Random Assignments on Sequentially Dichotomous Domains *

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Abstract

We present a possibility result on the existence of a random assignment rule satisfying sd-strategy-proofness, sd-efficiency, and equal treatment of equals. In particular, we introduce a class of preference domains: sequentially dichotomous domains. On any such domain, the probabilistic serial rule (Bogomolnaia and Moulin (2001)) is sd-strategy-proof. Moreover, any sequentially dichotomous domain is maximal for this rule to be sd-strategy-proof.

Keywords: Random assignment; probabilistic serial rule; sequentially dichotomous domain; sd-strategy-proof

JEL Classification: C78, D71.

1 Introduction

The random assignment problem (Bogomolnaia and Moulin (2001)) deals with the situation where n indivisible objects are to be allocated to n agents. Each agent receives exactly one object. Each agent reports to the planner a strict preference relation on objects. The planner then assigns a lottery to each agent according to some prescribed random assignment rule.[1] A preference relation on objects is extended to a preference relation on lotteries by first-order

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[1]Examples include allocating houses to residents (Shapley and Scarf (1974)), tasks to workers (Hylland and Zeckhauser (1979)), and college seats to applicants (Gale and Shapley (1962)).

stochastic dominance. Specifically, a lottery is perceived to be at least as good as another if the former first-order stochastically dominates the latter according to the given preference relation.[2]

Based on the extension of preference relations by first-order stochastic dominance, several axioms of a rule are defined. The first deals with efficiency. A rule is *sd-efficient* if it always specifies a random assignment which cannot be Pareto improved. The second deals with incentive compatibility. A rule is *sd-strategy-proof* if, for each agent, reporting his true preference relation always delivers a lottery which first-order stochastically dominates the lottery delivered by any misrepresentation. The third deals with fairness. A rule satisfies *equal treatment of equals* if whenever two agents report the same preference relation, they get the same lottery. We call a rule desirable if it satisfies all three axioms.

Unfortunately, finding such a rule is an impossible mission in many economic environments. Initially, Bogomolnaia and Moulin (2001) proved that there is no such rule on the universal domain.[3] This impossibility was strengthened to the single-peaked domain (Kasajima (2013)), and to the subset of the single-peaked domain where agents have a common peak (Chang and Chun (2017)). Recently, Liu and Zeng (2018) strengthened this impossibility to almost all connected domains.[4] These impossibilities raise the following question: *Is there a desirable rule on some reasonably restricted domains?*

We answer this question by introducing a class of domains: sequentially dichotomous domains. To define such a domain, we need some preliminary notions. A partition of the object set is called a "dichotomous refinement" of another partition, if from the latter to the former, exactly one block breaks into two smaller blocks and all the other blocks remain the same. A sequence of partitions is then called a "dichotomous path" if it satisfies three conditions. First, it starts from the coarsest partition. Second, it ends at the finest partition. Third, along the sequence, each partition is a dichotomous refinement of its previous one.

A preference relation is said to "respect a partition" if, for every pair of blocks in this partition, every object in one block is better than every object in the other. Further, a preference relation "respects a dichotomous path" if it respects every partition on the path. A collection of preference relations is a "sequentially dichotomous domain" if we can find a dichotomous path such that a preference relation is included if and only if it respects this dichotomous path. Hence the preference relations included in a sequentially dichotomous domain respect the same dichotomous path. This requirement is analogous to the requirement, imposed when specifying a single-peaked domain (Moulin (1980)), that all agents' preference relations are single-peaked with respect to the same linear order. Indeed, a sequentially dichotomous domain has the same cardinality as a single-peaked domain. Given that the number of objects is n, a sequentially dichotomous domain contains 2^{n-1} preference relations. Another interesting feature of sequentially dichotomous domains is that every such domain turns out to be a maximal "Condorcet

[2]A lottery first-order stochastically dominates another if and only if the former gives an expected utility that is at least as high as the expected utility delivered by the latter, according to *every* cardinal utility representing the given preference relation.

[3]The universal domain in this paper refers to the collection of all strict preference relations, namely the linear orders on object set. In addition, we assume $n \geqslant 4$. For the cases where $n < 4$, desirable rule exists.

[4]Connectedness is a domain richness condition due to Monjardet (2009). Many well-studied domains of preference relations are connected, including the universal domain, the single-peaked domain, the single-dipped domain, the maximal single-crossing domain, and so on.

domain," a well-studied domain in the classical social choice literature. We discuss this in subsection 1.1.

Our first main result, Theorem 1, states that on any sequentially dichotomous domain, the probabilistic serial rule (or the PS rule, see Bogomolnaia and Moulin (2001)) is *sd-strategy-proof*. Note that the PS rule is *sd-efficient* and treats equals equally on any domain. Hence, we have a possibility result: The PS rule is desirable on any sequentially dichotomous domain.

The next question we address is whether one can expand a sequentially dichotomous domain while preserving the *sd-strategy-proofness* of PS rule? The answer is in the negative. Theorem 2 states that every sequentially dichotomous domain is maximal for the PS rule to be desirable. In particular, whenever a new preference relation is added to a sequentially dichotomous domain, the PS rule becomes manipulable.

The remainder of this section discusses in detail this paper's relation and contribution to the literature. Thereafter, section 2 defines the random assignment model. Section 3 formally defines the sequentially dichotomous domains. Section 4 presents the results. Section 5 concludes. Omitted proofs are gathered in the appendix.

1.1 Relation and Contribution to the literature

Following the series of impossibilities by Bogomolnaia and Moulin (2001), Kasajima (2013), Chang and Chun (2017), and Liu and Zeng (2018), this paper provides a possibility result in the literature on designing a desirable random assignment rule.

In the literature on random assignment problems, two random assignment rules have been intensively studied: the random priority rule (or the RP rule, see Abdulkadiroğlu and Sönmez (1998)) and the PS rule. The RP rule is the uniform randomization of serial dictatorship rules. It is *sd-strategy-proof* and treats equals equally, but it is not *sd-efficient*. Quite a few papers are devoted to understanding why the RP rule is *sd-inefficient* and under what conditions it becomes *sd-efficient* (Abdulkadiroğlu and Sönmez (2003); Manea (2008); Kesten (2009); Manea (2009)).

The PS rule is *sd-efficient* and treats equals equally, but it is not *sd-strategy-proof*. Relative to the investigation on why the RP rule is not *sd-efficient*, a study of why the PS rule is not *sd-strategy-proof*, and under what conditions it becomes *sd-strategy-proof*, has been largely neglected. The only paper on the subject, to the author's knowledge, is Kojima and Manea (2010). However they dealt with a variant of the model, where each object has sufficiently many copies. For the baseline model discussed here, not much is known. By showing that a sequentially dichotomous domain is maximal for the PS rule to be *sd-strategy-proof*, the current paper helps in filling this gap.

This paper is also related to the literature on "Condorcet domains."[5] A set of preference relations is called a Condorcet domain if the majority rule does not generate Condorcet cycles. Classical papers in this literature include Black (1948), Black et al. (1958), Abello (1981), Fishburn (1997), Fishburn (2002), and so on. An excellent survey is by Monjardet (2009). The structure of sequentially dichotomous domains has been investigated by Danilov and Koshevoy (2013), who describe such a structure in a much more abstract manner from the view point

[5] I thank Hervé Moulin and John Weymark for pointing out this link.

of operations research. They proved that each sequentially dichotomous domain is a maximal Condorcet domain. Moreover, the size of such a domain is the largest in a class of maximal Condorcet domains. This class is called the symmetric domains, requiring that whenever a preference relation is included, its reversal is also included. Moreover, due to the fact that a sequentially dichotomous domain is a Condorcet domain, the sd-strategy-proofness of some random voting rules is guaranteed, for example the maximal lotteries (see Fishburn (1984) and Brandl et al. (2016)). This suggests that there may be some underlying link between the classical voting problem and the assignment problem, which deserves further research.

2 The Random Assignment Model

Let $A \equiv \{a, b, c, d, \dots\}$ be a finite set of objects and $I \equiv \{1, 2, \dots, n\}$ a finite set of agents. We assume $|A| = |I| = n \geqslant 4$. Each agent i has a strict preference P_i on A, namely a complete, transitive and antisymmetric binary relation on A.[6] Let $\mathbb{P}$ denote the set of *all* strict preferences on A. The set of admissible preferences is a set $\mathbb{D} \subseteq \mathbb{P}$, referred to as the **domain**. In particular, $\mathbb{P}$ is called the universal domain. Given $P_i \in \mathbb{D}$ and $k \in \{1, \cdots, n\}$, let $r_k(P_i)$ denote the k-th ranked object according to P_i. A preference profile $P \equiv (P_1, \dots, P_n) \in \mathbb{D}^n$ is an n-tuple of admissible preferences.

A **random assignment** is a bi-stochastic matrix, namely a non-negative square matrix whose elements in each row and each column sum to one. Formally, a random assignment is denoted as $L \equiv [L_{ia}]_{i\in I, a\in A}$ satisfying (i) $L_{ia} \geqslant 0, \forall i \in I, a \in A$, (ii) $\sum_{a\in A} L_{ia} = 1, \forall i \in I$, and (iii) $\sum_{i\in I} L_{ia} = 1, \forall a \in A$. Let $\mathcal{L}$ denote the set of all random assignments. The i-th row of a random assignment L, denoted as L_i, is the lottery assigned to agent i. Let $\Delta(A)$ denote the set of lotteries on A. Then $L_i \in \Delta(A)$ and L_{ia} specifies the probability that agent i gets a. The Birkhoff-von Neumann theorem guarantees that a random assignment can be decomposed as a lottery over permutation matrices, each of which in our setting can be interpreted as a deterministic assignment which indicates who gets which object.

Agents compare lotteries by first-order stochastic dominance. A lottery $L_i \in \Delta(A)$ **first-order stochastically dominates** $L'_i \in \Delta(A)$ according to P_i, denoted as $L_i\ P_i^{sd}\ L'_i$, if $\sum_{l=1}^{k} L_{ir_l(P_0)} \geqslant \sum_{l=1}^{k} L'_{ir_l(P_0)}$ for all $1 \leqslant k \leqslant n$. Given $P \in \mathbb{D}^n$, a random assignment L first-order stochastically dominates L' according to P, denoted as $L\ P^{sd}\ L'$, if $L_i\ P_i^{sd}\ L'_i$ for all $i \in I$.

A **rule** is a mapping $\varphi : \mathbb{D}^n \to \mathcal{L}$ which selects a random assignment for each profile of admissible preferences. Given $P \in \mathbb{D}^n$, $\varphi_{ia}(P)$ denotes the probability of agent i receiving object a, and thus $\varphi_i(P)$ denotes the lottery assigned to agent i.

A rule is called **desirable** if it satisfies the following three axioms.

First, for every agent, her lottery under truth-telling always first-order stochastically dominates her lottery induced by any misrepresentation, according to her true preference. Formally, $\varphi : \mathbb{D}^n \to \mathcal{L}$ is **sd-strategy-proof (sd-SP)** if for all $i \in I$, $P_i, P'_i \in \mathbb{D}$, and $P_{-i} \in \mathbb{D}^{n-1}$, $\varphi_i(P_i, P_{-i})\ \ P_i^{sd}\ \ \varphi_i(P'_i, P_{-i})$.

Second, a rule always selects a random assignment that cannot be Pareto improved. Formally, $\varphi : \mathbb{D}^n \to \mathcal{L}$ is **sd-efficient (sd-Eff)** if, for all $P \in \mathbb{D}^n$ and all $L' \in \mathcal{L}$, $[L'\ P^{sd}\ \varphi(P)] \Rightarrow$

[6] Rigorously speaking, P_i should be called a preference relation. For short, we call it a preference.

$[L' = \varphi(P)]$.

Last, whenever two agents report the same preference they receive the same lottery. Formally, a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ satisfies **equal treatment of equals (ETE)** if for all $P \in \mathbb{D}^n$, $[P_i = P_j] \Rightarrow [\varphi_i(P) = \varphi_j(P)]$.[7]

3 Sequentially Dichotomous Domains

This section defines the sequentially dichotomous domains.

A **partition** $\mathbf{A}$ of the object set A is a set of nonempty subsets of A such that every object is in exactly one of these subsets.[8] Formally, $\mathbf{A} \subset 2^A \backslash \{\emptyset\}$ such that $\bigcup_{A_k \in \mathbf{A}} A_k = A$ and $A_k \cap A_l = \emptyset$ for all distinct $A_k, A_l \in \mathbf{A}$. A typical element of a partition is called a **block** and it is denoted by $A_k \in \mathbf{A}$. We denote the set of partitions by $\mathcal{A}$, on which a binary relation is defined. A partition is called a dichotomous refinement of another if exactly one block of the latter partition breaks into two smaller blocks in the former partition and all the other blocks remain the same.

Definition 1. *A partition $\mathbf{A}' \in \mathcal{A}$ is a **dichotomous refinement** of another partition $\mathbf{A} \in \mathcal{A}$, if there are blocks $A_k \in \mathbf{A}$ and $A'_i, A'_j \in \mathbf{A}'$ such that $A_k = \mathbf{A} \backslash \mathbf{A}'$ and $\{A'_i, A'_j\} = \mathbf{A}' \backslash \mathbf{A}$.*

A sequence $(\mathbf{A}_t)_{t=1}^T \subset \mathcal{A}$ is a **dichotomous path** if it satisfies (i) $\mathbf{A}_1 = \{A\}$, (ii) $\mathbf{A}_T = \{\{a\} : a \in A\}$, and (iii) $\mathbf{A}_{t+1}$ is a dichotomous refinement of $\mathbf{A}_t$ for every $t = 1, \cdots, T-1$. Let $\mathbf{A}'$ be a dichotomous refinement of $\mathbf{A}$, $|\mathbf{A}'| = |\mathbf{A}| + 1$ and hence $T = n$ for any dichotomous path. Henceforth we denote a dichotomous path as $(\mathbf{A}_t)_{t=1}^n$.

