Strategy-Proofness and Efficiency for Non-Quasi-linear and Common-Tiered-Object Preferences: Characterization of Minimum Price Rule[1]

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Abstract

We consider the allocation problem of assigning heterogenous objects to a group of agents and determining how much they should pay. Each agent receives at most one object. Agents have non-quasi-linear preferences over bundles, each consisting of an object and a payment. In particular, we focus on the cases: (i) objects are linearly ranked and, as long as objects are equally priced, agents commonly prefer a higher ranked object to a lower ranked one, or (ii) objects are partitioned into several tiers and, as long as objects are equally priced, agents commonly prefer an object in the higher tier to an object in the lower tier. The minimum price rule assigns a minimum price (Walrasian) equilibrium to each preference profile. First, we analyze the equilibrium structures for common-object-ranking and common-tiered-object preference profiles, with special attention to the minimum price ones. Second, by assuming various conditions, we show that *on the four domains: common-object-ranking, common-tiered-object, normal and common-object-ranking, and normal and common-tiered-object domains, (i) for each preference profile, agents' welfare under rules satisfying efficiency, strategy-proofness, individual rationality, and no subsidy (or no subsidy for losers) is bounded below by the allocation selected by a minimum price rule, and (ii) only minimum price rules satisfy the four properties.*

Keywords: efficiency, strategy-proofness, non-quasi-linearity, minimum price rule, equilibrium structure, common-object-ranking domain, common-tiered-object domain

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1 Introduction

1.1 Motivating examples and main results

We consider allocation problems exemplified below, which motivate the present research:

Example A (Spectrum license allocation problem): Since the 1990s, governments in many countries have conducted auctions to allocate spectrum licenses. In many cases, each firm is admitted to obtain at most one license. By suitability for the technology of their businesses, firms have common ranking over spectrum licenses of different frequency bands. If licenses are equally priced, firms commonly prefer a spectrum license in the higher rank to the one in the lower rank. Spectrum license auctions often make a large amount of government revenue. Such large-scale auction payments generally influence a firm's abilities to utilize the license, and make firms' preferences over licenses and payments non-quasi-linear.

Example B (House allocation problem in Alonso-type housing market): Central business districts are located in the city center where households are employed and commute everyday with the same public transportation system. Houses are the same in qualities and sizes, but different in the distances to the city center. Each household needs at most one house. As long as houses are equally priced, households prefer a house with a shorter distance to the city center to the one with a longer distance, since longer distances necessitate more commuting cost and time. Since purchasing a house generates a great impact on the household's budget, each household has non-quasi-linear preferences over houses and payments.[1]

Example C (Flat allocation problem in condominium): Several flats belonging to a condominium are to be sold. These flats are similar in qualities and sizes, but different in orientations and floors. Each household needs at most one flat. If flats are equally priced, households commonly prefer flats in higher floors to those in lower floors. However, households might have different preferences on flats in the same floor due to different orientations even if they are equally priced. Such preferences create the phenomenon called "higher-floor premium."[2] It says that in a condominium, flats in higher floors have higher sale prices than those in lower floors. Similarly to Example B, each household has non-quasi-linear preferences over flats and payments.

The common features of the above allocation problems are as follows: Several heterogenous objects are assigned to a group of agents. Each agent receives at most one object, and obtaining the object needs some monetary payment. Agents have non-quasi-linear preferences over bundles, each consisting of an object and a payment. Non-quasi-linearity describes the environment where changing the same amount of money at different payments for a given object exerts different impacts on the benefit deriving from consuming that bundle. In addition, objects are linearly ranked, and if they are equally priced, agents commonly prefer a higher ranked object to a lower ranked one. Or objects are partitioned into several tiers, and if they are equally

[1] See Subsection 1.2.2 for details.

[2] See Subsection 1.2.3 for empirical works that document this phenomenon. This phenomenon is also reported by newspaper (e.g., The New York Times, "The Stratospherians," May 10, 2013).

priced, agents commonly prefer an object in the higher tier to an object in the lower tier, but agents may have different preferences over objects in the same tier.

We investigate the object allocation model with money transfer featured with the above-mentioned characteristics. An *allocation* specifies how the objects are allocated to agents and how much each agent should pay. A *rule* is a mapping from a class of agents' preference profiles (called "domain") to the set of allocations.

A *common-object-ranking domain* is a class of preference profiles such that an object-ranking is prespecified, and for each preference profile, an individual preference satisfies money monotonicity, object monotonicity, possibility of compensation, and more importantly, ranks objects according to the prespecified ranking. A *common-tiered-object domain* is a class of preference profiles such that objects are partitioned into several tiers in a prespecified manner, and for each preference profile, an individual preference satisfies the above first four conditions, and ranks objects according to the tiers. A preference is *normal* if, as payments of objects decrease, the agent benefits more from the object that he prefers initially. A *normal and common-object-ranking domain* is a subclass of a *common-object-ranking domain* such that for each preference profile, individual preferences are normal. A *normal and common-tiered-object domain* can be similarly defined.

An allocation is *efficient* if no one can be better off without reducing others' welfare or reducing the total amount of the payments. *Efficiency* describes the property of a rule that for each preference profile in its domain, the rule always selects an efficient allocation. *Strategy-proofness* says that for each agent and each preference profile, revealing their true preference is a weakly dominant strategy. *Individual rationality* says that for each agent and each preference profile, everyone should be no worse off than getting and paying nothing. This property guarantees the agents' voluntary participation. *No subsidy* says that the payment for each object is non-negative. *No subsidy for losers* says that for each preference profile, agents who get nothing cannot receive any subsidy.

In our settings, the set of equilibrium prices forms a non-empty complete lattice and the minimum (Walrasian) equilibrium price vector is well defined.[3] First, we investigate the structures of the equilibrium prices and object assignments for common-object-ranking and common-tiered-object preference profiles, with special focus on the minimum price equilibria. We show that the equilibrium prices for common-object-ranking preference profiles are monotonic along the object-ranking, and the equilibrium prices for common-tiered-object preference profiles exhibit the "higher-floor premium" documented by empirical works on housing markets.

Next, we analyze the "minimum price (Walrasian) rule." A *minimum price rule* is a rule that, given each preference profile, it always selects an equilibrium with the minimum price vector. By assuming various conditions, we show that *on the four domains: common-object-ranking, common-tiered-object, normal and common-object-ranking, and normal and common-tiered-object domains, (i) for each preference profile, agents' welfare under rules satisfying efficiency, strategy-proofness, individual rationality, and no subsidy (or no subsidy for losers) is bounded below by the allocation selected by a minimum price rule, and (ii) only minimum price rules satisfy the four properties.*

[3] See Facts 1 and 2 in Section 3 for details.

1.2 Related literature

Our results are related to the following three strands of literature.

1.2.1 Efficient and strategy-proof rules for non-quasi-linear preferences

The literature in this strand focuses on the identification of *efficient* and *strategy-proof* rules for non-quasi-linear preferences.[4] In the one-to-one two-sided matching model with money transfer, no rule satisfies *efficiency* and *strategy-proofness*, in addition to *individual rationality* and *no pair-wise budget deficit*. However, if *strategy-proofness* is weakened to *one-sided strategy-proofness*, the one-sided optimal core rule satisfies those properties (Demange and Gale, 1985; Morimoto, 2016).

As a special case of the one-to-one two-sided matching model with money transfer, object assignment models with money transfer have also been studied.[5] In these models, the minimum price rule is well defined. When objects are identical, the minimum price rule is equivalent to the Vickrey rule (Vickrey, 1961), and only the minimum price rule satisfies *efficiency* and *strategy-proofness*, in addition to *individual rationality* and *no subsidy* (Saitoh and Serizawa, 2008; Sakai, 2008). However, when objects are heterogenous, the outcome of the minimum price rule for non-quasi-linear preferences does not coincide to that of the Vickrey rule, and only the minimum price rule satisfies *efficiency*, *strategy-proofness*, *individual rationality* and *no subsidy for losers* (Morimoto and Serizawa, 2015). By imposing price restrictions on objects or introducing existing tenants, with some mild domain restriction, there are still some (constraint) *efficient* and *strategy-proof* rules (Andersson and Svensson, 2014; Andersson and Svensson, 2016; Andersson et al, 2016).

Our characterization results can be regarded as a further study of *efficient* and *strategy-proof* rules on non-quasi-linear domains. Saitoh and Serizawa (2008), Sakai (2008), and Morimoto and Serizawa (2015) are related to our study.

When objects are homogenous, Saitoh and Serizawa (2008) and Sakai (2008) characterize the Vickrey rule by using similar axioms for non-quasi-linear preferences, and Saitoh and Serizawa (2008) further show that the same characterization holds even on the non-quasi-linear preferences exhibiting normality. As mentioned above, when objects are heterogenous, the Vickrey rule is different from the minimum price rule. Thus their characterizations do not imply ours and their proof techniques can not be used to establish our results.

When objects are heterogenous, Morimoto and Serizawa (2015) have already characterized the minimum price rule by using similar properties, our results are different from theirs by three points.

[4]Some authors also investigated the strategy-proof and fair rules for the non-quasi-linear preferences, for example, Alkan et al, (1991), Sun and Yang (2003), Andersson, et al, (2010), Adachi (2014), and Tierney (2015) etc. Recently, Baisa (2015, 2016a) investigated the auction models for the non-quasi-linear preferences.

[5]Assuming each agent at most receives one object is important for identifying *efficient* and *strategy-proof* rules for non-quasi-linear preferences. Recently, Kazumura and Serizawa (2016) relaxed this assumption and showed some impossibility results.

First, we focus on common-object-ranking, common-tiered-object, normal and common-object-ranking, and normal and common-tiered-object domains, which are all smaller than the one they focus on. The smaller a domain is, the weaker the properties of rules are, such as *efficiency* and *strategy-proofness*. Thus, smaller domains imply that there may be better rules tailored to our concerned allocation problems, exemplified by Examples A, B and C. Thus Morimoto and Serizawa (2015)'s result does not imply ours.

Second, we owe some proof techniques to Morimoto and Serizawa (2015). But their proofs use preferences excluded from our domains. In some parts of the proofs, we need to carefully modify their proofs, and in many parts we need to develop new proof techniques. Particularly, their proof techniques crucially depend on the uses of non-normal preferences. Thus, to establish the characterization results for the preferences exhibiting normality, new proof techniques are proposed.[6]

Third, Morimoto and Serizawa (2015) assume that the number of agents is larger than that of objects. This assumption plays an important rule in their proofs. In some of our results, this assumption can be replaced by strengthening *no subsidy for losers* to *no subsidy*, and sometimes accompanied by some restrictions on the object tier.

1.2.2 Alonso-type housing market

The literature in this strand centers on the study of equilibria in Alonso-type housing market (e.g., Alonso, 1956; Kaneko, 1983; Kaneko et al, 2006; Sai, 2015). This literature assumes that agents have preferences over the bundles, each consisting of an object and "money", where money is defined as residual income after payment, and that all the agents have the same normal and common-object-ranking preference but have different income levels, which differentiate agents' preferences over the bundles, each consisting of an object and *payment*. We show the results of the equilibrium structures that are parallel to those of the literature by assuming weaker conditions on preferences. Since our concerned domains are larger than those analyzed by the literature, our results in turn imply the existing ones. Furthermore, we admit agents have different rankings of objects in the same tier. None of the literature in this strand investigates such cases. Thus our investigation of equilibrium structures for common-tiered-object preferences provides additional insights to the existing literature.

1.2.3 Common-tiered-object domains in the object assignment models

Theoretical works in this strand of literature study *efficient* and *strategy-proof* rules in the object assignment models without money transfer, such as the two-sided matching and probabilistic assignment models (Kandori et al., 2010; Kesten, 2010; Kesten and Kurino, 2013; Akahoshi, 2014). In this literature, the common-tiered-object preference structure is shown to be important in identifying the *efficient* and *strategy-proof* rules. However, since our model involves money transfers, the results of this literature do not imply our characterization results.

[6] See Section A2 of the Appendix for detailed explanations.

To the best of our knowledge, our paper is the first one that theoretically analyzes *efficient* and *strategy-proof* rules on the common-tiered-object domain with money transfer.

In addition, some empirical works, such as Ong, 2000; Chin et al, 2004; Conroy et al, 2013, investigate the "higher-floor premium" phenomena occurring in selling flats in condominiums in U.S.A., Singapore, and Malaysia.

1.3 Organization

The remaining parts are organized as follows. Section 2 introduces concepts and establishes the model. Section 3 defines the minimum price equilibria and centers on the investigation of the equilibrium structures for common-object-ranking and common-tiered-object preference profiles. Section 4 establishes the main results for common-object-ranking and common-tiered-object domains. Section 5 gives concluding remarks. Omitted proofs and those of the main results are placed in the Appendix.

2 The model and definitions

Consider an economy with $2 \leq n < \infty$ agents and $1 \leq m < \infty$ objects. Denote the set of agents by $N \equiv \{1, 2, \cdots, n\}$ and the set of (real) **objects** by $M \equiv \{1, 2, \cdots, m\}$. Not receiving an object is called receiving a **null** object. We call it object 0. Let $L \equiv M \cup \{0\}$. Each agent receives at most one object. We denote the object that agent $i \in N$ receives by $x_i \in L$. We denote the amount that agent i pays by $t_i \in \mathbb{R}$. The agents' common **consumption set** is $L \times \mathbb{R}$, and a generic (consumption) **bundle** for agent i is a pair $z_i \equiv (x_i, t_i) \in L \times \mathbb{R}$. Let $\mathbf{0} \equiv (0,0)$.

Each agent i has a complete and transitive preference R_i over $L \times \mathbb{R}$. Let P_i and I_i be the strict and indifference relations associated with R_i. A generic class of preferences is denoted by $\mathcal{R}$. We call $(\mathcal{R})^n$ a **domain**.

The following are basic properties of preferences, which we assume throughout the paper:

Money monotonicity: For each $x_i \in L$ and each pair $t_i, t'_i \in \mathbb{R}$, if $t_i < t'_i$, $(x_i, t_i) \, P_i \, (x_i, t'_i)$.

Object monotonicity: For each $x_i \in M$ and each $t_i \in \mathbb{R}$, $(x_i, t_i) \, P_i \, (0, t_i)$.

Possibility of compensation: For each $t_i \in \mathbb{R}$ and each pair $x_i, x_j \in L$, there is a pair $t_j, t'_j \in \mathbb{R}$ such that $(x_i, t_i) \, R_i \, (x_j, t_j)$ and $(x_j, t'_j) \, R_i \, (x_i, t_i)$.

Continuity: For each $z_i \in L \times \mathbb{R}$, the **upper contour set at** z_i, $UC(R_i, z_i) \equiv \{z'_i \in L \times \mathbb{R} : z'_i \, R_i \, z_i\}$ and the **lower contour set at** z_i, $LC(R_i, z_i) \equiv \{z'_i \in L \times R : z_i \, R_i \, z'_i\}$, are closed.

A preference R_i is **classical** if it satisfies the four properties just defined. Let $\mathcal{R}^C$ be the class of classical preferences. We call $(\mathcal{R}^C)^n$ the **classical domain**. Note that by money monotonicity, the possibility of compensation and continuity, for each $R_i \in \mathcal{R}^C$, each $z_i \in L \times \mathbb{R}$ and each $y \in L$, there is a unique amount $V_i(y; z_i) \in \mathbb{R}$ such that $(y, V_i(y; z_i)) \, I_i \, z_i$. We call $V_i(y; z_i)$ the **valuation of** y **at** z_i **for** R_i**.**

A preference $R_i \in \mathcal{R}^C$ satisfies **normality** if for each pair $x, y \in L$, each pair $t_i, t'_i \in \mathbb{R}$ with $t_i > t'_i$, if $(x, t_i)\, I_i\, (y, t'_i)$ and $d > 0$, then $(x, t_i - d)\, P_i\, (y, t'_i - d)$.[7] Let $\mathcal{R}^N$ denote the class of classical preferences satisfying normality. Obviously, $\mathcal{R}^N \subsetneq \mathcal{R}^C$. In some parts of the paper, we assume this property.

Figure 1 illustrates a preference R_i satisfying normality for $M = \{A, B\}$.

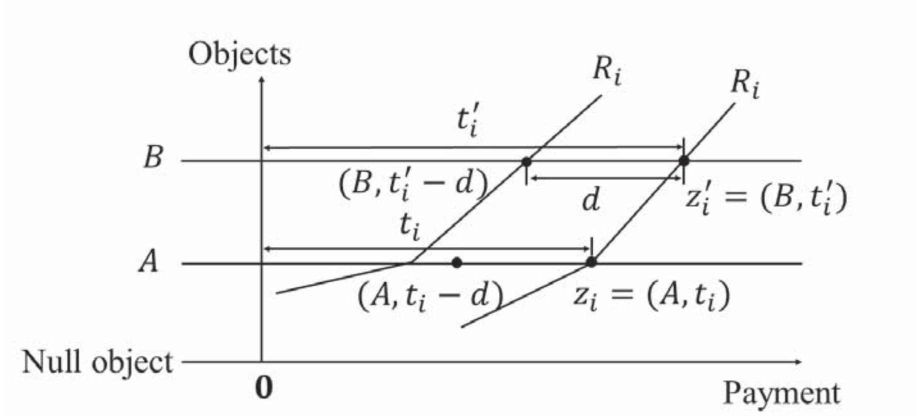

Figure 1 Illustration of preference satsifying normality

In Figure 1, there are three horizontal lines. The bottom line corresponds to the null object 0, the middle line to object A and the top line to object B, respectively. The intersection of the vertical line and each horizontal line denotes the bundle consisting of the corresponding object and no payment. For example, the origin **0** denotes the bundle consisting of the null object and no payment. For each point on one of three horizontal lines, the distance from that point to the vertical line denotes the payment. For example, z_i denotes the bundle consisting of object A and payment t_i. By money monotonicity, moving leftward along the same line makes the agent better off, i.e., if $d > 0$, then $(A, t_i - d)\, P_i\, (A, t_i)$. If the bundles are connected by a indifference curve, for example, z_i and z'_i, it means that agent i is indifferent between these two bundles, i.e., $z_i\, I_i\, z'_i$. In Figure 1, for $t_i < t'_i$ and $d > 0$, $(A, t_i)\, I_i\, (B, t'_i)$ implies $(B, t'_i - d)\, P_i\, (A, t_i - d)$. Thus, R_i satisfies normality.

