Probabilistic assignment of indivisible objects when agents have single-peaked preferences with a common peak*

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Abstract

Bogomolnaia and Moulin (2001) show that there is no rule satisfying *equal treatment of equals, stochastic dominance efficiency,* and *stochastic dominance strategyproofness* for a probabilistic assignment problem of indivisible objects. Later, Kasajima (2013) shows that the incompatibility result still holds when agents are restricted to have single-peaked preferences. In this paper, we further restrict the domain by requiring that all agents have single-peaked preferences with a common peak and investigate the existence of rules satisfying the three properties. As it turns out, the three properties are still incompatible even if all agents have the same preferences except three least preferred objects.

JEL classification: C71, D63.

Keywords: Probabilistic assignment, indivisible goods, single-peaked preferences with a common peak, impossibility result.

1. Introduction

We consider the problem of allocating indivisible objects to a group of agents when each agent is supposed to receive exactly one object. This problem arises in many situations: an allocation of time slots to golfers, an assignment of dormitories to students, a placement of students to public schools, etc.

It is obvious that the indivisibility of objects causes a difficulty in achieving fairness. For instance, suppose that there are two agents with identical preferences who want to allocate two objects. Each of the two possible allocations would not satisfy any reasonable notion of fairness. To overcome the difficulty, two possibilities are proposed in the literature, monetary transfers or lotteries. In this paper, we use lotteries to assign the objects, called a *probabilistic assignment*.

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This type of assignment problems was introduced by Hylland and Zeckhauser (1979).[1] Following Bogomolnaia and Moulin (2001), we investigate the existence of rules satisfying three properties: *equal treatment of equals* which requires that if two agents have the same preferences, then they should end up with the same assignment; *stochastic dominance efficiency,* which requires that a probabilistic allocation selected by a rule is not stochastically Pareto dominated by any other probabilistic allocation; *stochastic dominance strategyproofness,* which requires that a probabilistic assignment selected by the rule under truth-telling stochastically dominates her assignment obtained by lying.

Bogomolnaia and Moulin (2001) show that for a problem with three agents, these three conditions are compatible, but for a problem with more than three agents, they are not. Kasajima (2013) shows that the three conditions are incompatible on the restricted domain of single-peaked preference profiles. In this paper, we further restrict the preference domain and consider the compatibility of the three conditions on the domain of single-peaked preference profiles with a common peak. As it turns out, these three conditions are still incompatible even though all agents have the same preferences except three least preferred objects.

The paper is organized as follows. Section 2 introduces the model and three properties. Section 3 presents our main impossibility result with the proof when the number of agents is even. Section 4 discusses how our result can be modified with variations in our assumptions. Appendix provides the proof for our main result when the number of agents is odd.

2. The model

Let $N = \{1, 2, \ldots, n\}$ be a finite set of agents. A typical agent is denoted by $i \in N$. Let $O = \{o_1, o_2, \ldots, o_m\}$ be a finite set of distinct indivisible objects. Each agent is supposed to receive exactly one object. We assume that $|N| = |O| = n$. Each agent $i \in N$ has a complete, transitive, and strict binary relation R_i over objects. We refer to R_i as agent i's *preference relation.* Let $\mathcal{R}$ be a domain of preferences. Let $R = (R_i)_{i \in N}$ be a preference profile and $\mathcal{R}^N$ be a domain of preference profiles.

We consider the following restriction on preferences. Without loss of generality, let O be ordered in such a way that

$$o_1 \prec o_2 \prec \cdots \prec o_n.$$

For each $i \in N$, $R_i \in \mathcal{R}$ is *single-peaked* on O (with respect to $\prec$) if one of the following three conditions holds:

(i) there is $t \in \{2, \ldots, n-1\}$ such that $o_t \; R_i \; o_{t-1} \; R_i \cdots R_i \; o_1$ and $o_t \; R_i \; o_{t+1} \; R_i \cdots R_i \; o_n$,

(ii) $o_n \; R_i \; o_{n-1} \; R_i \cdots R_i \; o_1$,

(iii) $o_1 \; R_i \; o_2 \; R_i \cdots R_i \; o_n$.

[1]There are many papers on the probabilistic assignment. For example, Hylland and Zeckhauser 1979; Abdulkadiroğlu and Sönmez 1998, 2003; Bogomolnaia and Moulin 2001, 2002, 2004, 2015; Katta and Sethuraman 2006; Kojima 2009; Che and Kojima 2010; Bogomolnaia and Heo 2012; Kasajima 2013; Erdil 2014; Hashimoto et al. 2014; Bogomolnaia 2015.

For each $i \in N$ and each $R_i \in \mathcal{R}$, let $p_i(R_i)$ be the most preferred object, or peak. A preference profile is single-peaked on O if for each $i \in N$, R_i is single-peaked on $\mathcal{R}$. Let $\mathcal{R}_{SP}^N$ be the domain of single-peaked preference profiles. A single-peaked preference profile has a common peak if $R \in \mathcal{R}_{SP}^N$ and for each $i \in N$, $p_i = p \in O$.

A *deterministic allocation* is a one-to-one correspondence between the set of agents and the set of objects. It is represented as a 0 - 1 matrix, with rows indexed by agents and columns indexed by objects: a 0 - 1 matrix represents a deterministic allocation if and only if it contains exactly one 1 in each row and each column.

A *probabilistic allocation* is a probability distribution over deterministic allocations. It is also represented as a matrix, whose (i, j)th entry represents the probability that agent i receives object j. Formally, a probabilistic allocation is a matrix $M = [M_{ik}]_{i\in N, k\in O}$ such that

(i) for each $i \in N$ and each $k \in O$, $M_{ik} \in [0, 1]$,

(ii) for each $i \in N$, $\sum_{k\in O} M_{ik} = 1$, and

(iii) for each $k \in O$, $\sum_{i\in N} M_{ik} = 1$.

Let $\mathcal{M}$ be the set of all probabilistic allocations. For each $i \in N$, her *probabilistic assignment* in $M \in \mathcal{M}$ is a vector $M_i = [M_{ik}]_{k\in O}$, i.e. the ith row of M. A *rule* is a function which associates with each problem a matrix in $\mathcal{M}$. A generic rule is denoted by φ.

We introduce three requirements on rules. First is *equal treatment of equals,* which requires that if two agents have the same preferences, then they should end up with the same assignment.

equal treatment of equals: For each $R \in \mathcal{R}^N$ and each i, $j \in N$, if $R_i = R_j$, then $\varphi_i(R) = \varphi_j(R)$.

Next, we specify how an agent compares two assignments. For each $o \in O$, let $\tilde{o}$ be obtained by rearranging the objects from the best to the worst according to R_i, that is, $\tilde{o_1}\ R_i\ \tilde{o_2}\ R_i \dots R_i\ \tilde{o_n}$. An assignment $M_i = [M_{ik}]_{k\in O}$ *stochastically dominates* another assignment $M'_i = [M'_{ik}]_{k\in O}$ at R_i, which we write $M_i\ R_i^{sd}\ M'_i$, if

$$\sum_{k=1}^{t} M_{i\tilde{o}_k} \geq \sum_{k=1}^{t} M'_{i\tilde{o}_k} \text{ for } t = 1, \dots, n.$$

If strict inequality holds for some k, then we write $M_i\ P_i^{sd}\ M'_i$. An allocation $M \in \mathcal{M}$ stochastically Pareto dominates another allocation $M' \in \mathcal{M}$ if for each $i \in N$, $M_i\ R_i^{sd}\ M'_i$, and for some $i \in N$, $M_i\ P_i^{sd}\ M'_i$.

Stochastic dominance efficiency (simply, *sd-efficiency*) requires that a probabilistic allocation selected by the rule is not stochastically Pareto dominated by any other probabilistic allocation. For each $R \in \mathcal{R}^N$, let $Eff^{sd}(R) = \{M \in \mathcal{M}|$ there is no $M' \in \mathcal{M}$ such that M' stocastically dominates M at $R\}$.

sd-efficiency: For each $R \in \mathcal{R}^N$, $\varphi(R) \in Eff^{sd}(R)$.

Finally, *stochastic dominance strategyproofness* (simply, *sd-strategyproofness*) requires that a probabilistic assignment selected by the rule under truth-telling stochastically dominates her assignment obtained under lying.

sd-strategyproofness: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $\varphi_i(R) R_i^{sd} \varphi_i(R'_i, R_{-i})$.

3. Result

As shown in Bogomolnaia and Moulin (2001), for a problem with three agents, *equal treatment of equals, sd-efficiency,* and *sd-strategy-proofness* are compatible, but for a problem with four or more agents, they are not. Kasajima (2013) shows that these three conditions are incompatible on a restricted domain of single-peaked preference profiles. In this paper, we further restrict the preference domain and consider the compatibility of the three conditions on the domain of single-peaked preference profiles with a common peak.

We note that if the common peak is either the first or the last object, that is, either for all $i \in N$, $o_1 \; R_i \; o_2 \; R_i \cdots R_i \; o_n$ or for all $i \in N$, $o_n \; R_i \; o_{n-1} \; R_i \cdots R_i \; o_1$, then all preferences are the same and the equal division satisfies the three requirements. Therefore, from now on, we assume that all agents have a common peak $o_t \in O$ such that $t \in \{2, \ldots, n-1\}$. Among many possible preference profiles with a common peak o_t, we use in the proof the following three preferences which are identical except three least preferred objects. For each $i \in N$,

- R'_i: $o_t \; R_i \; \ldots \; R_i \; o_1 \; R_i \; o_{n-1} \; R_i \; o_n$,
- R''_i: $o_t \; R_i \; \ldots \; R_i \; o_{n-1} \; R_i \; o_1 \; R_i \; o_n$,
- R'''_i: $o_t \; R_i \; \ldots \; R_i \; o_{n-1} \; R_i \; o_n \; R_i \; o_1$.

When o_{n-1} is a common peak, we choose o_1, o_2, and o_n as three least preferred alternatives.

To simplify the notation, we denote each preference profile by a matrix where each row represents an agent's preference. For example, a matrix $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ represents the preference profile (R'_1, R'''_2, R'''_3).

Theorem *On the domain of single-peaked preference profiles with a common peak, if* $n \geq 4$, *then there is no rule satisfying equal treatment of equals, sd-efficiency, and sd-strategyproofness.*

Proof. Let $N = \{1, 2, \ldots, n\}$ be such that $n \geq 4$. Here, we give a proof for an even number of agents. The proof for an odd number of agents is given in the Appendix. Suppose, by way of contradiction, that there is a rule φ that satisfies the three requirements. Let $k = \frac{n}{2}$.

$$\overset{\textbf{Profile 1-0}}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \Rightarrow \overset{\textbf{Profile 1-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \Rightarrow \overset{\textbf{Profile 1-2}}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \Rightarrow \cdots \Rightarrow \overset{\textbf{Profile 1-k}}{\begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} 1 \\ \vdots \\ 1 \end{matrix}}\right\} \frac{n}{2} \; agents \\ \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}\right\} \frac{n}{2} \; agents \end{matrix}}$$

Profile 1-0, R^{10}: For each $i \in N$, $R^{10}_i = R''_i$. By *equal treatment of equals,* for each $i \in N$ and each $k \in O$, $\varphi_{ik}(R^{10}) = \frac{1}{n}$.