For each $t \in \{1, \cdots, n-1\}$, let $A_{t*} \equiv \mathbf{A}_t \backslash \mathbf{A}_{t+1}$ be the block in $\mathbf{A}_t$ that breaks into two smaller blocks. For each $t \in \{2, \cdots, n\}$, let $\{A_{t1}, A_{t2}\} \equiv \mathbf{A}_t \backslash \mathbf{A}_{t-1}$ be the two blocks whose union is a block in $\mathbf{A}_{t-1}$. Hence from $\mathbf{A}_1$ to $\mathbf{A}_2$, A_{1*} breaks into A_{21} and A_{22}; from $\mathbf{A}_2$ to $\mathbf{A}_3$, A_{2*} breaks into A_{31} and A_{32}, and so on.

Figure 1 below presents all dichotomous refinements when A contains four objects. In particular, two dichotomous paths are indicated, one by darkened arrows and the other by darkened and dotted arrows.

We say a block $A_k \subset A$ **clusters** in a preference $P_i \in \mathbb{P}$, if the objects in A_k are ranked next to each other in P_i. Formally, for all $a, b \in A_k$, there is no $x \in A \backslash A_k$ such that $a \ P_i \ x$ and $x \ P_i \ b$. By definition, the grand set A clusters in every preference $P_i \in \mathbb{P}$. In addition, every singleton set clusters in every preference. For these two extreme cases, the requirement of clustering becomes vacuous. The example below presents another instance.

Example 1. *Consider the preference $P_i : o_1 \succ o_2 \succ a \succ b \succ c \succ x \succ y \succ o_3 \succ o_4$. Let $A_1 = \{a, b, c\}$, $A_2 = \{x, y\}$, and $A_3 = A_1 \cup A_2$. Then A_1, A_2, and A_3 cluster in P_i.* ■

A preference $P_i \in \mathbb{P}$ **respects a partition** $\mathbf{A} \in \mathcal{A}$ if every $A_k \in \mathbf{A}$ clusters in P_i. Given $\mathbf{A}$, let $\mathbb{D}_{\mathbf{A}}$ denote the set of preferences respecting it, namely, $\mathbb{D}_{\mathbf{A}} \equiv \{P_i \in \mathbb{P} | P_i \text{ respects } \mathbf{A}\}$.

[7] The abbreviations of the axioms are used as both adjectives and nouns. For example, we write "φ satisfies ETE" as well as "φ is an ETE rule."

[8] I denote a partition with bold $\mathbf{A}$. Subscripts, superscripts, and accent symbols are used to differentiate particular instances. I denote subsets of objects by A with subscripts, superscripts, and accent symbols.

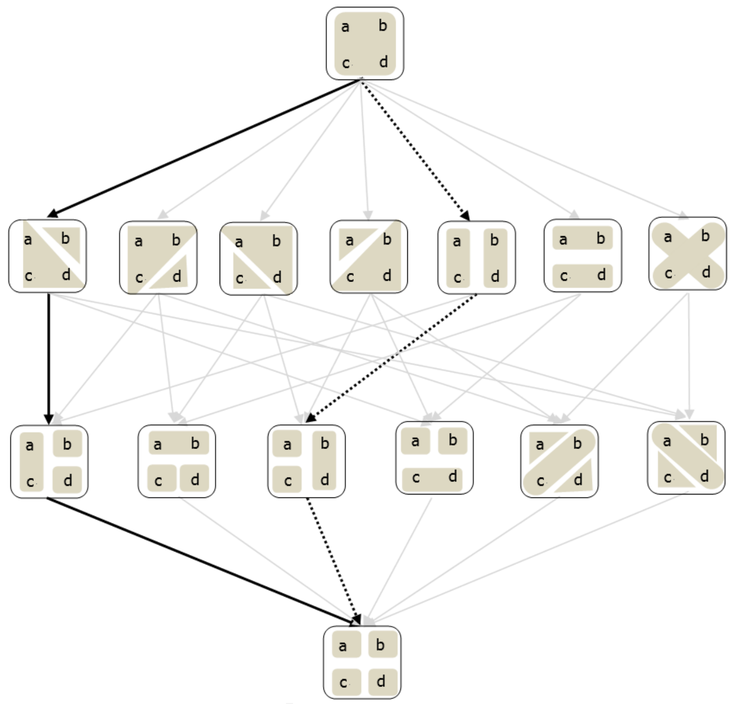


Figure 1: Dichotomous refinement relation when $A = \{a, b, c, d\}$. A shaded collection of objects represents a block containing these objects. A square containing shaded blocks represents a partition. An arrow pointing from one partition to another indicates that the latter is a dichotomous refinement of the former.

Let $\mathbf{A}_1 = \{A\}$ and $\mathbf{A}_2 = \{\{a\} : a \in A\}$ be respectively the coarsest and the finest partitions. Then every preference respects both $\mathbf{A}_1$ and $\mathbf{A}_2$. Put otherwise, $\mathbb{D}_{\mathbf{A}_1} = \mathbb{D}_{\mathbf{A}_2} = \mathbb{P}$. Consider another partition $\mathbf{A}_3 \equiv \{\{o_1, o_2\}, \{a, b, c\}, \{x, y\}, \{o_3, o_4\}\}$. Then P_i in Example 1 respects $\mathbf{A}_3$, denoted as $P_i \in \mathbb{D}_{\mathbf{A}_3}$.

We say a preference **respects a dichotomous path** if it respects every partition along the path. A sequentially dichotomous domain is hence defined as the set of preferences that respect a common dichotomous path.

Definition 2. *A domain $\mathbb{D} \subset \mathbb{P}$ is a **sequentially dichotomous domain (SDD)** if there is a dichotomous path $(\mathbf{A}_t)_{t=1}^n$ such that $P_i \in \mathbb{D}$ if and only if P_i respects $(\mathbf{A}_t)_{t=1}^n$.*

Alternatively, a domain $\mathbb{D}$ is an SDD if there is a dichotomous path $(\mathbf{A}_t)_{t=1}^n$ such that $\mathbb{D} = \bigcap_{t=1}^n \mathbb{D}_{\mathbf{A}_t}$. Note that the definition of SDDs imposes a richness condition: *every* preference respecting the path needs to be included. This facilitates our analysis because we focus on verifying sd-SP of the PS rule and the fact that if a rule is sd-SP on a domain, it is sd-SP on every sub-domain. When a domain $\mathbb{D}$ is defined as an SDD with respect to a dichotomous path $(\mathbf{A}_t)_{t=1}^n$, we say $\mathbb{D}$ **is induced by** $(\mathbf{A}_t)_{t=1}^n$.

Remark 1. *To define an SDD, a particular dichotomous path needs to be fixed. Hence the preferences included respect the same dichotomous path. As mentioned in the introduction, this is analogous to the assumption (all agents' preferences are single-peaked with respect to* *the same linear order) that underlies the single-peaked domains. Continuing with the analogy with the single-peaked domain, the cardinality of an SDD coincides with the cardinality of the single-peaked domain. In particular, denoting by n the number of objects, both an SDD and the single-peaked domain contain 2^{n-1} preferences. For instance, in Example 2, $n = 4$ and an SDD includes 8 preferences.* ■

Fixing a dichotomous path, the induced SDD is identified as the intersection of the domains respecting respectively the partitions in the path. The following are two examples of SDDs.

Example 2. *Let $A = \{a, b, c, d\}$. We claim that the domain $\mathbb{D} \equiv \{P_1, \cdots, P_8\}$ is an SDD, where the preferences are as follows.*

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
a	c	d	d	b	b	b	b
c	a	a	c	a	c	d	d
d	d	c	a	c	a	a	c
b	b	b	b	d	d	c	a

In particular, we verify $\mathbb{D} = \bigcap_{t=1}^{4} \mathbb{D}_{\mathbf{A}_t}$, where $(\mathbf{A}_t)_{t=1}^{4}$ is a dichotomous path with $\mathbf{A}_1 = \{\{a, b, c, d\}\}$, $\mathbf{A}_2 = \{\{a, c, d\}, \{b\}\}$, $\mathbf{A}_3 = \{\{a, c\}, \{d\}, \{b\}\}$, and $\mathbf{A}_4 = \{\{a\}, \{c\}, \{d\}, \{b\}\}$. This path is indicated by the darkened arrows in Figure 1.

First, since every preference respects $\mathbf{A}_1$ and $\mathbf{A}_4$, $\mathbb{D}_{\mathbf{A}_1} = \mathbb{D}_{\mathbf{A}_4} = \mathbb{P}$. Second, a preference respects $\mathbf{A}_2$ if and only if b is ranked either at the top or at the bottom. Hence $\mathbb{D}_{\mathbf{A}_2} = \{P_i \in \mathbb{P} : r_1(P_i) = b \text{ or } r_4(P_i) = b\}$. Last, a preference $P_i \in \mathbb{D}_{\mathbf{A}_2}$ respects $\mathbf{A}_3$ if and only if $\{a, c\}$ clusters in P_i. Hence $\mathbb{D}_{\mathbf{A}_2} \cap \mathbb{D}_{\mathbf{A}_3} = \{P_i \in \mathbb{D}_{\mathbf{A}_2} : \{a, c\} \text{ clusters in } P_i\}$. In summary, $\mathbb{D} = \bigcap_{t=1}^{4} \mathbb{D}_{\mathbf{A}_t}$.

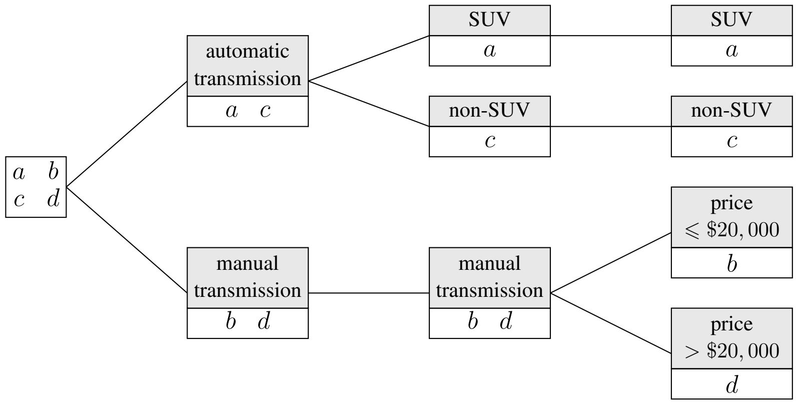
Similarly, the domain $\mathbb{D}' \equiv \{P'_1, \cdots, P'_8\}$ is also an SDD. In particular, $\mathbb{D}'$ is induced by $(\mathbf{A}'_t)_{t=1}^{4}$, where $\mathbf{A}'_1 = \{\{a, b, c, d\}\}$, $\mathbf{A}'_2 = \{\{a, c\}, \{b, d\}\}$, $\mathbf{A}'_3 = \{\{a\}, \{c\}, \{b, d\}\}$, and $\mathbf{A}'_4 = \{\{a\}, \{c\}, \{b\}, \{d\}\}$. This path is indicated by darkened and dotted arrows in Figure 1.

P'_1	P'_2	P'_3	P'_4	P'_5	P'_6	P'_7	P'_8
a	c	a	c	b	d	b	d
c	a	c	a	d	b	d	b
b	b	d	d	a	a	c	c
d	d	b	b	c	c	a	a

■

We provide in the following example an interpretation of the SDDs. One begins with a set of attributes and a linear order of them. Given an attribute, each object can be classified as either possessing the attribute or not. In particular, if an agent treats the first attribute as desirable, every object possessing this attribute is better than every object that does not (and vice versa), and so on using the lexicographic rationale with respect to the pre-specified order. Thus, a dichotomous path is interpreted as the result of sequentially classifying the objects according to the fixed order of attributes and the induced SDD is interpreted as being generated by lexicographically expressing the desirability of the attributes.[9] Example 3 below illustrates.

Example 3. *Let $\{a, b, c, d\}$ be a collection of four cars. The cars will be judged based on three attributes in the following order: first, whether or not it has an automatic transmission, second,whether or not it is an SUV, and finally, whether its price is no higher than 20,000 US dollars.*



A dichotomous path, in particular $(\mathbf{A}'_t)_{t=1}^4$ in Example 2, can be identified as the result of sequentially checking these three attributes in the given order. The figure above illustrates the procedure. First, the grand set $\{a, b, c, d\}$ is divided into $\{a, c\}$ and $\{b, d\}$ according to the first attribute. Second, $\{a, c\}$ is further divided into $\{a\}$ and $\{c\}$ according to the second attribute. Last, $\{b, d\}$ is further divided into $\{b\}$ and $\{d\}$ according to the third attribute.

The domain $\mathbb{D}'$ in Example 2, which is induced by the dichotomous path above, can now be interpreted as being generated by the lexicographic rationale with respect to varying attitudes towards the fixed order of attributes. For instance, let an agent be one who treats all three attributes as desirable. First, since she treats automatic transmissions as desirable, all the cars with automatic transmissions are better than the ones with manual transmissions. Hence a and c are better than b and d. Second, between a and c, since she treats SUVs as desirable, a is better than c. Last, between b and d, since she treats the ones with prices no higher than 20,000 US dollars as desirable, b is better than d. Consequently, her preference would be $P'_1 \in \mathbb{D}'$. The reader can verify that, if an agent treats the last attribute as desirable but the other two not, her preference would be $P'_7 \in \mathbb{D}'$. By varying the combinations of desirable attributes, all eight preferences in $\mathbb{D}'$ can be constructed by the lexicographic rationale. ■

Remark 2. *The fixed dichotomous path that underlies an SDD corresponds to a fixed order on the set of attributes. The multi-agent model we study hence implicitly assumes that all the agents evaluate the importance of the attributes in the same way. This may appear to be somewhat demanding. For example, in situations where comparisons between objects require professional*

[9] Preferences generated by the lexicographic rationale have been studied by decision theorists. See, for instance, Mandler et al. (2012).

inputs, this assumption requires that the agents refer to the same professional expert. In the centralized allocation problems we consider, the authority who implements the assignment could be such an expert. ■

In the next section we will show that the PS rule is sd-SP on any SDD. The verification of sd-SP for a random assignment rule is in general a difficult task. In the following, we present a key feature of an SDD, which simplifies this task. In particular, it turns out that any deviation on an SDD can be decomposed as a sequence of preferences, where, between every two successive preferences, exactly two adjacently ranked blocks are flipped. Such block flippings are called **block-adjacent reversals** and the preference resulting from a block-adjacent reversal is said to be block-adjacent to the initial one.

