is a conflict of interests between the two agents over a and b in state $\theta^{'}$, i.e., while agent 1 prefers a to b, agent 2 prefers b to a. This conflict of interests allows us to have the unique pure strategy Nash equilibrium in the game $\Gamma(\theta^{'})$, which results in outcome c. At the same time, this logic for the uniqueness of the pure-strategy equilibrium is extremely dubious because outcomes a and b are strictly better for both agents than outcome c.

Jackson (1992) further shows that his argument applies to *any* finite implementing mechanism. That is, for any finite mechanism which implements the SCF f in pure-strategy Nash equilibria, there must also exist a compelling mixed-strategy Nash equilibrium at state θ' inducing a lottery different from c, which is the socially desirable outcome by the SCF f at state $\theta^{'}$. Therefore, the SCF f is "not" C-implementable with respect to $\mathcal{U}$, which is the set of "all" cardinal utility representations, or any of its subsets. It thus follows that the identified compelling mixed strategy equilibrium persists independently of any cardinal representation.

4 Illustration of the Main Result

The main objective of this paper is to identify a class of environments where the issue of mixed strategy equilibria can be avoided by carefully designing an implementing mechanism. In this section, we illustrate how we resolve this issue in the slightly modified version of Example 4 of Jackson (1992).

One crucial feature Jackson's Example 4 has is that its argument seems to rely heavily on the extreme inefficiency of the SCF, i.e., the SCF f assigns the common worst outcome in state $\theta^{'}$.[10] To investigate how robust Jackson's argument is, we only make the following modification: both agents now strictly prefer c to d in state $\theta^{'}$, i.e., $c \succ_i^{\theta^{'}} d$ for each $i = 1, 2$.

We summarize the basic setup. Agent 1 has the state-independent preference $a \succ_1 b \succ_1 c \succ_1 d$ and agent 2 has the preference $a \succ_2^{\theta} b \succ_2^{\theta} d \succ_2^{\theta} c$ at state θ and preference $b \succ_2^{\theta'} a \succ_2^{\theta'} c \succ_2^{\theta'} d$ at state θ'. Consider the same SCF f such that $f(\theta) = a$ and $f(\theta') = c$. This way the SCF never assigns the worst outcome for any agent in either state (a feature that will also be implied by our sufficient condition).

With this modification, we are able to construct a mechanism that not only implements the SCF in pure-strategy Nash equilibrium but also guarantees that

[10] Jackson (1992, p.770) is well aware of this point.

all mixed-strategy equilibria of the constructed mechanism give each agent the expected payoff arbitrarily close to that of d, which is worse than that of c, the outcome induced by the SCF f at state θ'. Hence, we essentially overturn the implication of Jackson's Example 4 by assuming that there is a uniform bound for the utility difference.[11]

For each integer $k \geq 2$, we define $\Gamma^k = (M^k, g^k)$ as a mechanism with the following properties: (i) for each $i \in N$, $M_i^k = \{0, 1, \ldots, k\}$ and (ii) the outcome function $g^k : M^k \to A$ is given by the following rules: for each $m \in M^k$,

- If $m = (k, k)$, then $g^k(m) = c$;
- If there exists an integer h with $0 \leq h \leq k-1$ such that $m = (h, h)$, then $g^k(m) = a$;
- If there exists an integer h with $0 \leq h \leq k-1$ such that $m = (h, (h + 1 \bmod k))$, then $g^k(m) = b$; and
- Otherwise, $g^k(m) = d$.

We illustrate this mechanism as follows:

$g^k(m)$		Agent 2								
		k	$k-1$	$k-2$	$k-3$	$\cdots$	3	2	1	0
Agent 1	k	c	d	d	d	$\cdots$	d	d	d	d
	$k-1$	d	a	d	d	$\cdots$	d	d	d	b
	$k-2$	d	b	a	d	$\cdots$	d	d	d	d
	$k-3$	d	d	b	a	$\cdots$	d	d	d	d
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	3	d	d	d	d	$\cdots$	a	d	d	d
	2	d	d	d	d	$\cdots$	b	a	d	d
	1	d	d	d	d	$\cdots$	d	b	a	d
	0	d	d	d	d	$\cdots$	d	d	b	a

Table 3: $\Gamma^k = (M^k, g^k)$ where $k \geq 3$.

When $k = 2$, our mechanism is reduced to the one introduced by Jackson (1992) where we set $m_i^1 = 2; m_i^2 = 1$; and $m_i^3 = 0$ for each $i \in \{1, 2\}$.

[11]The mechanism presented here differs slightly from the canonical mechanism presented in Section 5. Specifically, the mechanism here has been tailored to the particular example at hand and simplified for ease of exposition. Nevertheless, the main insights of this paper are still obtained in this illustration section.

$g(m)$		Agent 2		
		2	1	0
Agent 1	2	c	d	d
	1	d	a	b
	0	d	b	a

Table 4: $\Gamma^k = (M^k, g^k)$ where $k = 2$.

For each $\theta \in \Theta$, $i \in \{1, 2\}$, and $\varepsilon > 0$, we define $\mathcal{U}_i^{\theta,\varepsilon}$ as a subset of $\mathcal{U}_i^\theta$ as follows:

$$\mathcal{U}_i^{\theta,\varepsilon} = \left\{ u_i \in \mathcal{U}_i^\theta \middle| \ |u_i(a,\theta) - u_i(a',\theta)| \geq \varepsilon, \ \forall a \in A, \forall a' \in A \backslash \{a\}, \ \forall \theta \in \Theta \right\}.$$

Let $\mathcal{U}^{\theta,\varepsilon} \equiv \times_{i \in N} \mathcal{U}_i^{\theta,\varepsilon}$ and $\mathcal{U}^\varepsilon \equiv \times_{\theta \in \Theta} \mathcal{U}^{\theta,\varepsilon}$. We observe that $\mathcal{U}^\varepsilon$ possesses the following monotonicity:

$$\varepsilon > \varepsilon' > 0 \Rightarrow \mathcal{U}^\varepsilon \subsetneq \mathcal{U}^{\varepsilon'} \subseteq \mathcal{U} \subseteq \mathcal{U}^0.$$

Loosely speaking, if we choose $\varepsilon > 0$ small enough, we can approximate $\mathcal{U}$ by $\mathcal{U}^\varepsilon$ to an arbitrary degree. We are now ready state the main result of this section.

Proposition 1 *For any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ large enough such that the SCF f is C-implementable with respect to $\mathcal{U}^\varepsilon$ by the mechanism Γ^K.*

Proof: The proof is completed by a series of lemmas. For the moment, we fix k in the proof and we ignore the dependence of the mechanism on k. We first show pure Nash implementation by the mechanism Γ^k.

Lemma 1 *The mechanism Γ^k implements the SCF in pure-strategy Nash equilibrium.*

Proof: The message profile $(1, 1)$ is a Nash equilibrium of the game $\Gamma^k(\theta)$, as it yields a which is their most preferred outcome for both agents so that no agent can find a profitable deviation. We claim that a is the unique Nash equilibrium outcome of the game $\Gamma^k(\theta)$. Let m be a message profile such that $g(m) \neq a$. We will show that m is "not" a Nash equilibrium in the game $\Gamma^k(\theta)$:

- If $g(m) = b$, there exists an integer h with $0 \leq h \leq k - 1$ such that $m = (h, (h + 1 \bmod k))$. Then agent 1 has an incentive to send a message $h + 1 \bmod k$ so that outcome a is induced.

- If $g(m) = c$, then $m = (k, k)$. Then, agent 2 has an incentive to send any message other than k so that outcome d is induced, as he strictly prefers outcome d to outcome c at state θ.

- If $g(m) = d$, then we have $m = (m_1, m_2)$ where $m_1 \neq m_2$. If $m_1 > m_2$ then, agent 1 has an incentive to deviate from m_1 to m_2 so that outcome a is induced. Conversely, if $m_2 > m_1$, then agent 2 has an incentive to deviate from m_2 to m_1, so that outcome a is induced.

We next claim that (k, k) is a Nash equilibrium of the game $\Gamma^k(\theta^{'})$ because any unilateral deviation from (k, k) yields d, which is inferior to c induced by (k, k) for both agents. Moreover, no other outcome can be induced by a Nash equilibrium in this game: every message profile $m = (m_1, m_2)$ where $m_2 < k$ and $g(m) \neq a$ has a profitable deviation for player 1 at $m_1' = m_2$, while every message profile $m = (m_1, m_2)$ where $m_1 < k$ and $g(m) \neq b$ has a profitable deviation for player 2 at $m_2' = m_1 + 1 \text{ mod } k$. Since $g(m) = a$ implies $m_1 < k$ and $g(m) = b$ implies $m_2 < k$, we have that there are no possible Nash equilibria with either $m_1 < k$ or $m_2 < k$. Thus, the only possible Nash equilibrium in pure strategies for this game is (k, k). ■

The following lemma is our key result, characterizing the set of Nash equilibria of the mechanism Γ^k in state $\theta^{'}$.

Lemma 2 *For each $i \in \{1, 2\}$, let $\sigma_i = (\sigma_i(0), \sigma_i(1), ..., \sigma_i(k))$ denote agent i's strategy and for each $x \in \{0, 1, \ldots, k\}$, let $\sigma_i(x)$ denote the probability that agent i chooses x. If $\sigma = (\sigma_1, \sigma_2)$ is a Nash equilibrium in the game $\Gamma^k(\theta^{'})$, then, for each $i \in \{1, 2\}$, there is a number $p^i \in [0, 1]$ such that $\sigma_i(x) = p^i/k$ for each $x \in \{0, \ldots, k-1\}$. Moreover, $p^1 = 0$ if and only if $p^2 = 0$.*

Proof: Recall that we set $u_i(d; \theta^{'}) = 0$ for each $u_i \in \mathcal{U}_i^{\theta^{'}}$ and $i \in \{1, 2\}$. Let σ be a Nash equilibrium of the game $\Gamma^k(\theta^{'})$. If $\sigma_i(k) = 1$ for each $i \in \{1, 2\}$, such p^i in the lemma is guaranteed to exist by setting $p^i = 0$. Thus, we assume that there exists $i \in \{1, 2\}$ for whom $\sigma_i(k) < 1$. We divide the proof into a series of steps, whose proofs will be found in the Appendix:

Step 1a: If there exists $x \in \{0, \ldots, k-1\}$ such that $\sigma_1(x) > 0$, then $\sigma_2(x) > 0$.

Step 1b: If there exists $x \in \{1, \ldots, k-1\}$ such that $\sigma_2(x) > 0$, then $\sigma_1(x-1) > 0$. Moreover, if $\sigma_2(0) > 0$, then $\sigma_1(k-1) > 0$.

Step 1c: If there exist $i \in \{1, 2\}$ and $x^{'} \in \{0, \ldots, k-1\}$ for whom $\sigma_i(x^{'}) > 0$, then $\sigma_1(x) > 0$ and $\sigma_2(x) > 0$ for all $x \in \{0, \ldots, k-1\}$.

Step 2: If there exist $i \in \{1, 2\}$ and $x, x^{'} \in \{0, \ldots, k-1\}$ such that $\sigma_i(x) > 0$ and $\sigma_i(x^{'}) > 0$, then $\sigma_i(x) = \sigma_i(x^{'})$.

It follows from both Steps 2 and 1c that $\sigma_i(x) = \sigma_i(x^{'})$ for every $x, x^{'} \in \{0, \ldots, k-1\}$ and $i \in \{1,2\}$. Thus, we can set $p^i = \sum_{x=0}^{k-1} \sigma_i(x)$. Since we assume $\sigma_i(k) < 1$ for each $i \in \{1,2\}$, we have $p^i > 0$. This completes the proof of Lemma 2. ■

As we can easily see in the proof of Lemma 1, there are no (compelling) mixed strategy Nash equilibria of the game $\Gamma^k(\theta)$ because, in state θ, the unique Nash equilibrium outcome is a, which is the best outcome for both agents. It thus remains to prove that there are no compelling mixed strategy equilibria in the game $\Gamma^k(\theta^{'})$.

If $k \geq 3$, we let σ^k be a nontrivial mixed-strategy Nash equilibrium in the game $\Gamma^k(\theta^{'})$. Then, the resulting outcome distribution induced by σ^k is given by

$$g \circ \sigma^k = \begin{cases} c & \text{w.p. } (1-p^1)(1-p^2) \\ a & \text{w.p. } (p^1 p^2)/k \\ b & \text{w.p. } (p^1 p^2)/k \\ d & \text{w.p. } ((k - 2p^1 p^2)/k) - ((1-p^1)(1-p^2)), \end{cases}$$

where $p^1, p^2 \in (0,1]$ and $p^i = \sum_{x=0}^{k-1} \sigma_i(x)$ for each $i \in \{1,2\}$. Recall the following pieces of notation:

$$\begin{aligned} \mathcal{U}_1^{\theta^{'}} &= \left\{ u_1(\cdot;\theta^{'}) \in [0,1]^A \middle| \, 1 = u_1(a;\theta^{'}) > u_1(b;\theta^{'}) > u_1(c;\theta^{'}) > u_1(d;\theta^{'}) = 0 \right\}; \\ \mathcal{U}_2^{\theta^{'}} &= \left\{ u_2(\cdot;\theta^{'}) \in [0,1]^A \middle| \, 1 = u_2(b;\theta^{'}) > u_2(a;\theta^{'}) > u_2(c;\theta^{'}) > u_2(d;\theta^{'}) = 0 \right\}. \end{aligned}$$

Let $\mathcal{U}^{\theta^{'}} \equiv \mathcal{U}_1^{\theta^{'}} \times \mathcal{U}_2^{\theta^{'}}$. For each $\varepsilon \in (0,1)$, we have

$$\begin{aligned} \mathcal{U}_1^{\theta^{'},\varepsilon} &= \left\{ u_1(\cdot;\theta^{'}) \in \mathcal{U}_1^{\theta^{'}} \middle| \, u_1(c;\theta^{'}) \geq \varepsilon \right\}; \\ \mathcal{U}_2^{\theta^{'},\varepsilon} &= \left\{ u_2(\cdot;\theta^{'}) \in U_2^{\theta^{'}} \middle| \, u_2(c;\theta^{'}) \geq \varepsilon \right\}. \end{aligned}$$

Similarly, let $\mathcal{U}^{\theta^{'},\varepsilon} \equiv \mathcal{U}_1^{\theta^{'},\varepsilon} \times \mathcal{U}_2^{\theta^{'},\varepsilon}$.

By the lemma below, we show that for each $\varepsilon > 0$, there exists $K \in \mathbb{N}$ large enough so that, for any $u \in \mathcal{U}^{\theta^{'},\varepsilon}$, the game $\Gamma^K(\theta^{'}, u)$ has no compelling mixed strategy equilibria.

Lemma 3 *For each $\varepsilon > 0$, there exists an integer $K \in \mathbb{N}$ large enough so that for any $k \geq K$, $i \in \{1,2\}$, and $(u_1(\cdot,\theta^{'}), u_2(\cdot;\theta^{'})) \in \mathcal{U}^{\theta^{'},\varepsilon}$,*

$$U_i(\sigma^k;\theta^{'}) < u_i(c;\theta^{'}),$$

where $U_i(\sigma^k;\theta^{'}) = \sum_{x=0}^{k} \sigma_1^k(x) \sum_{x^{'}=0}^{k} \sigma_2^k(x^{'}) u_i(g(x,x^{'});\theta^{'})$.

Proof: Fix $\varepsilon > 0$ and $i \in \{1,2\}$. We compute

$$U_i(\sigma^k;\theta') = \frac{p^1p^2}{k}[u_i(a;\theta') + u_i(b;\theta')] + (1-p^1)(1-p^2)u_i(c;\theta').$$

For each $(p^1,p^2) \in [0,1]^2$, we define

$$k(p^1,p^2) = \frac{u_i(a;\theta') + u_i(b;\theta')}{u_i(c;\theta')}\left[\frac{1}{p^1} + \frac{1}{p^2} - 1\right]^{-1}.$$

In the rest of the proof, we make use of the following properties of $k(p^1,p^2)$:

- $k(\cdot,\cdot)$ is strictly increasing in both arguments over $[0,1]^2$.
- $k(p^1_h,p^2_h)$ converges to zero no matter how the sequence $\{(p^1_h,p^2_h)\}_{h=1}^{\infty}$ approaches $(0,0)$. Thus, $k(0,0) \equiv \lim_{(p^1,p^2)\to(0,0)} k(p^1,p^2) = 0$.
- $k(1,1) = [u_i(a;\theta') + u_i(b;\theta')]/u_i(c;\theta') = \max_{(p^1,p^2)\in[0,1]^2} k(p^1,p^2)$.
- We can conveniently rewrite $k(p^1,p^2)$ as

$$k(p^1,p^2) = \frac{u_i(a;\theta') + u_i(b;\theta')}{u_i(c;\theta')}\frac{p^1p^2}{[1-(1-p^1)(1-p^2)]}.$$

We set $K = \min\{k \in \mathbb{N} | k \geq 2/\varepsilon\}$. As $2/\varepsilon \geq [u_i(a;\theta') + u_i(b;\theta')]/u_i(c;\theta')$ for any $u_i(\cdot;\theta') \in \mathcal{U}_i^{\theta'}[\varepsilon]$, we have that $K \geq k(1,1)$. Due to the strict monotonicity of $k(p^1,p^2)$ with respect to p^1 and p^2, we have that $K \geq k(p^1,p^2)$ for any $(p^1,p^2) \in [0,1]^2$. Hence, for any $k \geq K$:

$$\begin{aligned}
U_i(\sigma^k;\theta') &= \frac{p^1p^2}{k}[u_i(a;\theta') + u_i(b;\theta')] + (1-p^1)(1-p^2)u_i(c;\theta') \\
&\leq \frac{p^1p^2}{k(p^1,p^2)}[u_i(a;\theta') + u_i(b;\theta')] + (1-p^1)(1-p^2)u_i(c;\theta') \\
&\quad (\because\ k \geq K \geq k(p^1,p^2)\ \forall (p^1,p^2) \in [0,1]^2) \\
&= u_i(c;\theta')[1-(1-p^1)(1-p^2)] + (1-p^1)(1-p^2)u_i(c;\theta') \\
&= u_i(c;\theta').
\end{aligned}$$

This completes the proof of Lemma 3. ■

Combining Lemmas 1, 2, and 3 together, we complete the proof of Proposition 1. ■

5 The Main Result When $n \geq 3$

Throughout this section, we assume that there are at least three agents, i.e., $n \geq 3$. We refer the reader to Section 7 where we extend the main result of this section to the case of two agents.

5.1 Acceptability and Forums

Let $\mathcal{G}$ be a pair of agents in I. We call $C : \mathcal{G} \rightrightarrows \Delta(A)$ a *choice correspondence* if it maps each agent in $\mathcal{G}$ into a nonempty, finite subset of lotteries in $\Delta(A)$. Given a pair of agents $\mathcal{G}$ and a choice correspondence C, we define the concept of C-acceptability:

Definition 2 *Let $\mathcal{G} \subseteq I$ be a pair of agents, C a choice correspondence, $\theta \in \Theta$ a state, and $u \in \hat{\mathcal{U}}$ a representation. We say that lottery $x \in \Delta(A)$ is* C***-acceptable at state*** θ ***and representation*** u *if the following two conditions are satisfied:*

- $x \in \bigcup_{i \in \mathcal{G}} C(i)$;
- *For every $i \in \mathcal{G}$ and $y \in C(i)$, $u_i(x, \theta) \geq u_i(y, \theta)$.*

Remark: Strictly speaking, the definition of C-acceptability depends upon $\mathcal{G}$. However, since such $\mathcal{G}$ is always clear from the context whenever we discuss C-acceptability, we omit C's dependence on $\mathcal{G}$. Thus, we simply say C-acceptability without mentioning $\mathcal{G}$. If there are only two agents, i.e., $n = 2$, then there is no ambiguity about $\mathcal{G}$ so that we always take $\mathcal{G} = \{1, 2\}$ in the definition of C-acceptability.

We say that $\mathcal{F} = (\mathcal{G}, w, C, z)$ constitutes a *forum* if it satisfies the following properties:

1. $\mathcal{G}$ is a pair of agents in I;
2. $w : \{0, 1\} \to \mathcal{G}$ is a bijection, where we denote w^{-1} by its inverse function so that $w(w^{-1}(i)) = i$;
3. $C : \mathcal{G} \rightrightarrows \Delta(A)$ is a choice correspondence; and
4. $z \in \Delta(A)$ is a lottery such that $z \in \bigcap_{i \in \mathcal{G}} C(i)$.

For each $\hat{\theta} \in \Theta$, we write $\mathcal{F}_{\hat{\theta}} = (\mathcal{G}_{\hat{\theta}}, w_{\hat{\theta}}, C_{\hat{\theta}}, z_{\hat{\theta}})$ as a forum indexed by the state $\hat{\theta} \in \Theta$. In the forum $\mathcal{F}_{\hat{\theta}}$, we have $z_{\hat{\theta}} \in C_{\hat{\theta}}(i)$ for each $i \in \mathcal{G}_{\hat{\theta}}$. If the forum $\mathcal{F}_{\hat{\theta}}$ is used when θ is the true state, we define

$$C^*_{\hat{\theta}}(j, \theta) \equiv \arg \max_{y \in C_{\hat{\theta}}(j)} u_j(y, \theta)$$

as the set of agent j's best lotteries in state θ within $C_{\hat{\theta}}(j)$.