Profile 1-1, R^{11}: $R^{11}_1 = R'_1$ and for each $i = 2, \dots, n$, $R^{11}_i = R''_i$. By *sd-strategyproofness,*

$$\varphi_1(R^{11}) R'^{sd}_1 \varphi_1(R^{10}) \quad \text{and} \quad \varphi_1(R^{10}) R''^{sd}_1 \varphi_1(R^{11}).$$

Therefore, $\varphi_{1o_2}(R^{11}) = \varphi_{1o_2}(R^{10}), \dots, \varphi_{1o_{n-2}}(R^{11}) = \varphi_{1o_{n-2}}(R^{10})$, and $\varphi_{1o_n}(R^{11}) = \varphi_{1o_n}(R^{10})$. Invoking our conclusion for Profile 1-0, $\varphi_{1o_2}(R^{11}) = \cdots = \varphi_{1o_{n-2}}(R^{11}) = \varphi_{1o_n}(R^{11}) = \frac{1}{n}$.

Next, we show that $\varphi_{1o_{n-1}}(R^{11}) = 0$. Suppose by way of contradiction that $\varphi_{1o_{n-1}}(R^{11}) > 0$. Since $\varphi_{1o_1}(R^{11}) < \frac{2}{n}$, there is $i \in N \setminus \{1\}$ such that $\varphi_{io_1}(R^{11}) > 0$. Let $\delta = \min\{\varphi_{1o_{n-1}}(R^{11}), \varphi_{io_1}(R^{11})\}$. Let $M \in \mathcal{M}$ be such that $M_{1o_1} = \varphi_{1o_1}(R^{11}) + \delta$, $M_{1o_{n-1}} = \varphi_{1o_{n-1}}(R^{11}) - \delta$, $M_{io_1} = \varphi_{io_1}(R^{11}) - \delta$, $M_{io_{n-1}} = \varphi_{io_{n-1}} + \delta$, and other entries are the same as the entries at $\varphi(R^{11})$. It is easy to see that M stochastically Pareto dominates $\varphi(R^{11})$ at R^{11}, in violation of *sd-efficiency.* Therefore, $\varphi_{1o_{n-1}}(R^{11}) = 0$, which implies that $\varphi_{1o_1}(R^{11}) = \frac{2}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{11}) = \begin{pmatrix} \frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{1}{n} \\ \frac{n-2}{n(n-1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{n-2}{n(n-1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n} \end{pmatrix}$$

Profile 1-2, R^{12}: For $i = 1, 2$, $R^{12}_i = R'_i$ and for $i = 3, \dots, n$, $R^{12}_i = R''_i$. By *sd-strategyproofness,*

$$\varphi_2(R^{12}) R'^{sd}_2 \varphi_2(R^{11}) \quad \text{and} \quad \varphi_2(R^{11}) R''^{sd}_2 \varphi_2(R^{12}).$$

Invoking our conclusion for Profile 1-1, $\varphi_{2o_2}(R^{12}) = \cdots = \varphi_{2o_{n-2}}(R^{12}) = \varphi_{2o_n}(R^{12}) = \frac{1}{n}$.

Next, we show that $\varphi_{2o_{n-1}}(R^{12}) = 0$. Suppose by way of contradiction that $\varphi_{2o_{n-1}}(R^{12}) > 0$, which implies that $\varphi_{2o_1}(R^{12}) < \frac{2}{n}$. By *equal treatment of equals,* $\varphi_{1o_1}(R^{12}) < \frac{2}{n}$. Then, there is $i \in \{3, 4, \dots, n\}$ such that $\varphi_{io_1}(R^{12}) > 0$. By the same argument as in Profile 1-1, we can construct $M \in \mathcal{M}$ such that M stochastically Pareto dominates $\varphi(R^{12})$ at R^{12}, in violation of *sd-efficiency.* Therefore, $\varphi_{2o_{n-1}}(R^{12}) = 0$, which implies that $\varphi_{2o_1}(R^{12}) = \frac{2}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{12}) = \begin{pmatrix} \frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{1}{n} \\ \frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{1}{n} \\ \frac{n-4}{n(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{n-4}{n(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{1}{n} \\ \frac{n-4}{n(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{1}{n} \end{pmatrix}$$

Repeat this process until we obtain profile 1-k.

Profile 1-k, R^{1k}: For $i = 1, \ldots, k$, $R^{1k}_i = R'_i$ and for $i = k+1, \ldots, n$, $R^{1k}_i = R''_i$. By *sd-strategyproofness,*

$$\varphi_k(R^{1k}) R'^{sd}_k \varphi_k(R^{1(k-1)}) \quad \text{and} \quad \varphi_k(R^{1(k-1)}) R''^{sd}_k \varphi_k(R^{1k}).$$

Invoking our conclusion for Profile 1-(k-1), $\varphi_{ko_2}(R^{1k}) = \cdots = \varphi_{ko_{n-2}}(R^{1k}) = \varphi_{ko_n}(R^{1k}) = \frac{1}{n}$.

Next, we show that $\varphi_{ko_{n-1}}(R^{1k}) = 0$. Suppose by way of contradiction that $\varphi_{ko_{n-1}}(R^{1k}) > 0$, which implies that $\varphi_{ko_1}(R^{1k}) < \frac{2}{n}$. By *equal treatment of equals,* for $i = 1, \ldots, k-1$, $\varphi_{io_1}(R^{1k}) < \frac{2}{n}$. Since $\sum_{i=1}^{k} \varphi_{io_1}(R^{1k}) < 1$, there is $i \in \{k+1, \ldots, n\}$ such that $\varphi_{io_1}(R^{12}) > 0$. By the same argument as in Profile 1-1, we can construct $M \in \mathcal{M}$ such that M stochastically Pareto dominates $\varphi(R^{1k})$ at R^{1k}, in violation of *sd-efficiency*. Therefore, by *sd-efficiency,* $\varphi_{ko_{n-1}}(R^{1k}) = 0$, which implies that $\varphi_{ko_1}(R^{12}) = \frac{2}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{1k}) = \begin{pmatrix} \frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

$$\overset{\textbf{Profile 2-0}}{\begin{bmatrix} 0&1&0\\ 0&1&0\\ 0&1&0\\ \vdots&\vdots&\vdots\\ 0&1&0\\ 0&1&0\\ 0&1&0 \end{bmatrix}} \Rightarrow \overset{\textbf{Profile 2-1}}{\begin{bmatrix} 0&1&0\\ 0&1&0\\ 0&1&0\\ \vdots&\vdots&\\ 0&1&0\\ 0&1&0\\ 0&0&1 \end{bmatrix}} \Rightarrow \overset{\textbf{Profile 2-2}}{\begin{bmatrix} 1&0&0\\ 0&1&0\\ 0&1&0\\ \vdots&\vdots&\vdots\\ 0&1&0\\ 0&1&0\\ 0&0&1 \end{bmatrix}} \Rightarrow \overset{\textbf{Profile 2-3}}{\begin{bmatrix} 1&0&0\\ 1&0&0\\ 0&1&0\\ 0&1&0\\ \vdots&\vdots&\vdots\\ 0&1&0\\ 0&0&1 \end{bmatrix}} \Rightarrow \cdots \Rightarrow \overset{\textbf{Profile 2-k}}{\begin{bmatrix} 1&0&0\\ \vdots&\vdots&\vdots\\ 1&0&0\\ 0&1&0\\ \vdots&\vdots&\vdots\\ 0&1&0\\ 0&0&1 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} \frac{n}{2}-1 \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} \frac{n}{2} \\ \}\, 1 \end{matrix}}$$

Profile 2-0, R^{20}: (same as R^{10}) For each $i \in N$, $R^{20}_i = R''_i$. By *equal treatment of equals,* for each $i \in N$ and $k \in O$, $\varphi_{ik}(R^{20}) = \frac{1}{n}$.

Profile 2-1, R^{21}: For $i = 1, \ldots, n-1$, $R^{21}_i = R''_i$ and $R^{21}_n = R'''_n$. By *sd-strategyproofness,*

$$\varphi_n(R^{21}) R'''^{sd}_n \varphi_n(R^{20}) \quad \text{and} \quad \varphi_n(R^{20}) R''^{sd}_n \varphi_n(R^{21}).$$

Invoking our conclusion for Profile 2-0, $\varphi_{no_2}(R^{21}) = \varphi_{no_3}(R^{21}) = \cdots = \varphi_{no_{n-1}}(R^{21}) = \frac{1}{n}$.

Next, we show that $\varphi_{no_1}(R^{21}) = 0$. Suppose by way of contradiction that $\varphi_{no_1}(R^{21}) > 0$. Since $\varphi_{no_n}(R^{21}) < \frac{2}{n}$, there is $i \in N \setminus \{n\}$ such that $\varphi_{io_n}(R^{21}) > 0$. By the same argument as in Profile 1-1, we can construct $M \in \mathcal{M}$ such that M stochastically Pareto dominates $\varphi(R^{21})$ at R^{21}, in violation of *sd-efficiency*. Therefore, $\varphi_{no_1}(R^{21}) = 0$, which implies that $\varphi_{no_n}(R^{21}) = \frac{2}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{21}) = \begin{pmatrix} \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{n-2}{n(n-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{n-2}{n(n-1)} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{2}{n} \end{pmatrix}$$

Profile 2-2, R^{22}: $R_1^{22} = R_1'$, for $i = 2, \ldots, n-1$, $R_i^{22} = R_i''$, and $R_n^{22} = R_n'''$. By *sd-strategyproofness,*

$$\varphi(R^{22}) R_1'^{sd} \varphi(R^{21}) \quad \text{and} \quad \varphi(R^{21}) R_1''^{sd} \varphi(R^{22}),$$
$$\varphi(R^{22}) R_n'''^{sd} \varphi(R^{11}) \quad \text{and} \quad \varphi(R^{11}) R_n''^{sd} \varphi(R^{22}).$$

Invoking our conclusion for Profiles 2-1 and 1-1, for $i = 1, n$, $\varphi_{io_2}(R^{22}) = \varphi_{io_3}(R^{22}) = \cdots = \varphi_{io_{n-2}}(R^{22}) = \frac{1}{n}$. Also, $\varphi_{1o_n}(R^{22}) = \frac{n-2}{n(n-1)}$ and $\varphi_{no_{n-1}}(R^{22}) = \frac{1}{n-1}$.