Definition 3. *A preference $\tilde{P}_i$ is **block-adjacent** to P_i if there are two nonempty and disjoint subsets $A_1, A_2 \subset A$ such that*

1. *A_1, A_2, and $A_1 \cup A_2$ cluster in both P_i and $\tilde{P}_i$,*
2. *$\forall\, a, b \in A$ such that $a \in A_1$ and $b \in A_2$, $a\ \tilde{P}_0\ b$ if and only if $b\ P_0\ a$,*
3. *$\forall\, a, b \in A$ such that either $a \notin A_1$ or $b \notin A_2$, $a\ \tilde{P}_0\ b$ if and only if $a\ P_0\ b$.*

Block-adjacency can be seen as a generalization of "adjacency" by Monjardet (2009), which refers to a pair of preferences which differ only in a flip between two adjacently ranked objects. It is important to note that a block-adjacent reversal between A_1 and A_2 does not change the ranking of objects contained in A_1, nor the ranking of objects contained in A_2. This is captured by the third requirement in the above definition.

The following example illustrates a decomposition of a deviation as a sequence of block-adjacent reversals.

Example 4. *Consider the domain $\mathbb{D} = \{P_1, \cdots, , P_8\}$ in Example 2. Let a deviation be from P_1 to P_8. Such a deviation can be decomposed into the sequence of block-adjacent reversals below.*

$$\begin{array}{lccccccc}
P_1: & a & \succ & c & \succ & d & \succ & b \\
\downarrow & & & & & & & \\
P_5: & b & \succ & a & \succ & c & \succ & d \\
\downarrow & & & & & & & \\
P_7: & b & \succ & d & \succ & a & \succ & c \\
\downarrow & & & & & & & \\
P_8: & b & \succ & d & \succ & c & \succ & a
\end{array}$$

Recall that $\mathbb{D}$ is an SDD induced by $(\mathbf{A}_t)_{t=1}^4$ where $\mathbf{A}_1 = \{\{a, b, c, d\}\}$, $\mathbf{A}_2 = \{\{a, c, d\}, \{b\}\}$, $\mathbf{A}_3 = \{\{a, c\}, \{d\}, \{b\}\}$, and $\mathbf{A}_4 = \{\{a\}, \{c\}, \{d\}, \{b\}\}$. First, P_5 is block-adjacent to P_1 with respect to the block-adjacent reversal between $\{b\}$ and $\{a, c, d\}$. Second, P_7 is block-adjacent to P_5 with respect to the block-adjacent reversal between $\{a, c\}$ and $\{d\}$. Third, P_8 is block-adjacent to P_7 with respect to the block-adjacent reversal between $\{a\}$ and $\{c\}$.

Note that these three reversals correspond to the dichotomous divisions along the dichotomous path. First, from $\mathbf{A}_1$ *to* $\mathbf{A}_2$*,* $\{a, b, c, d\}$ *breaks into* $\{b\}$ *and* $\{a, c, d\}$*. Second, from* $\mathbf{A}_2$ *to* $\mathbf{A}_3$*,* $\{a, c, d\}$ *breaks into* $\{a, c\}$ *and* $\{d\}$*. Third, from* $\mathbf{A}_3$ *to* $\mathbf{A}_4$*,* $\{a, c\}$ *breaks into* $\{a\}$ *and* $\{c\}$*. In general, a deviation may not involve all the dichotomous divisions but a subset of them. For example, the deviation from* P_1 *to* P_4 *involves only two reversals: one between* $\{a, c\}$ *and* $\{d\}$ *and the other between* $\{a\}$ *and* $\{c\}$*.* ■

The remainder of this section presents three remarks on the SDDs.

Remark 3. *An SDD satisfies minimal richness, namely, for each* $a \in A$*, there is a preference* P_i *in the domain such that* $r_1(P_i) = a$*. This is illustrated by the domains in Example 2.* ■

Remark 4. *Two different dichotomous paths may induce the same SDD. Consider the domain* $\mathbb{D}'$ *in Example 2. We have seen that this domain is an SDD induced by the dichotomous path* $(\mathbf{A}'_t)_{t=1}^4$*. It turns out that* $\mathbb{D}'$ *is also induced by* $(\mathbf{A}''_t)_{t=1}^4$*, where* $\mathbf{A}''_3 \equiv \{\{a, c\}, \{b\}, \{d\}\}$ *and* $\mathbf{A}''_t \equiv \mathbf{A}'_t$ *for* $t = 1, 2, 4$*. However this non-uniqueness turns out to be inconsequential for our analysis.* ■

Remark 5. *Liu and Zeng (2018) introduced the notion of a restricted tier domain and showed that if a domain admits an sd-SP, sd-Eff, and ETE rule, then it is a subset of the union of restricted tier domains. A well-known fact about the PS rule is that it satisfies sd-Eff and ETE on any domain. We will show in Theorem 1 that the PS rule on an SDD is sd-SP. Then, as a corollary, an SDD can always be expressed as a subset of the union of some restricted tier domains. To see this, we first introduce restricted tier domains. An ordered partition of* A*, denoted by* $\mathcal{P}$*, is a restricted tier structure if each block contains at most two objects. The corresponding restricted tier domain, denoted by* $\mathbb{D}(\mathcal{P})$*, is then defined as the collection of preferences that respect the order of the blocks. Four restricted tier structures are listed below.*

$\mathcal{P}^1$	$\mathcal{P}^2$	$\mathcal{P}^3$	$\mathcal{P}^4$
$\{a, c\}$	$\{d\}$	$\{b\}$	$\{b\}$
$\{d\}$	$\{a, c\}$	$\{a, c\}$	$\{d\}$
$\{b\}$	$\{b\}$	$\{d\}$	$\{a, c\}$

The corresponding restricted tier domains are respectively $\mathbb{D}(\mathcal{P}^1) = \{P_1, P_2\}$*,* $\mathbb{D}(\mathcal{P}^2) = \{P_3, P_4\}$*,* $\mathbb{D}(\mathcal{P}^3) = \{P_5, P_6\}$*, and* $\mathbb{D}(\mathcal{P}^4) = \{P_7, P_8\}$*, where the involved preferences are from Example 2. Recall that the set of these eight preferences is an SDD. Hence we have decomposed it as the union of four restricted tier domains. Generally, this decomposition can be identified by examining the dichotomous path that induces the given SDD. For details, see the extension part of Liu and Zeng (2018).*

In addition, Liu and Zeng (2018) introduced a condition called the local elevating property. This condition requires the existence of three preferences, three objects, and three consecutive positions such that (i) these three objects are ranked at these three positions in all three preferences, (ii) the set of objects ranked higher than these three positions is the same in all three preferences, (iii) one of the three objects takes three different positions in three preferences and the other two are ranked in the same way. They proved that if a domain admits a desirable rule, *it violates the local elevating property. By Theorem 1, an SDD admits a desirable rule. Thus it violates the local elevating property.* ■

4 The PS Rule on Sequentially Dichotomous Domains

This section presents two main results. Theorem 1 shows that the PS rule is *sd-strategy-proof* on a sequentially dichotomous domain. Theorem 2 shows that a sequentially dichotomous domain is maximal for the PS rule to be *sd-strategy-proof*.

Theorem 1 is proved by means of two lemmas. Recall that a deviation in an SDD can be decomposed as a sequence of block-adjacent reversals. Accordingly, we define an incentive compatibility notion weaker than sd-SP, called *block-adjacent sd-strategy-proofness*. Let L_i be agent i's lottery when she reports her true preference. Let L'_i be her lottery when she reports a different preference that is block-adjacent to the true one. Then *block-adjacent sd-strategy-proofness* requires that L_i first-order stochastically dominates L'_i according to her true preference. Lemma 1 shows that, on an SDD, a rule is sd-SP if and only if it is *block-adjacent sd-strategy-proof*. Next, Lemma 2 implies *block-adjacent sd-strategy-proofness* of the PS rule.

We start the discussion from the formal definition of *block-adjacent sd-strategy-proofness*.

Definition 4. *A rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ is **block-adjacent sd-strategy-proof (BA-sd-SP)** if $\forall\, i \in I$, $P_i, \tilde{P}_i \in \mathbb{D}$, and $P_{-i} \in \mathbb{D}^{n-1}$, such that $\tilde{P}_i$ is block-adjacent to P_i, $\varphi_i(P_i, P_{-i})\ P_i^{sd}\ \varphi_i(\tilde{P}_i, P_{-i})$.*

Although BA-sd-SP is a weaker condition than sd-SP, it turns out that these two conditions are equivalent on an SDD. We present this fact below in Lemma 1. Example 5 illustrates the idea of the proof.

Example 5. *Consider the deviation from P_1 to P_8 in Example 4. We have shown that this deviation can be decomposed as a sequence of block-adjacent reversals. Let L_1, L_5, L_7, L_8 denote respectively the deviating agent's lottery when she reports P_1, P_5, P_7, P_8. Given BA-sd-SP, what we need to establish is $L_1\ P_1^{sd}\ L_8$.*

$$\begin{array}{lccccccc}
P_1: & a & \succ & c & \succ & d & \succ & b \\
\downarrow & & & & & & & \\
P_5: & b & \succ & a & \succ & c & \succ & d \\
\downarrow & & & & & & & \\
P_7: & b & \succ & d & \succ & a & \succ & c \\
\downarrow & & & & & & & \\
P_8: & b & \succ & d & \succ & c & \succ & a
\end{array}$$

$$\begin{array}{ccccccc}
L_1\ P_1^{sd}\ L_5 & + & L_5\ P_1^{sd}\ L_7 & + & L_7\ P_1^{sd}\ L_8 & \Rightarrow & L_1\ P_1^{sd}\ L_8 \\
 & & \Uparrow & & \Uparrow & & \\
 & & L_5\ P_5^{sd}\ L_7 & & L_7\ P_5^{sd}\ L_8 & & \\
 & & L_7\ P_7^{sd}\ L_5 & & L_{7b} = L_{8b} & & \\
 & & & & L_{7d} = L_{8d} & & \\
 & & & & \Uparrow & & \\
 & & & & L_7\ P_7^{sd}\ L_8 & & \\
 & & & & L_8\ P_8^{sd}\ L_7 & &
\end{array}$$

First, by BA-sd-SP, we have $L_5\ P_5^{sd}\ L_7$ and $L_7\ P_7^{sd}\ L_5$. These two statements together imply $L_{5b} = L_{7b}$. Given this and the fact that P_1 and P_5 rank a, c, and d in the same way, $L_5\ P_5^{sd}\ L_7$ implies $L_5\ P_1^{sd}\ L_7$.

Second, by the same argument as above, BA-sd-SP implies $L_{7b} = L_{8b}$, $L_{7d} = L_{8d}$, and $L_7 \; P_5^{sd} \; L_8$. By noting that P_1 and P_5 rank a and c in the same way, we have further $L_7 \; P_1^{sd} \; L_8$. Finally the transitivity of P_1^{sd} establishes $L_1 \; P_1^{sd} \; L_8$. ■

Lemma 1. *On a sequentially dichotomous domain, a rule is sd-strategy-proof if and only if it is block-adjacent sd-strategy-proof.*

The proof, which follows the logic illustrated in Example 5, is in Appendix A.

Lemma 1 is related to the studies on the equivalence between local and global incentive compatibility. Sato (2013) focused on deterministic ordinal mechanisms, namely, mappings from profiles of ordinal preferences on a set of deterministic outcomes to these outcomes. Instances include the deterministic voting rules and the deterministic allocation rules. He identified a condition, called *non-restoration*, and proved that if a domain satisfies this condition, a deterministic ordinal mechanism is strategy-proof if and only if it satisfies a local notion of incentive compatibility. This notion is called AM-strategy-proofness and requires that a flip between two adjacently ranked objects not lead to a better outcome. Cho (2016a) studied random ordinal mechanisms and proved that non-restoration is also sufficient for the equivalence between sd-SP and AM-sd-strategy-proofness, which requires that a flip between two adjacently ranked objects not lead to a lottery which is not first-order stochastically dominated by the lottery delivered by the original preference.[10]

The non-restoration property requires that between any two preferences, there is a sequence of preferences connecting them such that (i) every pair of adjacent preferences differ only in a flip of two adjacently ranked objects, and (ii) along the sequence no pair of objects are flipped more than once. An SDD violates the condition (i). To see this, note that between P_2 and P_3 in Example 2, there is no sequence of preferences in the given SDD satisfying (i). Hence the equivalence provided by Cho (2016a) cannot be used to simplify the verification of sd-SP on SDDs.

Block-adjacent reversals include as special cases the flips between adjacently ranked objects. Hence BA-sd-SP is stronger than AM-sd-strategy-proofness. Although BA-sd-SP requires more, it still helps in the verification of sd-SP on SDDs.

Due to Lemma 1, to show sd-SP of the PS rule on an SDD, it suffices to show BA-sd-SP. Lemma 2 below asserts a slightly stronger property.

Lemma 2. *Let $P \in \mathbb{P}^n$, $\tilde{P}_i \in \mathbb{P}$, if there are two nonempty and disjoint subsets of objects $A_1, A_2 \subset A$ such that*

1. *for all $j \in I$, A_1, A_2, and $A_1 \cup A_2$ cluster in P_j,*
2. *$\tilde{P}_i$ is block-adjacent to P_i with respect to the reversal between A_1 and A_2,*
3. *$\forall \; a \in A_1$ and $b \in A_2$, $a \; P_1 \; b$ and $b \; \tilde{P}_1 \; a$,*

[10] In addition to the first-order stochastic dominance extension, Cho (2016a) studied two other extension methods. Independently, Carroll (2012) studied also the equivalence between local and global incentive compatibility. However, he examined the equivalence on specific preference domains, for instance, the single-peaked domain. In addition, he also investigated cardinal mechanisms.

then:

1. $\forall\, a \in A_1$, $PS_{1a}(P) \geqslant PS_{1a}(\tilde{P}_1, P_{-1})$,
2. $\forall\, b \in A_2$, $PS_{1b}(P) \leqslant PS_{1b}(\tilde{P}_1, P_{-1})$,
3. $\forall\, x \in A \backslash A_1 \cup A_2$, $PS_{1x}(P) = PS_{1x}(\tilde{P}_1, P_{-1})$.