An **object allocation** is an n-tuple $(x_1, \ldots, x_n) \in L^n$ such that for each pair $i, j \in N$, if $x_i \neq 0$ and $i \neq j$, then $x_i \neq x_j$. We denote the set of object allocations by X. A **(**feasible**) allocation** is an n-tuple $z \equiv (z_1, \ldots, z_n) \equiv ((x_1, t_1), \ldots, (x_n, t_n)) \in [L \times \mathbb{R}]^n$ such that $(x_1, \ldots, x_n) \in X$. We denote the set of feasible allocations by Z. Given $z \in Z$, we denote its object and payment components at z by $x \equiv (x_1, \ldots, x_n)$ and $t \equiv (t_1, \ldots, t_n)$, respectively.

A **preference profile** is an n-tuple $R \equiv (R_i)_{i \in N} \in \mathcal{R}^n$. Given $R \in \mathcal{R}^n$ and $N' \subseteq N$, let $R_{N'} \equiv (R_i)_{i \in N'}$ and $R_{-N'} \equiv R_{N \setminus N'} \equiv (R_i)_{i \in N \setminus N'}$.

Next, we introduce two properties of domains we focus on. The first property is "**common-object-ranking**". It says that objects are ranked linearly, and for each payment, agents commonly prefer the bundle consisting of the object that has the higher rank and that payment to the bundle consisting of the object that has the lower rank and that payment.

[7] Kaneko (1983) introduced this definition.

Let $\pi \equiv (\pi(1), \ldots, \pi(m+1))$ be a permutation of objects in L, where $\pi(1)$ denotes the object ranked first, $\pi(2)$ denotes the object ranked second, and so on. For each pair $x, y \in L$, $x >_\pi y$ means that x has a higher rank than y according to π.

A preference $R_i \in \mathcal{R}^C$ **ranks objects according to** π if for each $t_i \in \mathbb{R}$,

$$(\pi(1), t_i)\, P_i \cdots P_i\, (\pi(m+1), t_i).$$

Remark 1: Since $R_i \in \mathcal{R}^C$, object monotonicity implies $\pi(m+1) = 0$.

Figure 2 illustrates a preference R_i ranking objects according to π for $M = \{A, B, C\}$ and $\pi = (\pi(1), \pi(2), \pi(3), \pi(4)) = (C, B, A, 0)$.

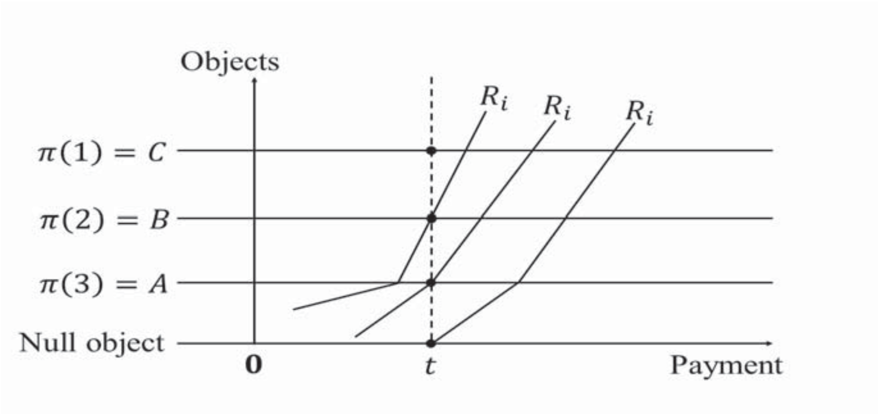

Figure 2 Illustration of preference ranking objects according to π

In Figure 2, for each $t_i \in \mathbb{R}$, $(C, t_i)\, P_i\, (B, t_i)\, P_i\, (A, t_i)\, P_i\, (0, t_i)$. Thus R_i ranks objects according to $\pi = (C, B, A, 0)$.

Remark 2:[8] (i) Let $R_i \in \mathcal{R}^N$ rank objects according to π only at some given $t_i \in \mathbb{R}$. Then, for each $t_i' \in \mathbb{R}$ such that $t_i' \leq t_i$, $(\pi(1), t_i')\, P_i \cdots P_i\, (\pi(m+1), t_i')$. Note that for each $t_i' \in \mathbb{R}$ such that $t_i' > t_i$, $(\pi(1), t_i')\, P_i \cdots P_i\, (\pi(m+1), t_i')$ may not hold.
(ii) Let $R_i \in \mathcal{R}^N$. Then normality does not imply that there is π such that R_i ranks objects according to π.
(iii) Let $R_i \in \mathcal{R}^N$. Then there is some π such that for each $t_i \in \mathbb{R}$, $(\pi(1), t_i)\, R_i \cdots R_i\, (\pi(m+1), t_i)$.

Let $\mathcal{R}^R(\pi)$ be the class of preferences ranking objects according to π. Note that $\mathcal{R}^R(\pi) \subsetneq \mathcal{R}^C$. A preference profile R **ranks objects according to** π if each individual preference in the preference profile all ranks objects according to π, i.e., for each $i \in N$, $R_i \in \mathcal{R}^R(\pi)$.

Figure 3 illustrates the preference profile R ranking objects according to π for $N = \{1, 2\}$, $M = \{A, B, C\}$, and $\pi = (\pi(1), \pi(2), \pi(3), \pi(4)) = (C, B, A, 0)$.

[8] The proof is relegated to the Appendix.

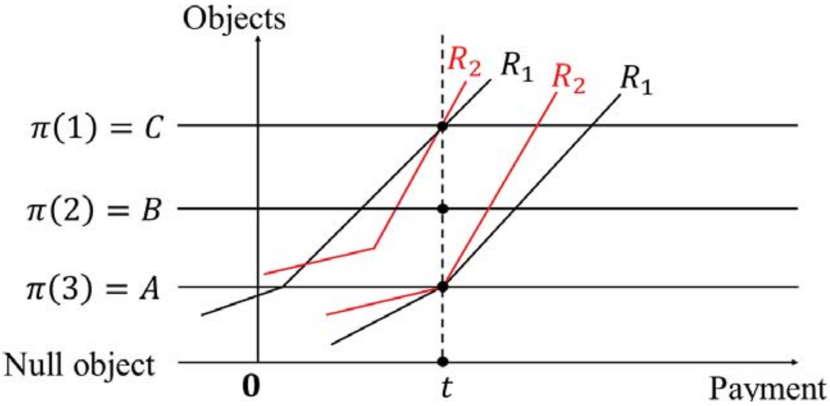

Figure 3 Illustration of preference profile ranking objects according to π

In Figure 3, for each $t \in \mathbb{R}$, $(C, t)\, P_1\, (B, t)\, P_1\, (A, t)\, P_1\, (0, t)$ and $(C, t)\, P_2\, (B, t)\, P_2\, (A, t)\, P_2\, (0, t)$. Thus, R ranks objects according to π.

Remark 3: (i) Let $R_0 \in \mathcal{R}^N$. For each $i \in N$, let $d_i \in \mathbb{R}$ and $R_i \in \mathcal{R}^C$ be a preference such that for each $x \in L$ and each $t_i \in \mathbb{R}$, $V_i(x; (0, t_i)) = V_0(x; (0, t_i - d_i)) + d_i$. Then, the preference profile $R = (R_1, \ldots, R_n)$ has the following properties:
(a) For each $i \in N$, we have $R_i \in \mathcal{R}^N$. Let d_i parameterize agent i's income. Since $R_0 \in \mathcal{R}^N$, for each pair $i, j \in N$, each $x \in M$, and each $t_0 \in \mathbb{R}$, if $d_i > d_j$, then $V_i(x; (0, t_0)) > V_j(x; (0, t_0))$. That is, the income effect on the valuation of x at $(0, t_0)$ is positive.
(b) For each $i \in N$, if R_i ranks objects according to some π at some given $t_i \in \mathbb{R}$, then the property in Remark 2 (i) holds for R_i.
(ii) Kaneko (1983), Kaneko et al (2006), and Sai (2015), etc., analyze the housing market by using the preference profile defined by Remark 3(i). Particularly, they set (b) as: for each $i \in N$, $(\pi(1), I_i)\, P_i \cdots P_i\, (\pi(m+1), I_i)$ for some π. Thus, by Remark 2(i), R_i ranks objects according to π for each $t_i \in \mathbb{R}$ such that $t_i \leq I_i$.

We call $(\mathcal{R}^R(\pi))^n$ a **common-object-ranking domain**. Whenever we mention a common-object-ranking domain $(\mathcal{R}^R(\pi))^n$, we assume that the permutation π is commonly known by all the agents. Let $\mathcal{R}^{NR}(\pi) \equiv \mathcal{R}^N \cap \mathcal{R}^R(\pi)$ be the class of preferences satisfying normality and ranking objects according to π. We call $(\mathcal{R}^{NR}(\pi))^n$ a **normal and common-object-ranking domain**.

The second property we focus is "**common-tiered-object-ranking**", a generalization of the first one. It says that objects are partitioned into tiers, and for each payment, agents commonly prefer the bundle consisting of the object in the higher tier and that payment to the bundle consisting of the object in the lower tier and that payment.

We describe a tier partition by an indexed family $\mathcal{T} = \{T_l\}_{l \in K}$ of non-empty subsets of L such that (i) $K \equiv \{1, 2, \cdots, k\}$ and $1 \leq k \leq m+1$, (ii) $\cup_{l \in K} T_l = L$ and (iii) for each $l, l' \in K$ with $l \neq l'$, $T_l \cap T_{l'} = \varnothing$, where T_l denotes the $l-$th tier for each $l \in K$. For every pair $x, y \in L$, $x >_{\mathcal{T}} y$ means that x is in a higher tier than y according to $\mathcal{T}$.

A preference $R_i \in \mathcal{R}^C$ **ranks objects according to $\mathcal{T}$** if for each $t_i \in \mathbb{R}$, each $x \in T_l$ and each $y \in T_{l'}$ with $l \neq l'$ and $l < l'$, $(x, t_i)\, P_i\, (y, t_i)$. Note that we do not impose any restrictions on the agent's preferences over objects within the same tier when those payments are same.

Remark 4: (i) Since $R_i \in \mathcal{R}^C$, object monotonicity implies $k \geq 2$ and $T_k = \{0\}$.

(ii) If a preference $R_i \in \mathcal{R}^C$ ranks objects according to π, then R_i also ranks objects according to $\mathcal{T}$ such that $T_1 = \{\pi(1)\}, \cdots, T_{m+1} = \{\pi(m+1)\}$.

Figure 4 illustrates a preference R_i ranking objects according to $\mathcal{T}$ for $M = \{A, B, C\}$ and $\mathcal{T} = T_1 \cup T_2 \cup T_3$ with $T_1 = \{B, C\}$, $T_2 = \{A\}$, and $T_3 = \{0\}$.

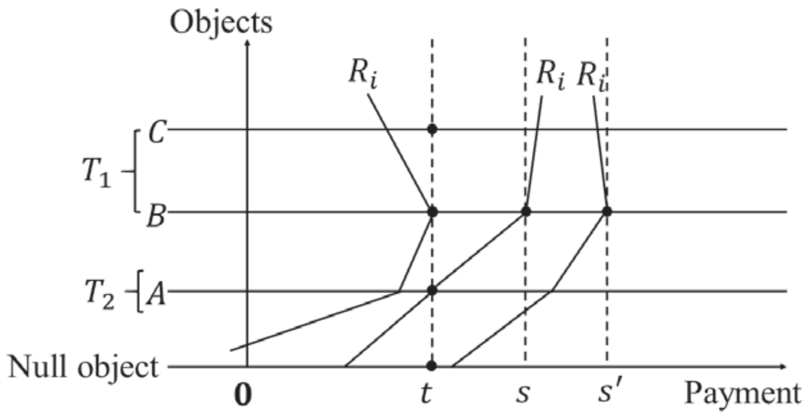

Figure 4 Illustration of preference ranking objects according to $\mathcal{T}$

In Figure 4, for each $t \in \mathbb{R}$, each $y \in T_1$, each $x \in T_2$, and $0 \in T_3$, we have $(y, t)\, P_i\, (x, t)\, P_i\, (0, t)$. Note that $(C, s)\, P_i\, (B, s)$ and $(B, s')\, P_i\, (C, s')$. Thus, R_i ranks objects according to $\mathcal{T}$, but does not rank objects according to any object permutation.

Let $\mathcal{R}^T(\mathcal{T})$ be the class of preferences ranking objects according to $\mathcal{T}$. Obviously, $\mathcal{R}^T(\mathcal{T}) \subseteq \mathcal{R}^C$. A preference profile R **ranks objects according to $\mathcal{T}$** if each individual preference in the preference profile all ranks objects according to $\mathcal{T}$, i.e., for each $i \in N$, $R_i \in \mathcal{R}^T(\mathcal{T})$.

Figure 5 illustrates the preference profile R ranking objects according to $\mathcal{T}$ for $N = \{1, 2\}$, $M = \{A, B, C\}$, and $\mathcal{T} = T_1 \cup T_2 \cup T_3$ with $T_1 = \{B, C\}$, $T_2 = \{A\}$, and $T_3 = \{0\}$.

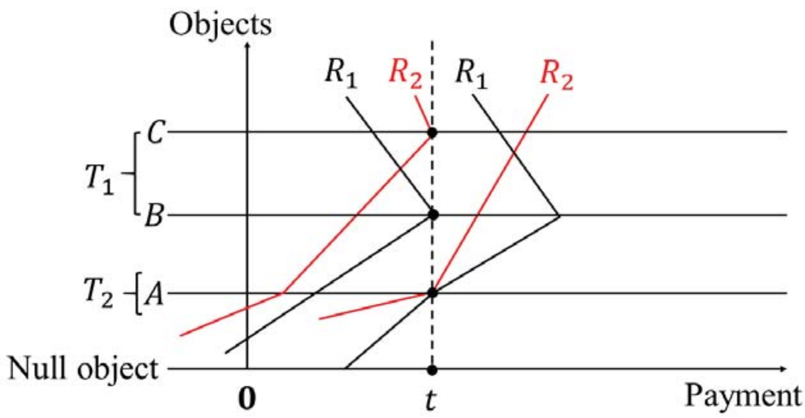

Figure 5 Illustration of preference profile ranking objects according to $\mathcal{T}$

In Figure 5, for each $t \in \mathbb{R}$, $(C,t)\,P_1\,(B,t)\,P_1\,(A,t)\,P_1\,(0,t)$ and $(B,t)\,P_2\,(C,t)\,P_2\,(A,t)\,P_2\,(0,t)$. Thus, R ranks objects according to $\mathcal{T}$.

We call $(\mathcal{R}^T(\mathcal{T}))^n$ a **common-tiered-object domain**. Whenever we mention a common-tiered-object domain $(\mathcal{R}^T(\mathcal{T}))^n$, we assume that only the permutation $\mathcal{T}$ is commonly known by all the agents. Let $\mathcal{R}^{NT}(\mathcal{T}) \equiv \mathcal{R}^N \cap \mathcal{R}^T(\mathcal{T})$ be the class of preferences satisfying normality and ranking objects according to $\mathcal{T}$. We call $(\mathcal{R}^{NT}(\mathcal{T}))^n$ a **normal and common-tiered-object-ranking domain**.

Remark 5: (i) Any common-object-ranking domain is included in the common-tiered-object domain with respect to some $\mathcal{T} = \{T_i\}_{i \in K}$. If $2 \leq k < m+1$, such a inclusion relation is strict.
(ii) If $k = m+1$, a common-tiered-object domain with k tiers is a common-object-ranking domain.
(iii) Consider two common-tiered-object domains with respect to $\mathcal{T} = \{T_l\}_{l \in K}$ and $\mathcal{T}' = \{T'_{l'}\}_{l' \in K'}$. If $\mathcal{T}'$ is coarser than $\mathcal{T}$, then the common-tiered-object domain with respect to $\mathcal{T}$ is a subset of the one with respect to $\mathcal{T}'$.[9]
(iv) Since the classical domain is the common-tiered-object domain with the coarsest class of tiers, i.e., $k = 2$, any common-tiered-object domain is a subset of the classical domain.

An (allocation) **rule** on $\mathcal{R}^n$ is a mapping f from $\mathcal{R}^n$ to Z. Given a rule f and $R \in \mathcal{R}^n$, we denote bundle assigned to agent i by $f_i(R) \equiv (x_i(R), t_i(R))$ where $x_i(R)$ denotes the assigned object and $t_i(R)$ the associated payment. We write,

$$f(R) \equiv (f_i(R))_{i \in N},\ x(R) \equiv (x_i(R))_{i \in N}, \text{ and } t(R) \equiv (t_i(R))_{i \in N}.$$

We define the concept of "efficiency".

An allocation $z' \in Z$ **dominates** $z \in Z$ for $R \in \mathcal{R}^n$ **in agents' welfare** if for each $i \in N$, $z'_i\,R_i\,z_i$, and for some $i \in N$, $z'_i\,P_i\,z_i$. An allocation $z' \in Z$ **weakly dominates** $z \in Z$ for $R \in \mathcal{R}^n$ **in agents' welfare** if for each $i \in N$, $z'_i\,R_i\,z_i$. An allocation $z' \in Z$ (**weakly**) **dominates** $z \in Z$ for $R \in \mathcal{R}^n$ **in revenue** if $\sum_{i \in N} t_i (\leq) < \sum_{i \in N} t'_i$. An allocation $z' \in Z$ (Pareto-)**dominates** $z \in Z$ for $R \in \mathcal{R}^n$ if (i) $z' \in Z$ dominates z for R in agents' welfare and $z' \in Z$ weakly dominates $z \in Z$ in revenue, or (ii) $z' \in Z$ weakly dominates z for R in agents' welfare and $z' \in Z$ dominates $z \in Z$ in revenue. An allocation z is (Pareto-)**efficient** for $R \in \mathcal{R}^n$ if there is no allocation $z' \in Z$ that dominates z.

Remark 6: (i) Dominance in revenue takes the perspective of object suppliers, i.e., governments or auctioneers, and implicitly assumes that their welfare is only monotonically increasing in the revenues.
(ii) *Dominance in revenue* is indispensable in the definition of *efficiency*. Without this condition, no allocation is efficient.

Efficiency of the rule says that for each preference profile, the rule chooses an efficient allocation.