Next, we show that $\varphi_{1o_{n-1}}(R^{22}) = 0$ and $\varphi_{no_1}(R^{22}) = 0$. Suppose by way of contradiction that $\varphi_{1o_{n-1}}(R^{22}) > 0$, which implies that $\varphi_{1o_1}(R^{22}) < \frac{2n-1}{n(n-1)}$. By the same argument as in Profile 1-1, we can construct $M \in \mathcal{M}$ such that M stochastically Pareto dominates $\varphi(R^{22})$ at R^{22}, in violation of *sd-efficiency.* Therefore, $\varphi_{1o_{n-1}}(R^{22}) = 0$, which implies that $\varphi_{1o_1}(R^{22}) = \frac{2n-1}{n(n-1)}$. Similarly, we can show that by *sd-efficiency,* $\varphi_{no_1}(R^{22}) = 0$, which implies that $\varphi_{no_n}(R^{22}) = \frac{2n-3}{n(n-1)}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{22}) = \begin{pmatrix} \frac{2n-1}{n(n-1)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n-2}{n(n-1)} \\ \frac{n^2-3n+1}{n(n-1)(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{n^2-4n+5}{n(n-1)(n-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{n^2-3n+1}{n(n-1)(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{n^2-4n+5}{n(n-1)(n-2)} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{2n-3}{n(n-1)} \end{pmatrix}$$

Profile 2-3, R^{23}: For $i = 1, 2$, $R_i^{22} = R_i'$, for $i = 3, \ldots, n-1$, $R_i^{22} = R_i''$, and $R_n^{22} = R_n'''$. By *sd-strategyproofness,*

$$\varphi_2(R^{23}) R_2'^{sd} \varphi_2(R^{22}) \quad \text{and} \quad \varphi_2(R^{22}) R_2''^{sd} \varphi_2(R^{23}),$$
$$\varphi_n(R^{23}) R_n'''^{sd} \varphi_n(R^{12}) \quad \text{and} \quad \varphi_n(R^{12}) R_n''^{sd} \varphi_n(R^{23}).$$

Invoking our conclusion for Profiles 2-2 and 1-2, for $i = 2, n$, $\varphi_{io_2}(R^{23}) = \varphi_{io_3}(R^{23}) = \cdots = \varphi_{io_{n-2}}(R^{23}) = \frac{1}{n}$. Also, $\varphi_{2o_n}(R^{23}) = \frac{n^2-4n+5}{n(n-1)(n-2)}$ and $\varphi_{no_{n-1}}(R^{23}) = \frac{1}{n-2}$. By *sd-efficiency,* $\varphi_{2o_{n-1}}(R^{23}) = 0$ and $\varphi_{no_1}(R^{23}) = 0$, which imply that $\varphi_{2o_1}(R^{23}) = \frac{2n^2-5n+1}{n(n-1)(n-2)}$ and $\varphi_{no_n}(R^{23}) = \frac{2n-6}{n(n-2)}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{23}) = \begin{pmatrix} \frac{2n^2-5n+1}{n(n-1)(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^2-4n+5}{n(n-1)(n-2)} \\ \frac{2n^2-5n+1}{n(n-1)(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^2-4n+5}{n(n-1)(n-2)} \\ \frac{(n-4)(n-3)n-2}{n(n-1)(n-2)(n-3)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{(n^2-5n+8)}{n(n-1)(n-3)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{(n-4)(n-3)n-2}{n(n-1)(n-2)(n-3)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{(n^2-5n+8)}{n(n-1)(n-3)} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{2n-6}{n(n-2)} \end{pmatrix}$$

We repeat this process until we obtain profile 2-k.

Profile 2-k, R^{2k}: For $i = 1, \ldots, k-1$, $R^{2k}_i = R'_i$, for $i = k, \ldots, n-1$, $R^{2k}_i = R''_i$, and $R^{2k}_n = R'''_n$. By *sd-strategyproofness,*

$$\varphi_{k-1}(R^{2k}) R'^{sd}_{k-1} \varphi_{k-1}(R^{2(k-1)}) \quad \text{and} \quad \varphi_{k-1}(R^{2(k-1)}) R''^{sd}_{k-1} \varphi_{k-1}(R^{2k}),$$

$$\varphi_n(R^{2k}) R'''^{sd}_n \varphi_n(R^{1(k-1)}) \quad \text{and} \quad \varphi_n(R^{1(k-1)}) R''^{sd}_n \varphi_n(R^{2k}).$$

Invoking our conclusion for Profile 2-(k-1) and Profile 1-(k-1), for $i = k-1, n$, $\varphi_{io_2}(R^{2k}) = \varphi_{io_3}(R^{2k}) = \cdots = \varphi_{io_{n-2}}(R^{2k}) = \frac{1}{n}$. Also, $\varphi_{(k-1)o_n}(R^{2k}) = \frac{n^2+n-8}{(n-1)n(n+2)}$ and $\varphi_{no_{n-1}}(R^{2k}) = \frac{2}{n+2}$. By *sd-efficiency,* $\varphi_{(k-1)o_{n-1}}(R^{2k}) = 0$ and $\varphi_{no_1}(R^{2k}) = 0$, which imply that $\varphi_{(k-1)o_1}(R^{2k}) = \frac{2(n^2+n+1)}{(n-1)n(n+2)}$ and $\varphi_{no_n}(R^{2k}) = \frac{n+6}{n(n+2)}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{2k}) = \begin{pmatrix} \frac{2n^2-(\frac{2n}{2}-1)n+1}{n(n-1)(n-(\frac{n}{2}-1))} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^2-(\frac{n}{2}+1)n+(\frac{3n}{2}-4)}{n(n-1)(n-(\frac{n}{2}-1))} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2n^2-(\frac{2n}{2}-1)n+1}{n(n-1)(n-(\frac{n}{2}-1))} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^2-(\frac{n}{2}+1)n+(\frac{3n}{2}-4)}{n(n-1)(n-(\frac{n}{2}-1))} \\ \frac{(n-2(\frac{n}{2}-1))(n-\frac{n}{2})n-(\frac{n}{2}-1)}{n(n-1)(n-(\frac{n}{2}-1))(n-\frac{n}{2})} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n+2} & \frac{n^2-(\frac{n}{2}+2)n+(\frac{3n}{2}-1)}{n(n-1)(n-\frac{n}{2})} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{(n-2(\frac{n}{2}-1))(n-\frac{n}{2})n-(\frac{n}{2}-1)}{n(n-1)(n-(\frac{n}{2}-1))(n-\frac{n}{2})} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n+2} & \frac{n^2-(\frac{n}{2}+2)n+(\frac{3n}{2}-1)}{n(n-1)(n-\frac{n}{2})} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n+2} & \frac{2n-3(\frac{n}{2}-1)}{n(n-(\frac{n}{2}-1))} \end{pmatrix}$$

$$\underset{\textbf{Profile 3-0}}{\begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}} \Rightarrow \underset{\textbf{Profile 3-1}}{\begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \Rightarrow \underset{\textbf{Profile 3-2}}{\begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \Rightarrow \cdots \Rightarrow \underset{\textbf{Profile 3-(k-1)}}{\begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\ \vdots\\ 1\end{matrix}}\right\} \frac{n}{2}+1 \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} \frac{n}{2}-1 \end{matrix}}$$

Profile 3-0, R^{30}: For each $i \in N$, let $R^{30}_i = R'_i$. By *equal treatment of equals,* for each $i \in N$ and $k \in O$, $\varphi_{ik}(R^{20}) = \frac{1}{n}$.

Profile 3-1, R^{31}: For $i = 1, \ldots, n-1$, $R^{31}_i = R'_i$ and $R^{31}_n = R''_n$. By *sd-strategyproofness,*

$$\varphi_n(R^{31}) R''^{sd}_n \varphi_n(R^{30}) \quad \text{and} \quad \varphi_n(R^{30}) R'^{sd}_n \varphi_n(R^{31}).$$

Invoking our conclusion for Profile 3-0, $\varphi_{no_2}(R^{31}) = \cdots = \varphi_{no_{n-2}}(R^{31}) = \varphi_{no_n}(R^{31}) = \frac{1}{n}$. By *sd-efficiency,* $\varphi_{no_1}(R^{31}) = 0$, which implies that $\varphi_{no_{n-1}}(R^{31}) = \frac{2}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{31}) = \begin{pmatrix} \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-2}{n(n-1)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-2}{n(n-1)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

Profile 3-2, R^{32}: For $i = 1, \ldots, n-2$, $R_i^{32} = R_i'$ and for $i = n-1, n$, $R_i^{31} = R_i''$. By *sd-strategyproofness,*

$$\varphi_{n-1}(R^{32}) R_{n-1}''^{sd} \varphi_{n-1}(R^{31}) \quad \text{and} \quad \varphi_{n-1}(R^{31}) R_{n-1}'^{sd} \varphi_{n-1}(R^{32}).$$

Invoking our conclusion for Profile 3-1, $\varphi_{(n-1)o_2}(R^{32}) = \cdots = \varphi_{(n-1)o_{n-2}}(R^{32}) = \varphi_{(n-1)o_n}(R^{32}) = \frac{1}{n}$. By *sd-efficiency,* $\varphi_{(n-1)o_1}(R^{32}) = 0$, which implies that $\varphi_{(n-1)o_{n-1}}(R^{31}) = \frac{2}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{32}) = \begin{pmatrix} \frac{1}{n-2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-4}{n(n-2)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-4}{n(n-2)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

We repeat this process until we obtain Profile 3-(k-1).

Profile 3-(k-1), $R^{3(k-1)}$: For $i = 1, \ldots, k+1$, $R_i^{3(k-1)} = R_i'$ and for $i = k+2, \ldots, n$, $R_i^{3(k-1)} = R_i''$. By *sd-strategyproofness,*

$$\varphi_{n-(k-2)}(R^{3(k-1)}) R_{n-(k-2)}''^{sd} \varphi_{n-(k-2)}(R^{3(k-2)}) \quad \text{and}$$

$$\varphi_{n-(k-2)}(R^{3(k-2)}) R_{n-(k-2)}'^{sd} \varphi_{n-(k-2)}(R^{3(k-1)}).$$

Invoking our conclusion for Profile 3-(k-2), $\varphi_{(n-(k-2))o_2}(R^{3(k-1)}) = \cdots = \varphi_{(n-(k-2))o_{n-2}}(R^{3(k-1)}) = \varphi_{(n-(k-2))o_n}(R^{3(k-1)}) = \frac{1}{n}$. By *sd-efficiency,* $\varphi_{(n-(k-2))o_1}(R^{3(k-1)}) = 0$, which implies that $\varphi_{(n-(k-2))o_{n-1}}(R^{3(k-1)}) = \frac{2}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{3(k-1)}) = \begin{pmatrix} \frac{2}{n+2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{4}{n(n+2)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2}{n+2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{4}{n(n+2)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

$$\begin{array}{c}\underline{\textbf{Profile 3-1}}\\ \begin{bmatrix}1&0&0\\ \vdots&\vdots&\vdots\\ 1&0&0\\1&0&0\\1&0&0\\1&0&0\\0&1&0\end{bmatrix}\end{array} \Rightarrow \begin{array}{c}\underline{\textbf{Profile 4-1}}\\ \begin{bmatrix}1&0&0\\ \vdots&\vdots&\vdots\\ 1&0&0\\1&0&0\\1&0&0\\1&0&0\\0&0&1\end{bmatrix}\end{array} \Rightarrow \begin{array}{c}\underline{\textbf{Profile 4-2}}\\ \begin{bmatrix}1&0&0\\ \vdots&\vdots&\vdots\\ 1&0&0\\1&0&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}\end{array} \Rightarrow \cdots \Rightarrow \begin{array}{c}\underline{\textbf{Profile 4-(k-1)}}\\ \begin{bmatrix}1&0&0\\ \vdots&\vdots&\vdots\\ 1&0&0\\0&1&0\\ \vdots&\vdots&\vdots\\ 0&1&0\\0&0&1\end{bmatrix}\begin{array}{l}\left.\begin{array}{c}\\ \\ \\ \end{array}\right\}\frac{n}{2}+1\\ \left.\begin{array}{c}\\ \\ \\ \end{array}\right\}\frac{n}{2}-2\\ \}\,1\end{array}\end{array}$$