The proof of Lemma 2 is in Appendix B.

Lemma 2 says that if one agent performs a block-adjacent reversal between $A_1, A_2 \subset A$ and it is known that in every other's preference A_1, A_2, and $A_1 \cup A_2$ cluster, then for every object that has been moved downward in the deviator's preference, the probability for the deviator to get this object is non-increasing. Note that there is no requirement on how the objects in A_1 (or the objects in A_2) are ranked in P_j for $j \neq i$. It is evident that this lemma implies BA-sd-SP on an SDD.

The statement of Lemma 2 is stronger than the statement that the PS rule is BA-sd-SP on an SDD for two reasons. First, the lemma does not require the preferences to be taken from a given SDD. Rather, the requirement is that A_1, A_2, and $A_1 \cup A_2$ cluster in the preferences involved. Second, BA-sd-SP does not require the probability of every object that is moved downwards to be non-increasing. Rather, it requires only the probability of every upper contour set to be non-increasing. For example, consider the block-adjacent deviation from $a \succ b \succ c \succ d$ to $c \succ a \succ b \succ d$. BA-sd-SP requires that the probability of getting a, b combined be non-increasing. In particular, the probability of getting b is allowed to increase as long as the decrease in a's probability exceeds the increase in b's probability. Although Lemma 2 states more than what is needed to show the theorem, we still present it, in the hope that it may prove useful for further research.

We are now ready to present the theorem.

Theorem 1. *The PS rule is sd-strategy-proof on a sequentially dichotomous domain.*

The theorem follows directly from Lemma 1, and Lemma 2. By Lemma 2, the PS rule on an SDD is BA-sd-SP. Then by Lemma 1 the PS rule is sd-SP.

Given Theorem 1, the next question we ask is: Can we expand an SDD while preserving the sd-SP of PS rule? The answer is in the negative, as stated in Theorem 2.

Theorem 2. *A sequentially dichotomous domain is maximal for the PS rule to be sd-strategy-proof.*

The proof of Theorem 2 is in Appendix C. In the proof, we fix an arbitrary SDD, denoted by $\mathbb{D}$, and an arbitrary preference $\tilde{P}_0 \in \mathbb{P} \backslash \mathbb{D}$. We first compare $\tilde{P}_0$ with an arbitrary dichotomous path which induces $\mathbb{D}$. This allows us to identify two preferences in the SDD, $P_0, \bar{P}_0 \in \mathbb{D}$. We then construct two preference profiles consisting of only $\tilde{P}_0, P_0, \bar{P}_0$. In these two preference profiles, one agent deviates. Finally we calculate the relevant probabilities specified by the PS rule and show that this deviation is profitable. The manipulability of the PS rule between the two constructed preference profiles is essentially the same as illustrated by the following example, although the proof is more complicated and involves the consideration of several cases.[11]

[11] This example is modified from the first example in Liu and Zeng (2018).

Example 6. *Let $A = \{a, b, c, d\}$. Let $P \equiv (P_{-4}, P_4)$ and $P' \equiv (P_{-4}, P'_4)$ be two preference profiles below. The consumption procedures of the PS rule are depicted in Figure 2.*

$$P = \begin{pmatrix} P_1 : a \succ c \succ b \succ d \\ P_2 : a \succ b \succ c \succ d \\ P_3 : b \succ a \succ c \succ d \\ P_4 : b \succ a \succ c \succ d \end{pmatrix} \qquad P' = \begin{pmatrix} P_1 : a \succ c \succ b \succ d \\ P_2 : a \succ b \succ c \succ d \\ P_3 : b \succ a \succ c \succ d \\ P'_4 : a \succ b \succ c \succ d \end{pmatrix}$$

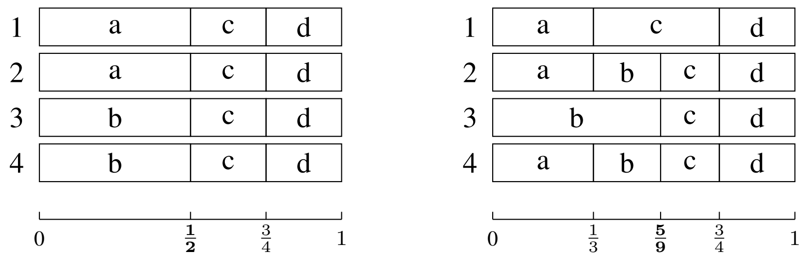


Figure 2: Consumption procedures under P and P' of the PS rule

Note that $PS_{4a}(P_{-4}, P_4) + PS_{4b}(P_{-4}, P_4) = \frac{1}{2} < \frac{5}{9} = PS_{4a}(P_{-4}, P'_4) + PS_{4b}(P_{-4}, P'_4)$. In other words, agent 4 gets a high probability to get an object that she prefers to c. This makes the manipulation profitable. Agent 4 manipulates in the following way: by flipping a and b, a reaches exhaustion earlier, at $\frac{1}{3}$ instead of $\frac{1}{2}$. Then since agent 1 would start to consume c as soon as a is exhausted, agent 4 excludes one agent from consuming a or b in the time period $(\frac{1}{3}, \frac{1}{2}]$. This means that she can enjoy a longer consumption of these two objects in combination. In the proof of Theorem 2, a and b are replaced with two blocks but the idea illustrated here still holds. ■

The remainder of this section discusses two related questions. The first is related to Cho (2016b). That paper identified a sufficient condition on a preference profile, called *recursive decomposability*, which ensures that the PS assignment is uniquely sd-Eff and sd-envy-free.[12] The sequential structure of an SDD then raises an question: Is the PS rule uniquely sd-Eff and sd-EF on an SDD? The following example however shows that these two axioms are not sufficient to pin down the PS assignments.

Example 7. *Let a dichotomous path be $(\mathbf{A}_t)_{t=1}^4$ such that $\mathbf{A}_1 = \{\{a, b, c, d\}\}$, $\mathbf{A}_2 = \{\{a, b\}, \{c, d\}\}$, $\mathbf{A}_3 = \{\{a, b\}, \{c\}, \{b\}\}$, and $\mathbf{A}_4 = \{\{a\}, \{b\}, \{c\}, \{d\}\}$. Let P be the preference profile as follows. One can verify that the preferences are contained in the SDD induced by $(\mathbf{A}_t)_{t=1}^4$.*

We construct a continuum of random assignments, which are parameterized by a real number $\alpha \in [1/4, 3/4]$.

$$P = \begin{pmatrix} a & \succ & b & \succ & c & \succ & d \\ c & \succ & d & \succ & a & \succ & b \\ c & \succ & d & \succ & a & \succ & b \\ c & \succ & d & \succ & a & \succ & b \end{pmatrix} \qquad L^\alpha = \begin{pmatrix} & a & b & c & d \\ 1: & \alpha & 1-\alpha & 0 & 0 \\ 2: & (1-\alpha)/3 & \alpha/3 & 1/3 & 1/3 \\ 3: & (1-\alpha)/3 & \alpha/3 & 1/3 & 1/3 \\ 4: & (1-\alpha)/3 & \alpha/3 & 1/3 & 1/3 \end{pmatrix}$$

[12] *Sd-envy-freeness* requires that an agent always weakly prefers her own lottery to any other's. Formally, a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ is **sd-envy-free (sd-EF)** if for all $P \in \mathbb{D}^n$ and all $i, j \in I$, $\varphi_i(P)\ P_i^{sd}\ \varphi_j(P)$.

Note first that $PS(P) = L^{3/4}$. We show that an assignment L is sd-Eff and sd-EF at P if and only if $L \in \{L^\alpha : \alpha \in [1/4, 3/4]\}$.

If: First, sd-EF can be verified by definition. We check sd-Eff. Suppose not, let $L' \neq L$ and $L'\ P^{sd}\ L$. We derive a contradiction for each of the following two cases. Case 1: $L'_1 \neq L_1$. Then $L'_1\ P_1^{sd}\ L_1$ implies $L'_{1a} > L_{1a} = \alpha$ and $L'_{1c} = L'_{1d} = 0$. Feasibility and $L'_i\ P_i^{sd}\ L_i$, $\forall i = 2,3,4$ then imply $L'_{ic} = L_{ic} = 1/3$, $L'_{id} = L_{id} = 1/3$, and $L'_{ia} \geqslant L_{ia} = (1-\alpha)/3$, $\forall\ i = 2,3,4$. We have hence a contradiction to feasibility: $\sum_{i\in I} L'_{ia} > \sum_{i \in I} L_{ia} = 1$. Case 2: $L'_1 = L_1$. Then feasibility and $L'_i\ P_i^{sd}\ L_i$, $\forall i = 2,3,4$ imply $L'_i = L_i$, $\forall i = 2,3,4$, which is a contradiction to $L' \neq L$.

Only if: Let L be sd-Eff and sd-EF. First sd-Eff implies either $L_{1c} = L_{1d} = 0$ or $L_{ia} = L_{ib} = 0$ for all $i = 2,3,4$. Feasibility rules out the later case. Given $L_{1c} = L_{1d} = 0$, feasibility and sd-EF imply $L_{ic} = L_{id} = 1/3$ for all $i = 2,3,4$. Given this, feasibility and sd-EF further imply that L is in the form of L^α, with $\alpha \in [0,1]$. Finally, sd-EF requires $\alpha \in [1/4, 3/4]$. In particular, $L_1\ P_1^{sd}\ L_2$ implies $\alpha \geqslant 1/4$ and $L_2\ P_2^{sd}\ L_1$ implies $\alpha \leqslant 3/4$. ■

The second question is whether the class of SDDs is uniquely maximal. Put otherwise, if we know that the PS rule is sd-SP on a given domain, can we structure it as a sub-domain of an SDD? The example below proves this to be wrong.

Example 8. *Let $n = 4$ and let a domain $\mathbb{D}$ consist of the following two preferences.*

$$P_0: \quad a \succ b \succ c \succ d$$

$$\bar{P}_0: \quad c \succ a \succ d \succ b$$

It is not difficult to check that the PS rule is sd-SP on $\mathbb{D}$. However $\mathbb{D}$ can never be a sub-domain of any SDD. The key is that we cannot find a binary partition that both P_0 and $\bar{P}_0$ respect. ■

Interestingly the pattern indicated by the preferences in the above example has been investigated, for example Rossin and Bouvel (2006). It seems that the exclusion of the above pattern is crucial for the computer either to generate permutations fast or to compare two sets of permutations fast. For CS studies, it is perfectly justified that the pattern is excluded artificially. However, there is no economically reasonable excuse, to my understanding, for excluding such a pattern. Hence, although by excluding such a pattern, we might be able to establish the uniqueness of SDDs for the PS rule to be sd-SP, this exercise is of little economic interest.

5 Conclusion

We identified a class of domains, called *sequentially dichotomous domains*, and proved that the PS rule is *sd-strategy-proof* on any such domain. In addition, each of these domains is maximal for the PS rule to be *sd-strategy-proof*.

We close the discussion by listing three questions for future research. First, are there interesting sets of axioms that characterize the PS rule on an SDD? Second, on an SDD, what is the set of all sd-Eff and sd-EF rules? Finally, as mentioned in the introduction, there appears to be some connection between the voting problem and the random assignment problem, which deserves further investigation.

Appendix

A Proof of Lemma 1

The necessity part is evident by definition. We prove the sufficiency part.

Let $\mathbb{D}$ be an SDD and let $(\mathbf{A}_t)_{t=1}^n$ be a dichotomous path that induces $\mathbb{D}$. In addition, let $\varphi : \mathbb{D}^n \to \mathcal{L}$ be a BA-sd-SP rule. Let $P \in \mathbb{D}^n$ and $P_1' \in \mathbb{D}\backslash\{P_1\}$. In addition, let $L \equiv \varphi(P)$ and $L' \equiv \varphi(P_1', P_{-1})$. It suffices to show $L_1 \; P_1^{sd} \; L_1'$.

Along the dichotomous path from $\mathbf{A}_1$ to $\mathbf{A}_n$, a block breaks into two smaller ones at each step. A preference P_0 respecting the dichotomous path figures the ranking between the two blocks in each step $t = 2, \cdots, n$. (For $t = 1$, there is only one major block.) Recall that for each $t = 2, \cdots, n$, $A_{(t-1)*}$ breaks into A_{t1} and A_{t2}. Without loss of generality, suppose $a \; P_1 \; b$ for all $t = 2, \cdots, n$, $a \in A_{t1}$, $b \in A_{t2}$. Consequently $P_1 \neq P_1'$ implies the existence of a subsequence $\{t_\gamma\}_{\gamma=1}^{\Gamma} \subset \{t\}_{t=2}^n$ such that $b \; P_1' \; a$ for all $\gamma = 1, \cdots, \Gamma$, $a \in A_{t_\gamma 1}$, $b \in A_{t_\gamma 2}$.

Let $P_{10} \equiv P_1$. We define a sequence of preferences $(P_{1\gamma})_{\gamma=1}^{\Gamma}$ such that for each γ, (i) $b \; P_{1\gamma} \; a$ if and only if $a \; P_{1\gamma-1} \; b$ for all $a \in A_{t_\gamma 1}$ and $b \in A_{t_\gamma 2}$; and (ii) $a \; P_{1\gamma} \; b$ if and only if $a \; P_{1\gamma-1} \; b$ otherwise. That is, for each $\gamma \in \{1, \cdots, \Gamma\}$, the difference between $P_{1\gamma}$ and $P_{1\gamma-1}$ is only a reversal between the blocks $A_{t_\gamma 1}$ and $A_{t_\gamma 2}$. Hence each $P_{1\gamma}$ results from a block-adjacent reversal of $P_{1\gamma-1}$. In addition, following this sequence of block-adjacent reversals, we approach P_1' and in the end $P_{1\Gamma} = P_1'$. Note that for each $\gamma \in \{1, \cdots, \Gamma\}$, $P_{1\gamma} \in \mathbb{D}$.