Efficiency: For each $R \in \mathcal{R}^n$, $f(R)$ is efficient for R.

[9] $\mathcal{T}'$ is **coarser than** $\mathcal{T}$ if for each $l \in K$, there is $l' \in K'$ such that $T_l \subseteq T_{l'}$.

We introduce other properties of rules. *Strategy-proofness* says that no agent ever benefits from misrepresenting his preference.

Strategy-proofness: For each $R \in \mathcal{R}^n$, each $i \in N$ and each $R'_i \in \mathcal{R}$, $f_i(\overset{truth}{R_i}, R_{-i}) \overset{truth}{R_i} f_i(\overset{lie}{R'_i}, R_{-i})$.[10]

Individual rationality says that no agent is ever assigned a bundle that makes him worse off than he would be if he had received the null object and paid nothing.

Individual rationality: For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R)\, R_i\, \mathbf{0}$.

No subsidy says that the payment of each agent is always nonnegative.

No subsidy: For each $R \in \mathcal{R}^n$ and each $i \in N$, $t_i(R) \geq 0$.[11]

No subsidy for losers is a weak variant of *no subsidy*: It says that if an agent receives the null object, his payment is nonnegative.

No subsidy for losers: For each $R \in \mathcal{R}^n$, if $x_i(R) = 0$, $t_i(R) \geq 0$.

3 Walrasian equilibria and minimum price equilibria

3.1 Definitions and fundamental properties

In this subsection, we define the equilibria and minimum price equilibria, and state several facts related to them. Throughout the subsection, let us fix $\mathcal{R} \equiv \mathcal{R}^C$ and all the facts hold on all of its subdomains, such as common-object-ranking and common-tiered-object domains, etc.

Let $p \equiv (p_1, \cdots, p_m) \in \mathbb{R}^m_+$ be a price vector. The **budget set at** p is defined as $B(p) \equiv \{(x, p_x) : x \in L\}$, where if $x = 0$, then $p_x = 0$. Given $R_i \in \mathcal{R}$, the **demand set at** p **for** R_i is defined as $D(R_i, p) \equiv \{x \in L : \text{for each } y \in L,\ (x, p_x)\, R_i\, (y, p_y)\}$.

Definition: Let $R \in \mathcal{R}^n$. A pair $((x,t), p) \in Z \times \mathbb{R}^m_+$ is a (Walrasian) **equilibrium** for R if

$$\text{for each } i \in N,\ x_i \in D(R_i, p) \text{ and } t_i = p_{x_i}, \quad \text{(E-i)}$$

$$\text{for each } y \in M,\ \text{if for each } i \in N,\ x_i \neq y, \text{ then } p_y = 0. \quad \text{(E-ii)}$$

Condition (E-i) says that each agent receives an object from his demand set and pays its price. Condition (E-ii) says that the prices of unassigned objects are zero.

Fact 1 (Alkan and Gale, 1990; Alaei et al, 2016) (**Existence**). For each $R \in \mathcal{R}^n$, there is an equilibrium.

Given $R \in \mathcal{R}^n$, we denote the **set of equilibria** for R by $W(R)$, the **set of equilibrium allocations** for R by $Z(R)$, and the **set of equilibrium price vectors** for R by $P(R)$,

[10] Thomson (2015) introduced this notation.

[11] In many real-life allocation problems, imposing *no subsidy* on the rules is mild and suitable. For example, in the car license auction in Singapore, the participants who get the car licenses need to pay their bids and those who cannot get the car licenses do not obtain any subsidy. In the fish auction in the Tsukiji fish market in Japan, the merchants who get the fish need to pay their bids and those who cannot get the fish do not obtain any subsidy. In the auction literature, Saitoh and Serizawa (2008) and Baisa (2016b) also imposed *no subsidy* on the rules. Same argument holds for the justification of *no subsidy for losers.*

respectively, i.e.,

$$\begin{aligned} Z(R) &\equiv \{z \in Z : \text{for some } p \in \mathbb{R}^m_+, (z,p) \in W(R)\}, \text{ and} \\ P(R) &\equiv \{p \in \mathbb{R}^m_+ : \text{for some } z \in Z, (z,p) \in W(R)\}. \end{aligned}$$

Fact 2 (Demange and Gale, 1985) (**Lattice property**). For each $R \in \mathcal{R}^n$, $P(R)$ is a complete lattice and there is a unique equilibrium price vector $p \in P(R)$ such that for each $p' \in P(R)$, $p \leq p'$.

A minimum price equilibrium is an equilibrium whose price vector is minimum. Given $R \in \mathcal{R}^n$, let $p^{\min}(R)$ be the minimum equilibrium price vector for R, $W^{\min}(R)$ the **set of minimum price equilibria associated with** $p^{\min}(R)$, and $Z^{\min}(R)$ the **set of minimum price equilibrium allocations associated with** $p^{\min}(R)$, respectively, i.e.,

$$Z^{\min}(R) \equiv \{z \in Z : (z, p^{\min}(R)) \in W^{\min}(R)\}.$$

Although there might be several minimum price equilibria, they are indifferent for each agent, i.e., for each $R \in \mathcal{R}^n$, each pair $z, z' \in Z^{\min}(R)$, and each $i \in N$, $z_i \, I_i \, z'_i$.

Since a preference profile R is fixed in the rest of this section, we write $p^{\min}$ instead of $p^{\min}(R)$ for simplicity.

Figure 6 illustrate a minimum price equilibrium for $N = \{1, 2, 3\}$, $M = \{A, B, C, D\}$, and $R \in (\mathcal{R}^C)^n$.

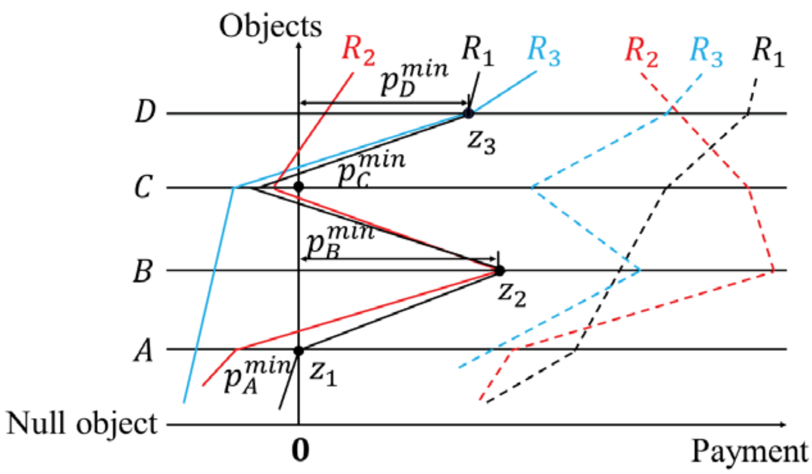

Figure 6 Illustration of minimum price equilibrium for preference profile from classical domain

In Figure 6, a minimum price equilibrium allocation is as follows: agent 1 receives object A and pays 0. Agent 2 receives object B and pays $p_B^{\min}$. Agent 3 receives object D and pays $p_D^{\min}$. The prices of objects A and C are 0.

Let's see why $z \equiv (z_1, z_2, z_3)$ is a minimum price equilibrium allocation. First, for each agent $i = 1, 2, 3$, z_i is maximal for R_i in the budget set $\{\mathbf{0}, (A, p_A^{\min}), (B, p_B^{\min}), (C, p_C^{\min}), (D, p_D^{\min})\}$. Thus, z is an equilibrium allocation. Let $p^{\min} \equiv (p_A^{\min}, p_B^{\min}, p_C^{\min}, p_D^{\min})$.

Next, let $p \equiv (p_A, p_B, p_C, p_D)$ be an equilibrium price. We show $p \geq p^{\min}$. By the nonnegativity of prices, $p_A \geq 0 = p_A^{\min}$ and $p_C \geq 0 = p_C^{\min}$.

If $p_B < p_B^{\min}$ and $p_D < p_D^{\min}$, then by $p_A \geq 0$ and $p_C \geq 0$, all three agents prefer (B, p_B) or (D, p_D) to $\mathbf{0}$, (A, p_A) and (C, p_C). In such a case, at least one agent cannot receive an object from his demand set, contradicting (E-i). Thus, $p_B \geq p_B^{\min}$ or $p_D \geq p_D^{\min}$. To prove $p_B \geq p_B^{\min}$ and $p_D \geq p_D^{\min}$, we derive the contradiction to the cases, $p_B < p_B^{\min}$ and $p_D \geq p_D^{\min}$, and $p_B \geq p_B^{\min}$ and $p_D < p_D^{\min}$, separately.

If $p_B < p_B^{\min}$, then by $p_A \geq 0$, $p_C \geq 0$, and $p_D \geq p_D^{\min}$, both agents 1 and 2 prefer (B, p_B) to $\mathbf{0}$, (A, p_A), (C, p_C) and (D, p_D). In such a case, one of agents 1 and 2 cannot receive the demanded object, contradicting (E-i).

If $p_D < p_D^{\min}$, then by $p_A \geq 0$, $p_B \geq p_B^{\min}$, and $p_C \geq 0$, both agents 1 and 3 prefer (D, p_D) to $\mathbf{0}$, (A, p_A), (B, p_B) and (C, p_C). In such a case, one of agents 1 and 3 cannot receive the demanded object, contradicting (E-i).

Thus, $p_B \geq p_B^{\min}$ and $p_D \geq p_D^{\min}$. Thus, $p \geq p^{\min}$ and $(z, p^{\min})$ is a minimum price equilibrium.

In the following, we state the characterization of minimum price equilibria by means of the "overdemanded" and "(weakly) underdemanded" sets. These concepts are important to derive the structure of minimum price equilibria for common-object-ranking and common-object-ranking preference profiles in the next subsection.

Given p and $M' \subseteq M$, let $N^D(p, M') \equiv \{i \in N : D(R_i, p) \subseteq M'\}$ and $N^{WD}(p, M') \equiv \{i \in N : D(R_i, p) \cap M' \neq \varnothing\}$.

Example 1. Figure 6 illustrates $N^D(p, M')$ and $N^{WD}(p, M')$ for $M' = \{A\}$, $\{A, B\}$ and $\{A, B, D\}$. For $M' = \{A\}$, we have $N^D(p^{\min}, \{A\}) = \varnothing$ and $N^{WD}(p^{\min}, \{A\}) = \{1\}$. For $M' = \{A, B\}$, we have $N^D(p^{\min}, \{A, B\}) = \{2\}$ and $N^{WD}(p^{\min}, \{A, B\}) = \{1, 2\}$. For $M' = \{A, B, D\}$, we have $N^D(p^{\min}, \{A, B, D\}) = \{1, 2, 3\}$ and $N^{WD}(p^{\min}, \{A, B, D\}) = \{1, 2, 3\}$.

Given a set S, $|S|$ denotes the cardinality of S.

Definition: (i) A non-empty set $M' \subseteq M$ of objects is **overdemanded at** p for R if $\left|N^D(p, M')\right| > |M'|$.
(ii) A non-empty set $M' \subseteq M$ of objects is (**weakly**) **underdemanded at** p for R if

$$[\forall x \in M', p_x > 0] \Rightarrow \left|N^{WD}(p, M')\right| (\leq) < |M'|\,.$$

By using "overdemanded set" and "(weakly) underdemanded set", we can characterize the minimum equilibrium price vector.

Fact 3 (Morimoto and Serizawa, 2015).[12] Let $R \in \mathcal{R}^n$. A price vector p is a minimum equilibrium price vector for R if and only if no set is overdemanded and no set is weakly underdemanded at p for R.

Example 2. Figure 6 illustrates Fact 3. First, $\left|N^D(p^{\min}, \{A\})\right| = 0 < |\{A\}| = 1$ and $\left|N^D(p^{\min}, \{C\})\right| = 0 < |\{C\}| = 1$. Similarly, $\{B\}$ nor $\{D\}$ are overdemanded. Second, $\left|N^D(p^{\min}, \{A, B\})\right| = 1 < |\{A, B\}| = 2$ and $\left|N^D(p^{\min}, \{A, C\})\right| = 0 < |\{A, C\}| = 2$. Similarly, neither of $\{A, D\}$, $\{B, D\}$, $\{B, C\}$, and $\{C, D\}$ are overdemanded. Third, $\left|N^D(p^{\min}, \{A, B, C\})\right| =$

[12] Mishra and Talman (2010) established the parallel characterization for quasi-linear preferences.

$1 < |\{A,B,C\}| = 3$ and $\left|N^D(p^{\min},\{A,B,D\})\right| = 3 \leq |\{A,B,D\}| = 3$. Similarly, $\{A,C,D\}$ nor $\{B,C,D\}$ are overdemanded. Thus, no set is overdemanded. For the objects with positive prices, namely, B and D, $\left|N^{WD}(p^{\min},\{B\})\right| = 2 > |\{B\}| = 1$, $\left|N^{WD}(p^{\min},\{D\})\right| = 2 > |\{D\}| = 1$, and $\left|N^{WD}(p^{\min},\{B,D\})\right| = 3 > |\{B,D\}| = 2$. Thus, no set of is weakly underdemanded.

3.2 Structures of equilibria for common-object-ranking and common-tired-object preference profiles

In this subsection, we state the structures of the equilibria for common-object-ranking and common-object-ranking preference profiles, with special attention to the minimum price ones. The results in this subsection will help us better understand the equilibrium properties of Alonso-type housing market in urban economics,[13] and theoretically support the observations of the "higher-floor premium" for condominiums documented by empirical works.[14] Higher-floor premium says that in the same condominium, flats in higher floors are priced higher than those in lower floors. Note that these features in general do not hold for the classical preference profiles. The results in this subsection highlight the importance of the study of common-object-ranking and common-object-ranking preference profiles.

First, we focus on the structure of the equilibria for the common-object-ranking preference profile. In such equilibria, prices are monotonic along object-ranking.

Proposition 1 (Price monotonicity along object-ranking): Let $\mu \equiv \min\{n, m+1\}$. In an equilibrium (z,p) for $R \in (\mathcal{R}^R(\pi))^n$,
(i) if $m+1 \leq n$, then $p_{\pi(1)} > \cdots > p_{\pi(\mu)} = 0$, and all the objects are assigned,
(ii) if $m+1 > n$, then $p_{\pi(1)} > \cdots > p_{\pi(\mu)} \geq p_{\pi(\mu+1)} = \cdots = p_{\pi(m+1)} = 0$, objects ranked no lower than $\pi(\mu)$ are assigned, and objects ranked lower than $\pi(\mu)$ are unassigned, and
(iii) in the minimum price equilibrium, $p^{\min}_{\pi(\mu)} = \cdots = p^{\min}_{\pi(m+1)} = 0$.

The proof of Proposition 1 is relegated to the Appendix. Figure 7 illustrates the structure of minimum price equilibrium for $N = \{1,2,3\}$, $M = \{A,B,C,D\}$, $\pi = (\pi(1),\pi(2),\pi(3),\pi(4),\pi(5)) = (D,C,B,A,0)$, and $R \in (\mathcal{R}^R(\pi))^n$.

[13] See Subsection 1.2.2 for the literature that theoretically analyzes this model.

[14] See Subsection 1.2.3 for the empirical works that document higher-floor premium.

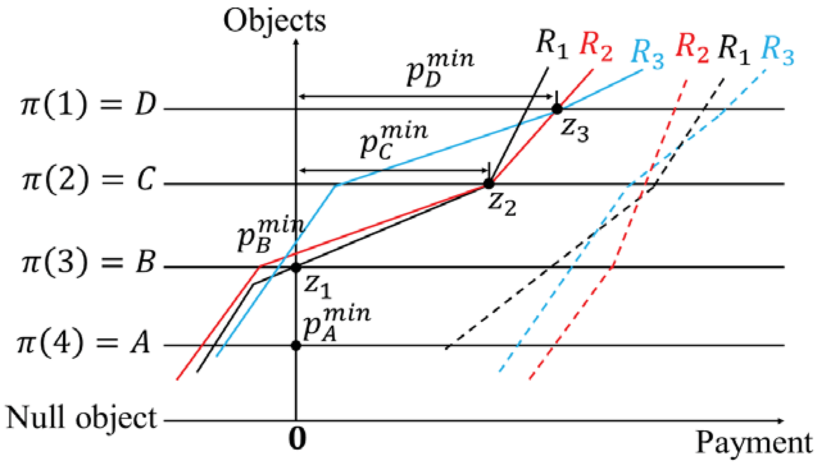

Figure 7 Illustration of minimum price equilibrium for preference profile from common-object-ranking domain

Similarly to Figure 6, $(z, p^{\min})$ is a minimum price equilibrium in Figure 7. Note that $\mu = 3$ and $p_D^{\min} > p_C^{\min} > p_B^{\min} = p_A^{\min} = 0 (= p_0^{\min})$. In addition, objects $\pi(1)$, $\pi(2)$, and $\pi(3)$ are assigned, but $\pi(4)$ and $\pi(5)$ are unassigned.

In a case where a preference profile is defined by Remark 3, it is straightforward to see that Proposition 1 holds. Fact 4 states that objects are assigned according to agents' incomes.

Fact 4 (Kaneko, 1983; Kaneko et al, 2006; Sai, 2015) (**Assortative assignment**). Let $\mu \equiv \min\{n, m+1\}$ and $R = (R_1, \ldots, R_n)$ be the preference profile defined in Remark 3 such that $d_1 > d_2 > \cdots > d_n$. Then, in an equilibrium (z, p) for R,
(i) if $m + 1 \leq n$, then $x_1 = \pi(1), \cdots, x_m = \pi(m)$ and $x_{m+1} = \cdots = x_n = 0$, and
(ii) if $m + 1 > n$, then $x_1 = \pi(1), \cdots, x_n = \pi(n)$.

The proof of Fact 4 is relegated to the Appendix. By Remark 5(ii), the common-object-ranking preference profile is a special case of common-tiered-object preference profile. In the following, we extend the results of Proposition 1 to a more general case, the structure of the equilibria for common-tired-object preference profiles.