5.2 Condition COM

Definition 3 *The environment $\mathcal{E} = \left(I, A, \Theta, (\succeq_i^\theta)_{i \in I, \theta \in \Theta}\right)$ satisfies **Condition** COM with respect to the SCF f and $\hat{\mathcal{U}}$ if there exists a collection of forums $\{\mathcal{F}_{\tilde{\theta}}\}_{\tilde{\theta} \in \Theta} = \{\mathcal{G}_{\tilde{\theta}}, w_{\tilde{\theta}}, C_{\tilde{\theta}}, z_{\tilde{\theta}}\}_{\tilde{\theta} \in \Theta}$ with the following properties:*

1. *This property has two parts:*
   - 1-i. *For every $\theta \in \Theta$ and $u \in \hat{\mathcal{U}}$, $f(\theta)$ is C_θ-acceptable at state θ and representation u.*
   - 1-ii. *For every $\theta, \hat{\theta} \in \Theta$, $f(\theta) \in C_{\hat{\theta}}(i) \Leftrightarrow f(\theta) \in C_{\hat{\theta}}(j)$, where $i = w_{\hat{\theta}}(0)$ and $j = w_{\hat{\theta}}(1)$. When $\hat{\theta} = \theta$, we have $f(\theta) \in C_\theta(i) \cap C_\theta(j)$ for all $\theta \in \Theta$.*
2. *For every $\theta, \hat{\theta} \in \Theta$ and $u \in \hat{\mathcal{U}}$, if $x \in \Delta(A)$ is $C_{\hat{\theta}}$-acceptable at state θ and representation u, then $x = f(\theta)$.*
3. *There exists $\varepsilon > 0$ such that for each $\theta \in \Theta$, $u_i(f(\theta), \theta) - u_i(z, \theta) \geq \varepsilon$ for all $i \in I$, all $u \in \hat{\mathcal{U}}$, and all $z \in \bigcup_{\tilde{\theta} \in \Theta}\{z_{\tilde{\theta}}\}$.*
4. *For all $\theta, \hat{\theta} \in \Theta$, $u \in \hat{\mathcal{U}}$, $i \in \mathcal{G}_{\hat{\theta}}$, if $f(\theta)$ is $C_{\hat{\theta}}$-acceptable at state θ and representation u, then $u_i(x, \theta) = u_i(f(\theta), \theta)$ implies $x = f(\theta)$ for all $x \in C_{\hat{\theta}}(i)$.*

Remark: There is a redundancy in Property 1. We do not need to assume $f(\theta) \in C_\theta(i) \cap C_\theta(j)$ for all $\theta \in \Theta$ in Property 1-ii because it follows from $f(\theta)$ is C_θ-acceptable at state θ and representation u in Property 1-i and $f(\theta) \in C_\theta(i) \Leftrightarrow f(\theta) \in C_\theta(j)$ when we set $\hat{\theta} = \theta$ in Property 1-i. Nevertheless, for the convenience of writing the proofs in the paper, we explicity add this property to part of Property 1-ii.

Property 2 admits a possibility that agent $i \in \mathcal{G}_{\hat{\theta}}$ is indifferent between $f(\theta)$ and some $x \neq f(\theta)$, while agent $j \in \mathcal{G}_{\hat{\theta}} \backslash \{i\}$ prefers $f(\theta)$ to x. However, Property 4 excludes this very possibility. This shows that Property 2 does not imply Property 4.

We present now an example of an environment with three players in which Condition COM is satisfied.

Example 1 *There are three agents, i.e.,* $I = \{1, 2, 3\}$. *Let* $A = \{a, b, c, d, z\}$ *be the set of pure alternatives,* $\Theta = \{\theta_a, \theta_b, \theta_c, \theta_d\}$ *be the set of states, and* f *be the SCF such that* $f(\theta_x) = x$ *for any* $x \in A\backslash\{z\}$. *Agents' preferences over* $A\backslash\{z\} = \{a, b, c, d\}$ *are summarized in Table 5. In addition, for any* $x \in A\backslash\{z\}$, *any* $i \in I$, *and any* $\theta \in \Theta$, *we assume* $x \succ_i^\theta z$, *so* z *is the common worst outcome across states.*

State	Agent 1	Agent 2	Agent 3
θ_a	$a \succ_1^{\theta_a} b \succ_1^{\theta_a} d \succ_1^{\theta_a} c$	$a \succ_2^{\theta_a} b \succ_2^{\theta_a} d \succ_2^{\theta_a} c$	$a \succ_3^{\theta_a} b \succ_3^{\theta_a} c \succ_3^{\theta_a} d$
θ_b	$a \succ_1^{\theta_b} b \succ_1^{\theta_b} c \succ_1^{\theta_b} d$	$b \succ_2^{\theta_b} a \succ_2^{\theta_b} c \succ_2^{\theta_b} d$	$b \succ_3^{\theta_b} a \succ_3^{\theta_b} d \succ_3^{\theta_b} c$
θ_c	$a \succ_1^{\theta_c} b \succ_1^{\theta_c} c \succ_1^{\theta_c} d$	$b \succ_2^{\theta_c} a \succ_2^{\theta_c} c \succ_2^{\theta_c} d$	$a \succ_3^{\theta_c} b \succ_3^{\theta_c} c \succ_3^{\theta_c} d$
θ_d	$d \succ_1^{\theta_d} a \succ_1^{\theta_d} b \succ_1^{\theta_d} c$	$b \succ_2^{\theta_d} a \succ_2^{\theta_d} c \succ_2^{\theta_d} d$	$a \succ_3^{\theta_d} b \succ_3^{\theta_d} d \succ_3^{\theta_d} c$

Table 5: Agents' Preferences over $A\backslash\{z\}$

Fix $\varepsilon > 0$ *as an arbitrary small number. For each* $i \in I$ *and* $\tilde{\theta} \in \Theta$, *we define*

$$\mathcal{U}_i^{\hat{\theta},\varepsilon} = \left\{ u_i(\cdot, \tilde{\theta}) \in [0,1]^5 \middle| \; u_i(z, \tilde{\theta}) = 0, \max_{\tilde{a} \in A} u_i(\tilde{a}, \tilde{\theta}) = 1, \text{ and } u_i(\tilde{a}, \tilde{\theta}) \geq \varepsilon, \forall \tilde{a} \in A\backslash\{z\} \right\},$$

as the set of all possible cardinal representations $u_i(\cdot, \tilde{\theta})$ *that are consistent with ordinal preferences* $\succeq_i^{\tilde{\theta}}$ *given in Table 5. Let* $\mathcal{U}^{\tilde{\theta},\varepsilon} \equiv \times_{i \in I} \mathcal{U}_i^{\tilde{\theta}}$ *and* $\mathcal{U}^\varepsilon \equiv \times_{\tilde{\theta} \in \Theta} \mathcal{U}^{\tilde{\theta},\varepsilon}$. *We construct the following collection of forums* $\{\mathcal{F}_{\tilde{\theta}}\}_{\tilde{\theta} \in \Theta} = \{\mathcal{G}_{\tilde{\theta}}, w_{\tilde{\theta}}, C_{\tilde{\theta}}, z_{\tilde{\theta}}\}_{\tilde{\theta} \in \Theta}$ *with the following properties:*

- *for each* $\theta \in \Theta$, *we set* $z_\theta = z$.
- *At state* θ_a *we set* $\mathcal{G}_{\theta_a} = \{1, 2\}$, $w_{\theta_a}(0) = 1$, $w_{\theta_a}(1) = 2$, *and* $C_{\theta_a}(1) = C_{\theta_a}(2) = \{a, b, c, d, z\}$.
- *At state* θ_b, *we set* $\mathcal{G}_{\theta_b} = \{2, 3\}$, $w_{\theta_b}(0) = 2$, $w_{\theta_b}(1) = 3$, *and* $C_{\theta_b}(2) = C_{\theta_b}(3) = \{a, b, c, d, z\}$.
- *At state* θ_c, *we set* $\mathcal{G}_{\theta_c} = \{2, 3\}$, $w_{\theta_c}(0) = 2$, $w_{\theta_c}(1) = 3$, *and* $C_{\theta_c}(2) = C_{\theta_c}(3) = \{c, d, z\}$.
- *Finally, at state* θ_d, *we set* $\mathcal{G}_{\theta_d} = \{1, 3\}$, $w_{\theta_d}(0) = 1$, $w_{\theta_d}(1) = 3$, *and* $C_{\theta_d}(1) = C_{\theta_c}(3) = \{c, d, z\}$.
- *Property 1-i*

  $f(\theta_a) = a$ *is* C_{θ_a}*-acceptable at* θ_a *and any* $u \in \mathcal{U}^\varepsilon$ *because* a *is the best outcome within* $C_{\theta_a}(i) = A$ *for each* $i \in \{1, 2\}$. $f(\theta_b) = b$ *is* C_{θ_b}*-acceptable* *at θ_b and any $u \in \mathcal{U}^\varepsilon$ because b is the best outcome within $C_{\theta_b}(i) = A$ for each $i \in \{2,3\}$. $f(\theta_c) = c$ is C_{θ_c}-acceptable at θ_c and any $u \in \mathcal{U}^\varepsilon$ because c is the best outcome within $C_{\theta_c}(i) = \{c,d,z\}$ for each $i \in \{2,3\}$. $f(\theta_d) = d$ is C_{θ_d}-acceptable at θ_d and any $u \in \mathcal{U}^\varepsilon$ because d is the best outcome within $C_{\theta_d}(i) = \{c,d,z\}$ for each $i \in \{1,3\}$.*

- *Property 1-ii*

  By construction, we set $C_{\tilde{\theta}}(i) = C_{\tilde{\theta}}(j)$ for all $\tilde{\theta} \in \Theta$ and $i,j \in \mathcal{G}_{\tilde{\theta}}$. This guarantees that $f(\theta) \in C_\theta(i) \cap C_\theta(j)$ for all $\theta \in \Theta$ and $i,j \in \mathcal{G}_\theta$.

  When $\theta = \theta_a$ and $\hat{\theta} = \theta_b$, it follows that $f(\theta_a) = a \in C_{\theta_b}(i) = A$ for all $i \in \{2,3\}$. When $\theta = \theta_a$ and $\hat{\theta} = \theta_c$, it follows that $f(\theta_a) = a \notin C_{\theta_c}(i) = \{c,d,z\}$ for all $i \in \{2,3\}$. When $\theta = \theta_a$ and $\hat{\theta} = \theta_d$, it follows that $f(\theta_a) = a \notin C_{\theta_d}(i) = \{c,d,z\}$ for all $i \in \{1,3\}$.

  When $\theta = \theta_b$ and $\hat{\theta} = \theta_a$, it follows that $f(\theta_b) = b \in C_{\theta_a}(i) = A$ for all $i \in \{1,2\}$. When $\theta = \theta_b$ and $\hat{\theta} = \theta_c$, it follows that $f(\theta_b) = b \notin C_{\theta_c}(i) = \{c,d,z\}$ for all $i \in \{2,3\}$. When $\theta = \theta_b$ and $\hat{\theta} = \theta_d$, it follows that $f(\theta_b) = b \notin C_{\theta_d}(i) = \{c,d,z\}$ for all $i \in \{1,3\}$.

  When $\theta = \theta_c$ and $\hat{\theta} = \theta_a$, it follows that $f(\theta_c) = c \in C_{\theta_a}(i) = A$ for all $i \in \{1,2\}$. When $\theta = \theta_c$ and $\hat{\theta} = \theta_b$, it follows that $f(\theta_c) = c \in C_{\theta_b}(i) = A$ for all $i \in \{2,3\}$. When $\theta = \theta_c$ and $\hat{\theta} = \theta_d$, it follows that $f(\theta_c) = c \in C_{\theta_d}(i) = \{c,d,z\}$ for all $i \in \{1,3\}$.

  When $\theta = \theta_d$ and $\hat{\theta} = \theta_a$, it follows that $f(\theta_d) = d \in C_{\theta_a}(i) = A$ for all $i \in \{1,2\}$. When $\theta = \theta_d$ and $\hat{\theta} = \theta_b$, it follows that $f(\theta_c) = c \in C_{\theta_b}(i) = A$ for all $i \in \{2,3\}$. When $\theta = \theta_d$ and $\hat{\theta} = \theta_c$, it follows that $f(\theta_d) = d \in C_{\theta_c}(i) = \{c,d,z\}$ for all $i \in \{2,3\}$.

- *Property 2*

  When $\theta = \theta_a$, for any $\hat{\theta} \in \{\theta_a, \theta_b\}$ and $u \in \mathcal{U}^\varepsilon$, it follows that a is the unique outcome in $\Delta(A)$ that is $C_{\hat{\theta}}$-acceptable at θ_a and u. Then, we have $f(\theta_a) = a$.

  When $\theta = \theta_a$ and $\hat{\theta} = \theta_c$, for any $u \in \mathcal{U}^\varepsilon$, it follows that there is no $x \in \Delta(A)$ that is C_{θ_c}-acceptable at θ_a and u because $u_2(d,\theta_a) > u_2(c,\theta_a)$ and $u_3(d,\theta_a) < u_3(c,\theta_a)$ and $C_{\theta_c}(i) = \{c,d,z\}$ for each $i \in \mathcal{G}_{\theta_c}$.

  When $\theta = \theta_a$ and $\hat{\theta} = \theta_d$, for any $u \in \mathcal{U}^\varepsilon$, it follows that there is no $x \in \Delta(A)$ that is C_{θ_d}-acceptable at θ_a and u because $u_1(d,\theta_a) > u_1(c,\theta_a)$ and $u_3(d,\theta_a) < u_3(c,\theta_a)$ and $C_{\theta_d}(i) = \{c,d,z\}$ for each $i \in \mathcal{G}_{\theta_d}$.

  When $\theta = \theta_b$, for any $\hat{\theta} \in \{\theta_a, \theta_b\}$ and $u \in \mathcal{U}^\varepsilon$, it follows that b is the unique outcome in $\Delta(A)$ that is $C_{\hat{\theta}}$-acceptable at θ_b and u. Then, we have $f(\theta_b) = b$.

When $\theta = \theta_b$ *and* $\hat{\theta} = \theta_c$*, for any* $u \in \mathcal{U}^\varepsilon$*, it follows that there is no* $x \in \Delta(A)$ *that is* C_{θ_c}*-acceptable at* θ_b *and* u *because* $u_2(d, \theta_b) < u_2(c, \theta_b)$ *and* $u_3(d, \theta_b) > u_3(c, \theta_b)$ *and* $C_{\theta_c}(i) = \{c, d, z\}$ *for each* $i \in \mathcal{G}_{\theta_c}$*.*

When $\theta = \theta_b$ *and* $\hat{\theta} = \theta_d$*, for any* $u \in \mathcal{U}^\varepsilon$*, it follows that there is no* $x \in \Delta(A)$ *that is* C_{θ_d}*-acceptable at* θ_b *and* u *because* $u_1(d, \theta_a) < u_1(c, \theta_a)$ *and* $u_3(d, \theta_a) > u_3(c, \theta_a)$ *and* $C_{\theta_d}(i) = \{c, d, z\}$ *for each* $i \in \mathcal{G}_{\theta_d} = \{1, 3\}$*.*

When $\theta = \theta_c$ *and* $\hat{\theta} = \theta_a$*, for any* $u \in \mathcal{U}^\varepsilon$*, it follows that there is no* $x \in \Delta(A)$ *that is* C_{θ_a}*-acceptable at* θ_c *and* u *because* $u_1(a, \theta_c) > u_1(b, \theta_c)$ *and* $u_2(a, \theta_c) < u_2(b, \theta_c)$ *and* $C_{\theta_a}(i) = A$ *for each* $i \in \mathcal{G}_{\theta_a} = \{1, 2\}$*.*

When $\theta = \theta_c$ *and* $\hat{\theta} = \theta_b$*, for any* $u \in \mathcal{U}^\varepsilon$*, it follows that there is no* $x \in \Delta(A)$ *that is* C_{θ_b}*-acceptable at* θ_c *and* u *because* $u_2(a, \theta_c) < u_2(b, \theta_c)$ *and* $u_3(a, \theta_c) > u_3(b, \theta_c)$ *and* $C_{\theta_b}(i) = A$ *for each* $i \in \mathcal{G}_{\theta_b} = \{2, 3\}$*.*

When $\theta = \theta_c$ *and* $\hat{\theta} = \theta_d$*, for any* $u \in \mathcal{U}^\varepsilon$*, it follows that* $c \in \Delta(A)$ *is the unique outcome that is* C_{θ_d}*-acceptable at* θ_c *and* u*. Then, we have* $f(\theta_c) = c$*.*

When $\theta = \theta_d$ *and* $\hat{\theta} = \theta_a$*, for any* $u \in \mathcal{U}^\varepsilon$*, it follows that there is no* $x \in \Delta(A)$ *that is* C_{θ_a}*-acceptable at* θ_d *and* u *because* d *is agent 1's best outcome within* $C_{\theta_a}(1)$ *in state* θ_d*, while* b *is agent 2's best outcome within* $C_{\theta_a}(2)$ *in state* θ_b *and* $C_{\theta_a}(i) = A$ *for each* $i \in \mathcal{G}_{\theta_a} = \{1, 2\}$*.*

When $\theta = \theta_d$ *and* $\hat{\theta} = \theta_b$*, for any* $u \in \mathcal{U}^\varepsilon$*, it follows that there is no* $x \in \Delta(A)$ *that is* C_{θ_b}*-acceptable at* θ_d *and* u *because* b *is agent 2's best outcome within* $C_{\theta_b}(2)$ *in state* θ_d*, while* a *is agent 3's best outcome within* $C_{\theta_b}(3)$ *in state* θ_d *and* $C_{\theta_b}(i) = A$ *for each* $i \in \mathcal{G}_{\theta_b} = \{2, 3\}$*.*

When $\theta = \theta_d$ *and* $\hat{\theta} = \theta_c$*, for any* $u \in \mathcal{U}^\varepsilon$*, it follows that* $d \in \Delta(A)$ *is the unique outcome that is* C_{θ_c}*-acceptable at* θ_c *and* u*. Then, we have* $f(\theta_d) = d$*.*

- *Property 3*

  This property is satisfied due to the very construction of $\mathcal{U}^\varepsilon$*.*

- *Property 4*

  When $\theta = \theta_a$ *and* $\hat{\theta} = \theta_b$*, for any* $u \in \mathcal{U}^\varepsilon$*, it follows that* $f(\theta_a) = a$ *is* C_{θ_b}*-acceptable at* θ_a *and* u*. Since* a *is the best outcome for both agents 2 and 3 in state* θ_a *and* $\mathcal{G}_{\theta_b} = \{2, 3\}$*, this property holds.*

  When $\theta = \theta_a$*, for any* $\hat{\theta} \in \{\theta_c, \theta_d\}$ *and* $u \in \mathcal{U}^\varepsilon$*, it follows that* $f(\theta_a)$ *is not* $C_{\hat{\theta}}$*-acceptable at* θ_a *and* u*. Hence, the property holds.*

  When $\theta = \theta_b$ *and* $\hat{\theta} = \theta_a$*, for any* $u \in \mathcal{U}^\varepsilon$*, it follows that* $f(\theta_b) = b$ *is not* C_{θ_a}*-acceptable at* θ_b *and* u *because* $u_1(a, \theta_b) > u_1(b, \theta_b)$*, while* $u_2(a, \theta_b) < u_2(b, \theta_b)$ *and* $C_{\theta_a}(i) = A$ *for each* $i \in \mathcal{G}_{\theta_a} = \{1, 2\}$*. Hence, this property holds.*

When $\theta = \theta_b$, for any $\hat{\theta} \in \{\theta_c, \theta_d\}$ and $u \in \mathcal{U}^{\varepsilon}$, it follows that $f(\theta_b)$ is not $C_{\hat{\theta}}$-acceptable at θ_b and u. Hence, the property holds.

When $\theta = \theta_c$, for any $\hat{\theta} \in \{\theta_a, \theta_b\}$ and $u \in \mathcal{U}^{\varepsilon}$, it follows that $f(\theta_c) = c$ is not $C_{\hat{\theta}}$-acceptable at θ_c and u. Hence, the property holds.

When $\theta = \theta_c$ and $\hat{\theta} = \theta_d$, for any $u \in \mathcal{U}^{\varepsilon}$, it follows that $f(\theta_c) = c$ is C_{θ_d}-acceptable at θ_c and u. Since c is the best outcome within $\{c, d, z\}$ for both agents 1 and 3 in state θ_c and $C_{\theta_d}(i) = \{c, d, z\}$ for each $i \in \mathcal{G}_{\theta_d} = \{1, 3\}$, the property holds.

When $\theta = \theta_d$, for any $\hat{\theta} \in \{\theta_a, \theta_b\}$ and $u \in \mathcal{U}^{\varepsilon}$, it follows that $f(\theta_d) = d$ is not $C_{\hat{\theta}}$-acceptable at θ_c and u. Hence, the property holds.