Correspondingly we denote $L^\gamma \equiv PS(P_{1\gamma}, P_{-1})$ for each $\gamma = 1, \cdots, \Gamma$. In the following, we fix γ and show $L_1^{\gamma-1} \; P_1^{sd} \; L_1^\gamma$, in two steps.

Step 1: $L_1^{\gamma-1} \; P_{1\gamma-1}^{sd} \; L_1^\gamma$. Note that $P_{1\gamma}$ results from a block-adjacent reversal of $P_{1\gamma-1}$. Then BA-sd-SP establishes the step.

Step 2: $\forall \alpha = 2, \cdots, \gamma$, $L_1^{\gamma-1} \; P_{1\alpha-1}^{sd} \; L_1^\gamma \Rightarrow L_1^{\gamma-1} \; P_{1\alpha-2}^{sd} \; L_1^\gamma$.

First note that, since $\alpha \leqslant \gamma$, $A_{t_\gamma 1}$, $A_{t_\gamma 2}$, and $A_{t_\gamma 1} \cup A_{t_\gamma 2}$ cluster respectively in both $P_{1\alpha-1}$ and $P_{1\alpha-2}$. Note in addition that $P_{1\alpha-1}$ and $P_{1\alpha-2}$ differ only in block-adjacent reversal between $A_{t_{\alpha-1}1}$ and $A_{t_{\alpha-1}2}$. Note last that, by the definition of SDD, one of the following two cases occurs.

Case 1: $\left(A_{t_\gamma 1} \cup A_{t_\gamma 2}\right) \subset A_{t_{\alpha-1}1}$ or $\left(A_{t_\gamma 1} \cup A_{t_\gamma 2}\right) \subset A_{t_{\alpha-1}2}$. We show $L_1^{\gamma-1} \; P_{1\alpha-1}^{sd} \; L_1^\gamma \Rightarrow L_1^{\gamma-1} \; P_{1\alpha-2}^{sd} \; L_1^\gamma$ for the former sub-case and the argument applies to the latter sub-case as well. Note that according to the definitions of $(P_{1\gamma})_{\gamma=1}^{\Gamma}$, $a \; P_{1\alpha-2} \; b$ and $a \; P_{1\alpha-1} \; b$ for all $a \in A_{t_\gamma 1}$ and $b \in A_{t_\gamma 2}$. We illustrate the situation as follows.

$$P_{1\alpha-2}: \quad \cdots\cdots \succ \underbrace{\cdots \succ A_{t_\gamma 1} \succ A_{t_\gamma 2} \succ \cdots}_{A_{t_{\alpha-1}1}} \succ A_{t_{\alpha-1}2} \succ \cdots\cdots$$

$$P_{1\alpha-1}: \quad \cdots\cdots \succ A_{t_{\alpha-1}2} \succ \underbrace{\cdots \succ A_{t_\gamma 1} \succ A_{t_\gamma 2} \succ \cdots}_{A_{t_{\alpha-1}1}} \succ \cdots\cdots$$

Recall that BA-sd-SP implies that $L_1^{\gamma-1}$ and L_1^{γ} differ only in the assignments of the objects in $A_{t_\gamma 1}$ and $A_{t_\gamma 2}$. In addition, by construction, the ranking of the objects within these two blocks is the same in $P_{1\alpha-2}$ and $P_{1\alpha-1}$. Then by definition of first-order stochastic dominance, $L_1^{\gamma-1}\ P_{1\alpha-1}^{sd}\ L_1^{\gamma} \Rightarrow L_1^{\gamma-1}\ P_{1\alpha-2}^{sd}\ L_1^{\gamma}$.

Case 2: $\left(A_{t_\gamma 1} \cup A_{t_\gamma 2}\right) \cap \left(A_{t_{\alpha-1}1} \cup A_{t_{\alpha-1}2}\right) = \emptyset$. We illustrate the situation as follows.

$$\begin{aligned} P_{1\alpha-2}: &\quad \cdots\cdots \succ \cdots \succ A_{t_\gamma 1} \succ A_{t_\gamma 2} \succ \cdots \succ A_{t_{\alpha-1}1} \succ A_{t_{\alpha-1}2} \succ \cdots\cdots \\ P_{1\alpha-1}: &\quad \cdots\cdots \succ \cdots \succ A_{t_\gamma 1} \succ A_{t_\gamma 2} \succ \cdots \succ A_{t_{\alpha-1}2} \succ A_{t_{\alpha-1}1} \succ \cdots\cdots \end{aligned}$$

Recall that BA-sd-SP implies that $L_1^{\gamma-1}$ and L_1^{γ} differ only in the assignments of the objects in $A_{t_\gamma 1}$ and $A_{t_\gamma 2}$. In addition, by construction, the ranking of the objects within these two blocks is the same in $P_{1\alpha-2}$ and $P_{1\alpha-1}$. Then by definition of first-order stochastic dominance, $L_1^{\gamma-1}\ P_{1\alpha-1}^{sd}\ L_1^{\gamma} \Rightarrow L_1^{\gamma-1}\ P_{1\alpha-2}^{sd}\ L_1^{\gamma}$, which verifies step 2.

Finally, by the transitivity of P_1^{sd}, $L_1 = L_1^0\ P_1^{sd}\ L_1^1\ P_1^{sd} \cdots P_1^{sd}\ L_1^{\Gamma} = L_1'$.

B Proof of Lemma 2

For purpose of illustration, we start by introducing the definition of PS rule. Although the objects are indivisible, the PS rule treats them as divisible and selects the assignment for a given preference profile as follows. Starting from time 0, all agents consume their most preferred object at unit speed until some objects reach exhaustion. Then agents reformulate their preferences by removing the objects exhausted and resume consuming their most preferred objects in the available set, until there are some other objects reach exhaustion. This procedure is repeated until all objects reach exhaustion. The end time of this procedure is 1, since we have n agents consuming n objects at unit speed. Finally, the share of an object consumed by an agent is interpreted as the probability of this agent obtaining this object. Formally the PS rule is defined as follows. We borrow the notation from Kojima and Manea (2010).

Definition 5. *Fix a preference profile $P \in \mathbb{D}^n$, $PS(P)$ is the random assignment $[L_{ia}]_{i\in I, a\in A} \in \mathcal{L}$ calculated as follows. For any $a \in A' \subset A$, let $N(a, A') \equiv \{i \in I | a\ P_i\ b, \forall b \in A' \backslash \{a\}\}$ be the set of agents whose most preferred object in A' is a. Let $A^0 = A$, $t^0 = 0$, and $L_{ia}^0 = 0$ for any $i \in I$ and $a \in A$. For any $v \geqslant 1$, given t^{v-1}, A^{v-1}, and $[L_{ia}^{v-1}]_{i\in I, a\in A}$, we define*

$$t^v \equiv \min_{a\in A^{v-1}} \max \left\{ t \in [0,1] | \sum_{i\in I} L_{ia}^{v-1} + |N(a, A^{v-1})|(t - t^{v-1}) \leqslant 1 \right\}$$

$$A^v \equiv A^{v-1} \backslash \left\{ a \in A^{v-1} | \sum_{i\in I} L_{ia}^{v-1} + |N(a, A^{v-1})|(t^v - t^{v-1}) = 1 \right\}$$

$$L_{ia}^v \equiv \begin{cases} L_{ia}^{v-1} + t^v - t^{v-1} & \text{if } i \in N(a, A^{v-1}) \\ L_{ia}^{v-1} & \text{otherwise.} \end{cases}$$

Since A is a finite set, there exists $\bar{v}$ such that $A^{\bar{v}} = \emptyset$. Let $PS(P) = L^{\bar{v}}$.

Fix a preference profile P, we call the sequence generated by applying the PS rule to P, denoted by $\mathcal{E} \equiv (t^v, A^v, L^v)_{v=0}^{\bar{v}}$, the corresponding **consumption procedure**. Evidently, for each $v \in \{0, \cdots, \bar{v}\}$, A^v is the collection of objects which are still available at time t^v. In other words, if $a \in A^{v-1} \backslash A^v$, then a is available at t^{v-1} and reaches exhaustion at t^v. For each $a \in A$, let $t_a \equiv t^v$ where $a \in A^{v-1} \backslash A^v$ denote the time at which a reaches exhaustion.

Recall that an SDD is an intersection of domains, each of which respects a partition. Then for a better understanding of the consumption procedure specified by the PS rule when the preferences belong to an SDD, we investigate the consumption procedure subject to a given partition.

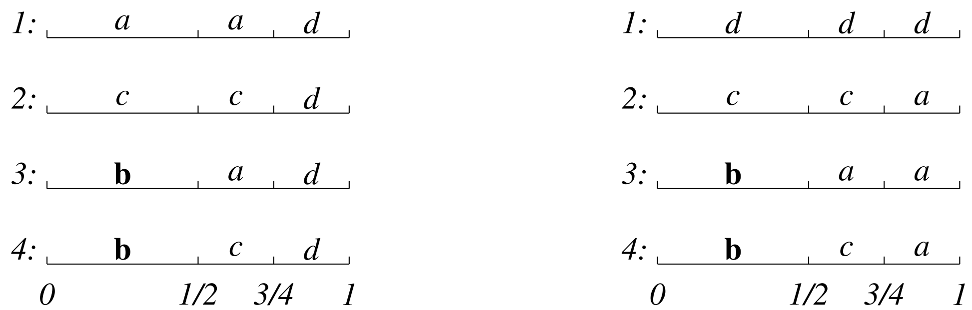
Given a preference profile $P \in \mathbb{D}_{\mathbf{A}}^n$, every block in $\mathbf{A}$ clusters in every agent's preference. Let for instance $\{a, b\} \in \mathbf{A}$ be a block, then for every agent either a is ranked just above b or b is ranked just above a. Hence if a reaches exhaustion before b ($t_a < t_b$), the agents who prefer a to b switch from a to b at t_a while all the others keep consuming b until t_b. If instead b reaches exhaustion before a ($t_b < t_a$), then the agents who prefer b to a switch from b to a at t_b and all the others keep consuming a until t_a. Consequently, if we focus on only blocks rather than objects, we can ignore the time at which a reaches exhaustion if this occurs before b and ignore the time at which b reaches exhaustion if this occurs before a. In other words, what we care about is only the time at which the whole block reaches exhaustion. For the example here, we need only to identify $\max\{t_a, t_b\}$. Formally, **the consumption procedure subject to** $\mathbf{A}$, denoted as $\mathcal{E}\big|_{\mathbf{A}} \equiv \left(t^v\big|_{\mathbf{A}}, A^v\big|_{\mathbf{A}}, L^v\big|_{\mathbf{A}}\right)$, is defined as follows.

Let $V\big|_{\mathbf{A}} \equiv \{v \in \{0, \cdots, \bar{v}\} | \exists A_k \in \mathbf{A} \text{ s.t. } t^v = \max_{a \in A_k} t_a\}$ be the collection of time points at which a block reaches exhaustion.

- $t^v\big|_{\mathbf{A}}$ is the subsequence of $(t^v)_{v=0}^{\bar{v}}$ involving elements in $V\big|_{\mathbf{A}}$. We record only the time points at which a block in $\mathbf{A}$ reaches exhaustion.
- $A^v\big|_{\mathbf{A}}$ is the subsequence of $(A^v)_{v=0}^{\bar{v}}$ involving elements in $V\big|_{\mathbf{A}}$.
- $L^v\big|_{\mathbf{A}}$ is a matrix $[L^v_{iA_k}]_{i \in I, A_k \in \mathbf{A}}$ defined only for elements in $V\big|_{\mathbf{A}}$, where $L^v_{iA_k} \equiv \sum_{a \in A_k} L^v_{ia}$.

When we focus on the consumption procedure subject to a partition, an important observation is that the consumption procedure subject to $\mathbf{A}$ should not change when the preference profile is changing in a way that the "ranking" of the blocks remain the same. That is, the change involves only permutations within blocks won't change the consumption procedure subject to the partition. Here is an example.

Example 9. *Let $P \equiv (P_1, P_2, P_5, P_6)$ and $\bar{P} \equiv (P_3, P_2, P_5, P_6)$ where the preferences are from Example 2. The consumption procedures of two profiles are illustrated as follows.*



Recall that the involved preferences respect $\mathbf{A}_2 = \{\{a,c,d\},\{b\}\}$*. In addition the consumption procedure subject to* $\mathbf{A}_2 = \{\{a,c,d\},\{b\}\}$ *is not changing: In time interval* $(0, 1/2]$ *agents* 1 *and* 2 *consume objects in block* $\{a,c,d\}$ *and agents* 3 *and* 4 *consume* $\{b\}$*. Then in time interval* $(1/2, 1]$ *all agent consume objects in* $\{a,c,d\}$*.* ■

We formalize the observation illustrated in Example 9. Given a partition $\mathbf{A}$, a preference respecting it, $P_0 \in \mathbb{D}_{\mathbf{A}}$, induces a (strict) preference on $\mathbf{A}$ in a natural way: a block is said to be ranked above another block if every object in the former is ranked above every object in the latter. Given a partition $\mathbf{A}$, $\mathbb{P}(\mathbf{A})$ denotes the collection of all strict preferences on $\mathbf{A}$.

Definition 6. *Fixing a partition* $\mathbf{A}$*, a preference on blocks* $P_0^{\mathbf{A}} \in \mathbb{P}(\mathbf{A})$ *is* ***induced*** *by* $P_0 \in \mathbb{D}_{\mathbf{A}}$ *if,* $\forall\ A_k, A_l \in \mathbf{A}$*,* $[A_k\ P_0^{\mathbf{A}}\ A_l] \Leftrightarrow [\forall a \in A_k, b \in A_l, a\ P_0\ b]$*.*

Evidently, a preference $P_0 \in \mathbb{D}_{\mathbf{A}}$ induces a *unique* preference $P_0^{\mathbf{A}}$ on $\mathbf{A}$. However the converse is not true: two different preferences $P_0, P_0' \in \mathbb{D}_{\mathbf{A}}$ may induce the same preference $P_0^{\mathbf{A}}$ on $\mathbf{A}$. For example, preferences P_1 and P_3 in Example 2 induce the same preference on $\mathbf{A}_2$: $\{a,c,d\} \succ \{b\}$.

Now we formalize the observation in Example 9 as the following lemma.