Profile 4-1, R^{41}: For $i = 1, \ldots, n-1$, $R_i^{41} = R_i'$ and $R_n^{41} = R_n'''$. By *sd-strategyproofness,*

$$\varphi_n(R^{41}) R_n'''^{sd} \varphi_n(R^{31}) \quad \text{and} \quad \varphi_n(R^{31}) R_n''^{sd} \varphi_n(R^{41}).$$

Invoking our conclusion for Profile 3-1, $\varphi_{no_2}(R^{31}) = \ldots = \varphi_{no_{n-2}}(R^{31}) = \varphi_{no_n}(R^{31}) = \frac{1}{n}$. By *sd-efficiency,* $\varphi_{no_1}(R^{41}) = 0$, which implies that $\varphi_{no_{n-1}}(R^{31}) = \frac{2}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{41}) = \begin{pmatrix} \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-2}{n(n-1)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-2}{n(n-1)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

Profile 4-2, R^{42}: For $i = 1, \ldots, n-2$, $R_i^{42} = R_i'$, $R_{n-1}^{42} = R_{n-1}''$, and $R_n^{42} = R_n'''$. By *sd-strategyproofness,*

$$\varphi_{n-1}(R^{42}) R_{n-1}''^{sd} \varphi_{n-1}(R^{41}) \quad \text{and} \quad \varphi_{n-1}(R^{41}) R_{n-1}'^{sd} \varphi_{n-1}(R^{42}),$$

$$\varphi_n(R^{42}) R_n'''^{sd} \varphi_n(R^{32}) \quad \text{and} \quad \varphi_n(R^{32}) R_n''^{sd} \varphi_n(R^{42}).$$

Invoking our conclusion from Profiles 4-1 and 3-2, for each $i = n-1, n$, $\varphi_{io_2}(R^{42}) = \cdots = \varphi_{io_{n-2}}(R^{42}) = \varphi_{(n-1)o_n}(R^{42}) = \frac{1}{n}$ and $\varphi_{no_{n-1}}(R^{42}) = \frac{2}{n}$. By *sd-efficiency,* $\varphi_{(n-1)o_1}(R^{42}) = \varphi_{no_1}(R^{42}) = 0$, which implies that $\varphi_{(n-1)o_{n-1}}(R^{42}) = 0$ and $\varphi_{no_n}(R^{42}) = \frac{1}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{42}) = \begin{pmatrix} \frac{1}{n-2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-4}{n(n-2)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-4}{n(n-2)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

Repeat this process until we have Profile 4-(k-1).

Profile 4-(k-1), $R^{4(k-1)}$: For $i = 1, \ldots, k+1$, $R_i^{4(k-1)} = R_i'$, for $i = k+2, \ldots, n-1$, $R_i^{4(k-1)} = R_i''$, and $R_n^{4(k-1)} = R_n'''$. By *sd-strategyproofness,*

$$\varphi_{n-(k-2)}(R^{4(k-1)}) R_{n-(k-2)}''^{sd} \varphi_{n-(k-2)}(R^{4(k-2)}) \quad \text{and}$$

$$\varphi_{n-(k-2)}(R^{4(k-2)}) R_{n-(k-2)}'^{sd} \varphi_{n-(k-2)}(R^{4(k-1)}),$$

$$\varphi_n(R^{4(k-1)}) R_n'''^{sd} \varphi_n(R^{3(k-1)}) \quad \text{and} \quad \varphi_n(R^{3(k-1)}) R_n''^{sd} \varphi_n(R^{4(k-1)}).$$

Invoking our conclusion from Profiles 4-(k-2) and 3-(k-1), for $i = n-(k-2), n$, $\varphi_{io_2}(R^{4(k-1)}) = \cdots = \varphi_{io_{n-2}}(R^{4(k-1)}) = \varphi_{(n-(k-2))o_n}(R^{4(k-1)}) = \frac{1}{n}$ and $\varphi_{no_{n-1}}(R^{4(k-1)}) = \frac{2}{n}$. By *sd-efficiency,* $\varphi_{(n-(k-2))o_1}(R^{4(k-1)}) = \varphi_{no_1}(R^{4(k-1)}) = 0$, which implies that $\varphi_{(n-(k-2))o_{n-1}}(R^{4(k-1)})$

$= 0$ and $\varphi_{no_n}(R^{4(k-1)}) = \frac{1}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{4(k-1)}) = \begin{pmatrix} \frac{2}{n+1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{4}{n(n+2)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2}{n+1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{4}{n(n+2)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

Profile 5-0, R^{50}: For $i = 1, \ldots, k$, $R_i^{50} = R_i'$, for $i = k+1, \ldots, n-1$, $R_i^{50} = R_i''$, and $R_n^{50} = R_n'''$. By *sd-strategy-proofness,*

$$\varphi_k(R^{50}) R_k'^{sd} \varphi(R^{2k}) \quad \text{and} \quad \varphi(R^{2k}) R_k''^{sd} \varphi_k(R^{50}),$$

$$\varphi_{k+1}(R^{50}) R_{k+1}''^{sd} \varphi(R^{4(k-1)}) \quad \text{and} \quad \varphi(R^{4(k-1)}) R_{k+1}'^{sd} \varphi_{k+1}(R^{50}),$$

$$\varphi_n(R^{50}) R_n'''^{sd} \varphi_n(R^{1k}) \quad \text{and} \quad \varphi_n(R^{1k}) R_n''^{sd} \varphi_n(R^{50}).$$

Invoking our conclusion for Profiles 2-k, 4-(k-1), and 1-k,

$$\varphi_{ko_n}(R^{50}) = \frac{n^2 - (\frac{n}{2} + 2)n + (\frac{3n}{2} - 1)}{n(n-1)(n - \frac{n}{2})} = \frac{n^2 - n - 2}{n^2(n-1)},$$

$$\varphi_{(k+1)o_n}(R^{50}) = \frac{1}{n}, \qquad \text{and}$$

$$\varphi_{no_{n-1}}(R^{50}) = \frac{2}{n}.$$

By *sd-efficiency,* $\varphi_{no_1}(R^{50}) = 0$, which implies that $\varphi_{no_n}(R^{50}) = \frac{1}{n}$. Now by adding up the assignment of o_n to all agents, we have

$$\underbrace{\frac{n^2 - n - 2}{n^2(n-1)}}_{\text{assignment for } i \in \{1, \ldots, \frac{n}{2}\}} \cdot \frac{n}{2} + \underbrace{\frac{1}{n}}_{i \in \{\frac{n}{2}+1, \ldots, n-1\}} \cdot (\frac{n}{2} - 1) + \frac{1}{n} < 1$$

which contracts to $\varphi(R^{50}) \in \mathcal{M}$. ■

Our impossibility result strengthens the results of both Bogomolnaia and Moulin (2001) and Kasajima (2013) since our result holds on a restricted domain of preferences than the ones they consider. Also, it is easy to check the independence of axioms in our theorem by using the same rules as in Kasajima (2013, p.213).

Remark 1: From our proof, it is clear that even if we further restrict our domain of problems with more than three agents by assuming that all agents have the same preferences except three least preferred objects, our impossibility result still holds.

4. Discussion

We discuss a variation of our problem. Suppose that each agent has a complete and transitive (not necessarily strict) binary relation over O. For each agent $i \in N$, her preference relation is denoted by R_i, the strict relation by P_i, and the indifference relation by I_i. Let $\mathcal{R}^N$ be a domain of preference profiles. We show that the impossibility result in Kasajima (2013) carries over to this domain.

Stochastic dominance envy-freeness (simply, *sd-envyfreeness*) requires that each agent's assignment stochastically dominates any other agent's assignment. This axiom is stronger than *equal treatment of equals*.

sd-envyfreenesss: For each $R \in \mathcal{R}^N$ and each $i, j \in N$, $\varphi_i(R) R_i^{sd} \varphi_j(R)$.

Next is a weakening of *sd-strategyproofness,* which requires that a probabilistic assignment selected by the rule should not be stochastically dominated by her assignment obtained under lying.

weak sd-strategyproofness: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, it is not the case $\varphi_i(R'_i, R_{-i}) P_i^{sd} \varphi_i(R)$.

We show that even if indifference is allowed, a similar result holds. When $n \geq 4$, there is no rule satisfying three requirements of *sd-envyfreeness, sd-efficiency,* and *weak sd-strategyproofness.*

Proof. Let $N = \{1, \ldots, n\}$ with $n \geq 4$ and $O = \{o_1, \ldots, o_n\}$. Let O be ordered in such a way that $o_1 \prec o_2 \prec \cdots \prec o_n$. As before, we assume that each agent has a common peak $o_t \in O \setminus \{o_1, o_n\}$. Consider the following two preference profiles.

Profile 1, R^1: $o_t P_1 o_{t-1} \cdots o_{n-2} P_1 o_{n-1} I_1 o_1 P_1 o_n$, for $i = 2, \ldots, n-1$, $o_t P_i o_{t-1} \cdots o_{n-2} P_i o_{n-1} P_i o_1 P_i o_n$, and $o_t P_n o_{t-1} \cdots o_{n-2} P_n o_{n-1} P_n o_n P_n o_1$.

Profile 2, R^2: $o_t P'_1 o_{t-1} \cdots o_{n-2} P'_1 o_{n-1} P'_1 o_1 P'_1 o_n$, for $i = 2, \ldots, n-1$, $o_t P_i o_{t-1} \cdots o_{n-2} P_i o_{n-1} P_i o_1 P_i o_n$, and $o_t P_n o_{t-1} \cdots o_{n-2} P_n o_{n-1} P_n o_n P_n o_1$.

Note that all profiles are identical except three least preferred objects. Suppose by a way of contradiction that there is a rule φ satisfying the three requirements. For Profile 1, by *sd-envyfreeness,* for all $i, j \in N$,

$$\begin{gathered}\varphi_{io_t}(R^1) \geq \varphi_{jo_t}(R^1) \\ \vdots \\ \varphi_{io_t}(R^1) + \cdots + \varphi_{io_{n-2}}(R^1) \geq \varphi_{jo_t}(R^1) + \cdots + \varphi_{jo_{n-2}}(R^1)\end{gathered}$$

which together imply that $\varphi_{io_t}(R^1) = \cdots = \varphi_{io_{n-2}}(R^1) = \frac{1}{n}$.