When $\theta = \theta_d$ and $\hat{\theta} = \theta_c$, for any $u \in \mathcal{U}^{\varepsilon}$, it follows that $f(\theta_d) = d$ is C_{θ_c}-acceptable at θ_d and u. For any $\tilde{a} \in \{c, z\}$, we have $u_i(\tilde{a}, \theta_d) \neq u_i(f(\theta_d), \theta_d)$ for each $i \in \{2, 3\}$. Hence, the property holds.

Therefore, this environment satisfies Condition COM with respect to the SCF f and $\mathcal{U}^{\varepsilon}$.

5.3 The Canonical Mechanism

Condition COM is utilized to construct our canonical mechanism that achieves compelling implementation. By Condition COM, we can fix a collection of forums $\{\mathcal{F}_{\tilde{\theta}}\}_{\tilde{\theta} \in \Theta}$ that satisfies all the properties in Condition COM. Recall that we assume that there are at least three agents, i.e., $n \geq 3$. Fix $\theta^* \in \Theta$ and $k \geq 2$ as a state and an integer, respectively. We write $\Gamma^k = (M^k, g^k)$ as a mechanism. We define $M_i^k \equiv M_i^1 \times M_i^2 \times M_i^3$ as agent i's message space in the mechanism Γ^k. Let $m_i = (m_i^1, m_i^2, m_i^3) \in M_i$ be agent i's generic message such that (i) $m_i^1 \in M_i^1 = \Theta$; (ii) $m_i^2 = (m_i^2[\tilde{\theta}])_{\tilde{\theta} \in \Theta} \in M_i^2 = \times_{\tilde{\theta} \in \Theta} M_i^2[\tilde{\theta}]$ where $m_i^2[\tilde{\theta}] \in \{0, \ldots, k-1\}$; and (iii) $m_i^3 = (m_i^3[\tilde{\theta}])_{\tilde{\theta} \in \Theta} \in M_i^3 \equiv \times_{\tilde{\theta} \in \Theta} M_i^3[\tilde{\theta}]$ where, for all $i \in I$, $M_i^3[\tilde{\theta}] = C_{\tilde{\theta}}(i)$ if $i \in \mathcal{G}_{\tilde{\theta}}$ and $M_i^3[\tilde{\theta}] = \{\emptyset\}$ if $i \notin \mathcal{G}_{\tilde{\theta}}$. In words, each agent i announces a state, a collection of state-contingent integers between 0 and $k-1$, and a collection of state-contingent outcomes such that each outcome in state $\tilde{\theta}$ is required to be chosen from $C_{\tilde{\theta}}(i)$. We thus define $M = \times_{i \in I} M_i$ as the set of message profiles in the mechanism Γ^k. For any $m \in M$, we define $\theta^m \in \Theta$ as follows:

$$\theta^m = \begin{cases} \theta' & \text{if there exists } \theta' \in \Theta \text{ such that } |\{j \in I | m_j^1 = \theta'\}| > n/2, \\ \theta^* & \text{otherwise.} \end{cases}$$

Note that θ^m is well defined because we assume $n \geq 3$.

For any $m \in M$, $g^k(m)$ induces the following two rules:

Rule 1: If $\sum_{j \in \mathcal{G}_{\theta^m}} m_j^2[\theta^m] \bmod k$ is either 0 or 1, then

$$g^k(m) = m_{i^*}^3[\theta^m],$$

where $i^* = w_{\theta^m}\left(\sum_{j \in \mathcal{G}_{\theta^m}} m_j^2[\theta^m] \bmod k\right)$.
Rule 2: If $\sum_{j \in \mathcal{G}_{\theta^m}} m_j^2[\theta^m] \bmod k > 1$, then

$$g^k(m) = z_{\theta^m}.$$

5.4 Main Theorem

Theorem 1 *Let f be an SCF. Suppose that the finite environment $\mathcal{E} = \left(I, A, \Theta, (\succeq_i^\theta)_{i \in I, \theta \in \Theta}\right)$ satisfies Condition COM with respect to f and $\hat{\mathcal{U}}$. Then, the SCF f is C-implementable with respect to $\hat{\mathcal{U}}$.*

Proof: Suppose that $\mathcal{E}$ satisfies Condition COM with respect to the SCF f and the set of representations $\hat{\mathcal{U}}$. Therefore, throughout the proof of the theorem, we fix a collection of forums $(\mathcal{F}_{\tilde{\theta}})_{\tilde{\theta} \in \Theta} = (\mathcal{G}_{\tilde{\theta}}, w_{\tilde{\theta}}, C_{\tilde{\theta}}, z_{\tilde{\theta}})$ that satisfies Properties 1 through 4 of Condition COM with respect to f and $\hat{\mathcal{U}}$. We prove this theorem through a series of steps.

Step 1: For any $k \geq 2$, the SCF f is pure Nash implementable by the mechanism Γ^k.

Proof of Step 1: Let $\theta \in \Theta$ be a true state. Fix $u \in \hat{\mathcal{U}}$ arbitrarily. By Property 1, $f(\theta) \in C_\theta(i)$ for some $i = w_\theta(0)$. Let $m \in M$ be a message profile with the following properties:

- $m_j^1 = \theta$ for all $j \in I$;
- $m_j^2[\theta] = 0$ for all $j \in \mathcal{G}_\theta$;
- $m_i^3[\theta] = f(\theta)$.

Since $i^* = w_\theta(\sum_{j \in \mathcal{G}_\theta} m_j^2[\theta] \bmod k) = w_\theta(0)$ under Rule 1, and we assume $i = w_\theta(0)$, it follows that $g(m) = f(\theta)$. We next claim that m is a pure strategy Nash equilibrium in the game $\Gamma^k(\theta)$. First, since there are at least three agents (i.e., $n \geq 3$), no agent can unilaterally change $\theta^m = \theta$. Thus, every agent i cannot find any profitable deviation from m when restricting her deviation strategy to M_i^1. Hence, any profitable unilateral deviation of agent i from m, if any, must involve the change of her message in either M_i^2, M_i^3, or both. By construction of the mechanism Γ^k, the only agents who can unilaterally change the outcome from

$g^k(m)$ are those who are in $\mathcal{G}_\theta$. We also know that each agent $j \in \mathcal{G}_\theta$ can only induce outcomes within $C_\theta(j)$ by her unilateral deviation from m. The first part of Property 1 ensures that any agent $j \in \mathcal{G}_\theta$ finds $f(\theta)$ as her best outcome within $C_\theta(j)$ in state θ. Therefore, no agent $j \in \mathcal{G}_\theta$ can find any profitable unilateral deviation by inducing Rules 1 or 2. Thus, m is indeed a pure strategy Nash equilibrium in the game $\Gamma^k(\theta)$.

Now we show that $m \in pureNE(\Gamma^k(\theta))$ implies $g(m) = f(\theta)$. We assume by way of contradiction that there exists $m \in pureNE(\Gamma(\theta))$ such that $g(m) \neq f(\theta)$.

Since $g(m) \neq f(\theta)$, it follows from Property 2 that, for any $\tilde{\theta} \in \Theta$, $g(m)$ is "not" $C_{\tilde{\theta}}$-acceptable at state θ. In particular, we have that $g(m)$ is not C_{θ^m}-acceptable at state θ. This implies that there exist $i \in \mathcal{G}_{\theta^m}$ and $x \in C_{\theta^m}(i)$ such that $u_i(x,\theta) > u_i(g(m),\theta)$. We define $\hat{m}_i$ to be identical to m_i except that $\hat{m}_i^3[\theta^m] = x$ and $\hat{m}_i^2[\theta^m]$ such that agent i is the modulo game winner, i.e.,

$$w_{\theta^m}\left(\left(\hat{m}_i^2[\theta^m] + m_j^2[\theta^m]\right) \mod k\right) = i.$$

Then, agent i has a profitable unilateral deviation from m. This shows that m is not a pure strategy Nash equilibrium in the game $\Gamma^k(\theta)$, which is a desired contradiction. Thus, f is pure Nash implementable by mechanism Γ^k. ■

Throughout the proof, we denote by θ the true state and by $\hat{\theta}$ the state determined by the agents' announcement in the mechanism. Let $\Gamma^k = (M^k, g^k)$ be our canonical mechanism where $k \geq 3$. We define $C_{\hat{\theta}} \equiv \bigcup_{i \in I} C_{\hat{\theta}}(i)$ for each $\hat{\theta} \in \Theta$, and $C \equiv \bigcup_{\hat{\theta} \in \Theta} C_{\hat{\theta}}$. Note that $C_{\hat{\theta}}$ and C are both finite. For each $\hat{\theta} \in \Theta$, $q \in \{0, \ldots, k-1\}$, $i \in I$, and $x \in C$, we define

$$M^*(\hat{\theta}, q, x) = \left\{ m \in M^k \;\middle|\; \theta^m = \hat{\theta},\ \sum_{j \in \mathcal{G}_{\hat{\theta}}} m_j^2[\hat{\theta}] \pmod{k} = q,\ g^k(m) = x \right\}$$

as a subset of M^k. Notice that as $M^*(\hat{\theta}, q, x)$ requires a lot structure on itself, it may well be empty; in particular, it is empty for all $(\hat{\theta}, q, x)$ with $q \geq 2$ and $x \neq z_{\hat{\theta}}$. By construction, we have

$$\bigcup_{\hat{\theta} \in \Theta} \bigcup_{q \in \{0,\ldots,k-1\}} \bigcup_{x \in C} M^*(\hat{\theta}, q, x) = M^k.$$

Let σ be a mixed strategy profile in the mechanism Γ^k. For any $\hat{\theta} \in \Theta$, $q \in \{0, \ldots, k-1\}$, and $x \in C$, we define

$$P^\sigma(\hat{\theta}, q, x) \equiv \sum_{m \in M^*(\hat{\theta}, q, x)} \sigma(m).$$

For each $\hat{\theta} \in \Theta$ and $q \in \{0, \ldots, k-1\}$, we define

$$M^*(\hat{\theta}, q) = \bigcup_{x \in C} M^*(\hat{\theta}, q, x) \text{ and } P^\sigma(\hat{\theta}, q) = \sum_{m \in M^*(\hat{\theta}, q)} \sigma(m).$$

For any $\hat{\theta} \in \Theta$, we define

$$M^*(\hat{\theta}) = \bigcup_{q \in \{0, \ldots, k-1\}} M^*(\hat{\theta}, q) \text{ and } P^\sigma(\hat{\theta}) = \sum_{m \in M^*(\hat{\theta})} \sigma(m).$$

We can now define conditional probabilities as well:

$$P^\sigma(q, x|\hat{\theta}) = \begin{cases} P^\sigma(\hat{\theta}, q, x)/P^\sigma(\hat{\theta}) & \text{if } P^\sigma(\hat{\theta}) > 0, \\ 0 & \text{if } P^\sigma(\hat{\theta}) = 0. \end{cases}$$

$$P^\sigma(q|\hat{\theta}) = \begin{cases} P^\sigma(\hat{\theta}, q)/P^\sigma(\hat{\theta}) & \text{if } P^\sigma(\hat{\theta}) > 0, \\ 0 & \text{if } P^\sigma(\hat{\theta}) = 0. \end{cases}$$

$$P^\sigma(x|q, \hat{\theta}) = \begin{cases} P^\sigma(q, x|\hat{\theta})/P^\sigma(q|\hat{\theta}) & \text{if } P^\sigma(q|\hat{\theta}) > 0, \\ 0 & \text{if } P^\sigma(q|\hat{\theta}) = 0. \end{cases}$$

We define the set of message profiles in which $\hat{\theta}$ is the agreed-upon state chosen by the mechanism and agent i sends $m_i^2[\hat{\theta}] = q$:

$$M^*(\hat{\theta}, q, i) = \left\{ m \in M^k \,\middle|\, \theta^m = \hat{\theta},\ m_i^2[\hat{\theta}] = q \right\}.$$

Using this set, we define $P_i^\sigma(q|\hat{\theta})$ as the probability that agent i sends q under σ conditional on $M^*(\hat{\theta}, q, i)$:

$$P_i^\sigma(q|\hat{\theta}) = \begin{cases} \sum_{m_i : (m_i, m_{-i}) \in M^*(\hat{\theta}, q, i)} \sigma_i(m_i)/P^\sigma(\hat{\theta}) & \text{if } P^\sigma(\hat{\theta}) > 0, \\ 0 & \text{if } P^\sigma(\hat{\theta}) = 0. \end{cases}$$

Lastly, for each $\hat{\theta} \in \Theta$ and each $q \in \{0, 1\}$, define the following lottery:

$$\ell_q^k(\hat{\theta}, \sigma) \equiv \sum_{x \in C} P^\sigma(x|q, \hat{\theta}) x.$$

Step 2: For any mixed strategy profile σ in the mechanism $\Gamma^k = (M^k, g^k)$, $g^k(\sigma)$ can be represented by the following multiple forms:

$$\begin{aligned} g^k(\sigma) &= \sum_{\hat{\theta}\in\Theta}\sum_{x\in C}\left[\sum_{q\in\{0,1\}} P^\sigma(\hat{\theta},q,x)x + \sum_{q\in\{2,\ldots,k-1\}} P^\sigma(\hat{\theta},q,x)z_{\hat{\theta}}\right] \\ &= \sum_{\hat{\theta}\in\Theta} P^\sigma(\hat{\theta})\sum_{x\in C}\left[\sum_{q\in\{0,1\}} P^\sigma(q,x|\hat{\theta})x + \sum_{q\in\{2,\ldots,k-1\}} P^\sigma(q,x|\hat{\theta})z_{\hat{\theta}}\right] \\ &= \sum_{\hat{\theta}\in\Theta} P^\sigma(\hat{\theta})\ell^k(\hat{\theta},\sigma), \end{aligned}$$

where, for each $\hat{\theta}\in\Theta$,

$$\begin{aligned} \ell^k(\hat{\theta},\sigma) &\equiv \sum_{x\in C}\left[\sum_{q\in\{0,1\}} P^\sigma(q,x|\hat{\theta})x + \sum_{q\in\{2,\ldots,k-1\}} P^\sigma(q,x|\hat{\theta})z_{\hat{\theta}}\right] \\ &= \sum_{q\in\{0,1\}}\sum_{x\in C} P^\sigma(q,x|\hat{\theta})x + \sum_{q\in\{2,\ldots,k-1\}}\sum_{x\in C} P^\sigma(q,x|\hat{\theta})z_{\hat{\theta}} \\ &= \sum_{q\in\{0,1\}} P^\sigma(q|\hat{\theta})\ell^k_q(\hat{\theta},\sigma) + \sum_{q\in\{2,\ldots,k-1\}} P^\sigma(q|\hat{\theta})z_{\hat{\theta}} \\ &= P^\sigma(q=0|\hat{\theta})\ell^k_0(\hat{\theta},\sigma) + P^\sigma(q=1|\hat{\theta})\ell^k_1(\hat{\theta},\sigma) + \left(1-\sum_{q\in\{0,1\}} P^\sigma(q|\hat{\theta})\right) z_{\hat{\theta}} \\ &= P^\sigma(q=0|\hat{\theta})(\ell^k_0(\hat{\theta},\sigma) - z_{\hat{\theta}}) + P^\sigma(q=1|\hat{\theta})(\ell^k_1(\hat{\theta},\sigma) - z_{\hat{\theta}}) + z_{\hat{\theta}} \end{aligned}$$

Proof of Step 2: This comes from the construction of our mechanism. ■

Step 3: Let $\sigma \in NE(\Gamma^k(\theta,u))$ for some $u \in \hat{\mathcal{U}}$. Then, for any $m \in \text{supp}(\sigma)$, $q \in \{0,1\}$, and $\hat{\theta}\in\Theta$, if $m \in M^*(\hat{\theta},q)$, then $m_i^3[\hat{\theta}] \in C^*_{\hat{\theta}}(i,\theta)$, where $i = w^{-1}_{\hat{\theta}}(q)$.

Proof of Step 3: Fix $u \in \hat{\mathcal{U}}$. Let $\sigma \in NE(\Gamma^k(\theta,u))$. Fix $m \in \text{supp}(\sigma)$, $q \in \{0,1\}$, and $\hat{\theta}\in\Theta$. Assume that $m \in M^*(\hat{\theta},q)$, and let $i = w^{-1}_{\hat{\theta}}(q)$. Suppose, by way of contradiction, that $m_i^3[\hat{\theta}] \notin C^*_{\hat{\theta}}(i,\theta)$. We define $\bar{m}_i$ to be identical to m_i except that $\bar{m}_i^3[\hat{\theta}] = c^*_{\hat{\theta}}(i,\theta)$, for some $c^*_{\hat{\theta}}(i,\theta) \in C^*_{\hat{\theta}}(i,\theta)$. By construction, we know that $\bar{m}_i$ weakly dominates m_i. We next define $\bar{\sigma}_i$ to be the following deviation strategy: for any $\tilde{m}_i \in M_i$,

$$\bar{\sigma}_i(\tilde{m}_i) = \begin{cases} \sigma_i(\bar{m}_i) + \sigma_i(m_i) & \text{if } \tilde{m}_i = \bar{m}_i, \\ 0 & \text{if } \tilde{m}_i = m_i, \\ \sigma_i(\tilde{m}_i) & \text{otherwise.} \end{cases}$$

We compute the following utility difference:

$$\begin{aligned}
& u_i(g^k(\bar{\sigma}_i, \sigma_{-i}), \theta) - u_i(g^k(\sigma_i, \sigma_{-i}), \theta) \\
= & \sum_{\tilde{m}_{-i}} \sigma_i(m_i) \left[u_i(g^k(\bar{m}_i, \tilde{m}_{-i}), \theta) - u_i(g^k(m_i, \tilde{m}_{-i}), \theta) \right] \\
= & \sigma(m) \left[u_i(g^k(\bar{m}_i, m_{-i}), \theta) - u_i(g^k(m_i, m_{-i}), \theta) \right] \\
& + \sum_{\tilde{m}_{-i} \neq m_{-i}} \sigma(m_i, \tilde{m}_{-i}) \left[u_i(g^k(\bar{m}_i, \tilde{m}_{-i}), \theta) - u_i(g^k(m_i, \tilde{m}_{-i}), \theta) \right] \\
\geq & \sigma(m) \left[u_i(g^k(\bar{m}_i, m_{-i}), \theta) - u_i(g^k(m_i, m_{-i}), \theta) \right] \\
= & \sigma(m) \left[u_i(c^*_{\hat{\theta}}(i, \theta), \theta) - u_i(m_i^3[\tilde{\theta}], \theta) \right] \\
> & 0,
\end{aligned}$$

where the weak inequality follows because $\bar{m}_i$ weakly dominates m_i, and the strict inequality follows because $\sigma(m) > 0$ and $u_i(c^*_{\hat{\theta}}(i, \theta)) > u_i(m_i^3[\hat{\theta}], \theta)$, as $c^*_{\hat{\theta}}(i, \theta) \in C^*_{\hat{\theta}}(i, \theta)$. This shows that σ is not a Nash equilibrium in the game $\Gamma^k(\theta, u)$, which is the desired contradiction. Thus, we complete the proof. ■

Step 4: Let σ be a mixed strategy profile in the mechanism Γ^K, where we later choose $K \geq 3$ large enough, and fix $\hat{\theta} \in \Theta$ such that $P^\sigma(\hat{\theta}) > 0$. Assume that $P^\sigma(q = 0|\hat{\theta}) + P^\sigma(q = 1|\hat{\theta}) \leq 2/K$. Then, there exists $K \in \mathbb{N}$ large enough so that $u_i(\ell^K(\hat{\theta}, \sigma), \theta) < u_i(f(\theta), \theta)$ for all $i \in I$ and $u \in \hat{\mathcal{U}}$.

Proof of Step 4: Fix $\hat{\theta} \in \Theta$ with $P^\sigma(\hat{\theta}) > 0$, $u \in \hat{\mathcal{U}}$, and $i \in I$ arbitrarily. Since the utility achievable is bounded above from 1 and the compound lottery $l^K(\hat{\theta}, \sigma)$ induces $z_{\hat{\theta}}$ with probability equal to at least $1 - 2/K$, we have

$$\begin{aligned}
u_i(\ell^K(\hat{\theta}, \sigma), \theta) & \leq \frac{2}{K} \cdot 1 + \frac{K-2}{K} u_i(z_{\hat{\theta}}, \theta) \\
& = u_i(z_{\hat{\theta}}, \theta) + \frac{2\,(1 - u_i(z_{\hat{\theta}}, \theta))}{K} \\
& \leq u_i(z_{\hat{\theta}}, \theta) + \frac{2}{K}
\end{aligned}$$

By Property 3 of Condition COM, for any $i \in I$ and $u \in \hat{\mathcal{U}}$,

$$u_i(z_{\hat{\theta}}, \theta) \leq u_i(f(\theta), \theta) - \varepsilon.$$

Hence, we combine this with the previous inequality obtained so that

$$u_i(l^K(\hat{\theta}, \sigma), \theta) \leq u_i(f(\theta), \theta) - \varepsilon + \frac{2}{K}.$$

If we choose K to be the smallest integer such that $K \geq 4/\varepsilon$, we have

$$u_i(\ell^K(\hat{\theta}, \sigma), \theta) \leq u_i(f(\theta), \theta) - \frac{\varepsilon}{2},$$

for any $i \in I$ and $u \in \hat{\mathcal{U}}$. Since $\varepsilon > 0$, we have that $u_i(\ell^K(\hat{\theta}, \theta)) < u_i(f(\theta), \theta)$ for any $i \in I$ and $u \in \hat{\mathcal{U}}$, as desired. This completes the proof. ■

From here till the end of the proof of Step 7, we fix $u \in \hat{\mathcal{U}}$, $\hat{\theta} \in \Theta$, and adopt the convention that $w_{\hat{\theta}}(0) = i$ and $w_{\hat{\theta}}(1) = j$. We focus now on the lotteries that emerge from the remaining components of a strategy profile σ after a particular state $\hat{\theta}$ is selected by the mechanism, which we denote by $\ell^k(\hat{\theta}, \sigma)$. Steps 5, 6 and 7 are all used to show that if σ is a Nash equilibrium of the game $\Gamma^K(\theta, u)$ and $f(\theta) \neq \ell^K(\hat{\theta}, \sigma)$, all agents must prefer $f(\theta)$ to $\ell^K(\hat{\theta}, \sigma)$ where we choose K large enough.