Proposition 2 (Higher-floor premium). Let $\mu \equiv \min\{n, m+1\}$. Let $R \in (\mathcal{R}^T(\mathcal{T}))^n$, and $l_0 \in K$ be such that $\sum_{l=1}^{l_0-1} |T_l| < \mu \leq \sum_{l=1}^{l_0} |T_l|$. In an equilibrium (z, p) for R,
(i) if $l < l_0$, then for each $x \in T_l$, $p_x > 0$ and x is assigned to some agent,
(ii) if $l < l' \leq l_0$, then $\min\{p_x : x \in T_l\} > \max\{p_x : x \in T_{l'}\}$,
(iii) if $l > l_0$, then for each $x \in T_l$, $p_x = 0$ and x is unassigned, and
(iv) in the minimum price equilibrium, there is $x \in T_{l_0}$ such that $p_x^{\min} = 0$ and x is assigned to some agent.

The proof of Proposition 2 is relegated to the Appendix. Figure 8 illustrates the structure of minimum price equilibrium for $N = \{1, 2, 3\}$, $M = \{A, B, C, D\}$, $\mathcal{T} = T_1 \cup T_2 \cup T_3$ with $T_1 = \{C, D\}$, $T_2 = \{A, B\}$, and $T_3 = \{0\}$, and $R \in (\mathcal{R}^T(\mathcal{T}))^n$.

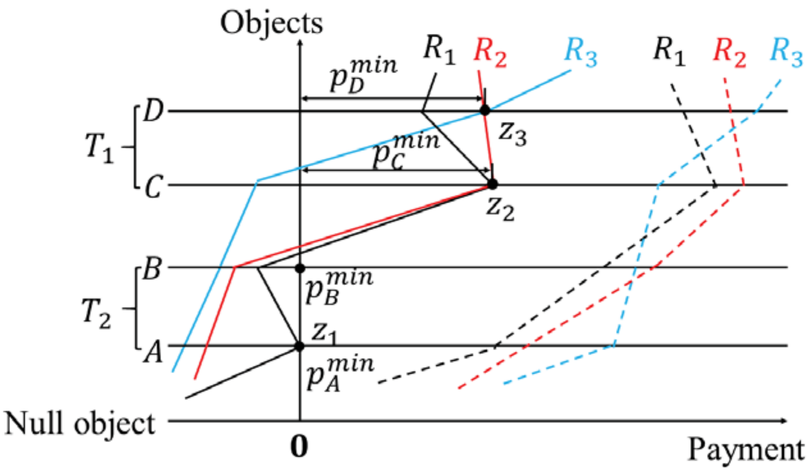

Figure 8 Illustration of minimum price equilibrium for preference profile from common-tiered-object domain

Similarly to Figure 6, $(z, p^{\min})$ is a minimum price equilibrium in Figure 8. In this case, $\mu = 3$ and $l_0 = 2$. Note that $\min\{p_C^{\min}, p_D^{\min}\} > p_A^{\min} = p_B^{\min} = 0(= p_0^{\min})$. In addition, objects in T_1 and T_2 are all assigned and objects in T_3 are unassigned.

An interesting case covered by Proposition 2 is the housing market economy analyzed by Kaneko (1983), Kaneko et al (2006), Sai (2015), etc., where objects in the same tier are the copies of the same objects. They assume that (a) K is the set of original objects ($|K| = k$), and for each $l \in K$, $|T_l|$ is the number of the copies of the original object l, and (b) a preference profile R is specified by Remark 3 and $d_1 > \cdots > d_n$. Particularly, for each $i \in N$, each $l \in K$, each pair $x, y \in T_l$, and each $t \in \mathbb{R}$, $(x, t)\, I_i\, (y, t)$. It is straightforward to see that Proposition 2 holds. In addition, let p be an equilibrium price for a preference profile satisfying this condition. Then for each $l \in K$ and each pair $x, y \in T_l$, we also have $p_x = p_y$. For the properties of the object assignment, we have Fact 5.

Fact 5 (Kaneko, 1983; Kaneko et al, 2006; Sai, 2015) (**Assortative assignment**).[15] Assume conditions (a) and (b) mentioned above. Let $l_0 \in K$ be such that if $n \leq \sum_{l=1}^{k} |T_l|$, then $\sum_{l=1}^{l_0-1} |T_l| < n \leq \sum_{l=1}^{l_0} |T_l|$ and if $n > \sum_{l=1}^{k} |T_l|$, then $l_0 = k$. In an equilibrium for R,
(i) for each $i = 1, \cdots, |T_1|$, we have $x_i \in T_1$,
(ii) for each $i = |T_1| + 1, \cdots, |T_1| + |T_2|$, we have $x_i \in T_2$, and
(iii) for each $i = \sum_{l=1}^{l_0-1} |T_l| + 1, \cdots, \sum_{l=1}^{l_0} |T_l|$, we have $x_i \in T_{l_0}$.

Remark 7: Kaneko (1983), Kaneko et al, (2006), and Sai (2015) show the parallel results of Propositions 1 and 2 by assuming the preference profile as defined by Remark 3. Our assumptions on preferences are weaker than theirs. Thus, their results do not imply ours but are special cases of ours.

[15] The proof is similar to Fact 4.

4 Efficient and strategy-proof rules

4.1 Equilibrium rules and dominance

First we define the rules based on the equilibrium selections and state their properties.

Definition: A rule f on $\mathcal{R}^n$ is called an **equilibrium rule** if for each $R \in \mathcal{R}^n$, $f(R) \in Z(R)$.

Definition: A rule f on $\mathcal{R}^n$ is called a **minimum price (MP) rule** if for each $R \in \mathcal{R}^n$, $f(R) \in Z^{\min}(R)$.

Definition: Let f and g be two rules on $\mathcal{R}^n$. Then f **weakly dominates** g in **agents' welfare** if for each $R \in \mathcal{R}^n$, $f(R)$ *weakly dominates* $g(R)$ *in agents' welfare*.

Fact 6 is straightforward from the definition of equilibrium.

Fact 6. Let $\mathcal{R} \subseteq \mathcal{R}^C$. An equilibrium rule on $\mathcal{R}^n$ satisfies *efficiency*, *individual rationality,* and *no subsidy (no subsidy for losers).*

Fact 7 distinguishes MP rules from other equilibrium rules by *dominance in agents' welfare.* It says that on the classical domain, MP rules are agent-optimal among all the equilibrium rules.

Fact 7 (**Agent-optimality of MP rule**). Let $\mathcal{R} \subseteq \mathcal{R}^C$. A minimum price rule *weakly dominates* any equilibrium rule *in agents' welfare* on $\mathcal{R}^n$.

The proof of Fact 7 is relegated to the Appendix. Next, we demonstrate that MP rules are also distinguished from other equilibrium rules by *strategy-proofness.* Fact 8 says that MP rules are *strategy-proof.*

Fact 8 (Demange and Gale, 1985). Let $\mathcal{R} \subseteq \mathcal{R}^C$. A minimum price rule on $\mathcal{R}^n$ satisfies *strategy-proofness.*

From now on, instead of paying our attention to equilibrium rules, we analyze general rules satisfying the above properties.

Fact 9 (Morimoto and Serizawa, 2015). Let $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$. Let f be a rule on $\mathcal{R}^n$ satisfies *strategy-proofness*, *efficiency*, *individual rationality,* and *no subsidy (no subsidy for losers).* Then, f *weakly dominates* a minimum price rule on $\mathcal{R}^n$ *in agents' welfare.*

Facts 7 and 9 imply Fact 10.

Fact 10. Let $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$. Let f be a rule on $\mathcal{R}^n$ satisfies *strategy-proofness*, *efficiency*, *individual rationality,* and *no subsidy (no subsidy for losers).* Then, f *weakly dominates* any equilibrium rule on $\mathcal{R}^n$ *in agents' welfare.*

Fact 10 says that on the classical domain, for a rule satisfying *strategy-proofness* in addition to *efficiency*, *individual rationality* and *no subsidy (no subsidy for losers)*, its selected allocation is as good as any equilibrium for each agent. Thus, when *strategy-proofness* is imposed together with *efficiency*, *individual rationality* and *no subsidy (no subsidy for losers)*, available candidates of rules are limited. By Facts 6 and 8, MP rules are such candidates and the selected allocations form the lower bounds of agents' welfare.

This is an important implication when one intends to design *strategy-proof* rules on the classical domain. Note that as exemplified in Subsection 3.2, there are interesting subdomains of the classical one, and the requirement of *strategy-proofness* is weaker on smaller domains. Thus, it is an important question whether this implication still holds on restricted domains, i.e., domains smaller than the classical one.

Thus, we consider the following restricted domains: common-object-ranking domains, $(\mathcal{R}^R(\pi))^n$, and common-tiered-object domains, $(\mathcal{R}^T(\mathcal{T}))^n$. In addition, we also consider the subdomains of $(\mathcal{R}^R(\pi))^n$ and $(\mathcal{R}^T(\mathcal{T}))^n$ that are restricted by normality of preferences, *i.e.*, the normal and common-object-ranking domains, $(\mathcal{R}^{NR}(\pi))^n$, and the normal and common-tiered-object-ranking domains, $(\mathcal{R}^{NT}(\mathcal{T}))^n$. Theorem 1 says that the implication of *strategy-proofness* in Fact 10 holds on such various restricted domains with some additional assumptions.

Theorem 1: Let $\mathcal{R} \subseteq \mathcal{R}^C$. Let f satisfy *efficiency*, *strategy-proofness*, *individual rationality* and *no subsidy for losers* on $\mathcal{R}^n$.
(i) Let $\mathcal{R} = \mathcal{R}^R(\pi)$ or $\mathcal{R}^{NR}(\pi)$. If $n > m$ or if f satisfies *no subsidy* on $\mathcal{R}^n$, then f *weakly dominates* any equilibrium rule on $\mathcal{R}^n$ *in agents' welfare*.
(ii) Let $\mathcal{T}$ be a tier partition such that $2 < k < m+1$. Let $\mathcal{R} = \mathcal{R}^T(\mathcal{T})$ or $\mathcal{R}^{NT}(\mathcal{T})$. If $n > m$, or if $l_0 \in K$ is such that $\sum_{l=1}^{l_0-1} |T_l| < \mu \leq \sum_{l=1}^{l_0} |T_l|$, $|T_{l_0}| = 1$ and f satisfies *no subsidy* on $\mathcal{R}^n$, then f *weakly dominates* any equilibrium rule on $\mathcal{R}^n$ *in agents' welfare*.

The proof of Theorem 1 is relegated to the Appendix. In Theorem 1 (ii), $|T_{l_0}| = 1$ implies that the worst object in the assigned objects is common to all the agents.

Two open questions relating to Theorem 1 remain. The first one is whether the above implication of *strategy-proofness* still holds when we further restrict agents' preferences by assuming that agents are indifferent between the same copies of each object. As we discussed in Fact 5, these preferences are used in the housing market economy analyzed by Kaneko (1983), Kaneko et al (2006), and Sai (2015), etc. Our proof of Theorem 1 depends on the assumption that objects are heterogeneous even in the same tier. Thus, although we believe that the implication of *strategy-proofness* still holds in their housing market economy, we have not yet established it.

The second one is on $n > m$ and *no subsidy*. Our proof of Theorem 1 (i) depends on $n > m$ or *no subsidy*. That of Theorem 1 (ii) also depends on $n > m$, or *no subsidy* combined with the assumption that the worst object in the assigned objects is common to all the agents, i.e., $|T_{l_0}| = 1$. Thus, although we also believe that the implication of *strategy-proofness* still holds without these restrictions, we have not yet established it.

4.2 Characterizations of minimum price rule on the common-object-ranking and common-tiered-object domains

By Fact 10, when *strategy-proofness* is imposed together with *efficiency*, *individual rationality* and *no subsidy (no subsidy for losers)*, available candidates for rules on the classical domain are limited. MP rules are such candidates. Fact 11 says that there is no other candidate than MP rules on the classical domain. When *efficiency*, *individual rationality* and *no subsidy (no subsidy for losers)* are required on the classical domain, MP rules are not only the lower bounds of agents' welfare under *strategy-proof* rules, but also their upper bounds.

Fact 11 (Morimoto and Serizawa, 2015). Let $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$. A rule f on $\mathcal{R}^n$ satisfies *efficiency*, *strategy-proofness*, *individual rationality,* and *no subsidy for losers (no subsidy)* if and only if it is a minimum price rule.

Although *strategy-proofness* is weaker on smaller domains, by Theorem 1, when *strategy-proofness* is imposed together with *efficiency*, *individual rationality,* and *no subsidy (no subsidy for losers)*, available candidates of rules on various restricted domains are still limited. MP rules are always such candidates. Thus, it is an interesting question whether there are any other candidates on restricted domains, where the requirement of *strategy-proofness* is weaker.

First, we consider the common-object-ranking domain $(\mathcal{R}^R(\pi))^n$.

Theorem 2: Let $\mathcal{R} \equiv \mathcal{R}^R(\pi)$. (i) Let $n > m$. A rule f on $\mathcal{R}^n$ satisfies *efficiency*, *strategy-proofness*, *individual rationality,* and *no subsidy for losers* if and only if it is a minimum price rule.
(ii) A rule f on $\mathcal{R}^n$ satisfies *efficiency*, *strategy-proofness*, *individual rationality,* and *no subsidy* if and only if it is a minimum price rule.

The proof of Theorem 2 is relegated to the Appendix.

Remark 8: (i) Theorem 2 (i) is the parallel result of Fact 11 on $(\mathcal{R}^R(\pi))^n$. In addition, similarly to Morimoto and Serizawa (2015), when $n > m$, we can also show that *efficiency*, *strategy-proofness*, *individual rationality,* and *no subsidy for losers* imply *no subsidy* on $(\mathcal{R}^R(\pi))^n$.
(ii) Theorem 2 (ii) is independent of Theorem 2 (i) in the following points. First, Theorem 2 (ii) does not assume $n > m$. Second, Theorem 2 (ii) uses *no subsidy*, which is stronger than *no subsidy for losers*. Third, in the case where $n \leq m$, *efficiency*, *strategy-proofness*, *individual rationality,* and *no subsidy for losers* do not imply *no subsidy* on $(\mathcal{R}^R(\pi))^n$. To see this, consider a MP rule with negative entry fee on a common-object-ranking domain. This rule satisfies *efficiency*, *strategy-proofness,* and *individual rationality.*[16] Since $n \leq m$, no agent is a loser and the rule satisfies *no subsidy for losers*. However, by Proposition 1, for the agent who receives $\pi(n)$, he receives a subsidy (the negative entry fee).

Next, we consider the common-tiered-object domain $(\mathcal{R}^T(\mathcal{T}))^n$ with respect to an indexed family of tiers $\mathcal{T} = \{T_i\}_{i \in K}$ with $|K| = k$. Recall that $2 \leq k \leq m + 1$. By Remark 5 (ii), Theorem 2 implies the characterization result of the MP rule on $(\mathcal{R}^T(\mathcal{T}))^n$ for $k = m + 1$. Similarly, Fact 11 implies the characterization result of the MP rule on $(\mathcal{R}^T(\mathcal{T}))^n$ for $k = 2$. Thus, we focus on the case where $2 < k < m + 1$. However, we need an additional assumption $n > m$ to establish the characterization result on $(\mathcal{R}^T(\mathcal{T}))^n$.

Theorem 3: Let $\mathcal{R} \equiv \mathcal{R}^T(\mathcal{T})$ be such that $2 < k < m + 1$. (i) Let $n > m$. Then, a rule f on $\mathcal{R}^n$ satisfies *efficiency*, *strategy-proofness*, *individual rationality,* and *no subsidy for losers* if and only if it is a minimum price rule.
(ii) Let $l_0 \in K$ be such that $\sum_{l=1}^{l_0-1} |T_l| < \mu \leq \sum_{l=1}^{l_0} |T_l|$. Let $|T_{l_0}| = 1$. Then, a rule f on $\mathcal{R}^n$ satisfies *efficiency*, *strategy-proofness*, *individual rationality,* and *no subsidy* if and only if it is a minimum price rule.

The proof of Theorem 3 is relegated to the Appendix. By the parallel argument to Remark

[16] See Morimoto and Serizawa (2015).

8 (ii), Theorem 3 (i) and (ii) are independent. By Fact 11, Theorems 2 (i) and Theorem 3 (i), we have:

Corollary 1: Let $n > m$ and $\mathcal{R} \equiv \mathcal{R}^T(\mathcal{T})$ such that $2 \leq k \leq m+1$. Then a rule f on $\mathcal{R}^n$ satisfies *efficiency*, *strategy-proofness*, *individual rationality,* and *no subsidy for losers* if and only if it is a minimum price rule.

Since *no subsidy* is stronger than *no subsidy for losers*, Fact 6 implies that Theorem 2 (ii) holds for the case of $k = 2$. Combining with Theorems 2 (ii) and 3 (ii), we have:

Corollary 2: Let $\mathcal{R} \equiv \mathcal{R}^T(\mathcal{T})$ such that $2 \leq k \leq m+1$ and $|T_{l_0}| = 1$. Then a rule f on $\mathcal{R}^n$ satisfies *efficiency*, *strategy-proofness*, *individual rationality,* and *no subsidy* if and only if it is a minimum price rule.

Next, we consider normal and common-object-ranking domains, $(\mathcal{R}^{NR}(\pi))^n$, and normal and common-tiered-object domains, $(\mathcal{R}^{NT}(\mathcal{T}))^n$. To characterize MP rules on those domains, we make an additional assumption that $n \leq m+1$.

Theorem 4: Let $\mathcal{R} \equiv \mathcal{R}^{NR}(\pi)$ and $n \leq m+1$. Then, a rule f on $\mathcal{R}^n$ satisfies *efficiency*, *strategy-proofness*, *individual rationality,* and *no subsidy* if and only if it is a minimum price rule.

Theorem 5: Let $\mathcal{R} \equiv \mathcal{R}^{NT}(\mathcal{T})$ be such that $2 < k < m+1$. Let $n \leq m+1$ and $|T_{l_0}| = 1$. Then, a rule f on $\mathcal{R}^n$ satisfies *efficiency*, *strategy-proofness*, *individual rationality,* and *no subsidy* if and only if it is a minimum price rule.

The proofs of Theorems 4 and 5 are relegated to the Appendix. By Theorems 4 and 5, we have:

Corollary 3: Let $\mathcal{R} \equiv \mathcal{R}^{NT}(\mathcal{T})$ such that $2 < k \leq m+1$ and $|T_{l_0}| = 1$. Then a rule f on $\mathcal{R}^n$ satisfies *efficiency*, *strategy-proofness*, *individual rationality,* and *no subsidy* if and only if it is a minimum price rule.