And for each $i, j \in N \setminus \{1\}$,

$$\varphi_{io_t}(R^1) + \cdots + \varphi_{io_{n-2}}(R^1) + \varphi_{io_{n-1}}(R^1) \geq \varphi_{jo_t}(R^1) + \cdots + \varphi_{jo_{n-2}}(R^1) + \varphi_{jo_{n-1}}(R^1),$$

which imply that $\varphi_{2o_{n-1}}(R^1) = \varphi_{3o_{n-1}}(R^1) = \cdots = \varphi_{no_{n-1}}(R^1)$. Finally, for each $i, j \in N \setminus \{n\}$,

$$\varphi_{io_{n-1}}(R^1) + \varphi_{io_1}(R^1) \geq \varphi_{jo_{n-1}}(R^1) + \varphi_{jo_1}(R^1)$$

which imply that $\varphi_{1o_{n-1}}(R^1)+\varphi_{1o_1}(R^1) = \cdots = \varphi_{n-1o_{n-1}}(R^1)+\varphi_{n-1o_1}(R^1)$. By *sd-efficiency*, $\varphi_{no_1}(R^1) = 0$ and $\varphi_{1o_{n-1}}(R^1) = 0$ so that $\varphi_{1o_n}(R^1) = \frac{n^2-3n+3}{n(n-1)^2}$.

Altogether,

$$\varphi(R^1) = \begin{pmatrix} \frac{2n-3}{(n-1)^2} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^2-3n+3}{n(n-1)^2} \\ \frac{n-2}{(n-1)^2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{n^2-3n+3}{n(n-1)^2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{n-2}{(n-1)^2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{n^2-3n+3}{n(n-1)^2} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{2n-3}{n(n-1)} \end{pmatrix}$$

For Profile 2, by *sd-envyfreeness,* for all i, $j \in N$,

$$\begin{gathered} \varphi_{io_t}(R^2) \geq \varphi_{jo_t}(R^2) \\ \vdots \\ \varphi_{io_t}(R^2) + \cdots + \varphi_{io_{n-1}}(R^2) \geq \varphi_{jo_t}(R^2) + \cdots + \varphi_{jo_{n-1}}(R^2) \end{gathered}$$

which together imply that $\varphi_{io_t}(R^2) = \cdots = \varphi_{io_{n-1}}(R^2) = \frac{1}{n}$.

And for each i, $j \neq n$,

$$\varphi_{io_t}(R^2) + \cdots + \varphi_{io_1}(R^2) \geq \varphi_{jo_t}(R^2) + \cdots + \varphi_{jo_1}(R^2),$$

which imply that $\varphi_{1o_1}(R^2) = \cdots = \varphi_{n-1o_1}(R^2)$. By *sd-efficiency*, $\varphi_{no_1}(R^2) = 0$.

Therefore,

$$\varphi(R^2) = \begin{pmatrix} \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{n-2}{n(n-1)} \\ \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{n-2}{n(n-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{n-2}{n(n-1)} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{2}{n} \end{pmatrix}$$

However, since $\frac{1}{n-1} + \frac{1}{n} > \frac{2n-3}{(n-1)^2}$, $\varphi_1(R_1', R_{-1}^1) P_1^{sd} \varphi_1(R^1)$, a contradiction. ■

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Appendix:

We give a formal for the problem with an odd number of agents. Let $n \geq 5$ be an odd number. As in the case of an even number of agents, we assume that all agents have a common peak o_t such that $t \in \{2, 3, \ldots n-1\}$. We use the following three preference profiles.

- R'_i: $o_t \ R_i \ \ldots \ R_i \ o_1 \ R_i \ o_{n-1} \ R_i \ o_n$,
- R''_i: $o_t \ R_i \ \ldots \ R_i \ o_{n-1} \ R_i \ o_1 \ R_i \ o_n$,
- R'''_i: $o_t \ R_i \ \ldots \ R_i \ o_{n-1} \ R_i \ o_n \ R_i \ o_1$.

When o_{n-1} is a common peak, we choose o_1, o_2, and o_n as three least preferred alternatives.

Suppose, by way of contradiction, that there is a rule φ that satisfies the three requirements. Since we obtain a contradiction when the number of agents for (R', R'', R''') is $(\frac{n-1}{2}, \frac{n-3}{2}, 2)$, we take the following steps of profiles to figure out the allocation for this profile. Let $k = \frac{n-1}{2}$.

$$
\begin{array}{ccccccc}
\textbf{Profile 1-0} & & \textbf{Profile 1-1} & & \textbf{Profile 1-2} & & \textbf{Profile 1-k} \\
\begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}
& \Rightarrow &
\begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
& \Rightarrow &
\begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}
& \Rightarrow \cdots \Rightarrow &
\begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \end{bmatrix}
\begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \frac{n+1}{2} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \frac{n-1}{2} \end{matrix}
\end{array}
$$

Profile 1-0, R^{10}: For each $i \in N, R_i^{10} = R_i'$. By *equal treatment of equals,* for each $i \in N$ and each $k \in O$, $\varphi_{ik}(R^{10}) = \frac{1}{n}$.

Profile 1-1, R^{11}: For each $i = 1, \ldots, n-1$, $R_i^{11} = R_i'$ and $R_n^{11} = R_n''$. By *sd-strategyproofness,*

$$\varphi_n(R^{11}) R_n''^{sd} \varphi_n(R^{10}) \quad \text{and} \quad \varphi_n(R^{10}) R_n'^{sd} \varphi_n(R^{11}).$$

Invoking our conclusion for Profile 1-0, $\varphi_{no_2}(R^{11}) = \varphi_{no_3}(R^{11}) = \cdots = \varphi_{no_{n-2}}(R^{11}) = \varphi_{no_n}(R) = \frac{1}{n}$. By *sd-efficiency,* $\varphi_{no_1}(R^{11}) = 0$, which implies that $\varphi_{no_{n-1}}(R^{11}) = \frac{2}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{11}) = \begin{pmatrix} \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-2}{n(n-1)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-2}{n(n-1)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

Profile 1-2, R^{12}: For $i = 1, \ldots, n-2$, $R_i^{12} = R_i'$ and for $i = n-1, n$, $R_i^{12} = R_i''$. By *sd-strategyproofness,*

$$\varphi_{n-1}(R^{12}) R_{n-1}''^{sd} \varphi_{n-1}(R^{11}) \quad \text{and} \quad \varphi_{n-1}(R^{11}) R_{n-1}'^{sd} \varphi_{n-1}(R^{12}).$$

Invoking our conclusion for Profile 1-1, $\varphi_{(n-1)o_2}(R^{12}) = \varphi_{(n-1)o_3}(R^{12}) = \cdots = \varphi_{(n-1)o_{n-2}}(R^{12}) = \varphi_{(n-1)o_n}(R^{12}) = \frac{1}{n}$. By *sd-efficiency,* $\varphi_{(n-1)o_1}(R^{12}) = 0$, which implies that $\varphi_{(n-1)o_{n-1}}(R^{12}) = \frac{2}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R) = \begin{pmatrix} \frac{1}{n-2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-4}{n(n-2)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-4}{n(n-2)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

Repeat this process until we obtain profile 1-k.

Profile 1-k, R^{1k}: For $i = 1, \ldots, k+1$, $R_i^{1k} = R_i'$ and for $i = k+2, \ldots, n$, $R_i^{1k} = R_i''$. By *sd-strategyproofness,*

$$\varphi_{n-(k-2)}(R^{1k}) R_{n-(k-2)}''^{sd} \varphi_{n-(k-2)}(R^{1(k-1)}) \quad \text{and}$$

$$\varphi_{n-(k-2)}(R^{1(k-1)})R'^{sd}_{n-(k-2)}\varphi_{n-(k-2)}(R^{1k})$$

Invoking our conclusion for Profile 1-(k-1), $\varphi_{(n-(k-2))o_2}(R^{1k}) = \cdots = \varphi_{(n-(k-2))o_{n-2}}(R^{1k}) = \varphi_{(n-(k-2))o_n}(R^{1k}) = \frac{1}{n}$. By *sd-efficiency,* $\varphi_{(n-(k-2))o_1}(R^{1k}) = 0$, which implies that $\varphi_{(n-(k-2))o_{n-1}}(R^{12}) = \frac{2}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{1k}) = \begin{pmatrix} \frac{2}{n+1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n(n+1)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2}{n+1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n(n+1)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

$$\begin{array}{c}\textbf{Profile 1-1}\\ \begin{bmatrix}1&0&0\\ \vdots&\vdots&\vdots\\ 1&0&0\\ 1&0&0\\ 1&0&0\\ 1&0&0\\ 0&1&0\end{bmatrix}\end{array} \Rightarrow \begin{array}{c}\textbf{Profile 2-1}\\ \begin{bmatrix}1&0&0\\ \vdots&\vdots&\vdots\\ 1&0&0\\ 1&0&0\\ 1&0&0\\ 1&0&0\\ 0&0&1\end{bmatrix}\end{array} \Rightarrow \begin{array}{c}\textbf{Profile 2-2}\\ \begin{bmatrix}1&0&0\\ \vdots&\vdots&\vdots\\ 1&0&0\\ 1&0&0\\ 1&0&0\\ 0&1&0\\ 0&0&1\end{bmatrix}\end{array} \Rightarrow \cdots \Rightarrow \begin{array}{c}\textbf{Profile 2-k}\\ \begin{bmatrix}1&0&0\\ \vdots&\vdots&\vdots\\ 1&0&0\\ 0&1&0\\ \vdots&\vdots&\vdots\\ 0&1&0\\ 0&0&1\end{bmatrix}\begin{matrix}\left.\vphantom{\begin{matrix}1\\ \vdots\\ 1\end{matrix}}\right\}\frac{n+1}{2}\\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\}\frac{n-3}{2}\\ \}\,1\end{matrix}\end{array}$$

Profile 2-1, R^{21}: For $i = 1, \ldots, n-1$, $R^{21}_i = R'_i$ and $R^{21}_n = R'''_n$. By *sd-strategyproofness,*

$$\varphi_n(R)R'''^{sd}_n\varphi_n(R^{11}) \quad \text{and} \quad \varphi_n(R^{11})R''^{sd}_n\varphi_n(R).$$

Invoking our conclusion for Profile 1-1, $\varphi_{no_2}(R^{21}) = \cdots = \varphi_{no_{n-2}}(R^{21}) = \frac{1}{n}$ and $\varphi_{no_{n-1}}(R^{21}) = \frac{2}{n}$. By *sd-efficiency,* $\varphi_{no_1}(R^{21}) = 0$, which implies that $\varphi_{no_n}(R^{21}) = \frac{1}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{21}) = \begin{pmatrix} \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-2}{n(n-1)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-2}{n(n-1)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

Profile 2-2, R^{22}: For $i = 1, \ldots, n-2$, $R^{22}_i = R'_i$, $R^{22}_{n-1} = R''_{n-1}$, and $R^{22}_n = R'''_n$. By *sd-strategyproofness,*

$$\varphi_{n-1}(R^{22})R''^{sd}_{n-1}\varphi_{n-1}(R^{21}) \quad \text{and} \quad \varphi_{n-1}(R^{21})R'^{sd}_{n-1}\varphi_{n-1}(R^{22}),$$

$$\varphi_n(R^{22})R'''^{sd}_n\varphi_n(R^{12}) \quad \text{and} \quad \varphi_n(R^{12})R''^{sd}_n\varphi_n(R^{22}).$$