Lemma 3. *For* $P, \bar{P} \in \mathbb{D}_{\mathbf{A}}^n$ *such that* $P_i^{\mathbf{A}} = \bar{P}_i^{\mathbf{A}}$ *for all* $i \in I$*, the consumption procedures subject to* $\mathbf{A}$ *for two preference profiles are the same, namely* $\mathcal{E}\big|_{\mathbf{A}} = \bar{\mathcal{E}}\big|_{\mathbf{A}}$*. This implies also* $\sum_{a\in A_k} PS_{ia}(P) = \sum_{a\in A_k} PS_{ia}(\bar{P})$ *for all* $i \in I$ *and* $A_k \in \mathbf{A}$*.*

We are now ready to prove Lemma 2.

Let $L \equiv PS(P)$ and $\tilde{L} \equiv PS(\tilde{P}_1, P_{-1})$. Let $\mathcal{E} \equiv (t^v, A^v, [L_{ia}^v]_{i\in I, a\in A})_{v=0}^{\bar{v}}$ and $\tilde{\mathcal{E}} \equiv \left(\tilde{t}^v, \tilde{A}^v, [\tilde{L}_{ia}^v]_{i\in I, a\in A}\right)_{v=0}^{\bar{\tilde{v}}}$ be respectively consumption procedures for P and $(\tilde{P}_1, P_{-1})$. In addition, let $B \equiv \{a \in A | a\ P_1\ x, \forall x \in A_1 \cup A_2\}$ and $C \equiv A\backslash (A \cup A_1 \cup A_2)$ be respectively the upper and lower contour sets of objects in $A_1 \cup A_2$ according to P_1. Evidently, these two sets are the same for $\tilde{P}_1$.

Before proceeding to the proof, we clarify some notation.

- For each $a \in A$, let t_a and $\tilde{t}_a$ denote respectively the times at which a reaches exhaustion in $\mathcal{E}$ and $\tilde{\mathcal{E}}$.
- For each $D \subset A$, let $t_D \equiv \max\{t_a : a \in D\}$ and $\tilde{t}_D \equiv \max\{\tilde{t}_a : a \in D\}$ be respectively the times at which D reaches exhaustion in $\mathcal{E}$ and $\tilde{\mathcal{E}}$.
- For each $a \in A$ and $v \in \{0, 1, \cdots, \bar{v}\}$, let $S_a(t^v) \equiv 1 - \sum_{i\in I} L_{ia}^v$ be what remains of a at time t^v in $\mathcal{E}$. Similarly, for each $a \in A$ and $\tilde{v} \in \{0, 1, \cdots, \bar{\tilde{v}}\}$, let $S_a(t^{\tilde{v}}) \equiv 1 - \sum_{i\in I} L_{ia}^{\tilde{v}}$ be what remains of a at time $t^{\tilde{v}}$ in $\tilde{\mathcal{E}}$.

We prove the lemma in three steps.

Step 1: $L_{1a} = \tilde{L}_{1a}$ for all $a \in A\backslash (A_1 \cup A_2)$.

Consider a partition $\mathbf{A} \equiv \{A_1 \cup A_2, \{a\} : a \in A\backslash (A_1 \cup A_2)\}$. Since P_1 and $\tilde{P}_1$ differ only in a flip between A_1 and A_2, $P_1^{\mathbf{A}} = \tilde{P}_1^{\mathbf{A}}$. Then, by Lemma 3, we are done. In addition, by Lemma 3, we have the following facts.

Fact 1. *An agent starts to consume objects in $A_1 \cup A_2$, if any, at the same time in $\mathcal{E}$ and $\tilde{\mathcal{E}}$.*

Fact 2. *An agent stops consuming objects in $A_1 \cup A_2$, if any, at the same time in $\mathcal{E}$ and $\tilde{\mathcal{E}}$.*

Fact 3. *$t_B = \tilde{t}_B$, namely B reaches exhaustion at the same time in $\mathcal{E}$ and $\tilde{\mathcal{E}}$. In addition, $S_a(t_B) = \tilde{S}_a(t_B)$ for each $a \in A_1 \cup A_2$, that is when B reaches exhaustion what remains of each object in $A_1 \cup A_2$ is the same in $\mathcal{E}$ and $\tilde{\mathcal{E}}$.*

Step 2: $L_{1a} \geqslant \tilde{L}_{1a}$ for all $a \in A_1$.

If $\sum_{a \in A_1} S_a(t_B) = 0$, namely, when B reaches exhaustion and agent 1 is about to consume objects in A_1, A_1 has already reached exhaustion, then Fact 3 implies $L_{1a} = \tilde{L}_{1a} = 0$ for all $a \in A_1$. In this case, the result holds trivially.

If $\sum_{a \in A_2} S_a(t_B) = 0$, namely, when B reaches exhaustion, A_2 has already reached exhaustion, then $\mathcal{E} = \tilde{\mathcal{E}}$, which then implies $L_{1a} = \tilde{L}_{1a}$ for all $a \in A_1$. In this case, the result holds trivially.

If $\sum_{a \in A_1} \tilde{L}_{1a} = 0$, the result holds trivially.

Hence we show Step 2 given $\sum_{a \in A_1} S_a(t_B) > 0$, $\sum_{a \in A_2} S_a(t_B) > 0$, and $\sum_{a \in A_1} \tilde{L}_{1a} > 0$.

Note that before deviation t_B is the time when agent 1 starts to consume A_1. After the deviation, when B reaches exhaustion, agent 1 starts to consume A_2 and will turn to A_1 when A_2 reaches exhaustion. Hence $\tilde{t}_{B \cup A_2} > \tilde{t}_B = t_B$ is the time at which agent 1 starts to consume A_1 after the deviation. In particular, the time at which agent 1 starts to consume A_1 is latter after the deviation.

Then to show Step 2, it suffices to show the following two Claims.

Claim 1. *For each $a \in A_1$, $S_a(t_B) \geqslant \tilde{S}_a(\tilde{t}_{B \cup A_2})$.*

In words, when agent 1 is about to consume objects in A_1, after the deviation, she finds that what remains of each object in A_1 is less than that before the deviation.

Claim 2. *For each $i \in \{j \in I \backslash \{1\} : \sum_{a \in A_1} L_{ja} > 0\}$, $t_{U(A_1, P_i)} \geqslant \tilde{t}_{U(A_1, P_i)}$, where $U(A_1, P_i) \equiv \{a \in A : a\ P_i\ x, \forall x \in A_1\}$ denotes the upper contour set of A_1 according to P_i.*

Recall that $t_{U(A_1, P_i)}$ and $\tilde{t}_{U(A_1, P_i)}$ denote respectively the points in time when agent i starts to consume A_1 before and after agent 1's deviation. Hence this statement says that, at every time point, all the agents who compete with agent 1 in consuming A_1 will still do so after the deviation.

Since $\tilde{t}_{B \cup A_2} > t_B$, to show Claim 1, it suffices to show the following:

For each $i \in \{j \in I \backslash \{1\} : \sum_{a \in A_1} L_{ja} > 0$ and $t_{U(A_1, P_j) < t_B}\}$, $t_{U(A_1, P_i)} \geqslant \tilde{t}_{U(A_1, P_i)}$.

In other words, every agent $i \in I \backslash \{1\}$ who started to consume A_1 at a time before t_B, before agent 1's deviation, starts to consume A_1 at a time no latter than that time, after agent 1's deviation. This new statement is implied by Claim 2. Hence what we need to show is just Claim 2.

To show this, recall first Fact 1 which says that each agent starts to consume $A_1 \cup A_2$ at the same time before and after agent 1's deviation. Then pick any $i \in I \backslash \{1\}$ with $\sum_{a \in A_1} L_{ia} > 0$, if $a\ P_i\ b$ for all $a \in A_1$ and $b \in A_2$, we know that she starts to consume A_1 at the same time before and after agent 1's deviation. Suppose instead, $b\ P_i\ a$ for all $a \in A_1$ and $b \in A_2$, since we know this agent starts to consume A_1 immediately when all the objects in A_2 reach exhaustion, what we need to show is just $t_{B \cup A_2} > \tilde{t}_{B \cup A_2}$. This can be seen from the consumption procedures for P and $(\tilde{P}_1, P_{-1})$ subject to the partition $\{A_1, A_2, \{a \in A : a \notin A_1 \cup A_2\}\}$. Hence we have shown that for each $i \in I \backslash \{1\}$ with $\sum_{a \in A_1} L_{ia} > 0$, if she prefers A_1 to A_2, she starts to consume A_1 at the same time before and after the deviation; if she prefers A_2 to A_1, she starts to consume A_1 after the deviation at a time earlier than before deviation, which proves Step 2.

Step 3: $L_{1a} \leqslant \tilde{L}_{1a}$ for all $a \in A_2$.

By exchanging the roles of P_1 and $\tilde{P}_1$, Step 3 is implied by Step 2.

C Proof of Theorem 2

Let $\mathbb{D}$ be an SDD and let $(\mathbf{A}_t)_{t=1}^n$ be a corresponding dichotomous path. Let $\tilde{P}_0 \in \mathbb{P} \backslash \mathbb{D}$ be an arbitrary preference not in $\mathbb{D}$, we show that the PS rule defined on the union of $\mathbb{D}$ and $\tilde{P}_0$ is manipulable. Formally, the rule is a mapping $PS : \left(\mathbb{D} \cup \{\tilde{P}_0\}\right)^n \to \mathcal{L}$.

Since $\mathbb{D} = \bigcup_{t=1}^n \mathbb{D}_{\mathbf{A}_t}$ and $P_0 \notin \mathbb{D}$, $\underline{t} \equiv \min\{t \in \{1, \cdots, n\} : \tilde{P}_0 \notin \mathbb{D}_{\mathbf{A}_t}\}$ is well-defined. In addition, since $\tilde{P}_0$ respects $\mathbf{A}_1 = \{A\}$ trivially, $\underline{t} \geqslant 2$. Recall that from $\mathbf{A}_{\underline{t}-1}$ to $\mathbf{A}_{\underline{t}}$, $A_{(\underline{t}-1)*}$ breaks into $A_{\underline{t}1}$ and $A_{\underline{t}2}$. Let $a \equiv r_1(\tilde{P}_0, A_{(\underline{t}-1)*})$ be the most preferred objects according to $\tilde{P}_0$ in $A_{(\underline{t}-1)*}$. Without loss of generality, let $a \in A_{\underline{t}1}$. In addition, let $c \equiv r_1(\tilde{P}_0, A_{\underline{t}2})$ be the most preferred objects according to $\tilde{P}_0$ in $A_{\underline{t}2}$. Since $\tilde{P}_0$ does not respect $\mathbf{A}_{\underline{t}}$, there is $b \in A_{\underline{t}1} \backslash \{a\}$ ranked below c. Let $b \equiv r_1(\tilde{P}_0, \{x \in A_{\underline{t}1} : c \ \tilde{P}_0 \ x\})$ be the most preferred object in $A_{\underline{t}1}$ that is ranked below c. In addition, let $C \equiv \{x \in A_{\underline{t}1} \backslash \{a\} : x \ \tilde{P}_0 \ c\}$. Note that C may be empty. Hence $\tilde{P}_0$ can be illustrated as follows.

$$\tilde{P}_0 : \quad \cdots\cdots \succ \overbrace{\underbrace{a \succ \cdots C \cdots}_{\subset A_{\underline{t}1}} \succ \underbrace{c \succ \cdots}_{\subset A_{\underline{t}2}} \succ \underbrace{b \succ \cdots}_{\subset A_{\underline{t}1}} \succ \cdots\cdots}^{A_{(\underline{t}-1)*}} \succ \cdots\cdots$$

There is a unique $\bar{t} > \underline{t}$ such that $a, b \in A_{(\bar{t}-1)*}$, $a \in A_{\bar{t}1}$, and $b \in A_{\bar{t}2}$ In other words, from $\mathbf{A}_{\bar{t}-1}$ to $\mathbf{A}_{\bar{t}}$, a and b are split into two separate blocks. Let $B \equiv A_{\bar{t}1} \backslash \{a\}$ and $D \equiv A_{\bar{t}2} \backslash \{b\}$. Note that B and D may be empty.

In the following, we identify two preferences $P_0, \bar{P}_0 \in \mathbb{D}$. Then we construct two preference profiles consisting of only $\tilde{P}_0$, P_0 and $\bar{P}_0$ and show that there is a profitable manipulation. To do this, we need to consider four cases.

Case 1: $C = \emptyset$ or $r_1(\tilde{P}_0, C) \notin B \cup D$.

Since $\tilde{P}_0$ respects $\mathbf{A}_{\underline{t}-1}$, $\tilde{P}_0^{\mathbf{A}_{\underline{t}-1}}$ is well-defined. Let $P_0^{\mathbf{A}_{\underline{t}-1}} = \bar{P}_0^{\mathbf{A}_{\underline{t}-1}} = \tilde{P}_0^{\mathbf{A}_{\underline{t}-1}}$. Second, let $A_{(\bar{t}-1)*}$ be the first-ranked block in $\mathbf{A}_{\bar{t}-1}$ according to both P_0 and $\bar{P}_0$. Third, let $A_{\bar{t}1} P_0^{\mathbf{A}_{\bar{t}}} A_{\bar{t}2}$ and $A_{\bar{t}2} \bar{P}_0^{\mathbf{A}_{\bar{t}}} A_{\bar{t}1}$. Fourth, let a be the first-ranked object in $A_{\bar{t}1}$ and b the first-ranked object in $A_{\bar{t}2}$ according to both P_0 and $\bar{P}_0$. Last, let P_0 and $\bar{P}_0$ rank the objects contained in the same block in the same way. It is evident that $P_0, \bar{P}_0 \in \mathbb{D}$. Hence, P_0 and $\bar{P}_0$ are illustrated below.

$$P_0: \quad \cdots\cdots \succ \underbrace{\overbrace{a \succ \cdots B \cdots}^{=A_{\bar{t}1}} \succ \overbrace{b \succ \cdots D \cdots}^{=A_{\bar{t}2}} \succ \cdots}_{=A_{\underline{t}1}} \succ \underbrace{c \cdots\cdots}_{=A_{\underline{t}2}} \succ \cdots\cdots$$

$$\bar{P}_0: \quad \cdots\cdots \succ \underbrace{\overbrace{b \succ \cdots D \cdots}^{=A_{\bar{t}2}} \succ \overbrace{a \succ \cdots B \cdots}^{=A_{\bar{t}1}} \succ \cdots}_{=A_{\underline{t}1}} \succ \underbrace{c \cdots\cdots}_{=A_{\underline{t}2}} \succ \cdots\cdots$$

Let $P \equiv (\tilde{P}_1, P_2, P_3, \cdots, P_n)$ and $P' \equiv (\tilde{P}_1, \bar{P}_2, P_3, \cdots, P_n)$. We calculate the probabilities specified by the PS rule as follows.