The "only if" parts of Theorems 2, 3, 4, and 5 fail if we drop any one of their axioms. In the following, we show the independence of axioms for Theorem 2(ii). Independence of axioms for other results can be shown similarly.

Example 3 (*Dropping Efficiency*). Let f be the "no-trade rule" such that for each preference profile, it assigns $(0, 0)$ to each agent. Then, f satisfies *strategy-proofness*, *individual rationality*, and *no subsidy*, but not *efficiency.*

Example 4 (*Dropping Strategy-proofness*). Let f be the "maximum equilibrium rule" such that for each preference profile, it selects the maximum price equilibrium. By Facts 1 and 2, for each preference profile, there is a unique maximum equilibrium price. Then, f satisfies *efficiency*, *individual rationality*, and *no subsidy*, but not *strategy-proofness.*

Example 5 (*Dropping Individual rationality*). Let f be the MP rule with positive entry fee for each agent and $n > m$. Then, f satisfies *efficiency*, *strategy-proofness*, and *no subsidy*, but not *individual rationality.*[17]

[17] $n \geq m+1$ implies that there is $i \in N$ such that $x_i = 0$. Since i pays a positive entry fee e_i, then, $(0, 0)\, P_i\, (0, e_i)$, violating individual rationality.

Example 6 (*Dropping No subsidy*). Let f be the MP rule with negative entry fee for each agent and $n \leq m$. Then, f satisfies *efficiency*, *strategy-proofness*, and *individual rationality*, but not *no subsidy*.[18]

In Subsection 4.1, we discussed two open questions related to Theorem 1. Since Theorems 2 and 3 depend on Theorem 1, the same types of questions remain open for these theorems. We mention additional open questions related to Theorems 4 and 5. It is an open question whether only MP rules satisfy the four properties of Theorems 4 and 5 on the normal and common-object-ranking domains, $(\mathcal{R}^{NR}(\pi))^n$, and the normal and common-tiered-object domains, $(\mathcal{R}^{\mathcal{NT}}(\mathcal{T}))^n$, when $n > m + 1$.

In spite of these open questions, Theorems 2, 3, 4, and 5 cover various interesting subdomains of the classical one. They imply that when one intends to design rules satisfying *efficiency*, *strategy-proofness*, *individual rationality* and *no subsidy (no subsidy for losers)* on such domains, the only possibility is MP rules.

5 Concluding Remark

Morimoto and Serizawa (2015) demonstrated that the minimum price rule is an important rule for governments to allocate objects where agents have non-quasi-linear preferences. They also showed that the outcome selected by the minimum price rule coincides with that of the simultaneous ascending auction, which is adopted by governments in many countries to allocate spectrum licenses, etc. Thus, their results support the adoption of simultaneous ascending auctions by governments. The implications of our results are similar to theirs, but cover practically important cases which are not covered by theirs, especially for the housing markets and some cases of the spectrum licenses allocation that are exemplified in Introduction and demonstrated in Subsection 3.2. Thus, our results also highly advocate the adoption of simultaneous ascending auction in those situations.

Furthermore, we show that the minimum price equilibria for common-object-ranking and common-object-ranking preference profiles have special structures. The parallel results have already been established by Kaneko (1983), Kaneko et al (2006), and Sai (2015), but by assuming much stronger conditions on preference profiles. Our results would be useful when we conduct the minimum price rule in more general environments.

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Appendix

The structure of Appendix is as follows: In Section A1, we provide the omitted proofs of Remark 2, Proposition 1, Fact 4, Proposition 2, and Fact 7. In Section A2, we show the proofs of our main theorems.

Section A1: Proofs of Remark 2, Proposition 1, Fact 4, Proposition 2, and Fact 7

Subsection A1.1: Proof of Remark 2

(i) First, we prove the first statement in (i). By contradiction, suppose that there are $t_i' < t_i$ and $i < j$ such that $(\pi(j), t_i')\, R_i\, (\pi(i), t_i')$. Since $(\pi(i), t_i)\, P_i\, (\pi(j), t_i)$, by $R_i \in \mathcal{R}^C$, there is $d > 0$ such that $(\pi(i), t_i + d)\, I_i\, (\pi(j), t_i)$. Since $t_i' < t_i$, we have $t_i - t_i' > 0$. By normality, we have $(\pi(i), t_i + d - (t_i - t_i'))\, P_i\, (\pi(j), t_i - (t_i - t_i'))$. That is $(\pi(i), d + t_i')\; P_i\, (\pi(j), t_i')$. By $R_i \in \mathcal{R}^C$, $(\pi(i), t_i')\; P_i\, (\pi(j), d + t_i')$. Thus $(\pi(i), t_i')\; P_i\, (\pi(j), t_i')$, a contradiction.

Then, we show the second statement in (i) by providing an example. Let $M = \{x\}$ and $R_i \in \mathcal{R}^C$ be such that for each $t_i \in \mathbb{R}$ such that $t_i < 1$, $V_i(x, (0, t_i)) = 1 - (1 - t_i)/2$, and otherwise, $V_i(x, (0, t_i)) = t_i$. Then, for each $t_i \in \mathbb{R}$ such that $t_i < 1$, $(x, t_i)\, P_i\, (0, t_i)$. However, for each $t_i \in \mathbb{R}$ such that $t_i \geq 1$, $(x, t_i)\, I_i\, (0, t_i)$.

(ii) We provide an example to show (ii). Let $R_i \in \mathcal{R}^C$ be a preference such that for each $t_i \in \mathbb{R}$ and each $x \in M$, $(x, t_i)\, I_i\, (0, t_i)$. Then, R_i satisfies normality, but does not rank objects according to any π.

(iii) To see this, first we show that for each $t_i \in \mathbb{R}$, each pair $x, y \in L$, if $(x, t_i)\, P_i\, (y, t_i)$, then for each $t_i' \in \mathbb{R}$, $(x, t_i')\, R_i\, (y, t_i')$. We ignore the trivial case of $t_i' = t_i$.

Case 1: $t_i' \in \mathbb{R}$ such that $t_i' < t_i$. Similarly to (i), we can show $(x, t_i')\, P_i\, (y, t_i')$. Thus we have $(x, t_i')\, R_i\, (y, t_i')$.

Case 2: $t_i' \in \mathbb{R}$ such that $t_i' > t_i$. By contradiction, suppose that $(y, t_i')\, P_i\, (x, t_i')$. Then we have $(x, t_i'')\, I_i\, (y, t_i')$ for some $t_i'' < t_i'$. By $(x, t_i)\, P_i\, (y, t_i)$, we have $(x, t_i''')\, I_i\, (y, t_i)$ for some $t_i''' > t_i$. Since

$$(x, t_i''')\, I_i\, (y, t_i) \underset{t_i' > t_i}{P_i} (y, t_i')\, I_i\, (x, t_i''),$$

we have $t_i''' < t_i''$. Let $d = t_i'' - t_i'''$. Then,

$$(y, t_i' - d) \underset{\text{normality}}{P_i} (x, t_i'' - d) = (x, t_i''')\, I_i\, (y, t_i).$$

Since $t_i' > t_i'' > t_i''' > t_i$ and $d = t_i'' - t_i'''$, we have $t_i' - d > t_i$. By $(y, t_i' - d)\, P_i\, (y, t_i)$, this contradicts money monotonicity.

Then, by the above result, for each pair $x, y \in L$, we have: (a) for each $t_i \in \mathbb{R}$, $(x, t_i)\, R_i\, (y, t_i)$, or (b) for each $t_i \in \mathbb{R}$, $(y, t_i)\, R_i\, (x, t_i)$. Thus, we can define a binary relation r_i over objects by: for each pair $x, y \in L$, if (a) holds, $x\, r_i\, y$, and otherwise $y\, r_i\, x$. Transitivity of R_i implies Transitivity of r_i. Thus, for some object permutation π, we have $\pi(1)\, r_i \cdots r_i\, \pi(m+1)$. Thus $(\pi(1), t_i)\, R_i \cdots R_i\, (\pi(m+1), t_i)$. **Q.E.D.**

Subsection A1.2: Proof of Proposition 1

(i) Let $m+1 \leq n$. Then, $\mu = m+1$ implies $\pi(\mu) = 0$ and $p_{\pi(\mu)} = 0$. $m+1 \leq n$ implies that there is $i \in N$ such that $x_i = 0$. If there is $x \in M$ such that $p_x = 0$, then $(x, p_x)\, P_i\, \mathbf{0}$, contradicting $0 \in D(R_i, p)$. Thus, for each $x \in M$, we have $p_x > 0$. Thus, by (E-ii), all the objects are assigned. To see $p_{\pi(1)} > \cdots > p_{\pi(\mu)}$, by contradiction, suppose that there is a pair $x, y \in M$ such that $y >_\pi x \geq_\pi \pi(\mu)$ and $p_x \geq p_y$. Let $j \in N$ be such that $x_j = x$. By (E-i), we have $x \in D(R_j, p)$. By $R_j \in \mathcal{R}^R(\pi)$, we have $(y, p_y)\, P_j\, (x, p_x)$, contradicting $x \in D(R_j, p)$. Thus $0 = p_{\pi(\mu)} < \cdots < p_{\pi(1)}$.

(ii) Let $m+1 > n$. Then, $\mu = n$. Suppose that there is $x \in M$ such that $x \geq_\pi \pi(\mu)$ and x is unassigned. Then, there are $i \in N$ and $y <_\pi \pi(\mu)$ such that $x_i = y$. By (E-i), $y \in D(R_i, p)$. Since x is unassigned, by (E-ii), we have $p_x(R) = 0$. Thus, by $R_i \in \mathcal{R}^R(\pi)$, we have $(x, p_x)\, P_i\, (y, p_y)$, contradicting $y \in D(R_i, p)$. Thus, for each $x \in M$ such that $x \geq_\pi \pi(\mu)$, x is assigned to some agent. By $\mu = n$, objects ranked lower than $\pi(\mu)$ are unassigned. Thus, by (E-ii), we have $p_{\pi(\mu+1)} = \cdots = p_{\pi(m+1)} = 0$. Similarly to (i), we can show $p_{\pi(1)} > \cdots > p_{\pi(\mu)}$ and obviously, $p_{\pi(\mu)} \geq 0$.

(iii) If $m+1 \leq n$, (iii) follows from (i). Thus, let $m+1 > n$. We only need to prove $p^{\min}_{\pi(\mu)} = 0$, since the proofs of other statements are same as (ii).

Suppose that $p^{\min}_{\pi(\mu)} > 0$. Since $m+1 > n$, by (ii), we have $p_{\pi(1)} > \cdots > p_{\pi(\mu)} > 0$ and

$$\left|\{i \in N : D(R_i, p) \cap \{\pi(1), \cdots, \pi(\mu)\} \neq \varnothing\}\right| = n = \left|\{\pi(1), \cdots, \pi(\mu)\}\right|.$$

Thus $\{\pi(1), \cdots, \pi(\mu)\}$ is weakly underdemanded, contradicting Fact 3. **Q.E.D.**

Subsection A1.3: Proof of Fact 4

Suppose that $x_1 \neq \pi(1)$. Let $i \in N$ be such that $x_i = \pi(1)$. By (E-i), for each $x \in L$, $(\pi(1), p_{\pi(1)})\ R_i\ (x, p_x)$. Thus, by normality and $d_1 > d_i$, for each $x \in L \backslash \{\pi(1)\}$, $(\pi(1), p_{\pi(1)})\ P_1\ (x, p_x)$. Thus, $D(R_1, p) = \{\pi(1)\}$. Since $x_1 \neq \pi(1)$, it contradicts (E-i). Thus, $x_1 = \pi(1)$.

Suppose that $x_2 \neq \pi(2)$. Let $i \in N$ be such that $x_i = \pi(2)$. By $x_1 = \pi(1)$, we have $i \neq 1$. Thus, $d_2 > d_i$. By (E-i), for each $x \in L$, $(\pi(2), p_{\pi(2)})\ R_i\ (x, p_x)$. Thus, by normality and $d_2 > d_i$, for each $x \in L \backslash \{\pi(1), \pi(2)\}$, we have $(\pi(2), p_{\pi(2)})\ P_2\ (x, p_x)$. Thus, for each $x \in L \backslash \{\pi(1), \pi(2)\}$, $x \notin D(R_2, p)$. By $x_2 \neq \pi(2)$ and $x_1 = \pi(1)$, this contradicts (E-i). Thus, $x_2 = \pi(2)$. By repeating this argument, we can show (i) and (ii). **Q.E.D.**

Subsection A1.4: Proof of Proposition 2

(i) Let $l < l_0$. By contradiction, suppose there is $x \in T_l$ such that $p_x(R) = 0$. Then, by $l < l_0$, $\sum_{l'=1}^{l_0-1} |T_{l'}| < \mu \leq n$, and (z, p) is an equilibrium, there are $y \in M$ and $i \in N$ such that $y <_{\mathcal{T}} x$ and $y \in D(R_i, p)$. By $p_x = 0$ and $R_i \in \mathcal{R}^T(\mathcal{T})$, we have $(x, p_x)\, P_i\, (y, p_y)$, contradicting $y \in D(R_i, p)$. Thus, if $l < l_0$, then for each $x \in T_l$, $p_x > 0$, and by (E-ii), x is assigned to some agent.

(ii) Let $l < l' \leq l_0$, $x \in T_l$, and $y \in T_{l'}$ be such that $p_x = \min\{p_{x'} : x' \in T_l\}$ and $p_y = \max\{p_{y'} : y' \in T_{l'}\}$. By contradiction, suppose $p_x \leq p_y$. By (i) and $l < l_0$, we have $0 < p_x \leq p_y$. Thus, by (E-ii), there is $j \in N$ such that $y \in D(R_j, p)$. By $x >_{\mathcal{T}} y$, $R_j \in \mathcal{R}^T(\mathcal{T})$, and $p_x \leq p_y$, we have $(x, p_x)\, P_j\, (y, p_y)$, contradicting $y \in D(R_j, p)$. Thus, $p_x > p_y$.

(iii) Let $l > l_0$. By contradiction, suppose there is $x \in T_l$ such that $p_x > 0$. Then, by (E-ii), there is $i \in N$ such that $x_i = x \in D(R_i, p)$. This implies that $m+1 > n$, $\mu = n$, and

$n \leq \sum_{l'=1}^{l_0} |T_{l'}|$. Thus there are $l' < l$ and $y \in T_{l'}$ such that y is unassigned. By (E-ii), $p_y = 0$. By $R_i \in \mathcal{R}^T(\mathcal{T})$, we have $(y, p_y)\, P_i\, (x, p_x)$, contradicting $x \in D(R_i, p)$. Thus, if $l > l_0$, then for each $x \in T_l$, we have $p_x = 0$. In the following, we prove that x is also unassigned. Let $l > l_0$. By contradiction, suppose there are $i \in N$ and $x \in T_l$ such that $x_i = x \in D(R_i, p)$. This implies that $m + 1 > n$, $\mu = n$, and $n \leq \sum_{l=1}^{l_0} |T_l|$. Thus there are $l' < l$ and $y \in T_{l'}$ such that y is unassigned. By (E-ii), $p_y = 0$. By $R_i \in \mathcal{R}^T(\mathcal{T})$, we have $(y, p_y)\, P_i\, (x, p_x)$, contradicting $x \in D(R_i, p)$.

(iv) **Case 1**: $n \geq m + 1$. Since $\mu = m + 1$, $T_{l_0} = \{0\}$ and $p_0^{\min} = 0$. By $n \geq m + 1$, there is $i \in N$ such that $x_i = 0$.

Case 2: $n < m + 1$. Then $\mu = n$. If there is $i \in N$ such that $x_i = 0$, then by $n < m + 1$, there is an unassigned $x \in M$. By (E-ii), $p_x^{\min} = 0$. By $R_i \in \mathcal{R}^T(\mathcal{T})$, we have $(x, p_x^{\min})\, P_i\, (x_i, p_{x_i}^{\min})$, contradicting $x_i \in D(R_i, p^{\min}(R))$. Thus, for each $i \in N$, we have $x_i \in M$.

Let $M' \equiv \{x \in M : x = x_i \text{ for some } i \in N\}$. Then, $|M'| = n$. Suppose that for each $x \in M'$, $p_x^{\min} > 0$. Thus, by $|M'| = n$, M' is weakly underdemanded, contradicting Fact 3. Thus, there is $x \in M'$ such that $p_x^{\min} = 0$. By the definition of M', there is $i \in N$ such that $x = x_i$ and $p_x^{\min} = 0$.

Let $l \in K$ be such that $x \in T_l$. Since x is assigned, by (iii), we have $l \leq l_0$. Since $p_x^{\min} = 0$, by (i), we have $l > l_0 - 1$. Thus $l = l_0$. **Q.E.D.**

Subsection A1.5: Proof of Fact 7

Let f be a MP rule and g be an equilibrium rule on $\mathcal{R}^n$. Let p and p' be the price vectors associated with $f(R)$ and $g(R)$, respectively. Since $R \in (\mathcal{R}^C)^n$, f is a MP rule, and g is an equilibrium rule, Fact 2 implies $p \leq p'$. Let $i \in N$, $f_i(R) = (x, t_i)$ and $g_i(R) = (y, t_i')$. Since f and g are both equilibrium rules, then we have $x \in D(R_i, p)$, $t_i = p_x$ and $t_i' = p_y'$. Thus,

$$f_i(R) = (x, p_x) \underset{x \in D(R_i, p)}{R_i} (y, p_y) \underset{p \leq p'}{R_i} (y, p_y') = g_i(R).$$

Q.E.D.

Section A2: Proofs of Theorems 1 to 5.

Subsection A2.1: Proof of Theorem 1

The proof of Theorem 1 consists of Parts A, B and C. Part A corresponds to the proof of Theorem 1 (i) for the case where f satisfies *no subsidy*. Part B corresponds to the proof of Theorem 1 (ii) for the case where there is $l_0 \in K$ such that $\sum_{l=1}^{l_0 - 1} |T_l| < \mu \leq \sum_{l=1}^{l_0} |T_l|$ and $|T_{l_0}| = 1$, and f satisfies *no subsidy*. Part C corresponds to the proof of Theorem 1 for the cases not covered by Parts A and B.

As mentioned by Fact 9, our theorem is a parellel result of Morimoto and Serizawa (2015) on the restricted domain. Morimoto and Serizawa (2015) provides new proof techniques to deal with non-quasi-linearity. We owe to them some of their proof techniques and methods. However, we emphasize that our domains are smaller than theirs and their proofs often employ preferences outside our domains. Thus, even in the cases where their proof techniques can be applied, we need to modify them carefully, and in many cases we need to develop new proof techniques.