Let $m_j \in \text{supp}(\sigma_j)$, $\bar{m}_j$ be agent j's arbitrary message sent, and σ_{-j} be other players' strategy profile. Let (m_j, σ_{-j}) denote the strategy profile in which agent j plays m_j and other agents play σ_{-j}, and $(\bar{m}_j, \sigma_{-j})$ be the strategy profile in which agent j plays $\bar{m}_j$ and other agents play σ_{-j}. These two strategy profiles will induce different lotteries, $\ell^k(\hat{\theta}, (m_j, \sigma_{-j}))$ and $\ell^k(\hat{\theta}, (\bar{m}_j, \sigma_{-j}))$, respectively. Using Step 2, we can compute the difference in expected payoff for agent j at state θ between these two lotteries as:

$$\begin{aligned}
& u_j(\ell^k(\hat{\theta}, (\bar{m}_j, \sigma_j)), \theta) - u_j(\ell^k(\hat{\theta}, (m_j, \sigma_j)), \theta) \\
= \ & P^{(\bar{m}_j, \sigma_{-j})}(q = 0|\hat{\theta}) \left(u_j(\ell_0^k(\hat{\theta}, (m_j, \sigma_{-j})), \theta) - u_j(z_{\hat{\theta}}, \theta) \right) \\
+ \ & P^{(\bar{m}_j, \sigma_{-j})}(q = 1|\hat{\theta}) \left(\bar{u}_1 - u_j(z_{\hat{\theta}}, \theta) \right) + u_j(z_{\hat{\theta}}, \theta) \\
- \ & \Big[P^{(m_j, \sigma_{-j})}(q = 0|\hat{\theta}) \left(u_j(\ell_0^k(\hat{\theta}, (m_j, \sigma_{-j})), \theta) - u_j(z_{\hat{\theta}}, \theta) \right) \\
+ \ & P^{(m_j, \sigma_{-j})}(q = 1|\hat{\theta}) \left(u_j(m_j^3[\hat{\theta}], \theta) - u_j(z_{\hat{\theta}}, \theta) \right) + u_j(z_{\hat{\theta}}, \theta) \Big]
\end{aligned}$$

To ease the notation, we will adopt the following set of conventions:

$$
\begin{aligned}
P^{(m_j,\sigma_{-j})}(q=0|\hat{\theta}) &= p_0 \\
P^{(\bar{m}_j,\sigma_{-j})}(q=0|\hat{\theta}) &= \bar{p}_0 \\
P^{(m_j,\sigma_{-j})}(q=1|\hat{\theta}) &= p_1 \\
P^{(\bar{m}_j,\sigma_{-j})}(q=1|\hat{\theta}) &= \bar{p}_1 \\
u_j(\ell_0^k(\hat{\theta},(m_j,\sigma_{-j})),\theta) &= u_0 \\
u_j(m_j^3[\hat{\theta}],\theta) &= u_1 \\
u_j(\bar{m}_j^3[\hat{\theta}],\theta) &= \bar{u}_1 \\
u_j(z_{\hat{\theta}},\theta) &= u_z
\end{aligned}
$$

This introduced notation allows us to simply the previous expression to

$$
\begin{aligned}
& u_j(\ell^k(\hat{\theta},(\bar{m}_j,\sigma_j)),\theta) - u_j(\ell^k(\hat{\theta},(m_j,\sigma_j)),\theta) \\
= \; & \bar{p}_0(u_0-u_z) + \bar{p}_1(\bar{u}_1-u_z) + u_z - [p_0(u_0-u_z) + p_1(u_1-u_z) + u_z.]
\end{aligned}
$$

For any $x \in \{0,1\}$ and $q \in \{0,\dots,K-1\}$, we define $b_x(q) \in \{0,\dots,K-1\}$ such that $q + b_1(q) \pmod K = 1$.

Step 5: Let $\sigma \in NE(\Gamma^K(\theta,u))$, where we later choose K large enough. For any $\hat{\theta} \in \Theta$ with $P^\sigma(\hat{\theta}) > 0$, we assume that there exists $q \in \{0,1\}$ such that $P^\sigma(q|\hat{\theta}) = 0$. Then, there exists $K \in \mathbb{N}$ large enough so that, for any $\hat{\theta} \in \Theta$ with $P^\sigma(\hat{\theta}) > 0$, if $\ell^K(\hat{\theta},\sigma) \neq f(\theta)$, then $u_i(\ell^K(\hat{\theta},\sigma),\theta) < u_i(f(\theta),\theta)$ for all $i \in I$.

Proof of Step 5: Fix $\sigma \in NE(\Gamma^K(\theta,u))$, and $\hat{\theta} \in \Theta$ with $P^\sigma(\hat{\theta}) > 0$ arbitrarily. We assume that there exists $q \in \{0,1\}$ such that $P^\sigma(q|\hat{\theta}) = 0$. We further assume that $\ell^K(\hat{\theta},\sigma) \neq f(\theta)$. We divide the proof into each of the following two cases.

Case 1: $P^\sigma(q|\hat{\theta}) = 0$ for all $q \in \{0,1\}$

This implies that $\ell^K(\hat{\theta},\sigma)$ induces $z_{\hat{\theta}}$ with probability one. It follows from Property 3 of Condition COM that $u_i(\ell^K(\hat{\theta},\sigma),\theta) < u_i(f(\theta),\theta)$ for any $i \in I$, $u \in \hat{\mathcal{U}}$, and $K \geq 3$.

Case 2: $P^\sigma(q=0|\hat{\theta}) > 0$ and $P^\sigma(q=1|\hat{\theta}) = 0$

The argument we provide below regarding Case 2 will make it clear that we can handle the case that $P^\sigma(q=1|\hat{\theta}) > 0$ and $P^\sigma(q=0|\hat{\theta}) = 0$ in a similar fashion. So we omit this case. With the help of Step 2, we can write

$$
\ell^K(\hat{\theta},\sigma) = P^\sigma(q=0|\hat{\theta})(\ell_0^K(\hat{\theta},\sigma) - z_{\hat{\theta}}) + z_{\hat{\theta}}
$$

Fix $c^*_{\hat{\theta}}(j,\theta) \in C^*_{\hat{\theta}}(j,\theta)$. We further divide Case 2 into four sub-cases:

Case 2-1: $u_0 \geq u_j(c^*_{\hat{\theta}}(j,\theta),\sigma),\theta) \geq u_z$

This implies that some $c^*_{\hat{\theta}}(i,\theta) \in \text{supp }(\ell^K_0(\hat{\theta},\sigma))$ is $C_{\hat{\theta}}$-acceptable at state θ. It then follows from Property 2 of Condition COM that $c^*_{\hat{\theta}}(i,\theta) = f(\theta)$. From Step 4, we also have that $\text{supp }(\ell^k_0(\hat{\theta},\sigma)) \subseteq C_{\hat{\theta}}(i,\theta)$. Thus, $f(\theta) \in C^*_{\hat{\theta}}(i,\theta)$, which further implies that $u_i(c,\theta) = u_i(f(\theta),\theta)$ for all $c \in C^*_{\hat{\theta}}(i,\theta)$. Finally, by Property 4 of Condition COM, we have $C^*_{\hat{\theta}}(i,\theta) = \{f(\theta)\}$, which further implies $\ell^k_0(\hat{\theta},\sigma) = f(\theta)$. Since we assume that $\ell^K(\hat{\theta},\sigma) \neq f(\theta)$, we must have $P^\sigma(q=0|\hat{\theta}) < 1$. By Property 3 of Condition COM, there exists $\varepsilon > 0$ such that $u_j(f(\theta),\theta) - u_j(z_{\hat{\theta}},\theta) \geq \varepsilon$ for all $j \in I$ and $u \in \hat{\mathcal{U}}$. Due to the construction of $\ell^K(\hat{\theta},\theta)$ and the fact that $\ell^k_0(\hat{\theta},\sigma) = f(\theta)$, we conclude that $u_j(\ell^K(\hat{\theta},\sigma),\theta) < u_j(f(\theta),\theta)$ for all $j \in I$.

Case 2-2: $u_j(c^*_{\hat{\theta}}(j,\theta),\theta) > u_z > u_0$.

In this case, we will show that there is no Nash equilibrium σ in the game $\Gamma^K(\theta,u)$ such that $P^\sigma(q=0|\hat{\theta}) > 0$ and $P^\sigma(q=1|\hat{\theta}) = 0$. Suppose, on the contrary, that such σ constitutes a Nash equilibrium in the game $\Gamma^K(\theta,u)$.

Fix $m \in \text{supp}(\sigma)$ as a message profile such that $\theta^m = \hat{\theta}$. The existence of such m is guaranteed because we have $P^\sigma(\hat{\theta}) > 0$. Note that $P^\sigma(q=1|\hat{\theta}) = 0$ implies that $P^{(m_j,\sigma_{-j})}(q=1|\hat{\theta}) = p_1 = 0$. Our goal is to find a message $\bar{m}_j$, which together with σ_{-j} induces a lottery $\ell^K(\hat{\theta},(\bar{m}_j,\sigma_{-j}))$ that first-order stochastically dominates the lottery induced by (m_j,σ_{-j}), showing that σ is not a Nash equilibrium of the game $\Gamma^K(\theta,u)$. To achieve this, we need to find $\bar{m}^2_j[\hat{\theta}] \in \{0,\ldots,K-1\}$, which together with σ_{-j} guarantees that $\bar{p}_1 > 0$ and $\bar{p}_0 = 0$. We shall propose an algorithm selecting such $\bar{m}^2_j[\hat{\theta}]$.

Recall the following notation:

$$M^*(\hat{\theta},q,i) = \left\{ m \in M^K \,\middle|\, \theta^m = \hat{\theta},\ m^2_i[\hat{\theta}] = q \right\}.$$

and

$$P^\sigma_i(q|\hat{\theta}) = \sum_{m_i:(m_i,m_{-i})\in M^*(\hat{\theta},q,i)} \sigma_i(m_i)/P^\sigma(\hat{\theta}).$$

We also use this notation in Case 2-3 later. Start the algorithm by setting $q_0 = b_1(m^2_j[\hat{\theta}])$, where, for any $q \in \{0,\ldots,K-1\}$, we define $b_1(q) \in \{0,\ldots,K-1\}$ such that $q + b_1(q) \pmod K = 1$. It follows from $P^\sigma(q=1|\hat{\theta}) = 0$ that $P^\sigma_i(q_0|\hat{\theta}) = 0$.

Next, for any $h \in \{1, \cdots, K-1\}$ and $q_{h-1} \in \{0, \ldots, K-1\}$, we define

$$q_h = \begin{cases} q_{h-1}+1 & \text{if } q_{h-1} \leq K-2, \\ 0 & \text{if } q_{h-1} = K-1. \end{cases}$$

Since $\sum_{q=0}^{K-1} P_i^{\sigma}(q|\hat{\theta}) = 1$, we can choose $h \in \{1, \ldots, K-1\}$ uniquely in such a way that $P_i^{\sigma}(q_h|\hat{\theta}) > 0$ and $P_i^{\sigma}(q_{h'}|\hat{\theta}) = 0$ for all $h' \in \{0, \ldots, h-1\}$. Then, we set $\bar{m}_j^2[\hat{\theta}] = b_1(q_h)$. Define $\bar{m}_j^3[\hat{\theta}]$ as follows:

$$\bar{m}_j^3[\hat{\theta}] = \begin{cases} c^*_{\hat{\theta}}(j,\theta) & \text{if } m_j^3[\hat{\theta}] \notin C^*_{\hat{\theta}}(j,\theta), \\ m_j^3[\hat{\theta}] & \text{if } m_j^3[\hat{\theta}] \in C^*_{\hat{\theta}}(j,\theta). \end{cases}$$

Thus, we define $\bar{m}_j$ to be identical to m_j except that $m_j^2[\hat{\theta}]$ is replaced by $\bar{m}_j^2[\hat{\theta}]$ and $m_j^3[\hat{\theta}]$ is replaced by $\bar{m}_j^3[\hat{\theta}]$. By the algorithm to find q_h and construction of $\bar{m}_j^2[\hat{\theta}]$, we have the following properties:

$$\begin{aligned} \bar{p}_1 \equiv P^{(\bar{m}_j,\sigma_{-j})}(q=1|\hat{\theta}) &= P^{(\bar{m}_j,\sigma_{-j})}\left(q_h + b_1(q_h) \pmod{K} \middle| \hat{\theta}\right) \\ &= P_i^{(\bar{m}_j,\sigma_{-j})}(q_h|\hat{\theta}) > 0, \\ \bar{p}_0 \equiv P^{(\bar{m}_j,\sigma_{-j})}(q=0|\hat{\theta}) &= P^{(\bar{m}_j,\sigma_{-j})}\left(q_{h-1} + b_1(q_h) \pmod{K} \middle| \hat{\theta}\right) \\ &= P_i^{(\bar{m}_j,\sigma_{-j})}(q_{h-1}|\hat{\theta}) = 0. \end{aligned}$$

It follows from $P^{\sigma}(q=1|\hat{\theta}) = 0$ that $p_1 \equiv P^{(m_j,\sigma_{-j})}(q=1|\hat{\theta}) = 0$, implying $\bar{p}_1 - p_1 > 0$. Since $\bar{p}_0 = 0$, we also have

$$p_0 \equiv P^{(m_j,\sigma_{-j})}(q=0|\hat{\theta}) \geq P^{(\bar{m}_j,\sigma_j)}(q=0|\hat{\theta}) \equiv \bar{p}_0,$$

which implies $p_0 - \bar{p}_0 \geq 0$. Due to the construction of $\bar{m}_j^3[\hat{\theta}]$, we obtain the following inequalities:

$$\begin{aligned} u_j(\bar{m}_j^3[\hat{\theta}],\theta) = u_j(c^*_{\hat{\theta}}(j,\theta),\theta) > u_j(z_{\hat{\theta}},\theta) &\Rightarrow \bar{u}_1 - u_z > 0, \\ u_j(\bar{m}_j^3[\hat{\theta}],\theta) = u_j(c^*_{\hat{\theta}}(j,\theta),\theta) \geq u_j(m_j^3[\hat{\theta}],\theta) &\Rightarrow \bar{u}_1 - u_1 \geq 0. \end{aligned}$$

Now, we claim that $u_j(g(\bar{m}_j,\sigma_j),\theta) - u_j(g(m_j,\sigma_j),\theta) > 0$. Since $P^{\sigma}(\hat{\theta}) > 0$, by Step 2 and the construction of $\bar{m}_j$, it suffices to show that $u_j(\ell^K(\hat{\theta},(\bar{m}_j,\sigma_j)),\theta) - u_j(\ell^K(\hat{\theta},(m_j,\sigma_j)),\theta) > 0$. Thus, we compute

$$\begin{aligned} & u_j(\ell^K(\hat{\theta},(\bar{m}_j,\sigma_j)),\theta) - u_j(\ell^K(\hat{\theta},(m_j,\sigma_j)),\theta) \\ = \; & \bar{p}_0(u_0 - u_z) + \bar{p}_1(\bar{u}_1 - u_z) + u_z - [p_0(u_0 - u_z) + p_1(u_1 - u_z) + u_z] \\ = \; & (\bar{p}_1 - p_1)(\bar{u}_1 - u_z) + (p_0 - \bar{p}_0)(u_z - u_0) + p_1(\bar{u}_1 - u_1) \\ > \; & 0, \end{aligned}$$

where the strict inequality follows because $(p_0 - \bar{p}_0)(u_z - u_0) \geq 0$, $p_1(\bar{u}_1 - u_1) \geq 0$, and $(\bar{p}_1 - p_1)(\bar{u}_1 - u_z) > 0$. This contradicts the hypothesis that σ is a Nash equilibrium in the game $\Gamma^K(\theta, u)$. ■

Case 2-3: $u_j(c^*_{\hat{\theta}}(j,\theta),\theta) > u_0 > u_z$

In this case, we will show that there is no Nash equilibrium σ in the game $\Gamma^K(\theta, u)$ such that $P^\sigma(q = 0|\hat{\theta}) > 0$ and $P^\sigma(q = 1|\hat{\theta}) = 0$. Suppose, on the contrary, that such σ constitutes a Nash equilibrium in the game $\Gamma^K(\theta, u)$.

Fix $m \in \text{supp}(\sigma)$ as a message profile such that $\theta^m = \hat{\theta}$. The existence of such m is guaranteed because we have $P^\sigma(\hat{\theta}) > 0$. Note that $P^\sigma(q = 1|\hat{\theta}) = 0$ implies that $P^{(m_j,\sigma_{-j})}(q = 1|\hat{\theta}) = p_1 = 0$. Our goal here is to find a message $\bar{m}_j$, which together with σ_{-j} induces a lottery $\ell^K(\hat{\theta}, (\bar{m}_j, \sigma_{-j}))$ that first-order stochastically dominates the lottery induced by (m_j, σ_{-j}), showing that σ is not a Nash equilibrium of the game $\Gamma^K(\theta, u)$. To achieve this, we need to find $\bar{m}^2_j[\hat{\theta}] \in \{0, \ldots, K-1\}$, which together with σ_{-j} guarantees that $\bar{p}_1 > 0$ and $\bar{p}_0 + \bar{p}_1 > p_0 + p_1$. Such $\bar{m}^2_j[\hat{\theta}]$ will be found by the following algorithm:

Start the algorithm by setting $q_0 = b_1(m^2_j[\hat{\theta}])$. It follows from $P^\sigma(q = 1|\hat{\theta}) = 0$ that $P^\sigma_i(q_0|\hat{\theta}) = 0$. Next, for any $h \in \{1, \cdots, K-1\}$ and $q_{h-1} \in \{0, \ldots, K-1\}$, we define

$$q_h = \begin{cases} q_{h-1} - 1 & \text{if } q_{h-1} \geq 1 \\ K - 1 & \text{if } q_{h-1} = 0. \end{cases}$$

Since $\sum_{q=0}^{K-1} P^\sigma_i(q|\hat{\theta}) = 1$, we can choose $h \in \{1, \ldots, K-1\}$ uniquely in such a way that $P^\sigma_i(q_h|\hat{\theta}) > 0$ and $P^\sigma_i(q_{h'}|\hat{\theta}) = 0$ for all $h' \in \{0, \ldots, h-1\}$. Then, we set $\bar{m}^2_j[\hat{\theta}] = b_1(q_h)$. Define also $\bar{m}^3_j[\hat{\theta}]$ as follows:

$$\bar{m}^3_j[\hat{\theta}] = \begin{cases} c^*_{\hat{\theta}}(j,\theta) & \text{if } m^3_j[\hat{\theta}] \notin C^*_{\hat{\theta}}(j,\theta), \\ m^3_j[\hat{\theta}] & \text{if } m^3_j[\hat{\theta}] \in C^*_{\hat{\theta}}(j,\theta), \end{cases}$$

Thus, we define $\bar{m}_j$ to be identical to m_j except that $m^2_j[\hat{\theta}]$ is replaced by $\bar{m}^2_j[\hat{\theta}]$ and $m^3_j[\hat{\theta}]$ is replaced by $\bar{m}^3_j[\hat{\theta}]$.