$PS(P):$	a	B	b	D
$1:$	$\frac{1}{n}$	0	0	0
$2 \cdots n:$	$\frac{1}{n}$	$\frac{\lvert B\rvert}{n-1}$	$\frac{1}{n-1}$	$\frac{\lvert D\rvert}{n-1}$

For P, all agents share a equally. Then agents 2 to n consume $B \cup \{b\} \cup D$ while agent 1 consumes c if C is empty and $r_1(\tilde{P}_0, C)$ if not. Since the PS assignment is sd-Eff, $PS(P)_{1x} = 0$ for all $x \in B \cup \{b\} \cup D$. Then the ETE of PS assignment determines the other entries.

$PS(P'):$	a	B	b	D
$1:$	$\frac{1}{n-1}$	0	0	0
$2:$	0	0	$\frac{1}{n-1} + \frac{\lvert B\rvert}{n-2} + \frac{1-\frac{1}{n-1}-\frac{\lvert B\rvert}{n-2}}{n-1}$	$\frac{\lvert D\rvert}{n-1}$
$3 \cdots n:$	$\frac{1}{n-1}$	$\frac{\lvert B\rvert}{n-2}$	$\frac{1-\frac{1}{n-1}-\frac{\lvert B\rvert}{n-2}}{n-1}$	$\frac{\lvert D\rvert}{n-1}$

For P', agents other than 2 share a equally. By sd-Eff, $PS(P)_{1x} = 0$ for all $x \in B \cup \{b\} \cup D$. During the period when agents other than 2 consume a, agent 2 consumes b. When a reaches exhaustion, agents 3 to n start to consume B if B is nonempty and b if not; agent 1 starts to consume c if C is empty and $r_1(\tilde{P}_0, C)$ if not; and agent 2 still consumes b. It is evident that B will reach exhaustion before b or $r_1(\tilde{P}_0, C)$. Then agents 3 to n join agent 2 in consuming b, so they share equally what remains of b. In particular, $1 - \frac{1}{n-1} - \frac{|B|}{n-2}$ of b. Last, agents 2 to n share D equally.

Now we have a contradiction to sd-SP:

$$\text{sd-SP} \Rightarrow \sum_{x \in \{a,b\} \cup B \cup D} PS_{2x}(P) = \sum_{x \in \{a,b\} \cup B \cup D} PS_{2x}(P')$$

$$\Rightarrow \frac{1}{n} + \frac{|B|}{n-1} + \frac{1}{n-1} + \frac{|D|}{n-1} = \frac{1}{n-1} + \frac{|B|}{n-2} + \frac{1 - \frac{1}{n-1} - \frac{|B|}{n-2}}{n-1} + \frac{|D|}{n-1}$$

$$\Rightarrow \frac{1}{n(n-1)^2} = 0 : \text{contradiction.}$$

Case 2: $r_1(\tilde{P}_0, C) \in B$.

In order to identify a contradiction, we need to consider two sub-cases. Given $r_1(\tilde{P}_0, C) \in B$, there is an upper contour set in C contained in B. Let B_1 be the largest such upper contour set. Formally, $B_1 \equiv \max_{k \leqslant |C|}\{U_k(\tilde{P}_0, C) \subset B\}$[13]. Hence either $B_1 = C$ or $r_1(\tilde{P}_0, C \backslash B_1)$ exists. We consider the sub-cases: (i) $B_1 = C$ or $r_1(\tilde{P}_0, C\backslash B_1) \notin D$, or (ii) $r_1(\tilde{P}_0, C\backslash B_1) \in D$.

Sub-case 2.1: $B_1 = C$ or $r_1(\tilde{P}_0, C\backslash B_1) \notin D$.

For this sub-case, we use the same preferences $\tilde{P}_0$, P_0, and $\bar{P}_0$ as in Case 1 and the same profiles $P \equiv (\tilde{P}_1, P_2, P_3, \cdots, P_n)$ and $P' \equiv (\tilde{P}_1, \bar{P}_2, P_3, \cdots, P_n)$.

Given B_1, we identify the largest lower contour set in B according to P_0 such that no object in B_1 is contained. Let $\bar{B}_1$ denote the set B excluding this lower contour set. Formally, let $\bar{B}_1 \equiv \min_{k \leqslant |B|}\{U_k(P_0, B) : (B\backslash U_k(P_0, B)) \cap B_1 = \emptyset\}$. Then $B_1 \subset \bar{B}_1$. The preferences are illustrated below with B_1 and $\bar{B}_1$ explicitly located.

$$\tilde{P}_0: \quad \cdots\cdots \succ \quad \overbrace{\underbrace{a \succ B_1 \succ C\backslash B_1}_{\subset A_{\underline{t}1}} \succ \underbrace{c \succ \cdots}_{\subset A_{\underline{t}2}} \succ \underbrace{b \succ \cdots}_{\subset A_{\underline{t}1}} \succ \cdots\cdots}^{A_{(\underline{t}-1)*}} \quad \succ \cdots\cdots$$

$$P_0: \quad \cdots\cdots \succ \quad \underbrace{\overbrace{a \succ \bar{B}_1 \succ B\backslash \bar{B}_1}^{=A_{\bar{t}1}} \succ \overbrace{b \succ \cdots D \cdots}^{=A_{\bar{t}2}} \succ \cdots}_{=A_{\underline{t}1}} \succ \underbrace{c\cdots\cdots}_{=A_{\underline{t}2}} \quad \succ \cdots\cdots$$

$$\bar{P}_0: \quad \cdots\cdots \succ \quad \underbrace{\overbrace{b \succ \cdots D \cdots}^{=A_{\bar{t}2}} \succ \overbrace{a \succ \bar{B}_1 \succ B\backslash \bar{B}_1}^{=A_{\bar{t}1}} \succ \cdots}_{=A_{\underline{t}1}} \succ \underbrace{c\cdots\cdots}_{=A_{\underline{t}2}} \quad \succ \cdots\cdots$$

We calculate the probabilities specified by the PS rule as follows.

$$\begin{array}{lcccc}
PS(P): & a & B & b & D \\
1: & \frac{1}{n} & \frac{|\bar{B}_1|}{n} & 0 & 0 \\
2\cdots n: & \frac{1}{n} & \frac{|\bar{B}_1|}{n} + \frac{|B\backslash\bar{B}_1|}{n-1} & \frac{1}{n-1} & \frac{|D|}{n-1}
\end{array}$$

For P, all agents share a equally. By construction, after a reaches exhaustion agent 1 consumes B_1 and all others consume $\bar{B}_1$. Note that the time at which agent 1 stops consuming B_1 is exactly the time when the other agents stop consuming $\bar{B}_1$. Hence all the agents start consuming $\bar{B}_1$ at the same time and stop consuming $\bar{B}_1$ also at the same time. Then each agent consumes exactly $\frac{|\bar{B}_1|}{n}$ share of $\bar{B}_1$. Then agent 1 consumes c if $B_1 = C$ and $C\backslash B_1$ if not while all the others consume $B\backslash\bar{B}_1$ and then b and D. Note that sd-Eff implies that agent 1 consumes no share of b or D.

$$\begin{array}{lcccc}
PS(P'): & a & B & b & D \\
1: & \frac{1}{n-1} & \frac{|\bar{B}_1|}{n-1} & 0 & 0 \\
2: & 0 & 0 & \frac{1}{n-1} + \frac{|\bar{B}_1|}{n-1} + \frac{|B\backslash\bar{B}_1|}{n-2} + \frac{1-\frac{1}{n-1}-\frac{|\bar{B}_1|}{n-1}-\frac{|B\backslash\bar{B}_1|}{n-2}}{n-1} & \frac{|D|}{n-1} \\
3\cdots n: & \frac{1}{n-1} & \frac{|\bar{B}_1|}{n-1} + \frac{|B\backslash\bar{B}_1|}{n-2} & \frac{1-\frac{1}{n-1}-\frac{|\bar{B}_1|}{n-1}-\frac{|B\backslash\bar{B}_1|}{n-2}}{n-1} & \frac{|D|}{n-1}
\end{array}$$

[13] $U_k(\tilde{P}_0, C)$ denotes the collection of the most preferred k objects in C according to $\tilde{P}_0$. Formally, $U_k(\tilde{P}_0, C) \equiv \{r_1(\tilde{P}_0, C), \cdots, r_k(\tilde{P}_0, C)\}$

For P', all agents other than 2 share a equally and at the same time agent 2 consumes b. Then agents other than 2 share $\bar{B}_1$ equally and agent 2 still consumes b. Then agent 1 starts to consume c if $C = B_1$ and $C\backslash B_1$ if not and agents 3 to n share $B\backslash\bar{B}_1$ equally. During this time period agent 2 is still consuming b. Then all agents other than 1 share equally what remains of b, and D after b reaches exhaustion. The zero entries in the above table are implied by sd-Eff.

Now we have a contradiction to sd-SP:

$$\begin{aligned}
\text{sd-SP} \Rightarrow & \sum_{x\in\{a,b\}\cup B\cup D} PS_{2x}(P) = \sum_{x\in\{a,b\}\cup B\cup D} PS_{2x}(P') \\
\Rightarrow & \frac{1}{n} + \frac{|\bar{B}_1|}{n} + \frac{|B\backslash\bar{B}_1|}{n-1} + \frac{1}{n-1} + \frac{|D|}{n-1} \\
= & \frac{1}{n-1} + \frac{|\bar{B}_1|}{n-1} + \frac{|B\backslash\bar{B}_1|}{n-2} + \frac{1 - \frac{1}{n-1} - \frac{|\bar{B}_1|}{n-1} - \frac{|B\backslash\bar{B}_1|}{n-2}}{n-1} + \frac{|D|}{n-1} \\
\Rightarrow & \frac{1+|\bar{B}_1|}{n(n-1)^2} = 0 : \text{contradiction.}
\end{aligned}$$

Sub-case 2.2: $r_1(\tilde{P}_0, C\backslash B_1) \in D$.

Given $r_1(\tilde{P}_0, C\backslash B_1) \in D$, let D_1 be the largest upper contour set in $C\backslash B_1$ contained in D. Formally, $D_1 \equiv \max_{k\leqslant |C\backslash B_1|}\{U_k(\tilde{P}_0, C\backslash B_1) \subset D\}$. Similarly, let $\bar{D}_1 \equiv \min_{k\leqslant |D|}\{U_k(P_0, D) : (D\backslash U_k(P_0, D)) \cap D_1 = \emptyset\}$. For this sub-case, we use the same preferences and profiles as in the previous sub-case except a small change in P_0: D is ranked above b.

The preferences are illustrated below with B_1, $\bar{B}_1$, D_1 and $\bar{D}_1$ explicitly located.

$$\tilde{P}_0 : \quad \cdots\cdots \succ \quad \overbrace{\underbrace{a \succ B_1 \succ D_1 \succ C\backslash(B_1 \cup D_1)}_{\subset A_{\underline{t}1}} \succ \underbrace{c \succ \cdots}_{\subset A_{\underline{t}2}} \succ \underbrace{b \succ \cdots}_{\subset A_{\underline{t}1}} \succ \cdots\cdots}^{A_{(\underline{t}-1)*}} \quad \succ \cdots\cdots$$

$$P_0 : \quad \cdots\cdots \succ \quad \underbrace{\overbrace{a \succ \bar{B}_1 \succ B\backslash\bar{B}_1}^{=A_{\bar{t}1}} \succ \overbrace{\bar{D}_1 \succ D\backslash\bar{D}_1 \succ b}^{=A_{\bar{t}2}} \succ \cdots}_{=A_{\underline{t}1}} \succ \underbrace{c\cdots\cdots}_{=A_{\underline{t}2}} \quad \succ \cdots\cdots$$

$$\bar{P}_0 : \quad \cdots\cdots \succ \quad \underbrace{\overbrace{b \succ \cdots D \cdots}^{=A_{\bar{t}2}} \succ \overbrace{a \succ \bar{B}_1 \succ B\backslash\bar{B}_1}^{=A_{\bar{t}1}} \succ \cdots}_{=A_{\underline{t}1}} \succ \underbrace{c\cdots\cdots}_{=A_{\underline{t}2}} \quad \succ \cdots\cdots$$

We calculate the probabilities specified by the PS rule as follows.

$PS(P)$:	a	B	b	D
1 :	$\frac{1}{n}$	$\frac{\lvert\bar{B}_1\rvert}{n}$	0	$\frac{\lvert B\backslash\bar{B}_1\rvert}{n-1} + \frac{\lvert\bar{D}_1\rvert - \frac{\lvert B\backslash\bar{B}_1\rvert}{n-1}}{n}$
$2\cdots n$:	$\frac{1}{n}$	$\frac{\lvert\bar{B}_1\rvert}{n} + \frac{\lvert B\backslash\bar{B}_1\rvert}{n-1}$	$\frac{1}{n-1}$	$\frac{\lvert\bar{D}_1\rvert - \frac{\lvert B\backslash\bar{B}_1\rvert}{n-1}}{n} + \frac{\lvert D\backslash\bar{D}_1\rvert}{n-1}$

For P, all agents share a and $\bar{B}_1$ equally. Then agent 1 consumes D_1 and all the others consumes $B\backslash\bar{B}_1$. It is evident that $B\backslash\bar{B}_1$ reaches exhaustion faster and then agents 2 to n join agent 1 in consuming $\bar{D}_1$. When $\bar{D}_1$ reaches exhaustion, agent 1 starts consume c if $C\backslash(B_1 \cup D_1) = \emptyset$ and $C\backslash(B_1 \cup D_1)$ is otherwise. Agents 2 to n share $D\backslash\bar{D}_1$ equally.