To be precise, Part A is different from Morimoto and Serizawa (2015) by three points. The first point is that we need to identify which objects should be allocated and the payment boundary of the lowest ranked object, by Lemmas A.1 to A.3. This is because we do not impose any assumptions on the number of agents and that of objects. Note that these lemmas are not needed in Morimoto and Serizawa (2015) and our results cannot be established in their domain and by their proof technique either.

The second point is on "semi-maskin transformation," which we use in our proofs. It is different from the z_i-favoring transformation used by them.[19] z_i-favoring transformations play important roles in Morimoto and Serizawa's (2015) proofs, but we cannot construct the z_i-favoring transformation on our domain. Besides, we use the property of the semi-maskin transformation to prove Lemmas A.6 and A.7. However, this is not the case in Morimoto and Serizawa (2015) and they use different logics and proof techniques to show the similar results of Lemmas A.5 to A.7.

The third point is that we use different proof techniques to show Proposition A.1. In Morimoto and Serizawa (2015), they prove the parallel result of Proposition A.1 on the basis of Lemma B.8 (in their paper). Their proof of Lemma B.8 uses non-normal preferences in a crucial way. However, non-normal preferences are excluded by our domain $\mathcal{R}^{NR}(\pi)$. To prove Proposition A.1, we need to construct a lemma with different contents, namely, Lemma A.8. We prove Lemma A.8 by using only normal preferences. Besides, using the price structure shown by Proposition 1 is also important for the proof of Proposition A.1, i.e., the construction of (1-iii-(b-3)).

Part B is a generalization of Part A. By same reasoning, it is also different from Morimoto and Serizawa (2015). Since the basic proof idea follows Part A, with some modifications, to avoid redundancy, we directly provide the main result, without writing down the detailed proofs.

Part C deals with a similar situation to Morimoto and Serizawa (2015), the basic proof logics follow theirs. However, since the preferences used by them are excluded by our domain, we combine both the proof techniques used by Morimoto and Serizawa (2015) and Part A. Still to avoid redundancy, we also just demonstrate the main result, without writing down the detailed proofs.

PART A: Let $\mathcal{R} = \mathcal{R}^R(\pi)$ or $\mathcal{R}^{NR}(\pi)$. We prove that if f satisfies *efficiency*, *strategy-proofness*, *individual rationality* and *no subsidy* on $\mathcal{R}^n$, then f *weakly dominates* any equilibrium rule on $\mathcal{R}^n$ *in agents' welfare*.

Recall that $\pi = (\pi(1), \ldots, \pi(m), \pi(m+1))$, $\pi(m+1) = 0$, $\mu \equiv \min\{n, m+1\}$, and for each pair $x, y \in L$, $x >_\pi y$ means that x has a higher rank than y according to π. Let $M_0 \equiv \{\pi(1), \cdots, \pi(\mu)\}$ and $M_1 \equiv \{\pi(1), \cdots, \pi(\mu - 1)\}$.

Lemma A.1: Let f satisfy *efficiency*. Let $R \in (\mathcal{R})^n$. Then, (a) for each $x \in M_0$, there is $i \in N$ such that $x_i(R) = x$, and (b) for each $i \in N$, $x_i(R) \in M_0$.

Proof: (a) By contradiction, suppose that there is $x \in M_0$ such that for each $i \in N$, $x_i(R) \neq x$. By the definition of M_0, there is $i \in N$ such that $x >_\pi x_i(R)$.

[19] A preference R'_i is a z_i-favoring transformation of R_i at z_i if for each $y \in L\backslash\{x_i\}$, $V'_i(y, z_i) < 0$.

Define z' by: (i) $z'_i \equiv (x, t_i(R))$, and (ii) for each $j \in N\backslash\{i\}$, $z'_j \equiv f_j(R)$. Then, by $R \in (\mathcal{R}^R(\pi))^n$, $(x, t_i(R))\ P_i\ (\pi(\mu), t_i(R))$. Thus, z' dominates $f(R)$, contradicting *efficiency.*

(b) If $n \geq m+1$ (i.e., $\mu = m+1$ and $\pi(\mu) = 0$), then $M_0 = L$ and (b) holds trivially. If $n \leq m$ (i.e., $\mu = n$ and $\pi(\mu) >_\pi 0$), then $|M_0| = n$ and (b) follows from (a). **Q.E.D.**

Lemma A.2: Let f satisfy *efficiency*, *strategy-proofness,* and *individual rationality.* Let $R \in (\mathcal{R})^n$. Then, for each $i \in N$, $f_i(R)\, R_i\ (\pi(\mu), 0)$.

Proof: By contradiction, suppose that there is $i \in N$ such that $(\pi(\mu), 0)\ P_i\, f_i(R)$.

Claim: For each $x \in M_0$, $V_i(x; f_i(R)) > 0$.

By contradiction, suppose that there is $x \in M_0$ such that $V_i(x; f_i(R)) \leq 0$. Then,

$$f_i(R)\ I_i\ (x, V_i(x; f_i(R))) \underset{V_i(x;f_i(R))\leq 0}{R_i} (x, 0) \underset{x\in M_0}{R_i} (\pi(\mu), 0),$$

contradicting $(\pi(\mu), 0)\ P_i\, f_i(R)$. Thus the Claim holds.

By the above Claim, there is $\widehat{R}_i$ such that for each $x \in M_0$, $\widehat{V}_i(x; \mathbf{0}) < V_i(x; f_i(R))$. By Lemma A.1(b), $x_i(\widehat{R}_i, R_{-i}) \in M_0$. Thus,

$$t_i(\widehat{R}_i, R_{-i}) \underset{individual\ rationality}{\leq} \widehat{V}_i(x_i(\widehat{R}_i, R_{-i}); \mathbf{0}) < V_i(x_i(\widehat{R}_i, R_{-i}); f_i(R)).$$

Thus $f_i(\overset{lie}{\widehat{R}_i}, R_{-i}) \overset{truth}{P_i} f_i(\overset{truth}{R_i}, R_{-i})$, contradicting *strategy-proofness.* **Q.E.D.**

Lemma A.3: Let f satisfy *efficiency*, *strategy-proofness*, *individual rationality* and *no subsidy.* Let $R \in (\mathcal{R})^n$. Then, for each $i \in N$, if $x_i(R) = \pi(\mu)$, $t_i(R) = 0$.

Proof: Let $i \in N$ be such that $x_i(R) = \pi(\mu)$. By Lemma A.2, $f_i(R)\, R_i\ (\pi(\mu), 0)$. Thus $t_i(R) \leq 0$ while *no subsidy* implies $t_i(R) \geq 0$. Thus, $t_i(R) = 0$. **Q.E.D.**

Lemma A.4 (Morimoto and Serizawa, 2015): Let $i, j \in N$ and $z \in Z$ be such that $z_i\, R_i\, z_j$ and $z_i\, P_j\, z_j$. Assume that $t_j - V_i(x_j; z_i) < V_j(x_i; z_j) - t_i$. Then, there is $z' \in Z$ that dominates z.

Proof: Let $t'_i \equiv V_i(x_j; z_i)$ and $t'_j \equiv t_i + t_j - V_i(x_j; z_i)$.

Define z' by: (i) $z'_i \equiv (x_j, t'_i)$, (ii) $z'_j \equiv (x_i, t'_j)$, and (iii) for each $k \in N\backslash\{i, j\}$, $z'_k \equiv z_k$. Then, $z'_i\, I_i\, z_i$, and for each $k \in N\backslash\{i, j\}$, $z'_k\, I_k\, z_k$. Since $t_j + t_i - V_i(x_j; z_i) < V_j(x_i; z_j)$, then $z'_j\, P_j\, z_j$. Moreover,

$$\sum_{k\in N} t'_k = \sum_{k\in N\backslash\{i,j\}} t'_k + t'_i + t'_j = \sum_{k\in N\backslash\{i,j\}} t'_k + t_i + t_j = \sum_{k\in N} t_k.$$

Thus, z' dominates z. **Q.E.D.**

Given $z_i \equiv (x_i, t_i) \in L \times \mathbb{R}$ and $R_i \in \mathcal{R}$, $R'_i \in \mathcal{R}$ is a *semi-Maskin monotonic transformation* of R_i at z_i if (i) for each $y <_\pi x_i$, $V'_i(y; z_i) < 0$, and (ii) for each $y >_\pi x_i$, $V'_i(y; z_i) < V_i(y; z_i)$. Let $R_{SMM}(R_i, z_i)$ be the set of *semi-Maskin monotonic transformations* of R_i at z_i.

Lemma A.5: Let f satisfy *strategy-proofness* and *no subsidy.* Let $R \in (\mathcal{R})^n$ and $R'_i \in R_{SMM}(R_i, f_i(R))$. Then, $f_i(R'_i, R_{-i}) = f_i(R_i, R_{-i})$.

Proof: *Strategy-proofness* implies

$$f_i(\overset{truth}{R'_i}, R_{-i})\ \overset{truth}{R'_i}\ f_i(\overset{lie}{R_i}, R_{-i}).$$

Thus, $t_i(R'_i, R_{-i}) \leq V'_i(x_i(R'_i, R_{-i}); f_i(R)))$.

If $x_i(R'_i, R_{-i}) <_\pi x_i(R)$, then by $R'_i \in \mathcal{R}_{SMM}(R_i, f_i(R))$, we have: $t_i(R'_i, R_{-i}) \leq V'_i(x_i(R'_i, R_{-i}); f_i(R))) < 0$, contradicting *no subsidy*. Thus, $x_i(R'_i, R_{-i}) \geq_\pi x_i(R)$.

Suppose $x_i(R'_i, R_{-i}) >_\pi x_i(R)$. Then by $t_i(R'_i, R_{-i}) \leq V'_i(x_i(R'_i, R_{-i}); f_i(R)))$,

$$f_i(R'_i, R_{-i})\, R_i\, (x_i(R'_i, R_{-i}), V'_i(x_i(R'_i, R_{-i}); f_i(R))).$$

Thus by $R'_i \in \mathcal{R}_{SMM}(R_i, f_i(R))$,

$$(x_i(R'_i, R_{-i}), V'_i(x_i(R'_i, R_{-i}); f_i(R)))\, P_i\, (x_i(R'_i, R_{-i}), V_i(x_i(R'_i, R_{-i}); f_i(R)))\, I_i\, f_i(R).$$

Thus, $f_i(\overset{lie}{R'_i}, R_{-i})\ \overset{truth}{P_i}\ f_i(\overset{truth}{R_i}, R_{-i})$, violating *strategy-proofness*. Thus $x_i(R'_i, R_{-i}) = x_i(R)$.

Again, by *strategy-proofness*, $f_i(\overset{truth}{R_i}, R_{-i})\ \overset{truth}{R_i}\ f_i(\overset{lie}{R'_i}, R_{-i})$ and $f_i(\overset{truth}{R'_i}, R_{-i})\ \overset{truth}{R'_i}\ f_i(\overset{lie}{R_i}, R_{-i})$. Thus, by $x_i(R'_i, R_{-i}) = x_i(R)$, we have $t_i(R'_i, R_{-i}) = t_i(R)$. **Q.E.D.**

Given $R \in (\mathcal{R})^n$, $x \in L$ and $z \in [L \times \mathbb{R}]^n$, let $\rho^x(R) \equiv (\rho^x_1(R), \cdots, \rho^x_n(R))$ be the permutation on N defined by $V_{\rho^x_n(R)}(x; z_{\rho^x_n(R)}) \leq \cdots \leq V_{\rho^x_1(R)}(x; z_{\rho^x_1(R)})$. For each $k \in N$, let $C^k(R, x; z)$ be the k-th highest valuation of x from z for R, *i.e.*, $C^k(R, x; z) \equiv V_{\rho^x_k(R)}(x; z_{\rho^x_k(R)})$.

Lemma A.6: Let f satisfy *efficiency*, *strategy-proofness*, *individual rationality* and *no subsidy*. Let $R \in (\mathcal{R})^n$, $i \in N$ and $x \equiv x_i(R)$. Then, $t_i(R) \geq C^\mu(R, x; (\pi(\mu), 0))$.

Proof: By Lemma A.1(b), $x \in M_0$ and $x \geq_\pi \pi(\mu)$.

Case 1: $x = \pi(\mu)$. By Lemma A.3, $t_i(R) = 0 = C^\mu(R, \pi(\mu); (\pi(\mu), 0))$.

Case 2: $x >_\pi \pi(\mu)$. By contradiction, suppose that $t_i(R) < C^\mu(R, x; (\pi(\mu), 0))$. By $x >_\pi \pi(\mu)$, there is $R'_i \in \mathcal{R}_{SMM}(R_i, f_i(R))$ such that $-V'_i(\pi(\mu); f_i(R)) < C^\mu(R, x; (\pi(\mu), 0)) - t_i(R)$. By Lemma A.5, $f_i(R'_i, R_{-i}) = f_i(R_i, R_{-i})$. Thus

$$-V'_i(\pi(\mu); f_i(R'_i, R_{-i})) < C^\mu(R, x; (\pi(\mu), 0)) - t_i(R'_i, R_{-i}).$$

By Lemmas A.1(b) and A.3, there is $j \in N \backslash \{i\}$ such that $f_j(R'_i, R_{-i}) = (\pi(\mu), 0)$ and $V_j(x; (\pi(\mu), 0)) \geq C^\mu(R, x; (\pi(\mu), 0))$. Thus,

$$t_i(R'_i, R_{-i}) - V'_i(\pi(\mu); f_i(R'_i, R_{-i})) < C^\mu(R, x; (\pi(\mu), 0)) \leq V_j(x; (\pi(\mu), 0)).$$

Define z' by:
(i) $z'_i \equiv (\pi(\mu), V'_i(\pi(\mu); f_i(R'_i, R_{-i})))$,
(ii) $z'_j \equiv (x, t_i(R'_i, R_{-i}) - V'_i(\pi(\mu); f_i(R'_i, R_{-i})))$, and
(iii) for each $k \in N \backslash \{i, j\}$, $z'_k \equiv f_k(R'_i, R_{-i})$.

Then, $z'_i\, I'_i\, f_i(R'_i, R_{-i})$ and $z'_j\, P_j\, f_j(R'_i, R_{-i})$. Furthermore,

$$V'_i(\pi(\mu); f_i(R'_i, R_{-i})) + t_i(R'_i, R_{-i}) - V'_i(\pi(\mu); f_i(R'_i, R_{-i})) + \sum_{k \in N \backslash \{i,j\}} t_k(R'_i, R_{-i}) = \sum_{k \in N} t_k(R'_i, R_{-i}).$$

Thus, z' dominates $f(R'_i, R_{-i})$, contradicting *efficiency.* **Q.E.D.**

Lemma A.7: Let f satisfy *efficiency*, *strategy-proofness*, *individual rationality* and *no subsidy.* Let $R \in (\mathcal{R})^n$ and $i \in N$ be such that $x \equiv x_i(R) >_\pi \pi(\mu)$. Then, $V_i(x; (\pi(\mu), 0)) \geq C^{\mu-1}(R, x; (\pi(\mu), 0))$.

Proof: By contradiction, suppose that $V_i(x; (\pi(\mu), 0)) < C^{\mu-1}(R, x; (\pi(\mu), 0))$. Then,

$$V_i(x; (\pi(\mu), 0)) \leq C^{\mu}(R, x; (\pi(\mu), 0)) \underset{\text{Lemma A.6}}{\leq} t_i(R) \underset{\text{Lemma A.2}}{\leq} V_i(x; (\pi(\mu), 0)).$$

Thus, $t_i(R) = V_i(x; (\pi(\mu), 0)) = C^{\mu}(R, x; (\pi(\mu), 0)) < C^{\mu-1}(R, x; (\pi(\mu), 0))$. Thus, $V_i(\pi(\mu); f_i(R)) = 0$ and

$$\left|\{j \in N\backslash\{i\} : V_j(x; (\pi(\mu), 0)) \geq C^{\mu-1}(R, x; (\pi(\mu), 0))\}\right| = \mu - 1.$$

By $x_i(R) = x >_\pi \pi(\mu)$, $\mu = \min\{n, m+1\} \geq 2$, and Lemmas A.1 (b) and A.3, there is $j \in N\backslash\{i\}$ such that $f_j(R) = (\pi(\mu), 0)$ and $V_j(x; (\pi(\mu), 0)) \geq C^{\mu-1}(R, x; (\pi(\mu), 0))$. Thus

$$V_j(x; (\pi(\mu), 0)) > C^{\mu}(R, x; (\pi(\mu), 0)) = t_i(R).$$

By $V_i(\pi(\mu); f_i(R)) = 0$ and $t_j(R) = 0$,

$$t_j(R) - V_i(\pi(\mu); f_i(R)) = 0 < V_j(x; (\pi(\mu), 0)) - t_i(R).$$

By Lemma A.4, $f_i(R)$ is not *efficient*, a contradiction. **Q.E.D.**

Given $R \in (\mathcal{R})^n$, let $Z^{\pi(\mu)}(R) \equiv \{z \in Z : z_i \, R_i \, (\pi(\mu), 0) \text{ for each } i \in N\}$.

Lemma A.8: Let f satisfy *efficiency*, *strategy-proofness*, *individual rationality* and *no subsidy.* Let $R \in (\mathcal{R})^n$, $i \in N$, $x \in M_1$ and $z \in Z^{\pi(\mu)}$. Assume that
(8-i) for each $j \in N\backslash\{i\}$, $f_j(R) \, R_j \, z_j$,
(8-ii) $V_i(x; ((\pi(\mu), 0)) > C^1(R_{-i}, x; z)$,
(8-iii) there is $\varepsilon > 0$ such that $V_i(x; ((\pi(\mu), 0)) - C^1(R_{-i}, x; z) > 2\varepsilon$,
and for each $y \in M_1$ such that $y <_\pi x$,

$$V_i(y; ((\pi(\mu), 0)) < \min\{C^{\mu-1}(R, y; ((\pi(\mu), 0)), V_i(x; ((\pi(\mu), 0)) - C^1(R_{-i}, x; z) - 2\varepsilon\},$$

and
(8-iv) for each $j \neq i$, each $t \in [0, V_i(m; \mathbf{0})]$, each $t' \in [0, V_j(m; \mathbf{0})]$ and each $y >_\pi x$,
$t' - V_i(x; (y, t')) < V_j(y; (x, t)) - t$.
Then $x_i(R) = x$.

