An impossibility under bounded response of social choice functions*

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Abstract

We introduce a new axiom called *bounded response* which states that for each "smallest" change of a preference profile, the change of the social choice must be the "smallest", if any, for the agent who induces the change of a preference profile. We show that *bounded response* is weaker than *strategy-proofness*, and that *bounded response* and *efficiency* imply dictatorship. This impossibility has a far-reaching negative implication. On the universal domain of preferences, it is hard to find a nonmanipulability condition which leads to a possibility result.

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1 Introduction

We consider a society which is to choose one among a finite set of alternatives based on the agents' preferences. A social choice function (SCF) maps each profile of agents' preferences to an alternative. We propose an axiom called *bounded response*. A SCF satisfies *bounded response* if for each "smallest" change of a preference profile, the change of the social choice must be the "smallest", if any, for the agent who induces the change of a preference profile.

We explain *bounded response* in detail. Given a preference profile $\boldsymbol{R} = (R_1, \ldots, R_n)$, let x be the alternative chosen at $\boldsymbol{R}$. Suppose that one agent, say agent i, exchanges the positions of two consecutively ranked alternatives in R_i. We regard this as the "smallest change" of a preference profile. Let y be the social choice after the agent i's preferences changing. Then, *bounded response* requires that either $x = y$ or x and y are consecutively ranked in R_i. This implies that a small error in announcing preferences does not make a big difference in the social choice. Also, it is possible that an agent wavers in deciding which preference to report among similar ones. In such a case, *bounded response* ensures that the agent's decision on which preference to report is not very crucial in the sense that the decision does not make a big difference in terms of the ranks of the social choices. In these senses, *bounded response* is a property on stability of social choice and it would be desirable from the viewpoint of agents. On the other hand, it would not be as desirable as widely accepted axioms such as *efficiency*. Nevertheless, as we discuss shortly, our result with *bounded response* has important implications.

Our main result is simple; A SCF satisfies *bounded response* and *efficiency* if and only if it is dictatorial. This impossibility has interesting and important implications.

First, our main result shows that the impossibility of the Gibbard–Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) is not necessarily due to the incentive requirement of *strategy-proofness*. By the Gibbard–Satterthwaite theorem, it is well-known that *strategy-proofness* and *efficiency* lead to dictatorship. It can be seen that *bounded response* is weaker than *strategy-proofness*. Thus, *bounded response*, which is a "side effect" of *strategy-proofness*, is sufficient for the impossibility. Note that *bounded response* is not a condition on incentives to misreport preferences. It just limits the extent to which the social choice can respond to changes in preferences. Thus, at an agent i's preferences changing from R_i to R_i', it is possible under *bounded response* that the social choice at R_i' is preferable (according to R_i) to the social choice at R_i.

Second, our result readily leads to a new interesting impossibility theorem. Following recent researches on weaker conditions than *strategy-proofness* (for example, Reffgen, 2011;

Carroll, 2012; Sato, 2013; Cho, 2016; Mishra, 2016), we consider a new incentive condition, called *weak AM-proofness*. Assume that the options of misrepresentation are restricted to preferences that are adjacent to the true one as in Sato (2013). Given a preference profile $\boldsymbol{R}$, let x be the chosen alternative at $\boldsymbol{R}$, and R'_i be a false preference of agent i which is adjacent to R_i. Let y and z be the alternatives whose ranks are exchanged in the passage from R_i to R'_i. *Weak AM-proofness* requires that (i) if y and z are "near" x in R_i, then the social choice at R'_i cannot be preferred to x according to R_i, and (ii) if y and z are "far" from x in R_i, then the social choice at R'_i can be preferred to x according to R_i, but in that case, the social choice at R'_i and x should be consecutively ranked in R_i. As a straightforward corollary of our main result, we can see that *weak AM-proofness* and *efficiency* lead to dictatorship. Even when we allow profitable misrepresentation, when the degree of the profit is restricted in a different way from Reffgen (2011), we cannot deviate from the impossibility.