By the algorithm to find q_h and construction of $\bar{m}^2_j[\hat{\theta}]$, we have the following properties:

$$\begin{aligned}
\bar{p}_1 \equiv P^{(\bar{m}_j,\sigma_{-j})}(q=1|\hat{\theta}) &= P^{(\bar{m}_j,\sigma_{-j})}(q_h + b_1(q_h) \pmod K|\hat{\theta}) \\
&= P_i^{(\bar{m}_j,\sigma_{-j})}(q_h|\hat{\theta}) > 0, \\
&\geq P_i^{(\bar{m}_j,\sigma_{-j})}(q_1|\hat{\theta}) \\
&= \begin{cases} P^{(m_j,\sigma_{-j})}(q_0 - 1 + m_j^2[\hat{\theta}] \pmod K|\hat{\theta}) & \text{if } q_0 \geq 1 \\ P^{(m_j,\sigma_{-j})}(K - 1 + m_j^2[\hat{\theta}] \pmod K|\hat{\theta}) & \text{if } q_0 = 0 \end{cases} \\
&= P^{(m_j,\sigma_{-j})}(q=0|\hat{\theta}) \equiv p_0 \ (\because q_0 + m_j^2[\hat{\theta}] \pmod K = 1) \\
P^{(\bar{m}_j,\sigma_{-j})}(q=2|\hat{\theta}) &= P^{(\bar{m}_j,\sigma_{-j})}(q_{h-1} + b_1(q_h) \pmod K|\hat{\theta}) \\
&= P_i^{(\bar{m}_j,\sigma_{-j})}(q_{h-1}|\hat{\theta}) = 0,
\end{aligned}$$

which implies $\bar{p}_1 - p_0 \geq 0$. Since $p_1 \equiv P^{(m_j,\sigma_{-j})}(q = 1|\hat{\theta}) = 0$ and $P^{(\bar{m}_j,\sigma_{-j})}(q = 0|\hat{\theta}) \geq 0$, we have $P^{(\bar{m}_j,\sigma_{-j})}(q = 0|\hat{\theta}) \geq P^{(m_j,\sigma_{-j})}(q = 1|\hat{\theta})$, which implies that $\bar{p}_0 - p_1 \geq 0$. In addition, since $p_1 \equiv P^{(m_j,\sigma_{-j})}(q = 1|\hat{\theta}) = 0$, we also have $P^{(\bar{m}_j,\sigma_{-j})}(q = 1|\hat{\theta}) > P^{(m_j,\sigma_{-j})}(q = 1|\hat{\theta})$, which implies that $\bar{p}_1 - p_1 > 0$.

Since $u_j(c^*_{\hat{\theta}}(j,\theta),\theta) > u_0 > u_z$, due to the construction of $\bar{m}_j^3[\hat{\theta}]$, we have the following inequality:

$$\bar{u}_1 \equiv u_j(\bar{m}_j^3[\hat{\theta}],\theta) = u_j(c^*_{\hat{\theta}}(j,\theta),\theta) > u_j(\ell_0^K(\hat{\theta},(m_j,\sigma_{-j})),\theta) \equiv u_0$$

Thus, $\bar{u}_1 - u_0 > 0$.

We claim now that $u_j(g(\bar{m}_j,\sigma_j),\theta) - u_j(g(m_j,\sigma_j),\theta) > 0$. Since $P^\sigma(\hat{\theta}) > 0$, by Step 2, it suffices to show that $u_j(\ell^k(\hat{\theta},(\bar{m}_j,\sigma_j)),\theta) - u_j(\ell^k(\hat{\theta},(m_j,\sigma_j)),\theta) > 0$. Thus, we compute the following:

$$\begin{aligned}
& u_j(\ell^k(\hat{\theta},(\bar{m}_j,\sigma_j)),\theta) - u_j(\ell^k(\hat{\theta},(m_j,\sigma_j)),\theta) \\
= \ & \bar{p}_0(u_0 - u_z) + \bar{p}_1(\bar{u}_1 - u_z) + u_z - [p_0(u_0 - u_z) + p_1(u_1 - u_z) + u_z] \\
= \ & (\bar{p}_1 - p_1)(\bar{u}_1 - u_0) + (\bar{p}_1 - p_0)(u_0 - u_z) + (\bar{p}_0 - p_1)(u_0 - u_z) + p_1(\bar{u}_1 - u_1) \\
> \ & 0,
\end{aligned}$$

where the strict inequality follows because $(\bar{p}_1 - p_0)(u_0 - u_z) \geq 0$, $(\bar{p}_0 - p_1)(u_0 - u_z) \geq 0$, $p_1(\bar{u}_1 - u_1) \geq 0$, and $(\bar{p}_1 - p_1)(\bar{u}_1 - u_0) > 0$. This contradicts the hypothesis that σ is a Nash equilibrium in the game $\Gamma^K(\theta,u)$. ■

Case 2-4: $u_j(c^*_{\hat{\theta}}(j,\theta),\theta) = u_z > u_0$

We will show that $P^\sigma(q = 0|\hat{\theta}) \leq 1/K$. Suppose not, that is, $P^\sigma(q = 0|\hat{\theta}) > 1/K$. We construct $\bar{\sigma}_j$ to be identical to σ_j except that $P_i^{(\bar{\sigma}_j,\sigma_{-j})}(q|\hat{\theta}) = 1/K$ for all $q \in \{0, \cdots, K-1\}$ and $\bar{m}_j^3[\hat{\theta}] = c^*_{\hat{\theta}}(j, \theta)$ for all $m_j \in \text{supp}\ (\bar{\sigma}_j)$. Then, since we have $u_j(c^*_{\hat{\theta}}(j, \theta), \theta) = u_z$, we compute agent j's payoff difference between $(\bar{\sigma}_j, \sigma_{-j})$ and σ:

$$
\begin{aligned}
& u_j(g^K(\bar{\sigma}_j, \sigma_{-j}), \theta) - u_j(g^K(\sigma), \theta) \\
= \ & P^\sigma(\hat{\theta}) \left[\frac{K-1}{K} u_z + \frac{1}{K} u_0 \right] \\
& - P^\sigma(\hat{\theta}) \left[P^\sigma(q = 0|\hat{\theta}) u_0 + (1 - P^\sigma(q = 0|\hat{\theta})) u_z \right] \\
= \ & P^\sigma(\hat{\theta}) \left(P^\sigma(q = 0|\hat{\theta}) - \frac{1}{K} \right) [u_z - u_0] \\
> \ & 0,
\end{aligned}
$$

where the strict inequality follows because $P^\sigma(\hat{\theta}) > 0$; $P^\sigma(q = 0|\hat{\theta}) > 1/K$; and $u_z > u_0$. This implies that σ is not a Nash equilibrium in the game $\Gamma^K(\theta, u)$, which is the desired contradiction. Thus, we have $P^\sigma(q = 0|\hat{\theta}) + P^\sigma(q = 0|\hat{\theta}) \leq 1/K$. We can then use Step 4 to conclude that there exists $K \in \mathbb{N}$ large enough so that $u_i(\ell^K(\hat{\theta}, \sigma), \theta) < u_i(f(\theta), \theta)$ for all $i \in I$. ■

Step 6: Let $\sigma \in NE(\Gamma^K(\theta, u))$, where we later choose K large enough. Assume that there exists $\hat{\theta} \in \Theta$ with $P^\sigma(\hat{\theta}) > 0$ such that $P^\sigma(q|\hat{\theta}) > 0$ for all $q \in \{0, 1\}$. If $f(\theta)$ is not $C_{\hat{\theta}}$-acceptable at state θ, then there exists $K \in \mathbb{N}$ large enough so that $u_i(\ell^K(\hat{\theta}, \sigma), \theta) < u_i(f(\theta), \theta)$ for all $i \in I$.

Proof of Step 6: Fix $\sigma \in NE(\Gamma^K(\theta, u))$ and $\hat{\theta} \in \Theta$ such that $P^\sigma(\hat{\theta}) > 0$ and $P^\sigma(q|\hat{\theta}) > 0$ for all $q \in \{0, 1\}$. We take the contrapositive statement of Property 2 of Condition COM: for any $x \in \bigcup_{i \in \mathcal{G}_{\hat{\theta}}} C_{\hat{\theta}}(i)$, if $x \neq f(\theta)$, x is not $C_{\hat{\theta}}$-acceptable at state θ and representation $u \in \hat{\mathcal{U}}$. Let $\mathcal{G}_{\hat{\theta}} = \{i, j\}$ in the rest of the proof. In addition, we know that either

$$
f(\theta) \in C_{\hat{\theta}}(i) \cup C_{\hat{\theta}}(j) \ \text{ or } \ f(\theta) \notin C_{\hat{\theta}}(i) \cup C_{\hat{\theta}}(j)
$$

holds. Since $f(\theta)$ is not $C_{\hat{\theta}}$-acceptable at state θ and representation $u \in \hat{\mathcal{U}}$, we therefore have the following property: for any $x \in C_{\hat{\theta}}(i) \cup C_{\hat{\theta}}(j)$, x is not $C_{\hat{\theta}}$-acceptable at state θ and representation $u \in \hat{\mathcal{U}}$. This implies that for any $c^*_{\hat{\theta}}(i, \theta) \in C^*_{\hat{\theta}}(i, \theta)$ and any $c^*_{\hat{\theta}}(j, \theta) \in C^*_{\hat{\theta}}(j, \theta)$, we have $u_i(c^*_{\hat{\theta}}(i, \theta), \theta) > u_i(c^*_{\hat{\theta}}(j, \theta), \theta)$ and $u_j(c^*_{\hat{\theta}}(j, \theta), \theta) > u_j(c^*_{\hat{\theta}}(i, \theta), \theta)$.

If either $u_i(c^*_{\hat{\theta}}(i, \theta), \theta) = u_i(z_{\hat{\theta}}, \theta)$ or $u_j(c^*_{\hat{\theta}}(j, \theta), \theta) = u_j(z_{\hat{\theta}}, \theta)$ holds, then we can appeal to an argument identical to the one employed in Case 2-4 of Step 5 to conclude that there exists $K \in \mathbb{N}$ large enough so that $u_i(\ell^K(\hat{\theta}, \sigma), \theta) < u_i(f(\theta), \theta)$ for all $i \in I$.

Therefore, we can assume that both $u_i(c^*_{\hat{\theta}}(i,\theta),\theta) > u_i(z_{\hat{\theta}},\theta)$ and $u_j(c^*_{\hat{\theta}}(j,\theta),\theta) > u_j(z_{\hat{\theta}},\theta) \equiv u_z$ hold. We define π_i^{min} and π_i^{max} as follows:

$$\begin{aligned}\pi_i^{min} &= \min\{P_i^\sigma(q|\hat{\theta}) \in [0,1] | q \in \{0,\ldots,K-1\}\},\\ \text{and } \pi_i^{max} &= \max\{P_i^\sigma(q|\hat{\theta}) \in [0,1] | q \in \{0,\ldots,K-1\}\}.\end{aligned}$$

We can also define π_j^{min} and π_j^{max} in a similar fashion. We will show that $\pi_i^{min} = \pi_i^{max}$ and $\pi_j^{min} = \pi_j^{max}$, which implies that both agents randomize uniformly in their choice of integer. We shall prove this through Steps 6.a and 6.b below.

Step 6.a: For any $q^* \in \{0,\ldots,K-1\}$, if $P_i^\sigma(q^*|\hat{\theta}) = \pi_i^{min}$ and $\pi_i^{min} < \pi_i^{max}$, then $P_j^\sigma(b_1(q^*)|\hat{\theta}) = 0$, where $b_1(q^*) \in \{0,\ldots,K-1\}$ such that $q^* + b_1(q^*) \pmod K = 1$.

Proof of Step 6.a: Fix $q^* \in \{0,\ldots,K-1\}$ such that $P_i^\sigma(q^*|\hat{\theta}) = \pi_i^{min} < \pi_i^{max}$. Assume, by way of contradiction, that $P_j^\sigma(b_1(q^*)|\hat{\theta}) > 0$. This implies that there exists $m \in \text{supp }(\sigma)$ such that $\theta^m = \hat{\theta}$ and $m_j^2[\hat{\theta}] = b_1(q^*)$. This further implies

$$p_1 \equiv P^{(m_j,\sigma_{-j})}(q=1|\hat{\theta}) = P^{(m_j,\sigma_{-j})}(q^* + b_1(q^*)|\hat{\theta}) = P_i^{(m_j,\sigma_{-j})}(q^*|\hat{\theta}) = \pi_i^{min}.$$

We claim that m_j is not a best response to σ_{-j}, contradicting the hypothesis that σ is a Nash equilibrium in the game $\Gamma^K(\theta,u)$. To prove this, we consider two possible cases:

Case 1: $u_j(c^*_{\hat{\theta}}(j,\theta),\theta) > u_z > u_0$.

Our goal here is to find a message $\bar{m}_j$, which together with σ_{-j} induces a lottery $\ell^K(\hat{\theta},(\bar{m}_j,\sigma_{-j}))$ that first-order stochastically dominates the lottery induced by (m_j,σ_{-j}), hence showing that σ is not a Nash equilibrium of the game $\Gamma^K(\theta,u)$. To achieve this, we need to find $\bar{m}_j^2[\hat{\theta}] \in \{0,\ldots,K-1\}$, which together with σ_{-j} guarantees that $\bar{p}_1 > \pi_i^{min}$ and $\bar{p}_0 = \pi_i^{min}$. We shall propose an algorithm selecting such $\bar{m}_j^2[\hat{\theta}]$.

Recall the following notation:

$$M^*(\hat{\theta},q,i) = \left\{ m \in M^K \,\middle|\, \theta^m = \hat{\theta},\ m_i^2[\hat{\theta}] = q \right\},$$

and

$$P_i^\sigma(q|\hat{\theta}) = \sum_{m_i:(m_i,m_{-i})\in M^*(\hat{\theta},q,i)} \sigma_i(m_i)/P^\sigma(\hat{\theta}).$$

We also use this notation in Case 2 later. Start the algorithm by setting $q_0 = q^*$. Next, for any $h \in \{1, \cdots, K-1\}$ and $q_{h-1} \in \{0, \ldots, K-1\}$, we define

$$q_h = \begin{cases} q_{h-1} + 1 & \text{if } q_{h-1} \leq K-2, \\ 0 & \text{if } q_{h-1} = K-1. \end{cases}$$

Since $\pi_i^{min} < \pi_i^{max}$, we can choose $h \in \{1, \ldots, K-1\}$ uniquely in such a way that $P_i^\sigma(q_h|\hat{\theta}) > \pi_i^{min}$ and $P_i^\sigma(q_{h'}|\hat{\theta}) = \pi_i^{min}$ for all $h' \in \{0, \ldots, h-1\}$. Then, we set $\bar{m}_j^2[\hat{\theta}] = b_1(q_h)$. Define $\bar{m}_j^3[\hat{\theta}]$ as follows:

$$\bar{m}_j^3[\hat{\theta}] = \begin{cases} c^*_{\hat{\theta}}(j, \theta) & \text{if } m_j^3[\hat{\theta}] \notin C^*_{\hat{\theta}}(j, \theta), \\ m_j^3[\hat{\theta}] & \text{if } m_j^3[\hat{\theta}] \in C^*_{\hat{\theta}}(j, \theta). \end{cases}$$

Thus, we define $\bar{m}_j$ to be identical to m_j except that $m_j^2[\hat{\theta}]$ is replaced by $\bar{m}_j^2[\hat{\theta}]$ and $m_j^3[\hat{\theta}]$ is replaced by $\bar{m}_j^3[\hat{\theta}]$. By the algorithm selecting q_h and construction of $\bar{m}_j^2[\hat{\theta}]$, we have the following properties:

$$\begin{aligned} \bar{p}_1 \equiv P^{(\bar{m}_j, \sigma_{-j})}(q = 1|\hat{\theta}) &= P^{(\bar{m}_j, \sigma_{-j})}\Big(q_h + b_1(q_h) \pmod K \Big| \hat{\theta}\Big) \\ &= P_i^{(\bar{m}_j, \sigma_{-j})}(q_h|\hat{\theta}) > \pi_i^{min}, \\ \bar{p}_0 \equiv P^{(\bar{m}_j, \sigma_{-j})}(q = 0|\hat{\theta}) &= P^{(\bar{m}_j, \sigma_{-j})}\Big(q_{h-1} + b_1(q_h) \pmod K \Big| \hat{\theta}\Big) \\ &= P_i^{(\bar{m}_j, \sigma_{-j})}(q_{h-1}|\hat{\theta}) = \pi_i^{min}. \end{aligned}$$

It follows from $p_1 \equiv P^{(m_j, \sigma_{-j})}(q = 1|\hat{\theta}) = \pi_i^{min}$ that $\bar{p}_1 - p_1 > 0$. Since $\bar{p}_0 = \pi_i^{min}$, we also have

$$p_0 \equiv P^{(m_j, \sigma_{-j})}(q = 0|\hat{\theta}) \geq P^{(\bar{m}_j, \sigma_j)}(q = 0|\hat{\theta}) \equiv \bar{p}_0,$$

which implies $p_0 - \bar{p}_0 \geq 0$. Due to the construction of $\bar{m}_j^3[\hat{\theta}]$, we obtain the following inequalities:

$$\begin{aligned} u_j(\bar{m}_j^3[\hat{\theta}], \theta) = u_j(c^*_{\hat{\theta}}(j, \theta), \theta) > u_j(z_{\hat{\theta}}, \theta) &\Rightarrow \bar{u}_1 - u_z > 0, \\ u_j(\bar{m}_j^3[\hat{\theta}], \theta) = u_j(c^*_{\hat{\theta}}(j, \theta), \theta) \geq u_j(m_j^3[\hat{\theta}], \theta) &\Rightarrow \bar{u}_1 - u_1 \geq 0. \end{aligned}$$

We claim now that $u_j(g(\bar{m}_j, \sigma_j), \theta) - u_j(g(m_j, \sigma_j), \theta) > 0$. Since $P^\sigma(\hat{\theta}) > 0$, by Step 2 and the construction of $\bar{m}_j$, it suffices to show that $u_j(\ell^K(\hat{\theta}, (\bar{m}_j, \sigma_j)), \theta) - u_j(\ell^K(\hat{\theta}, (m_j, \sigma_j)), \theta) > 0$. Thus, we compute

$$\begin{aligned} & u_j(\ell^K(\hat{\theta}, (\bar{m}_j, \sigma_j)), \theta) - u_j(\ell^K(\hat{\theta}, (m_j, \sigma_j)), \theta) \\ = \; & \bar{p}_0(u_0 - u_z) + \bar{p}_1(\bar{u}_1 - u_z) + u_z - [p_0(u_0 - u_z) + p_1(u_1 - u_z) + u_z] \\ = \; & (\bar{p}_1 - p_1)(\bar{u}_1 - u_z) + (p_0 - \bar{p}_0)(u_z - u_0) + p_1(\bar{u}_1 - u_1) \\ > \; & 0, \end{aligned}$$

where the strict inequality follows because $(p_0 - \bar{p}_0)(u_z - u_0) \geq 0$, $p_1(\bar{u}_1 - u_1) \geq 0$, and $(\bar{p}_1 - p_1)(\bar{u}_1 - u_z) > 0$. This contradicts the hypothesis that σ is a Nash equilibrium in the game $\Gamma^K(\theta, u)$. ■

Case 2: $u_j(c^*_{\hat{\theta}}(j,\theta),\theta) > u_0 > u_z$.

Our goal here is to find a message $\bar{m}_j$, which together with σ_{-j} induces a lottery $\ell^k(\hat{\theta}, (\bar{m}_j, \sigma_{-j}))$ that first-order stochastically dominates the lottery induced by (m_j, σ_{-j}), showing that σ is not a Nash equilibrium of the game $\Gamma^K(\theta, u)$. To achieve this, we need to find $\bar{m}^2_j[\hat{\theta}] \in \{0, \dots, K-1\}$, which together with σ_{-j} guarantees that $\bar{p}_1 > \pi^{min}_i$ and $\bar{p}_0 + \bar{p}_1 > p_0 + p_1$. Such $\bar{m}^2_j[\hat{\theta}]$ will be found by the following algorithm:

Start the algorithm by setting $q_0 = q^*$. Next, for any $h \in \{1, \cdots, K-1\}$ and $q_{h-1} \in \{0, \dots, K-1\}$, we define

$$q_h = \begin{cases} q_{h-1} - 1 & \text{if } q_{h-1} \geq 1, \\ K-1 & \text{if } q_{h-1} = 0. \end{cases}$$

Since $\pi^{min}_i < \pi^{max}_i$, we can choose $h \in \{1, \dots, K-1\}$ uniquely in such a way that $P^\sigma_i(q_h|\hat{\theta}) > \pi^{min}_i$ and $P^\sigma_i(q_{h'}|\hat{\theta}) = \pi^{min}_i$ for all $h' \in \{0, \dots, h-1\}$. Then, we set $\bar{m}^2_j[\hat{\theta}] = b_1(q_h)$. Define also $\bar{m}^3_j[\hat{\theta}]$ as follows:

$$\bar{m}^3_j[\hat{\theta}] = \begin{cases} c^*_{\hat{\theta}}(j,\theta) & \text{if } m^3_j[\hat{\theta}] \notin C^*_{\hat{\theta}}(j,\theta), \\ m^3_j[\hat{\theta}] & \text{if } m^3_j[\hat{\theta}] \in C^*_{\hat{\theta}}(j,\theta). \end{cases}$$

Thus, we define $\bar{m}_j$ to be identical to m_j except that $m^2_j[\hat{\theta}]$ is replaced by $\bar{m}^2_j[\hat{\theta}]$ and $m^3_j[\hat{\theta}]$ is replaced by $\bar{m}^3_j[\hat{\theta}]$.