$PS(P')$:

	a	B	b	D
$1:$	$\frac{1}{n-1}$	$\frac{\lvert\bar{B}_1\rvert}{n-1}$	0	$\frac{\lvert B\backslash\bar{B}_1\rvert}{n-2}+\frac{\lvert\bar{D}_1\rvert-\frac{\lvert B\backslash\bar{B}_1\rvert}{n-2}}{n-1}$
$2:$	0	0	$\alpha+\frac{1-\alpha}{n-1}$	0
$3\cdots n:$	$\frac{1}{n-1}$	$\frac{\lvert\bar{B}_1\rvert}{n-1}+\frac{\lvert B\backslash\bar{B}_1\rvert}{n-2}$	$\frac{1-\alpha}{n-1}$	$\frac{\lvert\bar{D}_1\rvert-\frac{\lvert B\backslash\bar{B}_1\rvert}{n-2}}{n-1}+\frac{\lvert D\backslash\bar{D}_1\rvert}{n-2}$

where $\alpha=\frac{1}{n-1}+\frac{|\bar{B}_1|}{n-1}+\frac{|B\backslash\bar{B}_1|}{n-2}+\frac{|\bar{D}_1|-\frac{|B\backslash\bar{B}_1|}{n-2}}{n-1}+\frac{|D\backslash\bar{D}_1|}{n-2}$.

For P', agents other than 2 share a equally. When they consume a, agent 2 consumes b. After a reaches exhaustion, agents other than 2 share $\bar{B}_1$ equally and agent 2 still consumes b. After $\bar{B}_1$ reaches exhaustion, agent 1 consumes D_1, agents 3 to n share $B\backslash\bar{B}_1$ equally, and agent 2 still consumes b. It is evident that $B\backslash\bar{B}_1$ reaches exhaustion faster. Then agents other than 2 share equally what remains of $\bar{D}_1$ and agent 2 still consumes b. Then agent 1 consumes c if $C\backslash(B_1\cup D_1)=\emptyset$ and $C\backslash(B_1\cup D_1)$ if otherwise, agents 3 to n share $D\backslash\bar{D}_1$ equally, and agent 2 still consumes b. Then agents other than 1 consume what remains of b.

Now we have a contradiction to sd-SP:

$$\text{sd-SP}\Rightarrow\sum_{x\in\{a,b\}\cup B\cup D}PS_{2x}(P)=\sum_{x\in\{a,b\}\cup B\cup D}PS_{2x}(P')$$

$$\Rightarrow\frac{1}{n}+\frac{|\bar{B}_1|}{n}+\frac{|B\backslash\bar{B}_1|}{n-1}+\frac{1}{n-1}+\frac{|\bar{D}_1|-\frac{|B\backslash\bar{B}_1|}{n-1}}{n}+\frac{|D\backslash\bar{D}_1|}{n-1}$$

$$=\alpha+\frac{1-\alpha}{n-1}$$

$$\Rightarrow\frac{|\bar{B}_1|+|B\backslash\bar{B}_1|+|\bar{D}_1|+1}{n(n-1)^2}=0:\text{contradiction.}$$

Case 3: $r_1(\tilde{P}_0,C)\in D$ and $B\neq\emptyset$.

Given $r_1(\tilde{P}_0,C)\in D$, let D_1 be the largest upper contour set in C contained in D. Formally, $D_1\equiv\max_{k\leqslant|C|}\{U_k(\tilde{P}_0,C)\subset D\}$. Similarly, let $\bar{D}_1\equiv\min_{k\leqslant|D|}\{U_k(P_0,D):(D\backslash U_k(P_0,D))\cap D_1=\emptyset\}$. For this sub-case, we use the same preferences and profiles as in Case 1.

For the reader to understand the consumption procedure better, preferences are illustrated below and D_1 and $\bar{D}_1$ are explicitly located.

$$\tilde{P}_0:\ \cdots\cdots\succ\ \overbrace{\underbrace{a\succ D_1\succ C\backslash D_1}_{\subset A_{\underline{t}1}}\succ\underbrace{c\succ\cdots}_{\subset A_{\underline{t}2}}\succ\underbrace{b\succ\cdots}_{\subset A_{\underline{t}1}}\succ\cdots\cdots}^{A_{(\underline{t}-1)*}}\ \succ\cdots\cdots$$

$$P_0:\ \cdots\cdots\succ\ \underbrace{\overbrace{a\succ\cdots B\cdots}^{=A_{\bar{t}1}}\succ\overbrace{b\succ\bar{D}_1\succ D\backslash\bar{D}_1}^{=A_{\bar{t}2}}\succ\cdots}_{=A_{\underline{t}1}}\succ\underbrace{c\cdots\cdots}_{=A_{\underline{t}2}}\ \succ\cdots\cdots$$

$$\bar{P}_0:\ \cdots\cdots\succ\ \underbrace{\overbrace{b\succ\bar{D}_1\succ D\backslash\bar{D}_1}^{=A_{\bar{t}2}}\succ\overbrace{a\succ\cdots B\cdots}^{=A_{\bar{t}1}}\succ\cdots}_{=A_{\underline{t}1}}\succ\underbrace{c\cdots\cdots}_{=A_{\underline{t}2}}\ \succ\cdots\cdots$$

$P = (\tilde{P}_0, P_2, P_3, \cdots, P_n)$ and $P' = (\tilde{P}_1, \bar{P}_2, P_3, \cdots, P_n)$

$$
\begin{array}{rccccc}
PS(P): & & a & B & b & D \\
 & 1: & \frac{1}{n} & 0 & 0 & \frac{|B|}{n-1} + \frac{1}{n-1} + \frac{|\bar{D}_1| - \frac{|B|}{n-1} - \frac{1}{n-1}}{n} \\
 & 2\cdots n: & \frac{1}{n} & \frac{|B|}{n-1} & \frac{1}{n-1} & \frac{|\bar{D}_1| - \frac{|B|}{n-1} - \frac{1}{n-1}}{n} + \frac{|D\backslash\bar{D}_1|}{n-1}
\end{array}
$$

For P, all agents share a equally. Then agent 1 consumes D_1 and all the others share B and b equally. It is evident that $B \cup \{b\}$ reaches exhaustion faster. Then all agents share what remains of $\bar{D}_1$. Then agent 1 consumes c if $C\backslash D_1 = \emptyset$ and $C\backslash D_1$ if otherwise and all the other agents share $D\backslash\bar{D}_1$.

$PS(P')$:

$$
\begin{array}{rcccc}
 & a & B & b & D \\
1: & \frac{1}{n-1} & 0 & 0 & \frac{|B|}{n-1} + \frac{1 - \frac{1}{n-1} - \frac{|B|}{n-1}}{n-1} + \frac{|\bar{D}_1| - \frac{|B|}{n-1} - \frac{1 - \frac{1}{n-1} - \frac{|B|}{n-1}}{n-1}}{n} \\
2: & 0 & 0 & \frac{1}{n-1} + \frac{|B|}{n-1} + \frac{1 - \frac{1}{n-1} - \frac{|B|}{n-1}}{n-1} & \frac{|\bar{D}_1| - \frac{|B|}{n-1} - \frac{1 - \frac{1}{n-1} - \frac{|B|}{n-1}}{n-1}}{n} + \frac{|D\backslash\bar{D}_1|}{n-1} \\
3\cdots n: & \frac{1}{n-1} & \frac{|B|}{n-1} & \frac{1 - \frac{1}{n-1} - \frac{|B|}{n-1}}{n-1} & \frac{|\bar{D}_1| - \frac{|B|}{n-1} - \frac{1 - \frac{1}{n-1} - \frac{|B|}{n-1}}{n-1}}{n} + \frac{|D\backslash\bar{D}_1|}{n-1}
\end{array}
$$

For P', all agents except 2 share a equally and during they do this agent 2 consumes b. After a reaches exhaustion, agent 1 consumes D_1, agent 2 still consumes b, and all the others consume B. It is evident that B reaches exhaustion faster. Then agents 3 to n join agent 2 in consuming what remains of b and agent 1 still consumes D_1. After b reaches exhaustion, all agents share equally what remains of $\bar{D}_1$. Then agent 1 consumes c if $C\backslash D_1 = \emptyset$ and $C\backslash D_1$ if otherwise and all the other agents share $D\backslash\bar{D}_1$.

Now we have a contradiction to sd-SP:

$$
\text{sd-SP} \Rightarrow \sum_{x\in\{a,b\}\cup B\cup D} PS_{2x}(P) = \sum_{x\in\{a,b\}\cup B\cup D} PS_{2x}(P')
$$

$$
\Rightarrow \frac{1}{n} + \frac{|B|}{n-1} + \frac{1}{n-1} + \frac{|\bar{D}_1| - \frac{|B|}{n-1} - \frac{1}{n-1}}{n} + \frac{|D\backslash\bar{D}_1|}{n-1}
$$

$$
= \frac{1}{n-1} + \frac{|B|}{n-1} + \frac{1 - \frac{1}{n-1} - \frac{|B|}{n-1}}{n-1} + \frac{|\bar{D}_1| - \frac{|B|}{n-1} - \frac{1 - \frac{1}{n-1} - \frac{|B|}{n-1}}{n-1}}{n} + \frac{|D\backslash\bar{D}_1|}{n-1}
$$

$$
\Rightarrow \frac{|B|}{n(n-1)} = 0 : \text{contradiction.}
$$

Case 4: $r_1(\tilde{P}_0, C) \in D$ and $B = \emptyset$.

In this case $A_{\bar{t}1} = \{a\}$. We use the same preferences and profiles as in Case 3 except a small change in P_0: D is ranked above b.

$$\tilde{P}_0: \quad \cdots\cdots \succ \overbrace{\underbrace{a \succ D_1 \succ C\backslash D_1}_{\subset A_{\underline{t}1}} \succ \underbrace{c \succ \cdots}_{\subset A_{\underline{t}2}} \succ \underbrace{b \succ \cdots}_{\subset A_{\underline{t}1}} \succ \cdots\cdots}^{A_{(\underline{t}-1)*}} \succ \cdots\cdots$$

$$P_0: \quad \cdots\cdots \succ \underbrace{\overbrace{a}^{=A_{\bar{t}1}} \succ \overbrace{\bar{D}_1 \succ D\backslash\bar{D}_1 \succ b}^{=A_{\bar{t}2}} \succ \cdots}_{=A_{\underline{t}1}} \succ \underbrace{c\cdots\cdots}_{=A_{\underline{t}2}} \succ \cdots\cdots$$

$$\bar{P}_0: \quad \cdots\cdots \succ \underbrace{\overbrace{b \succ \cdots D \cdots}^{=A_{\bar{t}2}} \succ \overbrace{a}^{=A_{\bar{t}1}} \succ \cdots}_{=A_{\underline{t}1}} \succ \underbrace{c\cdots\cdots}_{=A_{\underline{t}2}} \succ \cdots\cdots$$

$P = (\tilde{P}_0, P_2, P_3, \cdots, P_n)$ and $P' = (\tilde{P}_1, \bar{P}_2, P_3, \cdots, P_n)$

$PS(P)$:

	a	D	b
$1:$	$\frac{1}{n}$	$\frac{\lvert\bar{D}_1\rvert}{n}$	0
$2\cdots n:$	$\frac{1}{n}$	$\frac{\lvert\bar{D}_1\rvert}{n} + \frac{\lvert D\backslash\bar{D}_1\rvert}{n-1}$	$\frac{1}{n-1}$

For P, all agents share a equally. Then they share $\bar{D}_1$ equally. Then agent 1 consumes c if $C\backslash D_1 = \emptyset$ and $C\backslash D_1$ if otherwise and all the others share $D\backslash\bar{D}_1$ and then b equally.

$PS(P')$:

	a	D	b
$1:$	$\frac{1}{n-1}$	$\frac{\lvert\bar{D}_1\rvert}{n-1}$	0
$2:$	0	0	$\frac{\lvert\bar{D}_1\rvert}{n-1} + \frac{\lvert D\backslash\bar{D}_1\rvert}{n-2} + \frac{1-\frac{\lvert\bar{D}_1\rvert}{n-1}-\frac{\lvert D\backslash\bar{D}_1\rvert}{n-2}}{n-1}$
$3\cdots n:$	$\frac{1}{n-1}$	$\frac{\lvert\bar{D}_1\rvert}{n-1} + \frac{\lvert D\backslash\bar{D}_1\rvert}{n-2}$	$\frac{1-\frac{\lvert\bar{D}_1\rvert}{n-1}-\frac{\lvert D\backslash\bar{D}_1\rvert}{n-2}}{n-1}$

For P', all agents other than 2 share a equally and during they do this agent 2 consumes b. After a reaches exhaustion, agents other than 2 share $\bar{D}_1$ equally and agent 2 still consumes b. It is evident that $\bar{D}_1$ reaches exhaustion faster. Then agent 1 consumes c if $C\backslash D_1\emptyset$ and $C\backslash D_1$ if otherwise, agent 3 to n share $D\backslash\bar{D}_1$ equally, and agent 2 still consumes b. It is evident that $D\backslash\bar{D}_1$ reaches exhaustion faster. Then agents 3 to n join agent 2 in consuming b so they share equally what remains of b.

Now we have a contradiction to sd-SP:

$$\text{sd-SP} \Rightarrow \sum_{x\in\{a,b\}\cup D} PS_{2x}(P) = \sum_{x\in\{a,b\}\cup D} PS_{2x}(P')$$

$$\Rightarrow \frac{1}{n} + \frac{|\bar{D}_1|}{n} + \frac{|D\backslash\bar{D}_1|}{n-1} + \frac{1}{n-1}$$

$$= \frac{|\bar{D}_1|}{n-1} + \frac{|D\backslash\bar{D}_1|}{n-2} + \frac{1-\frac{|\bar{D}_1|}{n-1}-\frac{|D\backslash\bar{D}_1|}{n-2}}{n-1}$$

$$\Rightarrow \frac{(n-1)^2+|\bar{D}_1|}{n(n-1)^2} = 0 : \text{contradiction.}$$

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