A few axioms in the existing literature are related to the ideas of how "close" with each other the social choices are. *Preference proximity* introduced by Baigent (1987) requires that if a preference profile $\boldsymbol{R}$ is "closer" to some status quo $\boldsymbol{R}^0$ than another one $\boldsymbol{R'}$, there should be the same relation in the values of the SCF f, that is, $f(\boldsymbol{R})$ is "closer" to $f(\boldsymbol{R}^0)$ than $f(\boldsymbol{R'})$. Our *bounded response* is distinguished from *preference proximity* in that the metric in the set of alternatives is given exogenously in the definition of *preference proximity*, while it is endogenous and depends upon the preference of the deviating agent in *bounded response*. Topological social choice theory considers continuity of social welfare functions, rather than social choice functions, where topology in the set of preferences can be defined in various ways.[1] Muto and Sato (2016a) consider an axiom of social welfare functions stating that each "smallest" change of a preference profile leads to the "smallest" change, if any, of the social preference, and prove an impossibility result. In a context of "claims problems", Kasajima and Thomson (2016) consider axioms such that the degree of a change of an outcome is bounded by the degree of a change of inputs to a rule.

The remainder of the paper is organized as follows. In Section 2, we introduce notations and definitions, including our main axiom *bounded response*. In Section 3, we present a number of results. In Section 3.1 we show our main theorem after introducing a technical condition called *same-sidedness condition*. In Section 3.2, we present an application to *weak AM-proofness*. In Section 3.3, we discuss results when *efficiency* is weakened to *unanimity*. In Section 3.4, we discuss whether our impossibility result holds on restricted domains of preferences. In Section 4, we provide a complete proof of the main theorem. Section 5 concludes.

[1] See Baigent (2010) for a survey of topological social choice.

2 Model

We consider a society consisting of n *agents* in $N = \{1, \ldots, n\}$ where $n \geq 2$. Let X be a finite set of feasible *alternatives* with $|X| = m \geq 3$, and $\mathcal{L}$ be the set of all *linear orders* on X.[2] By definition, $x \; R \; x$ for each $R \in \mathcal{L}$ and each $x \in X$. Each agent $i \in N$ has a *preference* $R_i \in \mathcal{L}$. For each pair of distinct alternatives $x, y \in X$, $x \; R_i \; y$ means that i (strictly) prefers x to y. If each agent i has a preference $R_i \in \mathcal{L}$, the n-tuple $(R_1, \ldots, R_n)$ is denoted by $\boldsymbol{R}$, and if some agent i changes preference from R_i to R'_i, the new preference profile is written as $(R'_i, \boldsymbol{R}_{-i})$. For each preference $R \in \mathcal{L}$ and each integer k ($1 \leq k \leq m$), let $r^k(R) \in X$ be the kth-ranked alternative according to R. For each preference $R \in \mathcal{L}$ and each alternative $x \in X$, let $\rho_R(x)$ be the rank of x with respect to R, i.e., $\rho_R(x) = \big|\{y \in X \,|\, y \; R \; x\}\big|$. Two alternatives x and y are *adjacent in* $R \in \mathcal{L}$ if they are consecutively ranked in R, i.e., $|\rho_R(x) - \rho_R(y)| = 1$. Two preferences R and R' are *adjacent* if the only difference between them is the ranks of two adjacent alternatives. If R and R' are adjacent and two distinct alternatives $x, y \in X$ satisfy $x \; R \; y$ and $y \; R' \; x$, the set $\{x, y\}$ is denoted by $A(R, R')$.