Invoking our conclusion for Profiles 1-2 and 2-2, for $i = n-1, n$, $\varphi_{io_2}(R^{22}) = \cdots = \varphi_{(n-1)o_{n-2}}(R^{22}) = \varphi_{(n-1)o_n}(R^{22}) = \frac{1}{n}$, and $\varphi_{no_{n-1}}(R^{22}) = \frac{2}{n}$. By *sd-efficiency,* $\varphi_{(n-1)o_1}(R^{22}) =$

0 and $\varphi_{no_1}(R^{22}) = 0$, which imply that $\varphi_{(n-1)o_{n-1}}(R^{22}) = \frac{2}{n}$ and $\varphi_{no_n}(R^{22}) = \frac{1}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R) = \begin{pmatrix} \frac{1}{n-2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-4}{n(n-2)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-4}{n(n-2)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

Repeat this process until we obtain profile 2-k

Profile 2-k, R^{2k}: For $i = 1, \ldots, k+1$, $R_i^{2k} = R_i'$, for $i = k+2, \ldots, n-1$, $R_i^{2k} = R_i''$, and $R_n^{2k} = R_n'''$. By *sd-strategyproofness,*

$$\varphi_{n-(k-2)}(R^{2k}) R''^{sd}_{n-(k-2)} \varphi_{n-(k-2)}(R^{2(k-1)}) \quad \text{and}$$

$$\varphi_{n-(k-2)}(R^{2(k-1)} R'^{sd}_{n-(k-2)} \varphi_{n-(k-2)}(R^{2k})$$

$$\varphi_n(R^{2k}) R'''^{sd}_n \varphi_n(R^{1k}) \quad \text{and} \quad \varphi_n(R^{1k}) R''^{sd}_n \varphi_n(R^{2k})$$

Invoking our conclusion for Profiles 2-(k-1) and 1-k, for $i = n-(k-2), n$, $\varphi_{io_2}(R^{2k}) = \cdots = \varphi_{io_{n-2}}(R^{2k}) = \varphi_{(n-(k-2))o_n}(R^{2k}) = \frac{1}{n}$ and $\varphi_{no_{n-1}}(R^{2k}) = \frac{2}{n}$. By *sd-efficiency,* $\varphi_{no_1}(R^{2k}) = \varphi_{(n-(k-2))o_1}(R^{2k}) = 0$, which imply that $\varphi_{(n-(k-2))o_{n-1}}(R^{2k}) = \frac{2}{n}$ and $\varphi_{no_n}(R^{2k}) = \frac{1}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{2k}) = \begin{pmatrix} \frac{2}{n+1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n(n+1)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2}{n+1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n(n+1)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

$$\underset{\textbf{Profile 2-2}}{\begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \Rightarrow \underset{\textbf{Profile 3-2}}{\begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}} \Rightarrow \underset{\textbf{Profile 3-3}}{\begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}} \Rightarrow \underset{\textbf{Profile 3-4}}{\begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}} \Rightarrow \cdots \Rightarrow \underset{\textbf{Profile 3-(k-1)}}{\begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} \frac{n+1}{2} \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} \frac{n-5}{2} \\ \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\} 2 \end{matrix}$$

Profile 3-2, R^{32}: For $i = 1, \ldots, n-2$, $R_i^{32} = R_i'$, and for $i = n-1, n$, $R_i^{22} = R_n'''$. By *sd-strategyproofness,*

$$\varphi_{n-1}(R^{32}) R'''^{sd}_{n-1} \varphi_{n-1}(R^{22}) \quad \text{and} \quad \varphi_{n-1}(R^{22}) R''^{sd}_{n-1} \varphi_{n-1}(R^{32}).$$

Invoking our conclusion for Profile 2-2, $\varphi_{(n-1)o_2}(R^{32}) = \varphi_{(n-1)o_3}(R^{32}) = \cdots = \varphi_{(n-2)o_n}(R^{32}) = \frac{1}{n}$ and $\varphi_{(n-1)o_{n-1}}(R^{32}) = \frac{2}{n}$. By *sd-efficiency*, $\varphi_{(n-1)o_1}(R^{32}) = 0$, which implies that $\varphi_{(n-1)o_n}(R^{32}) = \frac{1}{n}$. Finally, by *equal treatment of equals*,

$$\varphi(R^{32}) = \begin{pmatrix} \frac{1}{n-2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-4}{n(n-2)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-4}{n(n-2)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

Profile 3-3, R^{33}: For $i = 1, \ldots, n-3$, $R_i^{33} = R_i'$, $R_{n-2}^{33} = R_{n-2}''$ and for $i = n-1, n$, $R_i^{33} = R_i'''$. By *sd-strategyproofness*,

$$\varphi_{n-2}(R^{33}) R''^{sd} \varphi_{n-2}(R^{32}) \quad \text{and} \quad \varphi_{n-2}(R^{32}) R'^{sd} \varphi_{n-2}(R^{33})$$

$$\varphi_{n-1}(R^{33}) R'''^{sd} \varphi_{n-1}(R^{23}) \quad \text{and} \quad \varphi_{n-1}(R^{23}) R_{n-1}''^{sd} \varphi_{n-1}(R^{33})$$

Invoking our conclusion for Profiles 3-2 and 2-3, for $i = n-2, n-1$, $\varphi_{io_2}(R^{33}) = \cdots = \varphi_{io_{n-2}} = \varphi_{(n-2)o_n}(R^{33}) = \frac{1}{n}$ and $\varphi_{(n-1)o_{n-1}}(R^{33}) = \frac{2}{n}$. By *sd-efficiency*, $\varphi_{(n-2)o_1}(R^{33}) = \varphi_{(n-1)o_1}(R^{33}) = 0$, which imply that $\varphi_{(n-2)o_{n-1}}(R^{33}) = \frac{2}{n}$ and $\varphi_{(n-1)o_n}(R^{33}) = \frac{1}{n}$. Finally, by *equal treatment of equals*,

$$\varphi(R^{33}) = \begin{pmatrix} \frac{1}{n-3} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-6}{n(n-3)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-3} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{n-6}{n(n-3)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

We repeat this process until we obtain profile 3-k.

Profile 3-k, R^{3k}: For $i = 1, \ldots, k+1$, $R_i^{3k} = R_i'$, for $i = k+2, \ldots, n-2$, $R_i^{3k} = R_i''$, and for $i = n-1, n$, $R_i^{3k} = R_i'''$. By *sd-strategyproofness,*

$$\varphi_{n-(k-2)}(R^{3k}) R_{n-(k-2)}''^{sd} \varphi_{n-(k-2)}(R^{3(k-1)}) \quad \text{and}$$

$$\varphi_{n-(k-2)}(R^{3(k-1)}) R_{n-(k-2)}'^{sd} \varphi_{n-(k-2)}(R^{3k}),$$

$$\varphi_{n-1}(R^{3k}) R_{n-1}'''^{sd} \varphi_{n-1}(R^{2k}) \quad \text{and} \quad \varphi_{n-1}(R^{2k}) R_{n-1}''^{sd} \varphi_{n-1}(R^{3k}).$$

Invoking our conclusion for Profile 3-(k-1), for $i = n-(k-2), n-1$, $\varphi_{io_2}(R^{3k}) = \cdots = \varphi_{io_{n-2}}(R^{3k}) = \varphi_{(n-(k-2))o_n}(R^{3k}) = \frac{1}{n}$ and $\varphi_{(n-1)o_{n-1}}(R^{3k}) = \frac{2}{n}$. By *sd-efficiency,* $\varphi_{(n-(k-2))o_1}(R^{3k}) =$

$\varphi_{(n-1)o_1}(R^{3k}) = 0$, which imply that $\varphi_{(n-(k-2))o_{n-1}}(R^{3k}) = \frac{2}{n}$ and $\varphi_{(n-1)o_n}(R^{3k}) = \frac{1}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{3k}) = \begin{pmatrix} \frac{2}{n+1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n(n+1)} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2}{n+1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n(n+1)} & \frac{1}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n} & \frac{1}{n} \end{pmatrix}$$

$$\underset{\textbf{Profile 4-0}}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \end{bmatrix}} \Rightarrow \underset{\textbf{Profile 4-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \\ 0 & 1 & 0 \end{bmatrix}} \Rightarrow \underset{\textbf{Profile 4-2}}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \end{bmatrix}} \Rightarrow \cdots \Rightarrow \underset{\textbf{Profile 4-k}}{\begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \end{bmatrix}} \begin{matrix} \left.\vphantom{\begin{matrix}1\\ \vdots \\ 1\end{matrix}}\right\} \frac{n-1}{2} \\ \left.\vphantom{\begin{matrix}1\\ \vdots \\ 1\end{matrix}}\right\} \frac{n+1}{2} \end{matrix}$$

Profile 4-0, R^{40}: For each $i \in N$, $R_i^{40} = R_i''$. By *equal treatment of equals,* for each $i \in N$ and each $k \in O$, $\varphi_{ik}(R^{40}) = \frac{1}{n}$.

Profile 4-1, R^{41}: $R_1^{41} = R_1'$ and for $i = 2, \ldots, n$, $R_i^{41} = R_i''$. By *sd-strategyproofness,*

$$\varphi_1(R^{41}) R_1'^{sd} \varphi_1(R^{40}) \quad \text{and} \quad \varphi_1(R^{40}) R_1''^{sd} \varphi_1(R^{41}).$$

Invoking our conclusion for Profile 4-0, $\varphi_{1o_2}(R^{41}) = \varphi_{1o_3}(R^{41}) = \cdots = \varphi_{1o_{n-2}}(R^{41}) = \varphi_{1o_n}(R^{41}) = \frac{1}{n}$. By *sd-efficiency,* $\varphi_{1o_{n-1}}(R^{41}) = 0$, which implies that $\varphi_{1o_1}(R^{41}) = \frac{2}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{41}) = \begin{pmatrix} \frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{1}{n} \\ \frac{n-2}{n(n-1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{n-2}{n(n-1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n} \end{pmatrix}$$

Profile 4-2, R^{42}: For $i = 1, 2$, $R_i^{42} = R_i'$ and for $i = 3, \ldots, n$, $R_i^{42} = R_i''$. By *sd-strategyproofness,*

$$\varphi_2(R^{42}) R_2'^{sd} \varphi_2(R^{41}) \quad \text{and} \quad \varphi_2(R^{41}) R_2''^{sd} \varphi_2(R^{42}).$$

Invoking our conclusion for Profile 4-1, $\varphi_{2o_2}(R^{42}) = \varphi_{2o_3}(R^{42}) = \cdots = \varphi_{2o_{n-2}}(R^{42}) = \varphi_{2o_n}(R^{42}) = \frac{1}{n}$. By *sd-efficiency,* $\varphi_{2o_{n-1}}(R^{42}) = 0$, which implies that $\varphi_{2o_1}(R^{42}) = \frac{2}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{42}) = \begin{pmatrix} \frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{1}{n} \\ \frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{1}{n} \\ \frac{n-4}{n(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{n-4}{n(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{1}{n} \end{pmatrix}$$