Proof: By contradiction, suppose $x_i(R) \neq x$. By Lemma A.1(b), there is $j \in N\backslash\{i\}$ such that $x_j(R) = x$.

Note

$$t_j(R) \underset{\text{(8-i)}}{\leq} V_j(x; z_j) \leq C^1(R_{-i}, x; z) \underset{\text{(8-ii)}}{<} V_i(x; ((\pi(\mu), 0)).$$

Thus, there is $R'_j \in R_{SMM}(R_j, f_j(R))$ such that
(i) $-V'_j((\pi(\mu); f_j(R)) = V_i(x; ((\pi(\mu), 0)) - C^1(R_{-i}, x; z) - \varepsilon$,

(ii) for each $y \in M_1$ such that $y <_\pi x_j(R)$, $V'_j(y;(\pi(\mu),0)) > V_i(x;(\pi(\mu),0)) - C^1(R_{-i},x;z) - 2\varepsilon$, and,
(iii) for each $y >_\pi x_j(R)$, (8-iv) holds with respect to the pair R_i and R'_j.

By $R'_j \in R_{SMM}(R_j, f_j(R))$ and Lemma A.5, $f_j(R'_j, R_{-j}) = f_j(R)$. Thus by (i),
(i') $-V'_j((\pi(\mu); f_j(R'_j, R_{-j})) = V_i(x;((\pi(\mu),0)) - C^1(R_{-i},x;z) - \varepsilon$.

Let $y \equiv x_i(R'_j, R_{-j})$. By $f_j(R'_j, R_{-j}) = f_j(R)$, $y \neq x$. If $y >_\pi x$, then by (iii),

$$t_j(R'_j, R_{-j}) - V_i(x; f_i(R'_j, R_{-j})) < V'_j(y; f_j(R'_j, R_{-j})) - t_i(R'_j, R_{-j}).$$

By Lemma A.4, $f(R'_j, R_{-j})$ is not *efficient*, a contradiction. Thus, by $y \neq x$, $y <_\pi x$.

If $y \in M_1$, then

$$V_i(y;(\pi(\mu),0)) \underset{\text{(8-iii)}}{<} V_i(x;((\pi(\mu),0)) - C^1(R_{-i},x;z) - 2\varepsilon \underset{\text{(ii)}}{<} V'_j(y;(\pi(\mu),0)).$$

Since (8-iii) also implies $V_i(y;(\pi(\mu),0)) < C^{\mu-1}(R,y;(\pi(\mu),0))$, then we have

$$V_i(y;(\pi(\mu),0)) < C^{\mu-1}((R'_j, R_{-j}), y; (\pi(\mu),0)).$$

By Lemma A.7, this contradicts $y \in M_1$. Thus, $y <_\pi x$ but $y \notin M_1$.

By Lemmas A.1(b) and $y \notin M_1$, $x_i(R'_j, R_{-j}) = \pi(\mu)$. Thus, by Lemma A.3, $t_i(R'_j, R_{-j}) = 0$. Thus, by (i') and $t_j(R'_j, R_{-j}) = t_j(R) \leq C^1(R_{-i},x;z)$,

$$\begin{aligned} t_i(R'_j, R_{-j}) - V'_j(\pi(\mu); f_j(R'_j, R_{-j})) &< V_i(x;((\pi(\mu),0)) - C^1(R_{-i},x;z) \\ &\leq V_i(x;(\pi(\mu),0)) - t_j(R'_j, R_{-j}). \end{aligned}$$

Thus, by Lemma A.4, $f(R'_j, R_{-j})$ is not *efficient*, a contradiction. Thus $x_i(R) = x$. **Q.E.D.**

Proposition A.1: Let f satisfy *efficiency*, *strategy-proofness*, *individual rationality* and *no subsidy*. Let $R \in (\mathcal{R})^n$ and $z \in Z^{\min}(R)$. Then, for each $i \in N$, $f_i(R)\, R_i\, z_i$.

Proof: Without loss of generality, let $\pi = (\pi(1), \pi(2) \cdots, \pi(m+1)) = (m, \cdots, 1, 0)$. Let $x_0 \equiv \max\{0, m-n+1\}$. By $\mu \equiv \min\{n, m+1\}$, we have $\mu = m - x_0 + 1$ and $\pi(\mu) = x_0$. If $m > n$, then $x_0 = m - n + 1$. If $m \leq n$, then $x_0 = 0$. Note $M_0 \equiv \{x_0, \cdots, m\}$ and $M_1 \equiv \{x_0 + 1, \cdots, m\}$. We only prove $f_1(R)\, R_1\, z_1$. For each $j \in N \backslash \{1\}$, $f_j(R)\, R_j\, z_j$ can be proved similarly. If $x_1 = x_0$, then by Lemma A.3, $z_1 = (x_0, 0)$, and so by Lemma A.2, $f_1(R)\, R_1\, z_1$. Thus, let $x_1 > x_0$. Let $N^{x_0} \equiv \{i \in N |\ x_i > x_0\}$. By contradiction, suppose that $z_1\, P_1\, f_1(R)$.

Claim: For each $k = 0, 1, 2, \ldots$, there are a set $N(k+1)$ of $k+1$ distinct agents, saying $N(k+1) \equiv \{1, 2, ..., k+1\}$, and $R'_{N(k+1)} \in (\mathcal{R})^{k+1}$ such that:
(1-i) $z_{k+1}\, P_{k+1}\, f_{k+1}(R'_{N(k)}, R_{-N(k)})$;
(1-ii) $p^{\min}_{x_{k+1}}(R) < V'_{k+1}(x_{k+1}; (x_0, 0)) < V_{k+1}(x_{k+1}; f_{k+1}(R'_{N(k)}, R_{-N(k)}))$;
(1-iii) for each $j \in N(k+1)$,

(1-iii-a) there is $\varepsilon_j > 0$ such that $V'_j(x_j; (x_0, 0)) - p^{\min}_{x_j}(R) > 2\varepsilon_j$, and for each $y \in M_1$ such that $y < x_j$,

$$\begin{aligned} & V'_j(y;(x_0,0)) \\ < \;& \min\{C^{m-x_0+1}((R'_{\{1,\ldots j-1\}}, R_{N\setminus\{1,\ldots j-1\}}), y; (x_0,0)), V'_j(x_j;(x_0,0)) - p^{\min}_{x_j}(R) - 2\varepsilon_j, V_j(y;(x_0,0))\}, \end{aligned}$$

(1-iii-b) for each $y > x_j$

(1-iii-(b-1)) for each $i \in \{1,\cdots,j-1\}$, each $t \in [0, V'_i(m;\mathbf{0})]$ and each $t' \in [0, V'_j(m;\mathbf{0})]$,

$$t' - V'_j(x_j;(y,t')) < V'_i(y;(x_j,t)) - t,$$

(1-iii-(b-2)) for each $i \in \{j+1,\cdots,n\}$, each $t \in [0, V_i(m;\mathbf{0})]$ and each $t' \in [0, V'_j(m;\mathbf{0})]$,

$$t' - V'_j(x_j;(y,t')) < V_i(y;(x_j,t)) - t,$$

and

(1-iii-(b-3)) $V'_j(y;(x_0,0)) < p^{\min}_y(R)$;

(1-iv) $N(k+1) \subsetneq N^{x_0}$.

We inductively prove Claim.

Step 1: We prove Claim for the case of $k = 0$.

Note $N(1) = \{1\}$. By $z_1 \, P_1 \, f_1(R)$, (1-i-1) holds and $p^{\min}_{x_1}(R) < V_1(x_1; f_1(R))$. Thus, there is $R'_1 \in \mathcal{R}$ such that

(1-ii-1) $p^{\min}_{x_1}(R) < V'_1(x_1;(x_0,0)) < V_1(x_1; f_1(R))$;

(1-iii) for $1 \in N(1)$,

(1-iii-a-1) there is $\varepsilon_1 > 0$ such that $V'_1(x_1;((\pi(\mu),0)) - p^{\min}_{x_1}(R) > 2\varepsilon$,

and for each $y \in M_1$ such that $y < x_1$,

$$V'_1(y;(x_0,0)) < \min\{C^{m-x_0+1}(R, y;(x_0,0)), V'_1(x_1;((\pi(\mu),0)) - p^{\min}_{x_1}(R) - 2\varepsilon_1, V_1(y;(x_0,0))\},$$

(1-iii-b-1) for each $y > x_1$

(1-iii-(b-2)-1) for each $i \in N\setminus\{1\}$, each $t \in [0, V_i(m;\mathbf{0})]$ and each $t' \in [0, V'_1(m;\mathbf{0})]$,

$$t' - V'_1(x_1;(y,t')) < V_i(y;(x_1,t)) - t,$$

and

(1-iii-(b-3)-1) $V'_1(y;(x_0,0)) < p^{\min}_y(R)$.[20]

By the construction of R'_1, (1-ii-1) and (1-iii-1) holds. Thus, we prove (1-iv-1), i.e., $N(1) \subsetneq N^{x_0}$. By Lemma A.2, $z_1 \, P_1 \, f_1(R) \, R_1 \, (x_0,0)$. Thus $1 \in N^{x_0}$. Thus, $N(1) = \{1\} \subseteq N^{x_0}$. By contradiction, suppose that $N^{x_0} = \{1\}$. Then by Lemma A.1, $\mu = \min\{n, m+1\} = 2$, which implies $n = 2$ or $m = 1$. Thus, by Lemma A.1, for each $j \in N\setminus\{1\}$, $z_j = (x_0, 0)$.

By $z \in Z^{\min}(R)$, $z \in Z^{x_0}$. We show (8-i), (8-ii), (8-iii) and (8-iv) of Lemma A.8 with respect to z to conclude $x_1(R'_1, R_{N\setminus\{1\}}) = x_1$.

By Lemma A.2, for each $j \in N\setminus\{1\}$, $f_j(R'_1, R_{N\setminus\{1\}}) \, R_j \, (x_0,0) = z_j$. Thus, (8-i) holds. (1-iii-b-1) implies (8-iv).

[20] (1-iii-(b-1)-1) is satisfied vacuously.

Note that for each $j \in N\backslash\{1\}$, by $z \in Z^{\min}(R)$, $(x_0, 0) = z_j \, R_j \, z_1$ and $V_j(x_1; (x_0, 0)) \leq p^{\min}_{x_1}(R)$. Thus $C^1(R_{-i}, x_1; z) \leq p^{\min}_{x_1}(R)$, and so by (1-ii-1), (8-ii) holds. By $C^1(R_{-i}, x_1; z) \leq p^{\min}_{x_1}(R)$ and (1-iii-a-1),

$$0 < V'_1(x_1; (x_0, 0)) - p^{\min}_{x_1}(R) - 2\varepsilon_1 \leq V'_1(x_1; (x_0, 0)) - C^1(R_{-i}, x_1; z) - 2\varepsilon_1,$$

which implies (8-iii).

Since (8-i), (8-ii), (8-iii) and (8-iv) of Lemma A.8 hold, $x_1(R'_1, R_{N\backslash\{1\}}) = x_1$.

Note that

$$t_1(R'_1, R_{N\backslash\{1\}}) \underset{\text{Lemma A.2}}{\leq} V'_1(x_1; (x_0, 0)) \underset{\text{(1-ii-1)}}{<} V_1(x_1; f_1(R)).$$

Thus, by $x_1(R'_1, R_{N\backslash\{1\}}) = x_1$,

$$f_1(\overset{lie}{R'_1}, R_{N\backslash\{1\}}) \overset{truth}{P_1} f_1(\overset{truth}{R_1}, R_{N\backslash\{1\}}).$$

This contradicts *strategy-proofness*. Thus, $N(1) \subsetneq N^{x_0}$. Thus (1-iv-1) holds.

Induction hypothesis: There are a set $N(k)$ of $k > 0$ distinct agents, saying $N(k) = \{1, 2, ..., k\}$, and $R'_{N(k)} \in (\mathcal{R}^R)^k$ such that:

(1-i-k) $z_k \, P_k \, f_k(R'_{N(k-1)}, R_{-N(k-1)})$;

(1-ii-k) $p^{\min}_{x_k}(R) < V'_k(x_k; (x_0, 0)) < V_k(x_k; f_k(R'_{N(k-1)}, R_{-N(k-1)}))$;

(1-iii-k) for each $j \in N(k)$,

(1-iii-a-k) there is $\varepsilon_j > 0$ such that $V'_j(x_j; (x_0, 0)) - p^{\min}_{x_j}(R) > 2\varepsilon_j$, and for each $y \in M_1$ such that $y < x_j$,

$$\begin{aligned} & V'_j(y; (x_0, 0)) \\ < \ & \min\{C^{m-x_0+1}((R'_{\{1,\dots,j-1\}}, R_{N\backslash\{1,\dots,j-1\}}), y; (x_0, 0)), V'_j(x_j; (x_0, 0)) - p^{\min}_{x_j}(R) - 2\varepsilon_j, V_j(x_j; (x_0, 0))\}, \end{aligned}$$

(1-iii-b-k) for each $y > x_j$

(1-iii-(b-1)-k) for each $i \in \{1, \cdots, j-1\}$, each $t \in [0, V'_i(m; \mathbf{0})]$ and each $t' \in [0, V'_j(m; \mathbf{0})]$,

$$t' - V'_j(x_j; (y, t')) < V'_i(y; (x_j, t)) - t,$$

(1-iii-(b-2)-k) for each $i \in \{j+1, \cdots, n\}$, each $t \in [0, V_i(m; \mathbf{0})]$ and each $t' \in [0, V'_j(m; \mathbf{0})]$,

$$t' - V'_j(x_j; (y, t')) < V_i(y; (x_j, t)) - t,$$

and

(1-iii-(b-3)-k) $V'_j(y; (x_0, 0)) < p^{\min}_y(R)$;

(1-iv-k) $N(k) \subsetneq N^{x_0}$.

Step 2: We prove Claim for the case of $k+1$.

Step 2-1: We prove that there is $i \in N^{x_0}\backslash N(k)$ such that $z_i \, P_i \, f_i(R'_{N(k)}, R_{-N(k)})$.

By (1-iv-k), $N^{x_0}\backslash N(k) \neq \varnothing$. By contradiction, suppose that for each $i \in N^{x_0}\backslash N(k)$, $f_i(R'_{N(k)}, R_{-N(k)}) \, R_i \, z_i$.

Let z' be such that for each $i \in N \setminus N(k)$, $z'_i \equiv z_i$ and for each $i \in N(k) \setminus \{k\}$, $z'_i \equiv (x_0, 0)$. Then $z' \in Z^{x_0}$. We show (8-i), (8-ii), (8-iii) and (8-iv) of Lemma A.8 with respect to z' to conclude $x_k(R'_{N(k)}, R_{-N(k)}) = x_k$.

For each $i \in N \setminus N^{x_0}$, by $z_i = (x_0, 0)$ and Lemma A.2, $f_i(R'_{N(k)}, R_{-N(k)})\, R_i\, z_i = z'_i$. For each $i \in N^{x_0} \setminus N(k)$, by $z'_i = z_i$, $f_i(R'_{N(k)}, R_{-N(k)})\, R_i\, z'_i$. For each $i \in N(k) \setminus \{k\}$, by Lemma A.2, $f_i(R'_{N(k)}, R_{-N(k)})\, R'_i\, (x_0, 0) = z'_i$. Thus, (8-i) holds. (1-iii-(b-1)-k) and (1-iii-(b-2)-k) imply (8-iv).

In the following, we show:

$$C^1((R'_{N(k)\setminus\{k\}}, R_{-N(k)}), x_k; z') \leq p^{\min}_{x_k}(R). \qquad (*)$$

For each $i \in N \setminus N(k)$, by $z \in Z^{\min}(R)$, $z'_i = z_i\, R_i\, z_k$, and so $V_i(x_k; z'_i) \leq p^{\min}_{x_k}(R)$. For each $i \in N(k) \setminus \{k\}$, if $x_i > x_k$, (1-iii-a-k) implies:

$$\begin{aligned}
V'_i(x_k; z'_i) &= V'_i(x_k; (x_0, 0)) && \text{by } z'_i = (x_0, 0) \\
&< C^{m-x_0+1}((R'_{\{1,..i-1\}}, R_{N\setminus\{1,..i-1\}}), x_k; (x_0, 0)) && \text{by (1-iii-a-}k) \\
&\leq C^{m-x_0+1}(R, x_k; (x_0, 0)) && \text{by (1-iii-a-}k) \\
&\leq p^{\min}_{x_k}(R), && \text{by (1-iii-(b–3)-}k)
\end{aligned}$$

and if $x_i < x_k$, (1-iii-(b-3)-k) implies $V'_i(x_k; z'_i) = V'_i(x_k; (x_0, 0)) \leq p^{\min}_{x_k}(R)$. Thus, $(*)$ holds.

By (1-ii-k) and $(*)$, (8-ii) holds.

By $(*)$ and (1-iii-a-k),

$$0 < V'_k(x_k; (x_0, 0)) - p^{\min}_{x_k}(R) - 2\varepsilon_k \leq V'_k(x_k; (x_0, 0)) - C^1((R'_{N(k)\setminus\{k\}}, R_{-N(k)}), x_1; z') - 2\varepsilon_k.$$

Thus, by (1-iii-a-k), (8-iii) holds.

Since (8-i), (8-ii), (8-iii) and (8-iv) of Lemma A.8 hold, $x_k(R'_{N(k)}, R_{-N(k)}) = x_k$.

Note that

$$t_k(R'_{N(k)}, R_{-N(k)}) \underset{\text{Lemma A.2}}{\leq} V'_k(x_k; (x_0, 0)) \underset{\text{(1-ii-}k)}{<} V_k(x_k; f_k(R'_{N(k)\setminus\{k\}}, R_{-((N(k)\setminus\{k\}))})).$$

Thus, by $x_k(R'_{N(k)}, R_{-N(k)}) = x_k$,

$$f_k(\overset{lie}{R'_k}, R'_{N(k)\setminus\{k\}}, R_{-N(k)})\ \overset{truth}{P_k}\ f_k(\overset{truth}{R_k}, R'_{N(k)\setminus\{k\}}, R_{-N(k)}).$$

This contradicts *strategy-proofness*. Thus, there is $i \in N^{x_0} \setminus N(k)$ such that $z_i\, P_i\, f_i(R'_{N(k)}, R_{-N(k)})$.