A *social choice function* (*SCF*) f is a function from the set of preference profiles $\mathcal{L}^n$ to the set of alternatives X. A SCF is *dictatorship* if there exists $i \in N$ such that $f(\boldsymbol{R}) = r^1(R_i)$ for each $\boldsymbol{R} \in \mathcal{L}^n$. Agent i is called a *dictator*. We introduce a few properties of a SCF. A SCF f satisfies

(i) *strategy-proofness* if $f(\boldsymbol{R}) \; R_i \; f(R'_i, \boldsymbol{R}_{-i})$ for each $\boldsymbol{R} \in \mathcal{L}^n$, each $i \in N$, and each $R'_i \in \mathcal{L}$.

(ii) *monotonicity* if $f(R'_i, \boldsymbol{R}_{-i}) = f(\boldsymbol{R})$ for each $\boldsymbol{R} \in \mathcal{L}^n$, each $i \in N$, and each $R'_i \in \mathcal{L}$ such that $\{x \in X \,|\, f(\boldsymbol{R}) \; R_i \; x\} \subseteq \{x \in X \,|\, f(\boldsymbol{R}) \; R'_i \; x\}$.

(iii) *efficiency* if $f(\boldsymbol{R}) \neq x$ for each $\boldsymbol{R} \in \mathcal{L}^n$ and each $x \in X$ such that there exists $y \in X \setminus \{x\}$ satisfying $y \; R_i \; x$ for each $i \in N$.

(iv) *bounded response* if for each $\boldsymbol{R} \in \mathcal{L}^n$, each $i \in N$, and each $R'_i \in \mathcal{L}$ which is adjacent to R_i, $f(R_i, \boldsymbol{R}_{-i})$ and $f(R'_i, \boldsymbol{R}_{-i})$ are adjacent in R_i or the same, i.e.,

$$\big|\rho_{R_i}(f(R_i, \boldsymbol{R}_{-i})) - \rho_{R_i}(f(R'_i, \boldsymbol{R}_{-i}))\big| \leq 1.$$

Strategy-proofness ensures that reporting the true preference is always an optimal strategy regardless of what the other agents report. *Monotonicity* says that expanding the lower contour set of the social choice does not change the social choice. Muller and Satterthwaite (1977) show that, as long as only strict preferences are allowed, *monotonicity* is a

[2] A binary relation is a *linear order* if it is complete, transitive, and antisymmetric.

necessary and sufficient condition of *strategy-proofness*. *Efficiency* is the standard axiom saying that an alternative cannot be a social choice if it is Pareto dominated by some other alternative. *Bounded response* is our main axiom.[3] It states that if an agent i's preferences changing is the smallest in the sense that R_i and R_i' are adjacent, then the change of the social choice must be the smallest, if any. This implies that when an agent makes an error in announcing his preferences, as long as the error is "small", the social choice at the "incorrect" preference is near the social choice at the "correct" preference. In this sense, *bounded response* is a normatively desirable property. On the other hand, we understand that plausibility of *bounded response*, or more generally, continuity-like conditions, much depends on one's subjective opinion. Still, our result with *bounded response* has interesting implications as discussed in the Introduction.

In the formulation of *bounded response*, the change of the social choice is measured by the difference in the ranks according to the initial preference R_i of agent i, and thus this condition imposes no requirement on the change of the ranks according to the other agents' preferences. *Bounded response* allows agent i to be either better off or worse off after the agent i's preferences changing. We will observe that *bounded response* is weaker than *strategy-proofness* in the next section.

3 Result

In Section 3.1, we show our main theorem: *bounded response* and *efficiency* imply dictatorship. Then, in Section 3.2, we propose a new incentive condition and show that our main theorem readily implies an impossibility involving the new incentive condition. In Sections 3.3 and 3.4, we examine the robustness of our impossibility result.

3.1 Main theorem

First, we show that *bounded response* follows from *strategy-proofness*.