Profile 4-k, R^{4k}: For $i = 1, \ldots, k$, $R_i^{4k} = R_i'$ and for $i = k+1, \ldots, n$, $R_i^{4k} = R_i''$. By *sd-strategyproofness,*

$$\varphi_k(R^{4k}) R_k'^{sd} \varphi_k(R^{4(k-1)}) \quad \text{and} \quad \varphi_k(R^{4(k-1)}) R_k''^{sd} \varphi_k(R^{4k}).$$

Invoking our conclusion for Profile 4-(k-1), $\varphi_{ko_2}(R^{4k}) = \cdots = \varphi_{ko_{n-2}}(R^{4k}) = \varphi_{ko_n}(R^{4k}) = \frac{1}{n}$. By *sd-efficiency,* $\varphi_{ko_{n-1}}(R^{4k}) = 0$, which implies that $\varphi_{ko_1}(R^{4k}) = \frac{2}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{4k}) = \begin{pmatrix} \frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{1}{n} \\ \frac{2}{n(n+1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n+1} & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2}{n(n+1)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n+1} & \frac{1}{n} \end{pmatrix}$$

$$\underline{\textbf{Profile 4-0}} \qquad \underline{\textbf{Profile 5-1}} \qquad \underline{\textbf{Profile 5-2}} \qquad \underline{\textbf{Profile 5-3}} \qquad \underline{\textbf{Profile 5-(k+1)}}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \cdots \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} 1 \\ \vdots \\ 1 \end{matrix}}\right\} \frac{n-1}{2} \\ \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}\right\} \frac{n-1}{2} \\ \} \, 1 \end{matrix}$$

Profile 5-1, R^{51}: For $i = 1, \ldots, n-1$, $R_i^{51} = R_i''$ and $R_n^{51} = R_n'''$. By *sd-strategyproofness,*

$$\varphi_n(R^{51}) R_n'''^{sd} \varphi_n(R^{40}) \quad \text{and} \quad \varphi_n(R^{40}) R_n''^{sd} \varphi_n(R^{51}).$$

Invoking our conlcusion for Profile 4-0, $\varphi_{no_2}(R^{51}) = \varphi_{no_3}(R^{51}) = \cdots = \varphi_{no_{n-1}}(R^{51}) = \frac{1}{n}$. By *sd-efficiency,* $\varphi_{no_1}(R^{51}) = 0$, which implies that $\varphi_{no_n}(R^{51}) = \frac{2}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{51}) = \begin{pmatrix} \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{n-2}{n(n-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{n-2}{n(n-1)} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{2}{n} \end{pmatrix}$$

Profile 5-2, R^{52}: $R_1^{52} = R_1'$, for $i = 2, \ldots, n-1$, $R_i^{52} = R_i''$, and $R_n^{52} = R_n'''$. By *sd-starategyproofness,*

$$\varphi_1(R^{52}) R_1'^{sd} \varphi_1(R^{51}) \quad \text{and} \quad \varphi_1(R^{51}) R_1''^{sd} \varphi_1(R^{52}),$$

$$\varphi_n(R^{52}) R_n'''^{sd} \varphi_n(R^{41}) \quad \text{and} \quad \varphi_n(R^{41}) R_n''^{sd} \varphi_n(R^{52}).$$

Invoking our conclusion for Profiles 4-1 and 5-1, for $i = 1, n$, $\varphi_{io_2}(R^{52}) = \cdots = \varphi_{io_{n-2}}(R^{52}) = \frac{1}{n}$, $\varphi_{1o_n}(R^{52}) = \frac{n-2}{n(n-1)}$, and $\varphi_{no_{n-1}}(R^{52}) = \frac{1}{n-1}$. By *sd-efficiency,* $\varphi_{1o_{n-1}}(R^{52}) = \varphi_{no_1}(R^{52}) =$

0, which imply that $\varphi_{1o_1}(R^{52}) = \frac{2n-1}{n(n-1)}$ and $\varphi_{no_n}(R^{52}) = \frac{2n-3}{n(n-1)}$. Finally, by *equal treatment of equals*,

$$\varphi(R^{52}) = \begin{pmatrix} \frac{2n-1}{n(n-1)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n-2}{n(n-1)} \\ \frac{n^2-3n+1}{n(n-1)(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{n^2-4n+5}{n(n-1)(n-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{n^2-3n+1}{n(n-1)(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{n^2-4n+5}{n(n-1)(n-2)} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{2n-3}{n(n-1)} \end{pmatrix}$$

Profile 5-3, R^{53}: For $i = 1, 2$, $R_i^{53} = R_i'$, for $i = 3, \ldots, n-1$, $R_i^{53} = R_i''$, and $R_n^{53} = R_n'''$. By *sd-strategyproofness,*

$$\varphi_2(R^{53}) R_2'^{sd} \varphi_2(R^{52}) \quad \text{and} \quad \varphi_2(R^{52}) R_2'^{sd} \varphi_2(R^{53}),$$

$$\varphi_n(R^{53}) R_n'''^{sd} \varphi_n(R^{43}) \quad \text{and} \quad \varphi_n(R^{43}) R_n''^{sd} \varphi_n(R^{53}).$$

Invoking our conclusion for Profiles 4-3 and 5-2, for $i = 2, n$, $\varphi_{io_2}(R^{53}) = \cdots = \varphi_{io_{n-2}}(R^{53}) = \frac{1}{n}$, $\varphi_{2o_n}(R^{53}) = \frac{n^2-4n+5}{n(n-1)(n-2)}$, and $\varphi_{no_{n-1}}(R^{53}) = \frac{1}{n-2}$. By *sd-efficiency,* $\varphi_{2o_{n-1}}(R^{53}) = \varphi_{no_1}(R^{53}) = 0$, which imply that $\varphi_{2o_1}(R^{53}) = \frac{2n^2-5n+1}{n(n-1)(n-2)}$ and $\varphi_{no_n}(R^{53}) = \frac{2n-6}{n(n-2)}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{53}) = \begin{pmatrix} \frac{2n^2-5n+1}{n(n-1)(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^2-4n+5}{n(n-1)(n-2)} \\ \frac{2n^2-5n+1}{n(n-1)(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^2-4n+5}{n(n-1)(n-2)} \\ \frac{(n-4)(n-3)n-2}{n(n-1)(n-2)(n-3)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{(n^2-5n+8)}{n(n-1)(n-3)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{(n-4)(n-3)n-2}{n(n-1)(n-2)(n-3)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{(n^2-5n+8)}{n(n-1)(n-3)} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{2n-6}{n(n-2)} \end{pmatrix}$$

We repeat this process until we obtain profile 5-(k+1).

Profile 5-(k+1), $R^{5(k+1)}$: For $i = 1, \ldots, k$, $R_i^{5(k+1)} = R_i'$, for $i = k+1, \ldots, n-1$, $R_i^{5(k+1)} = R_i''$, and $R_n^{5(k+1)} = R_n'''$. By *sd-strategyproofness,*

$$\varphi_k(R^{5(k+1)}) R_k'^{sd} \varphi_k(R^{5k}) \quad \text{and} \quad \varphi_k(R^{5k}) R_k''^{sd} \varphi_k(R^{5(k+1)}),$$

$$\varphi_n(R^{5(k+1)}) R_n'''^{sd} \varphi_n(R^{4k}) \quad \text{and} \quad \varphi_n(R^{4k}) R_n''^{sd} \varphi_n(R^{5(k+1)}).$$

Invoking our conclusion for Profiles 4-k and 5-k, for $i = k, n$, $\varphi_{ko_2}(R^{5(k+1)}) = \cdots = \varphi_{io_{n-2}}(R^{5(k+1)}) = \frac{1}{n}$, $\varphi_{ko_n}(R^{5(k+1)}) = \frac{5-n^2}{n-n^3}$, and $\varphi_{no_{n-1}}(R^{5(k+1)}) = \frac{2}{n+1}$. By *sd-efficiency,* $\varphi_{ko_{n-1}}(R^{5(k+1)}) = 0$ and $\varphi_{no_1}(R^{5(k+1)}) = 0$, which imply that $\varphi_{ko_1}(R^{5(k+1)}) = \frac{2(n^2+1)}{n(n-1)(n-3)}$ and $\varphi_{no_n}(R^{5(k+1)}) =$

$\frac{n+3}{n^2+n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{5(k+1)}) = \begin{pmatrix} \frac{2n^2-(2(\frac{1+n}{2})-1)n+1}{n(n-1)(n-(\frac{1+n}{2})-1))} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^2-((\frac{1+n}{2})+1)n+(3(\frac{1+n}{2})-4)}{n(n-1)(n-(\frac{1+n}{2})-1))} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2n^2-(2(\frac{1+n}{2})-1)n+1}{n(n-1)(n-(\frac{1+n}{2})-1))} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^2-((\frac{1+n}{2})+1)n+(3(\frac{1+n}{2})-4)}{n(n-1)(n-(\frac{1+n}{2})-1))} \\ \frac{(n-2((\frac{1+n}{2})-1))(n-(\frac{1+n}{2}))n-((\frac{1+n}{2})-1)}{n(n-1)(n-((\frac{1+n}{2})-1))(n-\frac{1+n}{2})} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n+1} & \frac{n^2-((\frac{1+n}{2})+2)n+(3(\frac{1+n}{2})-1)}{n(n-1)(n-\frac{1+n}{2})} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{(n-2((\frac{1+n}{2})-1))(n-(\frac{1+n}{2}))n-((\frac{1+n}{2})-1)}{n(n-1)(n-((\frac{1+n}{2})-1))(n-\frac{1+n}{2})} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n+1} & \frac{n^2-((\frac{1+n}{2})+2)n+(3(\frac{1+n}{2})-1)}{n(n-1)(n-\frac{1+n}{2})} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n+1} & \frac{n+3}{n^2+n} \end{pmatrix}$$

$$\underset{\textbf{Profile 5-1}}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \Rightarrow \underset{\textbf{Profile 6-2}}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}} \Rightarrow \underset{\textbf{Profile 6-3}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}} \Rightarrow \underset{\textbf{Profile 6-4}}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}} \Rightarrow \cdots \Rightarrow \underset{\textbf{Profile 6-(k+1)}}{\begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}} \begin{matrix} \left.\vphantom{\begin{matrix}1\\ \vdots\\1\end{matrix}}\right\} \frac{n-3}{2} \\ \left.\vphantom{\begin{matrix}1\\ \vdots\\1\end{matrix}}\right\} \frac{n-1}{2} \\ \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\} 2 \end{matrix}$$

Profile 6-2, R^{62}: For $i = 1, \ldots, n-2$, $R_i^{62} = R_i''$ and for $i = n-1, n$, $R_i^{62} = R_i'''$. By *sd-strategyproofness,*

$$\varphi_{n-1}(R^{62}) R_{n-1}^{'''sd} \varphi_1(R^{51}) \quad \text{and} \quad \varphi_{n-1}(R^{51}) R_1^{''sd} \varphi_{n-1}(R^{62}).$$