By the algorithm selection q_h and construction of $\bar{m}^2_j[\hat{\theta}]$, we have the following properties:

$$\begin{aligned}
\bar{p}_1 \equiv P^{(\bar{m}_j,\sigma_{-j})}(q=1|\hat{\theta}) &= P^{(\bar{m}_j,\sigma_{-j})}(q_h + b_1(q_h) \pmod K|\hat{\theta}) \\
&= P^{(\bar{m}_j,\sigma_{-j})}_i(q_h|\hat{\theta}) > \pi^{min}_i, \\
&\geq P^{(\bar{m}_j,\sigma_{-j})}_i(q_1|\hat{\theta}) \\
&= \begin{cases} P^{(m_j,\sigma_{-j})}(q_0 - 1 + m^2_j[\hat{\theta}] \pmod K|\hat{\theta}) & \text{if } q_0 \geq 1 \\ P^{(m_j,\sigma_{-j})}(K - 1 + m^2_j[\hat{\theta}] \pmod K|\hat{\theta}) & \text{if } q_0 = 0 \end{cases} \\
&= P^{(m_j,\sigma_{-j})}(q=0|\hat{\theta}) \equiv p_0 \ (\because q_0 + m^2_j[\hat{\theta}] \pmod K = 1) \\
P^{(\bar{m}_j,\sigma_{-j})}(q=2|\hat{\theta}) &= P^{(\bar{m}_j,\sigma_{-j})}(q_{h-1} + b_1(q_h) \pmod K|\hat{\theta}) \\
&= P^{(\bar{m}_j,\sigma_{-j})}_i(q_{h-1}|\hat{\theta}) = \pi^{min}_i,
\end{aligned}$$

which implies $\bar{p}_1 - p_0 \geq 0$. Since $p_1 \equiv P^{(m_j,\sigma_{-j})}(q=1|\hat{\theta}) = \pi_i^{min}$ and $P^{(\bar{m}_j,\sigma_{-j})}(q=0|\hat{\theta}) \geq \pi_i^{min}$, we have $P^{(\bar{m}_j,\sigma_{-j})}(q=0|\hat{\theta}) \geq P^{(m_j,\sigma_{-j})}(q=1|\hat{\theta})$, which implies that $\bar{p}_0 - p_1 \geq 0$. In addition, since $p_1 \equiv P^{(m_j,\sigma_{-j})}(q=1|\hat{\theta}) = \pi_i^{min}$, we also have $P^{(\bar{m}_j,\sigma_{-j})}(q=1|\hat{\theta}) > P^{(m_j,\sigma_{-j})}(q=1|\hat{\theta})$, which implies that $\bar{p}_1 - p_1 > 0$.

Since we have $u_j(c^*_{\hat{\theta}}(j,\theta),\theta) > u_0 > u_z$, due to the construction of $\bar{m}_j^3[\hat{\theta}]$, we have the following inequality:

$$\bar{u}_1 \equiv u_j(\bar{m}_j^3[\hat{\theta}],\theta) = u_j(c^*_{\hat{\theta}}(j,\theta),\theta) > u_j(\ell_0^K(\hat{\theta},(m_j,\sigma_{-j})),\theta) \equiv u_0$$

Thus, $\bar{u}_1 - u_0 > 0$.

Now, we claim that $u_j(g(\bar{m}_j,\sigma_j),\theta) - u_j(g(m_j,\sigma_j),\theta) > 0$. Since $P^{\sigma}(\hat{\theta}) > 0$, by Step 2, it suffices to show that $u_j(\ell^k(\hat{\theta},(\bar{m}_j,\sigma_j)),\theta) - u_j(\ell^k(\hat{\theta},(m_j,\sigma_j)),\theta) > 0$. Thus, we compute the following:

$$\begin{aligned}
& u_j(\ell^k(\hat{\theta},(\bar{m}_j,\sigma_j)),\theta) - u_j(\ell^k(\hat{\theta},(m_j,\sigma_j)),\theta) \\
= \; & \bar{p}_0(u_0-u_z) + \bar{p}_1(\bar{u}_1-u_z) + u_z - [p_0(u_0-u_z) + p_1(u_1-u_z) + u_z] \\
= \; & (\bar{p}_1-p_1)(\bar{u}_1-u_0) + (\bar{p}_1-p_0)(u_0-u_z) + (\bar{p}_0-p_1)(u_0-u_z) + p_1(\bar{u}_1-u_1) \\
> \; & 0,
\end{aligned}$$

where the strict inequality follows because $(\bar{p}_1-p_0)(u_0-u_z) \geq 0$, $(\bar{p}_0-p_1)(u_0-u_z) \geq 0$, $p_1(\bar{u}_1-u_1) \geq 0$, and $(\bar{p}_1-p_1)(\bar{u}_1-u_0) > 0$. This contradicts the hypothesis that σ is a Nash equilibrium in the game $\Gamma^K(\theta,u)$. ■

Step 6.b: $\pi_i^{min} = \pi_i^{\max}$.

Proof of Step 6.b: Assume, by way of contradiction, that $\pi_i^{min} < \pi_i^{\max}$. We then use Step 6.a to conclude that, for each $q \in \{0,\cdots,K-1\}$, $P_i^{\sigma}(q|\hat{\theta}) = \pi_i^{min}$ implies $P_j^{\sigma}(b_1(q)|\hat{\theta}) = 0$. This implies $\pi_j^{min} = 0$ so that $\pi_j^{min} < \pi_j^{\max}$. We then establish the counterpart of Step 6.a by swapping the roles of i and j and replacing the function $b_1(q)$ with the function $b_0(q)$, where we define $b_0(q) \in \{0,\ldots,K-1\}$ such that $q + b_0(q) \pmod K = 0$. Therefore, we conclude that, for each $q \in \{0,\cdots,K-1\}$, if $P_j^{\sigma}(q|\hat{\theta}) = \pi_j^{min}$, then $P_i^{\sigma}(b_0(q)|\hat{\theta}) = 0$. Hence, $\pi_i^{min} = 0$.

Then, we can set $q \in \{0,\ldots,K-1\}$ such that $P_i^{\sigma}(q|\hat{\theta}) = 0$. By Step 6.a, we have

$$P_j^{\sigma}(b_1(q)|\hat{\theta}) = \begin{cases} P_j^{\sigma}(1-q \pmod K|\hat{\theta}) = 0 & \text{if } q \leq 1, \\ P_j^{\sigma}(K+1-q \pmod K|\hat{\theta}) = 0 & \text{if } q > 2. \end{cases}$$

By Step 6.a, we also have

$$P_i^{\sigma}(b_0(b_1(q))|\hat{\theta}) = \begin{cases} P_i^{\sigma}(q-1 \pmod K|\hat{\theta}) = 0 & \text{if } q \geq 1, \\ P_i^{\sigma}(K-1 \pmod K|\hat{\theta}) = 0 & \text{if } q = 0. \end{cases}$$

We use Step 6.a repeatedly to conclude that $P_i^\sigma(q|\hat{\theta}) = P_j^\sigma(q|\hat{\theta}) = 0$ for each $q \in \{0, \ldots, K-1\}$. However, this is simply impossible, as we have $\sum_{q=0}^{K-1} P_i^\sigma(q|\hat{\theta}) = \sum_{q=0}^{K-1} P_j^\sigma(q|\hat{\theta}) = 1$. Therefore, we must have $\pi_i^{min} = \pi_i^{\max}$. ■

Since we can replace the role of agent i with that of agent j in the entire argument, we conclude that $\pi_i^{min} = \pi_i^{max}$ and $\pi_j^{min} = \pi_j^{max}$. Thus, we have $P_i^\sigma(q|\hat{\theta}) = P_j^\sigma(q|\hat{\theta}) = 1/K$ for each $q \in \{0, \ldots, K-1\}$. This implies that $P^\sigma(q = 0|\hat{\theta}) + P^\sigma(q = 1|\hat{\theta}) \leq 2/K$. By Step 4, we conclude that there exists $K \in \mathbb{N}$ large enough so that $u_i(\ell^K(\hat{\theta}, \sigma), \theta) < u_i(f(\theta), \theta)$ for all $i \in I$. This completes the proof. ■

Step 7: Let $\sigma \in NE(\Gamma^K(\theta, u))$, where we later choose K large enough. Assume that there exists $\hat{\theta} \in \Theta$ such that $P^\sigma(\hat{\theta}) > 0$ and $P^\sigma(q|\hat{\theta}) > 0$ for all $q \in \{0, 1\}$. If $\ell^K(\hat{\theta}, \sigma) \neq f(\theta)$ and $f(\theta)$ is $C_{\hat{\theta}}$-acceptable at state θ and representation $u \in \mathcal{U}$, then there exists $K \in \mathbb{N}$ large enough so that $u_i(\ell^K(\hat{\theta}, \sigma), \theta) < u_i(f(\theta), \theta)$ for all $i \in I$.

Proof of Step 7: Fix $\sigma \in NE(\Gamma^K(\theta, u))$. Assume that there exists $\hat{\theta} \in \Theta$ such that $P^\sigma(\hat{\theta}) > 0$ and $P^\sigma(q|\hat{\theta}) > 0$ for all $q \in \{0, 1\}$. Assume further that $f(\theta)$ is $C_{\hat{\theta}}$-acceptable at state θ and representation $u \in \mathcal{U}$. Then, there exist $q \in \{0, 1\}$ and $i \in \mathcal{G}_{\hat{\theta}}$ such that $f(\theta) \in C_{\hat{\theta}}(i)$, where $i = w_{\hat{\theta}}(q)$. We write $\mathcal{G}_{\hat{\theta}} = \{i, j\}$ in the rest of the proof. By Property 1-ii of Condition COM, we have $f(\theta) \in C_{\hat{\theta}}(i) \cap C_{\hat{\theta}}(j)$.[12] By Property 4 of Condition COM, $f(\theta)$ is agent i's unique maximal element in $C_{\hat{\theta}}(i)$ so that $C^*_{\hat{\theta}}(i, \theta) = \{f(\theta)\}$. Similarly, by Property 4 of Condition COM, $f(\theta)$ is also agent j's unique maximal element in $C_{\hat{\theta}}(j)$ so that $C^*_{\hat{\theta}}(j, \theta) = \{f(\theta)\}$. Step 3 implies that for each $q \in \{0, 1\}$, we have $\text{supp}(\ell_q^K(\hat{\theta}, \sigma)) \in C^*_{\hat{\theta}}(w_{\hat{\theta}}(q), \theta)$, hence we can conclude that both $\ell_0^K(\hat{\theta}, \sigma) = f(\theta)$ and $\ell_1^K(\hat{\theta}, \sigma) = f(\theta)$. Using the notation developed in Step 2, we can write

$$\begin{aligned} \ell^K(\hat{\theta}, \sigma) &= P^\sigma(q = 0|\hat{\theta})(\ell_0^K(\hat{\theta}, \sigma) - z_{\hat{\theta}}) + P^\sigma(q = 1|\hat{\theta})(\ell_1^K(\hat{\theta}, \sigma) - z_{\hat{\theta}}) + z_{\hat{\theta}} \\ &= \left(P^\sigma(q = 0|\ \hat{\theta}) + P^\sigma(q = 1|\ \hat{\theta})\right)(f(\theta) - z_{\hat{\theta}}) + z_{\hat{\theta}}. \end{aligned}$$

Moreover, since we assume $\ell^K(\hat{\theta}, \sigma) \neq f(\theta)$, $z_{\hat{\theta}}$ is induced with positive probability. By Property 3 of Condition COM, we have $u_i(f(\theta), \theta) - u_i(z_{\hat{\theta}}, \theta) \geq \varepsilon$

[12] When there are only two agents, we do not need to assume $f(\theta) \in C_{\hat{\theta}}(i) \cap C_{\hat{\theta}}(j)$. The hypothesis that $f(\theta)$ is $C_{\hat{\theta}}$-acceptable at state θ and representation $u \in \mathcal{U}$ and $f(\theta) \neq \ell^K(\hat{\theta}, \sigma)$ implies that both agents must strictly prefer $f(\theta)$ to lottery $\ell^K(\hat{\theta}, \sigma)$. When there are only two agents, this implies that $\ell^K(\hat{\theta}, \sigma)$ is strictly Pareto dominated by $f(\theta)$. Hence, we can conclude that we can ignore σ because it is "uncompelling" without the help of Property 1-ii of Condition COM.

for all $i \in I$ and $u \in \hat{\mathcal{U}}$. By the continuity of expected payoff, this implies that $u_i(\ell^K(\hat{\theta}, \sigma), \theta) < u_i(f(\theta), \theta)$ for all $i \in I$ and $u \in \hat{\mathcal{U}}$. ■

By Step 2, we have the following expressions:

$$g^K(\sigma) = \sum_{\hat{\theta} \in \Theta} P^\sigma(\hat{\theta}) \ell^K(\hat{\theta}, \sigma) \text{ and } u_i(g^K(\sigma), \theta) = \sum_{\hat{\theta} \in \Theta} P^\sigma(\hat{\theta}) u_i(\ell^K(\hat{\theta}, \sigma), \theta).$$

Steps 5, 6 and 7 show that, for a fixed $u \in \hat{\mathcal{U}}$, we can find a value of K such that, for every $\theta, \hat{\theta} \in \Theta$ and every $i \in I$, if $P^\sigma(\hat{\theta}) > 0$ and $\ell^K(\hat{\theta}, \sigma) \neq f(\theta)$, then we have $u_i(\ell^K(\hat{\theta}, \sigma), \theta) < u_i(f(\theta, \theta)$. By Property 3 of Condition COM, we have that there exists $\varepsilon > 0$ such that $u_i(f(\theta), \theta) - u_i(z_{\hat{\theta}}, \theta) \geq \varepsilon$ for all possible representations $u \in \hat{\mathcal{U}}$. Therefore, we can choose $K \in \mathbb{N}$ large enough such that $u_i(\ell^K(\hat{\theta}, \sigma), \theta) < u_i(f(\theta), \theta)$ hold for all $u \in \hat{\mathcal{U}}$ (and all $\theta, \hat{\theta} \in \Theta$, $i \in I$). We summarize this into the following step:

Step 8: There exists $K \in \mathbb{N}$ large enough such that, for any $u \in \hat{\mathcal{U}}$ and $\sigma \in NE(\Gamma^K(\theta, u))$, it follows that either $g^K(\sigma) = f(\theta)$ or $u_i(g^K(\sigma), \theta) < u_i(f(\theta), \theta)$ for all $i \in I$.

Combining Steps 1 and 8, we conclude that there exists $K \in \mathbb{N}$ large enough such that the SCF f is C-implementable with respect to $\hat{\mathcal{U}}$ by the mechanism Γ^K. This completes the proof of the theorem. ■

6 Indispensability of Condition COM

We show in this section that each of the properties in Condition COM is indispensable for the results in Theorem 1. We show this by arguing that Property 2 and the first part of Property 1 in Condition COM are in fact implied by the existence of a mechanism in our class capable of C-implementing f, while for each of Properties 3 and 4, as well as the second part of Property 1 we will give an example in which all properties are satisfied, and yet the mechanisms in our family still fail at C-implementing f.

Proposition 2 *If f is C-implementable with respect to $\hat{\mathcal{U}}$ by a Γ^k mechanism, then the finite environment ε satisfies Property 2 and the first part of Property 1 with respect to f and $\hat{\mathcal{U}}$.*

Proof: To verify Properties 1 and 2, we must first define a collection of forums. We define this collection of forums based on the mechanism as follows: for each state θ, there must exist a message profile m such that $g(m) = f(\theta)$ and m is a

Nash equilibrium in the game $\Gamma^k(\theta)$. Let then $\mathcal{F}_\theta = \{\mathcal{G}_{\theta^m}, w_{\theta^m}, C_{\theta^m}, z_{\theta^m}\}$, where θ^m is the forum implied by m, as described in Step 1 of the mechanism.

Assume now that for some $\theta \in \Theta$ and some $u \in \mathcal{U}^\varepsilon$ we have that $f(\theta)$ is not C_θ-acceptable at state θ and representation u. This means that there exists a player $i \in \mathcal{G}_\theta$ and a lottery $l \in C_\theta(i)$ such that $u_i(l,\theta) > u_i(f(\theta),\theta)$. But that implies that we can then find a profitable unilateral deviation for agent i at the message profile that yields $g(m) = f(\theta)$, contradicting the hypothesis that this message profile was a Nash equilibrium [13]. This proves that the first part of Property 1 must be satisfied if Γ^k C-implements f.

Next, assume that for some pair $\theta, \hat{\theta} \in \Theta$ and some $u \in \mathcal{U}^\varepsilon$ we can find a lottery $l \neq f(\theta)$ with the property that l is $C_{\hat{\theta}}$-acceptable at state θ and representation u. Then we can construct a message profile m' such that $g(m') = l$ and m' is a Nash equilibrium for $\Gamma^k(\theta)$, contradicting the hypothesis that Γ^k C-implements f. Hence, Property 2 is also indispensable. This shows that the first part of Property 1 and Property 2 are implied by C-implementation using a mechanism of the Γ^k class. ■

We now show an example of an environment in which all properties in Condition COM are satisfied, except Property 3, and show that C-implementation fails in this environment.

Consider the environment described in section 3.1. As Condition COM requires three agents, we include a third agent in this environment, whose preferences are equal to agent 1's preferences at state θ, and equal to agent 2's preferences at state θ'. Take now the following collection of forums: for all $\hat{\theta} \in \Theta$, $\mathcal{G}_{\hat{\theta}} = \{1,2\}$, $w_{\hat{\theta}}(0) = 1$, $w_{\hat{\theta}}(1) = 2$, $z_{\hat{\theta}} = c$. Then take $C_\theta(1) = C_\theta(2) = \{a,b,c,d\}$ and $C_\theta(1) = C_\theta(2) = \{c,d\}$.

It can be easily verified that the forums above satisfy Properties 1, 2, and 4 of Condition COM, while violating Property 3. We will show now that when using this collection of forums to construct mechanisms of the class Γ^k C-implementation fails in a similar way to the mechanism in that section. Consider a strategy profile σ^* with the following characteristics:

- $m_i^1 = \theta$ for all $i \in I$;
- $\sigma_i^2[\theta'](k) = 1/K$ for all $i \in \{1,2\}$;
- $m_1^3[\theta] = a$, $m_2^3[\theta] = b$.

It can be verified that σ^* is a Nash equilibrium at state θ' for any mechanism from the class Γ^k constructed from the collection of forums described above. The

[13] The deviation is fairly straightforward to obtain, requiring only agent i to adjust his integer sent so that it induces Rule 1 with $i^* = i$ and then selecting $x_i^\theta = l$.

outcome, $g(\sigma^*)$, will yield a with probability $1/k$, b with probability $1/k$ and c with the remaining probability. This is strictly preferable to $c = f(\theta')$ for all agents at state θ'. Thus, C-implementation fails in this example. ■

For the next two examples, we will first present an environment in which Condition COM holds, then modify it in a specific way in each example, so that only a specific property is violated. Consider the following environment ε^*, with $I = \{1,2,3,4\}$, $\Theta = \{\theta, \theta'\}$, $A = \{a,b,c\}$, $f(\theta) = (0.1, 0.8, 0.1)$, in which $(0.1, 0.8, 0.1)$ denotes the lottery whic yields a with probability 0.1, b with probability 0.8 and c with probability 0.1. Using the same notation, we have $f(\theta') = (0.8, 0.1, 0.1)$. Preferences are given by table 6.

Agent	$u_i(a,\theta)$; $u_i(a,\theta')$	$u_i(b,\theta)$; $u_i(b,\theta')$	$u_i(c,\theta)$; $u_i(c,\theta')$
Agent 1	1 ; 0.8	0.8 ; 0	0 ; 1
Agent 2	0 ; 1	1 ; 0	0.8 ; 0.8
Agent 3	0 ; 0.8	0.8 ; 1	1 ; 0
Agent 4	0 ; 1	1 ; 0	0.8 ; 0.8

Table 6: The environment $\mathcal{E}^*$

This environment satisfy Condition COM with the following two forums: at θ, we have $\mathcal{G}_\theta = \{2,3\}$, $w_\theta(0) = 2, w_\theta(1) = 3$, $C_\theta(2) = \{z_\theta; a; c; f(\theta)\}$ and $C_\theta(3) = \{z_\theta; f(\theta)\}$, with $z_\theta = (1/3, 1/3, 1/3)$. At θ', we have $\mathcal{G}_{\theta'} = \{2,1\}$, $w_{\theta'}(0) = 2, w_{\theta'}(1) = 1$, $C_{\theta'}(2) = \{z_{\theta'}; b; c; f(\theta')\}$ and $C_{\theta'}(1) = \{z_{\theta'}; f(\theta')\}$, with $z_{\theta'} = (1/3, 1/3, 1/3)$.

To construct an environment in which only Property 4 is violated, we modify the environment above in one way: we include the lottery $(0.16, 0.5, 0.34)$ in $C_\theta(3)$. Under representation u, we have that $u_3((0.16, 0.5, 0.34), \theta) = 0.74 = u_3(f(\theta), \theta)$, hence violating Property 4. With this modification, we claim that a strategy profile σ^* with the following characteristics is a Nash equilibrium:

- $m_i^1 = \theta$ for all $i \in I$;
- $P_2^{\sigma^*}(0|\theta) = 1$, $P_3^{\sigma^*}(0|\theta) = P_3^{\sigma^*}(1|\theta) = 1/2$, meaning that half of the time agent 2 wins the modulo game and the other half the winner is agent 3;
- $P^{\sigma^*}(f(\theta)|q = 0, \theta) = 1$, $P^{\sigma^*}(f(\theta)|q = 1, \theta) = 0.99$, and $P^{\sigma^*}((0.16, 0.5, 0.34)|q = 1, \theta) = 0.01$, meaning that whenever agent 2 wins the modulo game, he always selects $f(\theta)$, but when agent 3 is the winner $f(\theta)$ will be chosen with probability 0.99 and $(0.16, 0.5, 0.34)$ will be chosen with the remaining probability.