[3]Muto and Sato (2016b) introduce an axiom called *individual bounded response*: for each $\boldsymbol{R} \in \mathcal{L}^n$, each $i \in N$, and each $R_i' \in \mathcal{L}$ which is adjacent to R_i, $\left|\rho_{R_i}(f(R_i, \boldsymbol{R}_{-i})) - \rho_{R_i'}(f(R_i', \boldsymbol{R}_{-i}))\right| \leq 1$. Note that the rank of $f(R_i', \boldsymbol{R}_{-i})$ is measured according to R_i' in *individual bounded response* whereas it is measured according to R_i in *bounded response*. It is readily shown that *individual bounded response* follows from *bounded response* in this paper, and there exists a nondictatorial SCF satisfying *individual bounded response* and *efficiency*. An example of a nondictatorial SCF satisfying *individual bounded response* and *efficiency* is the following; For each $\boldsymbol{R} \in \mathcal{L}^n$, $f(\boldsymbol{R}) = r^1(R_1)$ if $r^1(R_1)\ R_2\ r^2(R_1)$, and $f(\boldsymbol{R}) = r^2(R_1)$ if $r^2(R_1)\ R_2\ r^1(R_1)$.

In Muto and Sato (2016a), we introduce an axiom with the same name, but in the context of preference aggregation. We note that although they have similar background motivations, there is no logical relation between *bounded response* in this paper and the axiom in Muto and Sato (2016a).

	R_i	R'_i
$U(R_i, R'_i)$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$
$A(R_i, R'_i)$	x	y
	y	x
$L(R_i, R'_i)$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$

Figure 1: A partition of X given by a pair of adjacent preferences (R_i, R'_i).

Proposition 3.1. Strategy-proofness *implies* bounded response.

Proof. Suppose that a SCF f is *strategy-proof*. Let $\boldsymbol{R} \in \mathcal{L}^n$, $i \in N$, and $R'_i \in \mathcal{L}$ which is adjacent to R_i. By *strategy-proofness*,

$$f(\boldsymbol{R})\ R_i\ f(R'_i, \boldsymbol{R}_{-i}), \text{ and} \tag{1}$$

$$f(R'_i, \boldsymbol{R}_{-i})\ R'_i\ f(\boldsymbol{R}). \tag{2}$$

It is obvious that (at least) one of the following three conditions is true: (i) $f(\boldsymbol{R}) \notin A(R_i, R'_i)$, (ii) $f(R'_i, \boldsymbol{R}_{-i}) \notin A(R_i, R'_i)$, or (iii) $f(\boldsymbol{R}) \in A(R_i, R'_i)$ and $f(R'_i, \boldsymbol{R}_{-i}) \in A(R_i, R'_i)$. We show $\left|\rho_{R_i}(f(\boldsymbol{R})) - \rho_{R_i}(f(R'_i, \boldsymbol{R}_{-i}))\right| \leq 1$ in each case. First, suppose (i). Then, the lower contour set at $f(\boldsymbol{R})$ is the same for R_i and R'_i. By (1), $f(\boldsymbol{R})\ R'_i\ f(R'_i, \boldsymbol{R}_{-i})$, and by (2), we have $f(\boldsymbol{R}) = f(R'_i, \boldsymbol{R}_{-i})$. Thus, $\left|\rho_{R_i}(f(\boldsymbol{R})) - \rho_{R_i}(f(R'_i, \boldsymbol{R}_{-i}))\right| = 0 \leq 1$. Second, suppose (ii). Then, the lower contour set at $f(R'_i, \boldsymbol{R}_{-i})$ is the same for R_i and R'_i. By (2), $f(R'_i, \boldsymbol{R}_{-i})\ R_i\ f(\boldsymbol{R})$, and by (1), we have $f(\boldsymbol{R}) = f(R'_i, \boldsymbol{R}_{-i})$. Thus, $\left|\rho_{R_i}(f(\boldsymbol{R})) - \rho_{R_i}(f(R'_i, \boldsymbol{R}_{-i}))\right| = 0 \leq 1$. Third, suppose (iii). Then, the conclusion is immediate because for each $x, y \in X$, if $x \in A(R_i, R'_i)$ and $y \in A(R_i, R'_i)$, then $\left|\rho_{R_i}(x) - \rho_{R_i}(y)\right| \leq 1$. □