Invoking our conclusion for Profile 5-1, $\varphi_{(n-1)o_2}(R^{62}) = \varphi_{(n-1)o_3}(R^{62}) = \cdots = \varphi_{(n-1)o_{n-1}}(R^{62}) = \frac{1}{n}$. By *sd-efficiency,* $\varphi_{(n-1)o_1}(R^{62}) = 0$, which implies that $\varphi_{(n-1)o_n}(R^{62}) = \frac{2}{n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{62}) = \begin{pmatrix} \frac{1}{n-2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{n-4}{n(n-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-2} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{n-4}{n(n-2)} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{2}{n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & \frac{2}{n} \end{pmatrix}$$

Profile 6-3, R^{63}: $R_1^{63} = R_1'$, for $i = 2, \ldots, n-2$, $R_i^{62} = R_i''$, and for $i = n-1, n$, $R_i^{62} = R_i'''$. By *sd-strategyproofness,*

$$\varphi_1(R^{63}) R_1^{'sd} \varphi_1(R^{62}) \quad \text{and} \quad \varphi_1(R^{62}) R_1^{''sd} \varphi_1(R^{63}),$$

$$\varphi_{n-1}(R^{63}) R_{n-1}^{'''sd} \varphi_{n-1}(R^{52}) \quad \text{and} \quad \varphi_{n-1}(R^{52}) R_{n-1}^{''sd} \varphi_{n-1}(R^{63}).$$

Invoking our conclusion for Profiles 5-2 and 6-2, for $i = 1, n-1$, $\varphi_{io_2}(R^{63}) = \cdots = \varphi_{io_{n-2}}(R^{63}) = \frac{1}{n}, \varphi_{1o_n}(R^{63}) = \frac{n-4}{n(n-2)}, \varphi_{(n-1)o_{n-1}}(R^{63}) = \frac{1}{n-1}$. By *sd-efficiency,* $\varphi_{(n-1)o_1}(R^{63}) = \varphi_{1o_{n-1}}(R^{63}) = 0$, which imply that $\varphi_{1o_1}(R^{63}) = \frac{2(n-1)}{n(n-2)}$ and $\varphi_{(n-1)o_n}(R^{63}) = \frac{2n-3}{n(n-1)}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{63}) = \begin{pmatrix} \frac{2(n-1)}{n(n-2)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n-4}{n(n-2)} \\ \frac{n^2-4n+2}{n(n-2)(n-3)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{n^3-8n^2+21n-16}{n(n-1)(n-2)(n-3)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{n^2-4n+2}{n(n-2)(n-3)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{n^3-8n^2+21n-16}{n(n-1)(n-2)(n-3)} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{2n-3}{n(n-1)} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-1} & \frac{2n-3}{n(n-1)} \end{pmatrix}$$

Profile 6-4, R^{64}: For $i = 1, 2$, $R_i^{64} = R'_i$, for $i = 3, \ldots, n-2$, $R_i^{64} = R''_i$, and for $i = n-1, n$, $R_i^{64} = R'''_i$. By *sd-strategyproofness,*

$$\varphi_2(R^{64}) R_2^{'sd} \varphi_2(R^{63}) \quad \text{and} \quad \varphi_2(R^{63}) R_2^{''sd} \varphi_2(R^{64}),$$

$$\varphi_{n-1}(R^{64}) R_{n-1}^{'''sd} \varphi_{n-1}(R^{53}) \quad \text{and} \quad \varphi_{n-1}(R^{53}) R_{n-1}^{''sd} \varphi_{n-1}(R^{64}).$$

Invoking our conclusion for Profiles 5-3 and 6-3, for $i = 2, n-1$, $\varphi_{io_2}(R^{64}) = \cdots = \varphi_{io_{n-2}}(R^{64}) = \frac{1}{n}$, $\varphi_{2o_n}(R^{64}) = \frac{n^3-8n^2+21n-16}{n(n-1)(n-2)(n-3)}$, and $\varphi_{(n-1)o_{n-1}}(R^{64}) = \frac{1}{n-2}$. By *sd-efficiency,* $\varphi_{2o_{n-1}}(R^{64}) = \varphi_{(n-1)o_1}(R^{64}) = 0$, which imply that $\varphi_{2o_1}(R^{64}) = \frac{2(n^3-5n^2+6n-1)}{n(n-1)(n-2)(n-3)}$ and $\varphi_{no_n}(R^{64}) = \frac{2n-6}{n(n-2)}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{64}) = \begin{pmatrix} \frac{2(n^3-5n^2+6n-1)}{n(n-1)(n-2)(n-3)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^3-8n^2+21n-16}{n(n-1)(n-2)(n-3)} \\ \frac{2(n^3-5n^2+6n-1)}{n(n-1)(n-2)(n-3)} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \frac{n^3-8n^2+21n-16}{n(n-1)(n-2)(n-3)} \\ \frac{n^4-10n^3+31n^2-30n+4}{n(n-1)(n-2)(n-3)(n-4)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{n^4-12n^3+55n^2-108n+68}{n(n-1)(n-2)(n-3)(n-4)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{n^4-10n^3+31n^2-30n+4}{n(n-1)(n-2)(n-3)(n-4)} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{n^4-12n^3+55n^2-108n+68}{n(n-1)(n-2)(n-3)(n-4)} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{2n-6}{n(n-2)} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n-2} & \frac{2n-6}{n(n-2)} \end{pmatrix}$$

We repeat this process until we obtain profile 6-(k+1).

Profile 6-(k+1), $R^{6(k+1)}$: For $i = 1, \ldots, k-1$, $R_i^{6(k+1)} = R'_i$, for $i = k, \ldots, n-2$, $R_i^{6(k+1)} = R''_i$, and for $i = n-1, n$, $R_i^{6(k+1)} = R'''_i$. By *sd-strategyproofness,*

$$\varphi_{k-1}(R^{6(k+1)}) R_{k-1}^{'sd} \varphi_{k-1}(R^{6k}) \quad \text{and} \quad \varphi_{k-1}(R^{6k}) R_{k-1}^{''sd} \varphi_{k-1}(R^{6(k+1)}),$$

$$\varphi_{n-1}(R^{6(k+1)}) R_k^{'''sd} \varphi_{n-1}(R^{5(k+1)}) \quad \text{and} \quad \varphi_{n-1}(R^{5(k+1)}) R_{n-1}^{''sd} \varphi_{n-1}(R^{6(k+1)}).$$

Invoking our conclusion for Profiles 5-(k+1), 6-k, for $i = k-1, n-1$, $\varphi_{io_2}(R^{6(k+1)}) = \cdots = \varphi_{io_{n-2}}(R^{6(k+1)}) = \frac{1}{n}$, $\varphi_{(k-1)o_n}(R^{6(k+1)}) = \frac{n^4+n^3-23n^2+3n+2}{n^5+n^4-7n^3-n^2+6n}$, and $\varphi_{(n-1)o_{n-1}}(R^{6(k+1)}) =$

$\frac{2}{n+3}$. By *sd-efficiency,* $\varphi_{(k-1)o_{n-1}}(R^{6(k+1)}) = 0$ and $\varphi_{(n-1)o_1}(R^{6(k+1)}) = 0$, which imply that $\varphi_{(k-1)o_1}(R^{6(k+1)}) = a$ and $\varphi_{(n-1)o_n}(R^{6(k+1)}) = \frac{n+9}{n^2+3n}$. Finally, by *equal treatment of equals,*

$$\varphi(R^{6(k+1)}) = \begin{pmatrix} a & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & b \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & b \\ c & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n+3} & d \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n+3} & d \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n+3} & \frac{n+9}{n^2+3n} \\ 0 & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{2}{n+3} & \frac{n+9}{n^2+3n} \end{pmatrix}$$

where

$a = \frac{3}{n} - b$

$c = \frac{3}{n} - \frac{2}{n+3} - d$

$b = \frac{n^4-(2(\frac{n-3}{2})+6)n^3+((\frac{n-3}{2}-2)^2+17(\frac{n-3}{2}-2)+37)n^2-(7(\frac{n-3}{2}-2)^2+43(\frac{n-3}{2}-2)+58)n+2(3(\frac{n-3}{2}-2)^2+15(\frac{n-3}{2}-2)+16)}{n(n-1)(n-2)(n-\frac{n-3}{2})(n-(\frac{n-3}{2}+1))}$

$\quad = \frac{n^4+n^3-23n^2+3n+2}{n^5+n^4-7n^3-n^2+6n}$

$d = \frac{n^4-(2(\frac{n-3}{2})+8)n^3+((\frac{n-3}{2}-1)^2+17(\frac{n-3}{2}-1)+37)n^2-(7(\frac{n-3}{2}-1)^2+43(\frac{n-3}{2}-1)+58)n+2(3(\frac{n-3}{2}-1)^2+15(\frac{n-3}{2}-1)+16)}{n(n-1)(n-2)(n-(\frac{n-3}{2}+1))(n-(\frac{n-3}{2}+2))}$

$\quad = \frac{n^4-3n^3-7n^2+23n-22}{(n-2)(n-1)^2n(n+1)}$

Profile 7-0, R^{70}: For $i = 1, \ldots, k$, $R_i^{70} = R_i'$, for $i = k+1, \ldots, n-2$, $R_i^{70} = R_i''$, and for $i = n-1, n$, $R_i^{70} = R_i'''$. By *sd-strategyproofness,*

$$\varphi_k(R^{70}) R'^{sd} \varphi_k(R^{6k}) \text{ and } \varphi_k(R^{6k}) R''^{sd} \varphi_k(R^{70}),$$

$$\varphi_{n-1}(R^{70}) R'''^{sd} \varphi_{n-1}(R^{5k}) \text{ and } \varphi_{n-1}(R^{5k}) R''^{sd} \varphi_{n-1}(R^{70}),$$

$$\varphi_{k+1}(R^{70}) R''^{sd} \varphi_{k+1}(R^{3(k-1)}) \text{ and } \varphi_{k+1}(R^{3(k-1)}) R'^{sd} \varphi_{k+1}(R^{70}).$$

Invoking our conclusion for Profiles 6-(k+1), 5-(k+1), and 3-k, $\varphi_{(n-1)o_{n-1}}(R^{70}) = \frac{2}{n+1}$, $\varphi_{ko_n}(R^{70}) = \frac{n^4-3n^3-7n^2+23n-22}{(n-2)(n-1)^2n(n+1)}$, and $\varphi_{(k+1)o_n}(R^{70}) = \frac{1}{n}$. By *sd-efficiency,* $\varphi_{(n-1)o_1}(R^{70}) = 0$, which implies that $\varphi_{(n-1)o_n}(R^{70}) = \frac{n+3}{n(n+1)}$. Now by adding up the assignment of o_n to all agents, we have

$$\underbrace{\frac{n^4 - 3n^3 - 7n^2 + 23n - 22}{(n-2)(n-1)^2 n(n+1)}}_{\text{assignment for } i \in \{1,\ldots,k\}} \cdot \frac{n-1}{2} + \underbrace{\frac{1}{n}}_{i \in \{k+1,\ldots,n-2\}} \cdot \frac{n-3}{2} + \frac{n+3}{n(n+1)} \cdot 2 = 1 - \frac{2}{n(n-1)(n-2)} < 1,$$

which contradicts to $\varphi(R^{70}) \in \mathcal{M}$.■