The key reason why this strategy profile is a Nash equilibrium is that agent 3 is indifferent between $(0.16, 0.5, 0.34)$ and $f(\theta)$, so he can randomize between sending each of them. At the same time, agent 2 has no profitable deviations, since sending any other integer increases the probability of a punishment and the outcome of this equilibrium is close enough to $f(\theta)$. Thus, σ^* is a Nash equilibrium, with $u_3(g(\sigma^*), \theta)) = u_3(f(\theta), \theta))$, showing that C-implementation also fails in this example. ■

Lastly, we need to construct an environment in which only the second part of Property 1 is violated. To do so, we modify environment ε^* by removing $f(\theta)$ from $C_\theta(2)$, thus violating the second part of Property 1. We also replace c in $C_\theta(2)$ for the following lottery $(2/9, 7/18, 7/18)$. Finally, we modify agent 4's preferences at state θ to be $u_4(a, \theta) = 0$, $u_4(b, \theta) = 0.51$, and $u_4(c, \theta) = 1$. With this modification, we claim that a strategy profile σ^* with the following characteristics is a Nash equilibrium:

- $m_i^1 = \theta$ for all $i \in I$;
- $P_2^{\sigma^*}(0|\theta) = 7/9$, $P_2^{\sigma^*}(1|\theta) = 2/9$, $P_3^{\sigma^*}(0|\theta) = 14/23$, $P_3^{\sigma^*}(1|\theta) = 9/23$, which induce $P^{\sigma^*}(0|\theta) = 98/207$, $P^{\sigma^*}(1|\theta) = 91/207$ $P^{\sigma^*}(2|\theta) = 18/207$;
- $P^{\sigma^*}((2/9, 7/18, 7/18)|q = 0, \theta) = 1$ and $P^{\sigma^*}(f(\theta)|q = 1, \theta) = 1$, meaning that whenever agent 2 wins the modulo game, he always selects $(2/9, 7/18, 7/18)$, and when agent 3 is the winner $f(\theta)$ will always be chosen.

In this equilibrium, while both agents prefer agent 3 to be the winner of the modulo game, the lack of coordination causes agent 2 to win with positive probability. When that happens, he picks the lottery in his choices set that offers him the highest utility at state θ, $(2/9, 7/18, 7/18)$, a lottery that agent 4 strictly prefers to $f(\theta)$ at that state. At the same time, while the lack of coordination does induce z_θ with positive probability, the miscoordination is not strong enough to induce z_θ with a high probability. Thus, $g(\sigma)$ is still preferred to $f(\theta)$ to agent 4, showing that C-implementation fails in this example.

7 The Main Result When $n = 2$

Given that each forum essentially determines a subgame with only two agents, it is natural to ask if that Condition is also sufficient to achieve compelling implementation in environments with $n = 2$. Unfortunately, as compelling implementation is a stronger form of Nash implementation, the same difficulties that arise for pure Nash implementation with only two agents are also present here. The main difficulty has to do with how θ^m is chosen. With three agents or more, if all players announce the same state $\hat{\theta}$, then no individual player is capable of changing θ^m. This, in turn, limits the possible outcomes that a player might achieve through a unilateral deviation to be restricted to the outcomes in $C_{\hat{\theta}}$ alone. However, with only two agents, this argument fails, as a single agent can always force a change in θ^m by changing his own message. As a result, the set of outcomes that agent can achieve through an unilateral deviation becomes (potentially) larger than $C_{\hat{\theta}}$. Thus, in order to adjust for that, we need to modify both our class of mechanisms as well as the objects we use to specify them.

Our first step is to modify the notion of forum that we used in the definition of our class of mechanisms, as well as in Condition COM. With $I = \{0, 1\}$, both $\mathcal{G}_{\hat{\theta}}$ and $w_{\hat{\theta}}$ become redundant. At the same time, instead of having a single lottery $z_{\hat{\theta}}$ associated with each state, we will have a lottery associated with each combination of θ_0 and θ_1 reported by the agents. We will represent this through a function $z : \Theta \times \Theta \to \delta(A)$, with its image denoted by Z. As before, the values of this function assumes must be contained in the intersections of values of $C_{\hat{\theta}}$: for all $\theta_0, \theta_1 \in \Theta$, we must have $z(\theta_0, \theta_1) \in C_{\theta_1}(0) \cup C_{\theta_0}(1)$. Because of this interdependence between z and the collection $\{C_\theta\}_{\theta\in\Theta}$, we will call the set $\mathcal{F}^2 = (\{C_\theta\}_{\theta\in\Theta}, z)$ a *forum-2* if the values of z are contained in those intersections in the manner we just described.

We now define our new condition, called Condition COM_2

Definition 4 *The environment* $\mathcal{E} = \big(\{0, 1\}, A, \Theta, (\succeq_i^\theta)_{i\in\{0,1\},\theta\in\Theta}\big)$ *satisfies* ***Condition*** COM_2 *with respect to the SCF* f *and* $\hat{\mathcal{U}}$ *if there exists a forum*$_2$ $\mathcal{F}^2 = (\{C_\theta\}_{\theta\in\Theta}, z)$ *such that:*

1. *For every* $\theta \in \Theta$ *and* $u \in \hat{\mathcal{U}}$, $f(\theta)$ *is* C_θ*-acceptable at state* θ *and representation* u.

2. *For every* $\theta, \hat{\theta} \in \Theta$ *and* $u \in \hat{\mathcal{U}}$, *if* $x \in A$ *is is* $C_{\hat{\theta}}$*-acceptable at state* θ *and representation* u, *then* $x = f(\theta)$.

3. *There exists an* $\varepsilon > 0$ *such that for each* $\theta, \theta_0, \theta_1 \in \Theta$, $u_i(f(\theta), \theta) - u_i(z(\theta_0, \theta_1), \theta) \geq \varepsilon$ *for all* $i \in \{0, 1\}$ *and all* $u \in \hat{\mathcal{U}}$.

4. *For all* $\theta, \hat{\theta} \in \Theta$, $u \in \hat{\mathcal{U}}$, $i \in \{0, 1\}$, *and* $x \in C_{\hat{\theta}}$, *if* $f(\theta)$ *is* $C_{\hat{\theta}}$*-acceptable at state* θ *and representation* u, *then* $u_i(x, \theta) = u_i(f(\theta), \theta)$ *implies* $x = f(\theta)$.

5. *For each* $\theta, \theta_0, \theta_1 \in \Theta$ *and each* $u \in \hat{\mathcal{U}}$, *there exists either* $a_{(\theta_0,\theta_1)} \in C_{\theta_1}(0)$ *such that* $u_0(a_{(\theta_0,\theta_1)}, \theta) > u_0(z(\theta_0, \theta_1), \theta)$ *or* $b_{(\theta_0,\theta_1)} \in C_{\theta_0}(1)$ *such that* $u_1(b_{(\theta_0,\theta_1)}, \theta) > u_1(z(\theta_0, \theta_1), \theta)$;

We use these sets to build a new class of mechanisms, $\Gamma^k = (M^k, g^k)$. For each agent $i \in \{0, 1\}$, the message space is given by $M_i^k \equiv M_i^1 \times M_i^2 \times M_i^3$.

Each component of the message space is defined in a similar fashion as before: let $m_i = (m_i^1, m_i^2, m_i^3) \in M_i$ be agent i's generic message such that (i) $m_i^1 \in M_i^1 = \Theta$; (ii) $m_i^2 = (m_i^2[\tilde{\theta}])_{\tilde{\theta}\in\Theta} \in M_i^2 = \times_{\tilde{\theta}\in\Theta} M_i^2[\tilde{\theta}]$ where $m_i^2[\tilde{\theta}] \in \{0, \ldots, k-1\}$; and (iii) $m_i^3 = (m_i^3[\tilde{\theta}])_{\tilde{\theta}\in\Theta} \in M_i^3 \equiv \times_{\tilde{\theta}\in\Theta} M_i^3[\tilde{\theta}]$ where $M_i^3[\tilde{\theta}] = C_{\tilde{\theta}}(i)$.

The outcome function is also similar as before, with two rules:

Rule 1: If there is a $\theta^m \in \Theta$ such that $m_0^1 = m_1^1 = \theta^m$ and an $i^* \in \{0, 1\}$ such that $m_0^2[\theta^m] + m_1^2[\theta^m] \bmod k = i^*$, then

$$g^k(m) = m_{i^*}^3[\theta^m]$$

Rule 2: For all other cases,

$$g^k(m) = z(\theta_1, \theta_2)$$

.

With this mechanism, we have our second theorem:

Theorem 2 *Let f be an SCF. Suppose that the finite environment $\mathcal{E} = \big(\{1,2\}, A, \Theta, (\succeq_i^\theta)_{i\in I,\theta\in\Theta}\big)$ satisfies Condition COM_2 with respect to f and $\hat{\mathcal{U}}$ and $I = \{1, 2\}$. Then, the SCF f is C-implementable with respect to $\hat{\mathcal{U}}$.*

As Properties 1, 2, 3 and 4 of Condition COM_2 imply Properties 2, 3, 4, and the first part of Property 1 of Condition COM, many of the arguments established in the proof of Theorem 1 can be readily applied to prove Theorem 2. The main difference between the two proofs lies on the fact that the outcome function $g^k(\sigma)$ is now slightly different from the previous function. This requires modified arguments for Steps 1 and 2, as the proof for pure implementation with only 2 agents is different, as is the decomposition of the outcome function into a series of associated lotteries. Therefore, we first need to establish counterparts for these two steps. Once that is done, we will show that the rest of the argument follows from the argument in Theorem 1.

Step 1: Let f be an SCF and $I = \{0, 1\}$. Suppose that the finite environment $\mathcal{E}$ satisfies Properties 1, 2 and 5 of Condition COM_2 with respect to f and $\hat{\mathcal{U}}$. Then, f is implementable in pure strategies by $\Gamma^{k,\bar{\theta}}$.

Proof of Step 1:

Fix $\theta, \bar{\theta} \in \Theta$ and some $u \in \hat{\mathcal{U}}$. By Property 1 $f(\theta)$ is C_θ-acceptable at state θ. Thus, $f(\theta) \in C_\theta(i)$ for some $i \in I$. For simplicity, assume that $i = 0$. Let m be a message profile with the following characteristics:

- $m_j^1 = \theta$ for both agents;

- $m_j^2[\theta] = 0$ for all $j \in \{0, 1\}$;
- $m_0^3[\theta] = f(\theta)$.

It is easy to verify that $g(m) = f(\theta)$. We will now check that m is indeed a Nash equilibrium for the game at state θ. First, notice that any possible deviation strategy for either agent will yield a lottery that is in either $C_\theta(0)$ (if agent 0 is deviating) or $C_\theta(1)$ (if the deviation is done by agent 1). This is immediate for the case in which the deviating strategies preserve $m_i^1 = \theta$, but even a deviation involving a different choice for either θ_0 or θ_1 will result in $z(\theta_0, \theta)$ or $z(\theta, \theta_1)$, respectively, both of which are in the sets $C_\theta(0)$ and $C_\theta(1)$. Then Property 1 ensures that no lottery in $\{C_\theta(i)\}_{i\in\{0,1\}}$ can be a profitable unilateral deviation, which includes all possible outcomes under Rules 1 and 2. Thus, m is indeed a pure strategy Nash equilibrium in the game induced by Γ^k at state θ, regardless of the choices of k.

Fix $\theta \in \Theta$. We shall show that $m \in pureNE(\Gamma^k(\theta))$ implies $g(m) = f(\theta)$. We assume by way of contradiction that there exists $m \in pureNE(\Gamma^k)$ such that $g(m) \neq f(\theta)$. There are two cases to consider. The first one is when there is some $\hat{\theta} \in \Theta$ such that $g(m) \in C_{\hat{\theta}}(0) \cup C_{\hat{\theta}}(1)$. The second is when no such $\hat{\theta}$ exists, which implies that $g(m) = z(\theta_0, \theta_1)$.

Suppose that the first case is true. By Property 2, we know that $g(m)$ is not $C_{\hat{\theta}}$-acceptable at state θ, regardless of the choice of $\hat{\theta}$. Thus, there must exist some agent $i \in \{0, 1\}$ and some choice $x \in C_{\hat{\theta}}(i)$ such that $u_i(x, \theta) > u_i(g(m), \theta)$. Then, agent i has a profitable deviation by sending message $\hat{m}_i$ in which $m_i^3[\hat{\theta}] = x$ and $m_i^2[\hat{\theta}] = -m_j^2[\hat{\theta}] + i \bmod k$. Thus, $g(m)$ cannot be a Nash equilibrium at state θ.

Now suppose that the second case is true. Then, by Property 5, one of the agents has a profitable deviation: either agent 0 is better off by announcing $m_0^1 = m_1^1 = \theta_1$, $m_0^3[\theta_1] = a_{(\theta_0,\theta_1)}$ and $m_0^2[\theta_1] = -m_1^2[\theta_1] \pmod k$ or agent 1 is better off by announcing $m_1^1 = m_0^1\theta_0$, $m_1^3[\theta_0] = b_{(\theta_0,\theta_1)}$ and $m_1^2[\theta_0] = -m_0^2[\theta_0] + 1 \pmod k$ This concludes the argument. ■

As the message space of the mechanism for two agents is identical to the message space of the mechanism for three or more agents, we can define the sets $M^*(\hat{\theta}, q, x)$, $M^*(\hat{\theta}, q)$, $M^*(\hat{\theta})$, and $M^*(\hat{\theta}, q, i)$ in the same way as we did in Theorem 1. Likewise, we can also define the probabilities $P^\sigma(\hat{\theta}, q, x)$, $P^\sigma(q, x|\hat{\theta})$, $P^\sigma(q|\hat{\theta})$, $P^\sigma(x|q, \hat{\theta})$, and $P_i^\sigma(q|\hat{\theta})$, as well as lottery $\ell_q^k(\hat{\theta}, \sigma)$.

To proceed, we need to define a new set, $M^*(\theta_0, \theta_1)$, a new probability, $P^\sigma(\theta_0, \theta_1)$, and a new lottery, $\ell_z^k(\sigma)$. We do so as follows:

$$M^*(\theta_0, \theta_1) \equiv \left\{ m \in M^k \middle|\, m_0^1 = \theta_0 \text{ and } m_1^1 = \theta_1 \right\};$$

$$P^\sigma(\theta_0, \theta_1) \equiv \sum_{m \in M^*(\theta_0,\theta_1)} \sigma(m);$$

$$\ell_z^k(\sigma) = \sum_{\theta_0 \neq \theta_1} P^\sigma(\theta_0, \theta_1) z(\theta_0, \theta_1) \left(1 - \sum_{\theta \in \Theta} P^\sigma(\theta)\right)^{-1}.$$

With this notation, we establish a counterpart for Step 2 of Theorem 1's proof: **Step 2:** For any mixed strategy profile σ in the mechanism $\Gamma^k = (M^k, g^k)$, $g^k(\sigma)$ can be represented as:

$$g^k(\sigma) = \sum_{\hat{\theta} \in \Theta} P^\sigma(\hat{\theta}) \ell^k(\hat{\theta}, \sigma) + \left(1 - \sum_{\hat{\theta} \in \Theta} P^\sigma(\hat{\theta})\right) \ell_z^k(\sigma),$$

where, for each $\hat{\theta} \in \Theta$,

$$\ell^k(\hat{\theta}, \sigma) = P^\sigma(0|\hat{\theta})(\ell_0^k(\hat{\theta}, \sigma) - z(\hat{\theta}, \hat{\theta})) + P^\sigma(1|\hat{\theta})(\ell_1^k(\hat{\theta}, \sigma) - z(\hat{\theta}, \hat{\theta})) + z(\hat{\theta}, \hat{\theta}).$$

Proof of Step 2:

This comes from how we construct our mechanism and the way the lotteries were defined above. For each $\hat{\theta} \in \Theta$, we will have $m_0^1 = m_1^1 = \hat{\theta}$ with probability $P^\sigma(\hat{\theta})$. When that happens, following the steps outlined in Step 2 of Theorem 1, we can show that the outcome must be given by $\ell^k(\hat{\theta}, \sigma)$. With the remaining probability, $1 - \sum_{\hat{\theta} \in \Theta} P^\sigma(\hat{\theta})$, we will have that $m_0^1 \neq m_1^1$ and the mechanism will default to Rule 2, which will yield $z(\theta_0, \theta_1)$ for each pair $\theta_0 \neq \theta_1$, happening with probability $P^\sigma(\theta_0, \theta_1)$. This is represented by lottery $\ell_z^k(\sigma)$. ■

With these counterparts established, we will now show how the rest of the argument follows from Steps 3 to 8 developed in Theorem 1.

First, we note that Property 3 of Condition COM_2 implies that $u_i(\ell_z^k(\sigma), \theta) < u_i(f(\theta), \theta)$ for all $i \in \{0, 1\}$, as $\ell_z^k(\sigma)$ is just a weighted average of different punishment outcomes. In turn, this allows us to focus on the set of lotteries $\ell^k(\hat{\theta}, \sigma)$, just as we did in Theorem 1, since it follows from Step 2 that if each of these lotteries is dominated by $f(\theta)$, then $g(\sigma)$ will also be dominated. From here, once we take $\mathcal{G}_\theta = \{0, 1\}$ for all $\theta \in \Theta$, $w_\theta(q) = q$ for all $q \in \{0, 1\}$, and $z_{\hat{\theta}} = z(\hat{\theta}, \hat{\theta})$, we can replicate all of the arguments in the remaining steps of Theorem 1. We will briefly go over each of them.

Step 3 does not require any of the properties in Condition COM. It relies purely on the fact that whenever an agent has a positive chance of dictating the outcome, he must choose pick one of his favourite alternatives in his choice set. The same logic applies to the modified mechanism of the two-agent case.

Step 4 uses Property 3 of Condition COM to show that if lottery $\ell^k(\hat{\theta}, \sigma)$ puts sufficient weight on the punishment z_θ, it will be dominated by $f(\theta)$. As the lotteries $\ell^k(\hat{\theta}, \sigma)$ are defined exactly the same way in Step 2 of Theorem 2 as they were in Theorem 1 and as Property 3 of Condition COM_2 has the same implications of Property 3 of Condition COM, we can reach the same results from Step 4 as well.

Steps 5 and 6 deal with showing that if σ is a Nash equilibrium and $\ell^k(\hat{\theta}, \sigma) \neq f(\theta)$, then $\ell^k(\hat{\theta}, \sigma)$ is dominated by $f(\theta)$. To do so, we often rely on showing that if this is not the case, we can prove that σ is not a Nash equilibrium by constructing an alternative message for an agent that constitutes a profitable deviation from his prescribed strategy in σ. We can replicate these arguments in the case of two agents because the two mechanisms work in the same way regarding changes in the second and third components of a message $m_j^2[\hat{\theta}]$ and $m_j^3[\hat{\theta}]$. In particular, $P^\sigma(q|\hat{\theta})$ and $P_i^\sigma(q|\hat{\theta})$ both work in the same way in each of the mechanisms regarding changes in $m_j^2[\hat{\theta}]$, as both mechanisms have a sum in modulo k to determine the outcome once the mechanism selects $\theta^m = \hat{\theta}$. This, along with the fact that Properties 2, 3 and 4 of Condition COM_2 imply the same Properties in Condition COM allows us to replicate the same arguments for those steps in Theorem 1.

Step 7 is also similar to Steps 5 and 6, but with one difference: it requires the second part of Property 1 in its proof. This second part of Property 1 is absent in Condition COM_2. However, as we explain in that Step, that property is not needed for the case of only two agents. With only two agents, the proof for Step 7 is greatly simplified, as lottery $\ell^k(\hat{\theta}, \sigma)$ must only contain lotteries that are dominated by $f(\theta)$, besides $f(\theta)$ itself. This comes from the assumption that $f(\theta)$ is $C_{\hat{\theta}}$-acceptable at state θ, which implies that no agent has any lottery better than $f(\theta)$ in his choice set. Thus, even if one of the agents - say, agent 0 - does not have $f(\theta)$ in his own choice set, whichever lotteries agent 0 considers the best in his choice set, agent 1 must also prefer $f(\theta)$ to them, otherwise those lotteries would also be $C_{\hat{\theta}}$-acceptable, violating Property 2. If there were more than two agents, we would need to check if any of these choices by agent 0 is still preferred to $f(\theta)$ by any of the remaining agents, which is why the second part of Property 1 is needed. However, with only two agents, this is enough to establish that $\ell^k(\hat{\theta}, \sigma)$ is dominated by $f(\theta)$. Hence, the result from Step 7 also holds for the case of two agents.

This allows us to conclude with Step 8 much in the same way as in Theorem 1. When combined with the counterpart of Step 1 above, this proves our result. ■

8 Relation with pure Nash Implementation

In Lemma 1, we show that our canonical mechanism achieves pure Nash implementation under Properties 1 and 2 of Condition COM. These properties are, therefore, sufficient for pure Nash implementation by our mechanism. It is then natural to ask how they compare to the necessary and sufficient conditions in Moore and Repullo (1990).