Let $N(k+1) \equiv N(k) \cup \{i\}$. Without loss of generality, let $i \equiv k+1$. By (1-iv-k), $N(k+1) \subseteq N^{x_0}$. $z_{k+1}\, P_{k+1}\, f_{k+1}(R'_{N(k)}, R_{-N(k)})$ implies that there is $R'_{k+1} \in \mathcal{R}$ satisfying(1-i-$(k+1)$), (1-ii-$(k+1)$), and (1-iii-$(k+1)$).

Step 2-2: We prove (1-iv-$(k+1)$), i.e., $N(k+1) \subsetneq N^{x_0}$.

By contradiction, suppose that $N(k+1) = N^{x_0}$.

Let $z' \in Z^{x_0}$ be such that for each $i \in N\backslash\{k+1\}$, $z'_i \equiv (x_0, 0)$. We show (8-i), (8-ii), (8-iii) and (8-iv) of Lemma A.8 with respect to z' to conclude $x_{k+1}(R'_{N(k+1)}, R_{-N(k+1)}) = x_{k+1}$.

By Lemma A.2, for each $i \in N\backslash N(k+1)$, $f_i(R'_{N(k+1)}, R_{-N(k+1)})\, R_i\, z'_i$. By Lemma A.2 again, for each $i \in N(k+1)\backslash\{k+1\}$, $f_i(R'_{N(k)}, R_{-N(k)})\, R'_i\, z'_i$. Thus, (8-i) holds. (1-iii-(b-1)-$(k+1)$) and (1-iii-(b-2)-$(k+1)$) imply (8-iv).

In the following, we show:

$$C^1((R'_{N(k+1)\backslash\{k+1\}}, R_{-N(k+1)}), x_{k+1}; z') \leq p^{\min}_{x_{k+1}}(R). \qquad (**)$$

For each $i \in N\backslash N(k+1)$, by $N(k+1) = N^{x_0}$, we have $z_i = (x_0, 0)$. Then, by $z \in Z^{\min}(R)$, $z'_i = z_i\, R_i\, z_{k+1}$. Thus $V_i(x_{k+1}; z'_i) \leq p^{\min}_{x_{k+1}}(R)$. For each $i \in N(k+1)\backslash\{k+1\}$, if $x_i > x_{k+1}$, (1-iii-a-$(k+1)$) implies:

$$\begin{aligned} V'_i(x_{k+1}; z'_i) &< C^{m-x_0+1}((R'_{\{1,2,3,..i-1\}}, R_{N\backslash\{1,2,3,..i-1\}}), x_{k+1}; (x_0,0)) \\ &\leq C^{m-x_0+1}(R, x_{k+1}; (x_0,0)) \leq p^{\min}_{x_{k+1}}(R), \end{aligned}$$

and if $x_i < x_{k+1}$, (1-iii-(b-3)-$(k+1)$) implies $V'_i(x_{k+1}; z'_i) = V'_i(x_{k+1}; (x_0, 0)) \leq p^{\min}_{x_{k+1}}(R)$. Thus, $(**)$ holds.

By (1-ii-$(k+1)$) and $(**)$, (8-ii) holds.

By $(**)$ and (1-iii-a-$(k+1)$),

$$\begin{aligned} 0 &< V'_{k+1}(x_{k+1}; (x_0,0)) - p^{\min}_{x_{k+1}}(R) - 2\varepsilon_{k+1} \leq \\ &\quad V'_{k+1}(x_{k+1}; (x_0,0)) - C^1((R'_{N(k+1)\backslash\{k+1\}}, R_{-N(k+1)}), x_{k+1}; z') - 2\varepsilon_{k+1}. \end{aligned}$$

Thus, by (1-iii-a-$(k+1)$), (8-iii) holds.

Since (8-i), (8-ii), (8-iii) and (8-iv) of Lemma A.8 hold, $x_{k+1}(R'_{N(k+1)}, R_{-N(k+1)}) = x_{k+1}$.

Note that

$$\begin{aligned} t_{k+1}(R'_{N(k+1)}, R_{-N(k+1)}) &\underset{\text{Lemma A.2}}{\leq} V'_{k+1}(x_{k+1}; (x_0,0)) \\ &\underset{\text{(1-ii-}(k+1))}{<} V_{k+1}(x_{k+1}; f_{k+1}(R'_{N(k)}, R_{-(N(k))})). \end{aligned}$$

Thus, by $x_{k+1}(R'_{N(k+1)}, R_{-N(k+1)}) = x_{k+1}$,

$$f_{k+1}(\overset{lie}{R'_{k+1}}, R'_{N(k+1)\backslash\{k+1\}}, R_{-N(k+1)})\, \overset{truth}{P_{k+1}}\, f_{k+1}(\overset{truth}{R_{k+1}}, R'_{N(k+1)\backslash\{k+1\}}, R_{-N(k)}).$$

This contradicts *strategy-proofness*. Thus, (1-iv-$(k+1)$) holds.

By Claim, for each $k \geq 0$, $N(k+1) \subsetneq N^{x_0}$. Let $k = m - x_0$. Then, $|N(k+1)| = k+1 > m - x_0 = |N^{x_0}|$, a contradiction. **Q.E.D.**

PART B: As mentioned at the beginning of Section A2, we only state our main result here.

Proposition B.1: Let $\mathcal{T}$ be a tier partition such that $2 < k < m+1$. Let $\mathcal{R} = \mathcal{R}^T(\mathcal{T})$ or $\mathcal{R}^{NT}(\mathcal{T})$ such that there is $l_0 \in K$ is such that $\sum_{l=1}^{l_0-1} |T_l| < \mu \leq \sum_{l=1}^{l_0} |T_l|$, and $|T_{l_0}| = 1$. Let

f satisfy the four axioms of Proposition A.1. Let $R \in (\mathcal{R})^n$ and $|T_{l_0}| = 1$. Let $z \in Z^{\min}(R)$. Then, for each $i \in N$, $f_i(R)\, R_i\, z_i$.

PART C: As mentioned at the beginning of Section A2, we only state our main result here.

Proposition C.1: Let $\mathcal{R} = \mathcal{R}^T(\mathcal{T})$ or $\mathcal{R}^{NT}(\mathcal{T})$ such that $2 < k \leq m+1$. Let f satisfy the four axioms of Proposition A.1. Let $n > m$ and $z \in Z^{\min}(R)$. Then, for each $i \in N$, $f_i(R)\, R_i\, z_i$.

Subsection A2.2: Proofs of Theorems 2 and 3

Proof of Theorem 2

Since $\mathcal{R} \equiv \mathcal{R}^R(\pi)$, then Proposition C.1 holds for Theorem 2(i) and Proposition A.1 holds for Theorem 2(ii). Note that with slight modification, Proposition 3 and the completion of the proof of Theorem 2 in Morimoto and Serizawa (2015) can be established on $(\mathcal{R}^R(\pi))^n$. Thus, by the same reasoning logic as Morimoto and Serizawa (2015), we can complete the proof of Theorem 2. Thus, to avoid redundancy, we just omit writing them.

Proof of Theorem 3

Since $\mathcal{R} \equiv \mathcal{R}^T(\mathcal{T})$ be such that $2 < k < m+1$, then Proposition C.1 holds for Theorem 4(i) and Proposition B.1 holds for Theorem 4(ii). Note that Proposition 3 and the completion of the proof of Theorem 2 in Morimoto and Serizawa (2015) can be also established on $(\mathcal{R}^T(\mathcal{T}))^n$ such that $2 < k < m+1$. For the same reasoning as the proof of Theorem 2, we can complete the proof of Theorem 3

Subsection A2.3: Proof of Theorem 4

Since $\mathcal{R} \equiv \mathcal{R}^{NR}(\pi)$, then Proposition A.1 holds for Theorem 4. Note that when $\mathcal{R} = \mathcal{R}^{NR}(\pi)$, Proposition 3 in Morimoto and Serizawa (2015) does not hold since their proofs depend on the uses of non-normal preferences, which are excluded by our domain. Thus, new results and proof techniques are needed to complete the whole proof.

Proposition A.2: Let f satisfy *efficiency*, *strategy-proofness*, *individual rationality,* and *no subsidy.* Let $R \in (\mathcal{R})^n$ and $z \in Z^{\min}(R)$. Assume that $n \leq m+1$. Then, for each $i \in N$, $p^{\min}_{x_i(R)}(R) \leq t_i(R)$.

Proof: We only prove $p^{\min}_{x_1(R)}(R) \leq t_1(R)$. For each $j \in N\backslash\{1\}$, $p^{\min}_{x_j(R)}(R) \leq t_j(R)$ can be proved similarly. If $x_1(R) = \pi(\mu)$, then by Lemma A.3, $p^{\min}_{\pi(\mu)}(R) = t_1(R) = 0$. Thus, let $x_1(R) >_\pi \pi(\mu)$. By contradiction, suppose $p^{\min}_{x_1(R)}(R) > t_1(R)$. Then, there is $R'_1 \in R_{SMM}(R_1, f_1(R))$ such that (i) for each $x >_\pi \pi(\mu)$, $(\pi(\mu), 0)\, P'_1\, (x, p^{\min}_x(R))$.

By Lemma A.5 and $R'_1 \in R_{SMM}(R_1, f_1(R))$, $f_1(R) = f_1(R'_1, R_{-1})$. By $x_1(R) >_\pi \pi(\mu)$, Lemmas A.1(a) and A.3, there is $j \in N\backslash\{1\}$ such that $f_j(R'_1, R_{-1}) = (\pi(\mu), 0)$. By Proposition 1, we have: (ii) for each $x \in M_1$, $p^{\min}_x(R) > 0$ and $p^{\min}_x(R'_1, R_{-1}) > 0$.

Step 1: There is $x \in M_1$ such that $(x, p^{\min}_x(R))\, R_j\, f_j(R'_1, R_{-1})$.

Note $z_j \underset{z \in Z^{\min}(R)}{R_j} (\pi(\mu), 0) = f_j(R'_1, R_{-1})$. Thus, if $z_j \neq (\pi(\mu), 0)$, then since $x_j \in M_1$, Step 1 holds. Thus, let $z_j = (\pi(\mu), 0)$. Then, $x_j \notin M_1$ and $z_j = f_j(R'_1, R_{-1})$.

First, we show: (iii) there is $x \in M_1$ such that $(x, p^{\min}_x(R))\, I_j\, z_j$. By $z \in Z^{\min}(R)$, for each $x \in M_1$, $z_j\, R_j\, (x, p^{\min}_x(R))$. By contradiction, suppose that (iii) does not hold. Then, for each

$x \in M_1$, $z_j\, P_j\, (x, p_x^{\min}(R))$. Thus by $x_j \notin M_1$, $D(R_j, p^{\min}(R)) \cap M_1 = \varnothing$. By $n \leq m+1$, $n = |M_1| + 1$. Thus,

$$\begin{aligned} & \left|\{i \in N : D(R_i, p^{\min}(R)) \cap M_1 \neq \varnothing\}\right| \\ = \; & \left|\{i \in N\backslash\{j\} : D(R_i, p^{\min}(R)) \cap M_1 \neq \varnothing\}\right| \\ \leq \; & n - 1 = |M_1|. \end{aligned}$$

Thus by (ii), M_1 is weakly underdemanded for $p^{\min}(R)$, contradicting Fact 3. Thus, (iii) holds.

Thus, by (iii) and $z_j = f_j(R_1', R_{-1})$, $(x, p_x^{\min}(R))\, R_j\, f_j(R_1', R_{-1})$ and $x \in M_1$, and so Step 1 holds.

Step 2: For each $y \in M_1$, $p_y^{\min}(R_1', R_{-1}) < p_y^{\min}(R)$.

First, we show: (iv) there is $y_0 \in M_1$ such that $p_{y_0}^{\min}(R_1', R_{-1}) < p_{y_0}^{\min}(R)$. By contradiction, suppose that for each $y \in M_1$, $p_y^{\min}(R_1', R_{-1}) \geq p_y^{\min}(R)$. By (i), for each $y \in M_1$, $V_1'(y; (\pi(\mu), 0)) < p_y^{\min}(R) \leq p_y^{\min}(R_1', R_{-1})$. Thus, $D(R_1', p^{\min}(R_1', R_{-1})) \cap M_1 = \varnothing$. By $n \leq m+1$, $n = |M_1| + 1$. Thus

$$\left|\{i \in N : D(R_i, p^{\min}(R_1', R_{-1})) \cap M_1 \neq \varnothing\}\right| \leq n - 1 = |M_1|.$$

Thus by (ii), M_1 is weakly underdemanded for $p^{\min}(R_1', R_{-1})$, contradicting Fact 3. Thus, (iv) holds.

Let $M' \equiv \{y \in M_1 : p_x^{\min}(R_1', R_{-1}) \geq p_x^{\min}(R)\}$. By (iv), $M_1\backslash M' \neq \varnothing$. If $M' = \varnothing$, Step 2 holds. Thus, let $M' \neq \varnothing$. Then by $M_1\backslash M' \neq \varnothing$, Fact 3 and (ii),

$$\left|\{i \in N : D(R_i, p^{\min}(R)) \cap (M_1\backslash M') \neq \varnothing\}\right| > |M_1\backslash M'|.$$

Thus,

$$\left|\{i \in N\backslash\{1\} : D(R_i, p^{\min}(R)) \cap (M_1\backslash M') \neq \varnothing\}\right| \geq |M_1\backslash M'|. \tag{v}$$

Since $p_{M_1\backslash M'}^{\min}(R_1', R_{-1}) < p_{M_1\backslash M'}^{\min}(R)$ and $p_{M'}^{\min}(R_1', R_{-1}) \geq p_{M'}^{\min}(R)$, for each $i \in N\backslash\{1\}$ such that $D(R_i, p^{\min}(R)) \cap (M_1\backslash M') \neq \varnothing$, we have $D(R_i, p^{\min}(R_1', R_{-1})) \subset (M_1\backslash M')$, which implies $D(R_i, p^{\min}(R_1', R_{-1})) \cap M' = \varnothing$. Thus,

$$\begin{aligned} & \{i \in N\backslash\{1\} : D(R_o, p^{\min}(R)) \cap (M_1\backslash M') \neq \varnothing\} \\ \subseteq \; & \{i \in N\backslash\{1\} : D(R_i, p^{\min}(R_1', R_{-1})) \cap M' = \varnothing\}. \end{aligned}$$

Thus,

$$\begin{aligned} & \left|\{i \in N\backslash\{1\} : D(R_i, p^{\min}(R_1', R_{-1})) \cap M' \neq \varnothing\}\right| \\ \leq \; & \left|\{i \in N\backslash\{1\} : D(R_i, p^{\min}(R)) \cap (M_1\backslash M') = \varnothing\}\right|. \end{aligned} \tag{vi}$$

By (i), $D(R_1', p^{\min}(R)) \cap M_1 = \varnothing$, which implies $D(R_1', p^{\min}(R_1', R_{-1})) \cap M' = \varnothing$. Since $n \leq m+1$, $n = 1 + |M_1| = 1 + |M_1\backslash M'| + |M'|$. Thus,

$$\begin{aligned} & \left|\{i \in N : D(R_i, p^{\min}(R_1', R_{-1})) \cap M' \neq \varnothing\}\right| & \\ \leq \; & n - 1 - \left|\{k \in N\backslash\{1\} : D(R_k, p^{\min}(R)) \cap (M_1\backslash M') \neq \varnothing\}\right| & \text{by (vi)} \\ \leq \; & n - 1 - |M_1\backslash M'| & \text{by (v)} \\ = \; & |M'|. & \end{aligned}$$

Thus by (ii), M' is weakly underdemanded for $p^{\min}(R_1', R_{-1})$, contradicting Fact 3. Thus Step 2 holds.

Step 3: Completeting Proof of Proposition A.2

Consider $j's$ bundle z_j' for $W^{\min}(R_1', R_{-1})$.

Case 1: $z_j' = (\pi(\mu), 0)$ Note

$$(x, p_x^{\min}(R_1', R_{-1})) \underset{\text{Step 2}}{P_j} (x, p_x^{\min}(R)) \underset{\text{Step 1}}{R_j} z_j'.$$

This contradicts (E-i).

Case 2: $z_j' \neq (\pi(\mu), 0)$ Note that $x_j' \neq \pi(\mu)$. Thus,

$$z_j' \underset{x_j' \in D(R_j, p^{\min}(R_1', R_{-1}))}{R_j} (x, p_x^{\min}(R_1', R_{-1})) \underset{\text{Step 2}}{P_j} (x, p_x^{\min}(R)) \underset{\text{Step 1}}{R_j} (\pi(\mu), 0) = f_j(R_1', R_{-1}).$$

This contradicts Proposition A.1.

Thus, $p_{x_1(R)}^{\min}(R) \leq t_1(R)$ holds. **Q.E.D.**

Completion of the proof of Theorem 4

Let $z \in Z^{\min}(R)$. By Propositions A.1 and A.2,

$$t_i(R) \underset{\text{Proposition A.1}}{\leq} V_i(R, x_i(R); z_i) \underset{\text{Definition of Equilibrium}}{\leq} p_{x_i(R)}^{\min}(R) \underset{\text{Proposition A.2}}{\leq} t_i(R).$$

Thus, for each $y \in L$, $f_i(R) = (x_i(R), p_{x_i(R)}^{\min}(R))\, I_i\, z_i\, R_i\, (y, p_y^{\min}(R))$. Thus (E-i) is satisfied.

Since $n \leq m + 1$, by Lemma A.1, (E-ii) is satisfied.

Thus, $f(R) \in Z^{\min}(R)$. **Q.E.D.**

Subsection A2.4: Proof of Theorem 5

Let $\mathcal{R} \equiv \mathcal{R}^{NT}(\mathcal{T})$. By the same reasoning as the proof of Theorem 4, Proposition 3 in Morimoto and Serizawa (2015) does not hold either. The following proofs are the generalized ones of those in the proof of Theorem 5. Note that Lemmas B.1 to B.8 and Proposition B.1 hold for $(\mathcal{R})^n$. Following the proof techniques of Proposition A.2 in Subsection A2.3 with slight modification, we can prove Theorem 5. Thus, to avoid redundancy, we just omit writing them.