It turns out that Properties 1 and 2 are more restrictive than the condition in Moore and Repullo [14]. The reason for this is that our mechanism is different from the mechanism used in M-R: essentially, some of the features the mechanism has in order to achieve C-implementation in environments satisfying Condition COM end up compromising pure implementation in the environments that satisfy M-R, but not Properties 1 and 2 of Condition COM.

We will show an example of an environment which does not satisfy Properties 1 (first part) and 2 of Condition COM, but in which pure implementation is still possible:

- $I = \{1, 2, 3\}$, $\Theta = \{\theta, \theta'\}$, $f(\theta) = a$, $f(\theta') = c$.
- $\succ_1^\theta = a \succ_1^\theta b \succ_1^\theta d \succ_1^\theta c$;
- $\succ_1^{\theta'} = a \succ_1^{\theta'} b \succ_1^{\theta'} c \succ_1^{\theta'} d$;
- $\succ_2^\theta = a \succ_1^\theta \succ_2^\theta b \succ_2^\theta d \succ_2^\theta c$;
- $\succ_2^{\theta'} = b \succ_2^{\theta'} a \succ_2^{\theta'} c \succ_2^{\theta'} d$;
- $\succ_3^\theta = \succ_1^\theta$, $\succ_3^{\theta'} = \succ_2^{\theta'}$;

We claim that for this environment any $C_{\theta'}$ that satisfies the first part of Property 1 of Condition COM will violate Property 2. To verify this, notice that Property 1 requires $C_{\theta'}(i)$ to be contained in the lower contour set of c at state θ'. For all three agents, this lower contour set is equal to $\{c, d\}$. At the same time, at state θ, all three agents have identical preferences over $\{c, d\}$, so one of these alternatives will be $C_{\theta'}$-acceptable at state θ [15]. This violates Property 2, as $f(\theta) \neq c, d$. In particular, if we tried to implement this function using a mechanism of the $\Gamma^{\bar{\theta},k}$ class, this would result in a pure m at state θ yielding $g(m) \in \{c, d\}$.

[14] This still holds true even if we consider only the first part of Property 1. Though our proof of Step 1 does make use of the second part of that property, it is possible to prove it with only the first part of Property 1. Nonetheless, when combined with Property 2, they are still more restrictive than the condition in Moore and Repullo.

[15] If $d \in C_{\theta'}(i)$ for some $i = 1, 2, 3$, we will have that d is $C_{\theta'}$-acceptable at state θ; otherwise, if $C_{\theta'}(i) = c$ for all i, then c is $C_{\theta'}$-acceptable at state θ.

Despite not satisfying Properties 1 and 2 of Condition COM, pure implementation is still possible in this environment, if we use other kinds of mechanisms. Consider this 2-agent mechanism, using only agents 1 and 2:

$g(m)$		Agent 2		
		m_2^1	m_2^2	m_2^3
Agent 1	m_1^1	c	d	d
	m_1^2	d	a	b
	m_1^3	d	b	a

This is the same mechanism featured in Example 4 of Jackson (as well as the same SCF, for the matter). Indeed, the example above features preferences that are very similar to the ones in Jackson. The main reason why pure implementation is possible, even though Property 2 is violated, is that in this mechanism the set $C_{\theta'}(i)$ does not represent the set possible outcomes from unilateral deviations for agent i, given that forum θ' was selected. Thus, even if d is the best alternative in set $C_{\theta'}(i)$ at state θ, this does not imply that d is a Nash equilibrium outcome at that state. Indeed, both agents can unilaterally deviate to a at any message profile that yields d in the mechanism above. This is not the case for the mechanisms in the class $\Gamma^{\bar{\theta},k}$. As mentioned, in these mechanisms, the set $C_{\theta'}(i)$ represents all possible outcomes agent i can achieve through an unilateral deviation given that forum θ' was selected. This is important, because it helps us to narrow down what that agent can achieve through a mixed strategy once that forum has been selected. The cost we pay for this is that it makes it easier for pure equilibria with undesirable results to arise, so we need to be more restrictive with the preferences of the agents.

9 Conclusion

We present a concept of compelling implementation, which strengthens the requirement of pure-strategy Nash implementation with an additional property that every mixed strategy equilibrium is either socially desirable or “uncompelling” in the sense that its outcome is strictly Pareto dominated by the socially desirable outcome. The main contribution of this paper is to propose Condition COM under which compelling implementation is possible by finite mechanisms in environments with at least three agents. We construct an example that satisfies Condition COM and show that Condition COM is indispensable for our result. We also propose Condition COM_2 to extend our compelling implementation result to the case of two agents. Our implementing mechanism has desirable properties: transfers are not needed at all; only finite mechanisms are used; integer games are not invoked; and agents' risk attitudes do not matter.

10 Appendix

In this appendix, we provide the proofs we omitted in the main body of the paper.

10.1 Proof of Lemma 2

Proof of Step 1a: Assume by way of contradiction that there exists an integer $x \in \{0, \ldots, k-1\}$ such that $\sigma_1(x) > 0$ and $\sigma_2(x) = 0$. Then, there are two possibilities: either there exists $x^{'} \in \{0, \ldots, k-1\} \backslash \{x\}$ such that $\sigma_2(x^{'}) > 0$ or $\sigma_2(k) = 1$.

In the first case, let $x^{'} \in \arg\max_{x^{''} \in \{0,\ldots,k-1\} \backslash \{x\}} \sigma_2(x^{''})$. The expected payoff for agent 1 when sending message x is

$$U_1(x, \sigma_2; \theta^{'}) = \begin{cases} \sigma_2(x+1)u_1(b;\theta^{'}) & \text{if } x < k-1 \\ \sigma_2(0)u_1(b;\theta^{'}) & \text{if } x = k-1, \end{cases}$$

where we take into account that $u_i(d;\theta^{'}) = 0$. On the other hand, the expected payoff for agent 1 when sending message $x^{'}$ is given by

$$U_1(x^{'}, \sigma_2; \theta^{'}) = \begin{cases} \sigma_2(x^{'})u_1(a;\theta^{'}) + \sigma_2(x^{'}+1)u_1(b;\theta^{'}) & \text{if } x^{'} < k-1 \\ \sigma_2(x^{'})u_1(a;\theta^{'}) + \sigma_2(0)u_1(b;\theta^{'}) & \text{if } x^{'} = k-1 \end{cases}$$

As $u_1(a, \theta^{'}) > u_1(b, \theta^{'})$ and $\sigma_2(x^{'}) \geq \sigma_2(x+1)$, sending message $x^{'}$ is strictly better for agent 1 than sending x against σ_2, thus contradicting the hypothesis that message x is played with positive probability in the Nash equilibrium σ.

Consider the second possibility where agent 2 sends k with probability 1. Then, agent 1's expected payoff of sending message x is $U_1(x, \sigma_2; \theta^{'}) = 0$, while agent 1's expected payoff of sending message k is $U_1(\sigma_2; k; \theta^{'}) = u_1(c, \theta^{'}) > 0$, contradicting the hypothesis that message x is played with positive probability in the Nash equilibrium σ. ■

Proof of Step 1b: Assume by way of contradiction that there exists $x \in \{0, \ldots, k-1\}$ such that $\sigma_2(x) > 0$ and $\sigma_1(x-1) = 0$ if $x \geq 1$ and $\sigma_1(k-1) = 0$ if $x = 0$. Then we decompose our argument into the following two cases: (i) there exists $x^{'} \in \{0, \ldots, k-1\}$ such that $\sigma_1(x^{'}) > 0$ or (ii) $\sigma_1(k) = 1$.

We first consider Case (i). We assume without loss of generality that $x^{'} \in \arg\max_{x^{''} \in \{0,\ldots,k-1\}} \sigma_1(x^{''})$. Agent 2's expected payoff of sending message x against σ_1 in the game $\Gamma(\theta^{'})$ is given by

$$U_2(\sigma_1, x; \theta^{'}) = \sigma_1(x)u_2(a;\theta'),$$

while agent 2's expected payoff of sending message $(x^{'}+1 \text{ mod } k)$ against σ_1 in the game $\Gamma(\theta^{'})$ is given by

$$U_2(\sigma_1, x^{'}+1 \text{ mod } k; \theta^{'}) = \begin{cases} \sigma_1(x^{'})u_2(b;\theta^{'}) + \sigma_1(x^{'}+1)u_2(a;\theta^{'}) & \text{if } x^{'} < k-1 \\ \sigma_1(x^{'})u_2(b;\theta^{'}) + \sigma_1(0)u_2(a;\theta^{'}) & \text{if } x^{'} = k-1, \end{cases}$$

where we take into account that $u_2(d;\theta^{'}) = 0$. Since $u_2(b;\theta^{'}) > u_2(a;\theta^{'}) > 0$, due to the way $x^{'}$ is defined, we have $U_2(\sigma_1, x^{'}+1 \text{ mod } k; \theta^{'}) > U_2(\sigma_1, x; \theta^{'})$, which contradicts the hypothesis that message x is sent with positive probability in the Nash equilibrium σ.

We next consider Case (ii). Agent 2's expected payoff of sending message x against σ_1 in the game $\Gamma(\theta^{'})$ is given by

$$U_2(\sigma_1, x; \theta^{'}) = 0,$$

where we take into account that $u_2(d;\theta^{'}) = 0$. On the contrary, agent 2's expected payoff of sending message k against σ_1 in the game $\Gamma(\theta^{'})$ is given by

$$U_2(\sigma_1, k; \theta^{'}) = u_2(c;\theta^{'}).$$

Since $u_2(c;\theta^{'}) > u_2(d;\theta^{'}) = 0$, we have $U_2(\sigma_1, k; \theta^{'}) > U_2(\sigma_1, x; \theta^{'})$, contradicting the hypothesis that message x is sent with positive probability in the Nash equilibrium σ in the game $\Gamma(\theta^{'})$. ■

Proof of Step 1c: Assume first that $i = 1$; that is, there exists $x^{'} \in \{0, \dots, k-1\}$ such that $\sigma_1(x^{'}) > 0$. By Step 1a, we first have that $\sigma_2(x^{'}) > 0$. Second, by Step 1b, $\sigma_2(x^{'}) > 0$ implies $\sigma_1(x^{'}-1) > 0$ if $x^{'} \geq 1$ and $\sigma_1(k) > 0$ if $x^{'} = 0$. Third, using Step 1a once again, we conclude that $\sigma_2(x^{'}-1) > 0$ if $x^{'} \geq 1$ and $\sigma_2(k) > 0$ if $x^{'} = 0$. Finally, iterating this argument, we are able to conclude that $\sigma_1(x) > 0$ and $\sigma_2(x) > 0$ for all $x \in \{0, \dots, k-1\}$.

The case where $i = 2$ is analogous to the previous one, only that we start the loop by applying Step 1b first, before Step 1a. This completes the proof of Step 1c. ■

Proof of Step 2: Assume by way of contradiction that there exist $i \in N$ and $x, x^{'} \in \{0, \dots, k-1\}$ such that $\sigma_i(x) > \sigma_i(x^{'}) > 0$. By Step 1c, we know that $\sigma_i(\tilde{x}) > 0$ for all $\tilde{x} \in \{0, \dots, k-1\}$. Then, we can choose x and $x^{'}$ satisfying the following property:

$$x \in \arg\max_{\tilde{x} \in \{0,\dots,k-1\}} \sigma_i(\tilde{x}) \text{ and } x^{'} \in \arg\min_{\tilde{x} \in \{0,\dots,k-1\}} \sigma_i(\tilde{x}).$$

By Step 1c, we also know that $\sigma_j(\tilde{x}) > 0$ for each $\tilde{x} \in \{0, \dots, k-1\}$, where $j \in \{1,2\} \backslash \{i\}$.

Assume that $i = 2$. The expected payoff for agent 1 of sending message $x^{'}$ against σ_2 in the game $\Gamma(\theta^{'})$ is given by

$$U_1(x^{'}, \sigma_2; \theta^{'}) = \begin{cases} \sigma_2(x^{'})u_1(a; \theta^{'}) + \sigma_2(x^{'}+1)u_1(b; \theta^{'}) & \text{if } x^{'} < k-1 \\ \sigma_2(x^{'})u_1(a; \theta^{'}) + \sigma_2(0)u_1(b; \theta^{'}) & \text{if } x^{'} = k-1 \end{cases}$$

On the other hand, The expected payoff for agent 1 of sending message x against σ_2 in the game $\Gamma(\theta^{'})$ is given by

$$U_1(x, \sigma_2; \theta^{'}) = \begin{cases} \sigma_2(x)u_1(a; \theta^{'}) + \sigma_2(x+1)u_1(b; \theta^{'}) & \text{if } x < k-1 \\ \sigma_2(x)u_1(a; \theta^{'}) + \sigma_2(0)u_1(b; \theta^{'}) & \text{if } x = k-1. \end{cases}$$

We compute

$$\begin{aligned} & U_1(x, \sigma_2; \theta^{'}) - U_1(x^{'}, \sigma_2; \theta^{'}) \\ = \;& [\sigma_2(x) - \sigma_2(x^{'})]u_1(a; \theta^{'}) + [\sigma_2(x+1 \bmod k) - \sigma_2(x'+1 \bmod k)]u_1(b; \theta^{'}) \\ \geq \;& [\sigma_2(x) - \sigma_2(x^{'})]u_1(a; \theta^{'}) - [\sigma_2(x) - \sigma_2(x^{'})]u_1(b; \theta^{'}) \\ & (\because [\sigma_2(x+1 \bmod k) - \sigma_2(x'+1 \bmod k)] \geq -[\sigma_2(x) - \sigma_2(x^{'})], u_1(b; \theta^{'}) > 0) \\ = \;& [\sigma_2(x) - \sigma_2(x^{'})](u_1(a; \theta^{'}) - u_1(b; \theta^{'}) \\ > \;& 0. \end{aligned}$$

This implies that message x is a strictly better response for agent 1 against σ_2 than $x^{'}$ in the game $\Gamma(\theta^{'})$, contradicting the hypothesis that $\sigma_1(x^{'}) > 0$.

We next assume $i = 1$. The expected payoff for agent 2 of sending message $x^{'}+1$ against σ_1 in the game $\Gamma(\theta^{'})$ is given by

$$U_2(\sigma_1, x^{'}+1; \theta^{'}) = \begin{cases} \sigma_1(x^{'}+1)u_2(a; \theta^{'}) + \sigma_1(x^{'})u_1(b; \theta^{'}) & \text{if } x^{'} < k-1 \\ \sigma_1(0)u_2(a; \theta^{'}) + \sigma_1(x')u_1(b; \theta^{'}) & \text{if } x^{'} = k-1 \end{cases}$$

On the other hand, The expected payoff for agent 2 of sending message $x+1$ against σ_1 in the game $\Gamma(\theta^{'})$ is given by

$$U_2(\sigma_1, x+1; \theta^{'}) = \begin{cases} \sigma_1(x+1)u_1(a; \theta^{'}) + \sigma_1(x)u_2(b; \theta^{'}) & \text{if } x < k-1 \\ \sigma_1(0)u_1(a; \theta^{'}) + \sigma_1(x)u_2(b; \theta^{'}) & \text{if } x = k-1. \end{cases}$$

We compute

$$\begin{aligned} & U_2(\sigma_1, x+1; \theta^{'}) - U_2(\sigma_1, x^{'}+1; \theta^{'}) \\ = \;& [\sigma_1(x+1) - \sigma_1(x^{'}+1)]u_2(a; \theta^{'}) + [\sigma_1(x) - \sigma_1(x')]u_2(b; \theta^{'}) \\ \geq \;& [\sigma_1(x+1) - \sigma_1(x^{'}+1)]u_2(b; \theta^{'}) - [\sigma_1(x) - \sigma_1(x^{'})]u_2(a; \theta^{'}) \\ & (\because [\sigma_1(x+1 \bmod k) - \sigma_1(x'+1) \bmod k)] \geq -[\sigma_1(x) - \sigma_1(x^{'})], u_2(a; \theta^{'}) > 0) \\ = \;& [\sigma_1(x) - \sigma_1(x^{'})](u_2(b; \theta^{'}) - u_2(a; \theta^{'})) \\ > \;& 0. \end{aligned}$$

This implies that message $x+1$ is a strictly better response for agent 2 against σ_2 than $x^{'}+1$ in the game $\Gamma(\theta^{'})$, contradicting the hypothesis that $\sigma_2(x^{'}+1) > 0$. This completes the proof of Step 2. ■

References

[1] Abreu, D. and H. Matsushima, (1992), "Virtual Implementation in Iteratively Undominated Strategies: Complete Information," *Econometrica*, vol. 60, 993-1008.

[2] Abreu, D. and H. Matsushima, (1994), "Exact Implementation," *Journal of Economic Theory*, vol. 64, 1-19.

[3] Aghion, P., D. Fudenberg, R. Holden, T. Kunimoto, and O. Tercieux, (2012), "Subgame-Perfect Implementation under Information Perturbations," *Quarterly Journal of Economics*, vol. 127(4), 1843-1881

[4] Bergemann, D., S. Morris, and O. Tercieux, (2011), "Rationalizable Implementation," *Journal of Economic Theory*, vol. 146, 1253-1274.

[5] Chen, Y-C., T. Kunimoto, Y. Sun, and S. Xiong, (2022), "Maskin Meets Abreu and Matsushima," *Theoretical Economics*, forthcoming.

[6] Chen, Y-C., T. Kunimoto, Y. Sun, and S. Xiong, (2021), "Rationalizable Implementation in Finite Mechanisms," *Games and Economic Behavior*, vol. 129, 181-197.

[7] Chung, K-S and J. Ely, (2003), "Implementation with Near-Complete Information," *Econometrica*, vol. 71, 857-87.

[8] Dutta, B. and A. Sen, (1991), "A Necessary and Sufficient Condition for Two-Person Nash Implementation," *Review of Economic Studies*, vol. 58, 121-128.

[9] Goltsman, M., (2011), "Nash Implementation using Simple Mechanisms without Undesirable Mixed-Strategy Equilibria," *Working Paper*.

[10] Jackson, M-O., (1991), "Bayesian Implementation," *Econometrica*, vol. 59, 461-477.

[11] Jackson, M-O., (1992), "Implementation in Undominated Strategies: A Look at Bounded Mechanisms," *Review of Economic Studies*, vol. 59, (1992), 757-775.

[12] Jackson, M-O, (2001), "A Crash Course in Implementation Theory," *Social Choice and Welfare*, vol. 39, 655-708.

[13] Jackson, M-O, T-R Palfrey, and S. Srivastava, (1994), "Undominated Nash Implementation in Bounded Mechanisms," *Games and Economic Behavior*, vol. 6, 474-501.

[14] Jain, R., (2021), "Rationalizable Implementation of Social Choice Correspondences," *Games and Economic Behavior*, vol. 127, 47-66.

[15] Kartik, N. and O. Tercieux, (2012), "A Note on Mixed-Nash Implementation," *Working Paper*.

[16] Korplela, V., (2016), "Nash Implementation with Finite Mechanisms," *Working Paper*.

[17] Kunimoto, T., (2019), "Mixed Bayesian Implementation in General Environments," *Journal of Mathematical Economics*, vol. 82, 247-263.

[18] Kunimoto, T. and R. Serrano, (2019), "Rationalizable Implementation of Correspondences," *Mathematics of Operations Research*, vol. 44, 1326-1344.

[19] Lee, J. and H. Sabourian, (2015), "Complexity and Repeated Implementation," *Journal of Economic Theory*, vol. 158, 259-292.

[20] Maskin, E., (1999), "Nash Equilibrium and Welfare Optimality," *Review of Economic Studies*, vol. 66, 23-38.

[21] Maskin, E. and T. Sjöström, (2002), "Implementation Theory," in *Handbook of Social Choice and Welfare*, vol 1, edited by K.J. Arrow, A.K. Sen, and K. Suzumura, (2002), 237-288.

[22] McKelvey, R.D., (1989), "Game Forms for Nash Implementation of General Social Choice Correspondences," *Social Choice and Welfare*, vol. 6, 139-156.

[23] Mezzetti, C. and L. Renou, (2012a), "Implementation in Mixed Nash Equilibrium," *Journal of Economic Theory*, vol. 147, 2357-2375.

[24] Mezzetti, C. and L. Renou, (2012b), "Mixed Nash Implementation with Finite Mechanisms," *Working Paper*.

[25] Moore, J. and R. Repullo, (1988), "Subgame Perfect Implementation," *Econometrica*, vol. 56, 1191-1220.

[26] Moore, J. and R. Repullo, (1990), "Nash Implementation: A Full Characterization," *Econometrica*, vol. 58, 1083-1099.

[27] Saijo, T., (1988), "Strategy Space Reduction in Maskin's Theorem," *Econometrica*, vol. 56, 693-700.

[28] Serrano, R., (2004), "The Theory of Implementation of Social Choice Rules," *SIAM Review*, vol. 46, 377-414.

[29] Serrano, R. and R. Vohra, (2010), "Multiplicity of Mixed Equilibria in Mechanisms: A Unified Approach to Exact and Approximate Implementation," *Journal of Mathematical Economics*, vol. 46, 775-785.

[30] Xiong, S., (2023), "Rationalizable Implementation of Social Choice Functions: Complete Characterization," *Theoretical Economics*, vol. 18, 197-230.