Next, we introduce a condition weaker than *bounded response* (and also *strategy-proofness* by Proposition 3.1). For each pair of adjacent preferences $R_i, R'_i \in \mathcal{L}$, the following partition of the set of alternatives is induced: (a) $U(R_i, R'_i)$, the alternatives preferred to those in $A(R_i, R'_i)$ with respect to R_i or R'_i, (b) $A(R_i, R'_i)$, the pair of alternatives whose ranks are exchanged between R_i and R'_i, and (c) $L(R_i, R'_i)$, the alternatives less preferred to those in $A(R_i, R'_i)$ with respect to R_i or R'_i. More formally, for each pair of adjacent preferences $R_i, R'_i \in \mathcal{L}$, let $U(R_i, R'_i) = \{x \in X \setminus A(R_i, R'_i) \mid x\ R_i\ y \text{ for each } y \in A(R_i, R'_i)\}$, and $L(R_i, R'_i) = \{x \in X \setminus A(R_i, R'_i) \mid y\ R_i\ x \text{ for each } y \in A(R_i, R'_i)\}$. (We note that $U(R_i, R'_i)$ or $L(R_i, R'_i)$ may be empty.)

This partition is illustrated by Figure 1, in which each column presents a preference, and each column with dots represent parts of the orderings that are common to the two preferences. The following condition, called *same-sidedness condition*, states that even if the social choice changes by the agent i's preferences changing from R_i to R_i', these choices should belong to the same partition element. Thus, two social choices should come on the same side of $A(R_i, R_i')$.

Definition 3.1 (*Same-sidedness condition*)**.** A SCF f satisfies *same-sidedness condition* if for each $\boldsymbol{R} \in \mathcal{L}^n$, each $i \in N$, and each $R_i' \in \mathcal{L}$ such that R_i and R_i' are adjacent, the following three conditions are true:

(a) $f(\boldsymbol{R}) \in U(R_i, R_i')$ implies $f(R_i', \boldsymbol{R}_{-i}) \in U(R_i, R_i')$,

(b) $f(\boldsymbol{R}) \in A(R_i, R_i')$ implies $f(R_i', \boldsymbol{R}_{-i}) \in A(R_i, R_i')$, and

(c) $f(\boldsymbol{R}) \in L(R_i, R_i')$ implies $f(R_i', \boldsymbol{R}_{-i}) \in L(R_i, R_i')$.

This condition is weak in that if $f(\boldsymbol{R}) \in U(R_i, R_i')$ or $f(\boldsymbol{R}) \in L(R_i, R_i')$, and the partition element has more than two alternatives, then the difference in the ranks of $f(\boldsymbol{R})$ and $f(R_i', \boldsymbol{R}_{-i})$ according to R_i may be larger than one. Indeed, we can show that *same-sidedness condition* is implied by *bounded response*.[4]

Lemma 3.2. Bounded response *implies* same-sidedness condition.

Proof. Suppose that a SCF f satisfies *bounded response*. Let $x \equiv f(\boldsymbol{R})$ and $y \equiv f(R_i', \boldsymbol{R}_{-i})$. By *bounded response*,

$$\left|\rho_{R_i}(y) - \rho_{R_i}(x)\right| \leq 1, \text{ and} \tag{3}$$
$$\left|\rho_{R_i'}(x) - \rho_{R_i'}(y)\right| \leq 1. \tag{4}$$

[4] *Bounded response* and *same-sidedness condition* are related to two conditions in the recent literature. First, *swap-monotonicity* introduced by Mishra (2016) says that at an agent i's preferences changing from R_i to an adjacent R_i', if $f(\boldsymbol{R}) \in A(R_i, R_i')$, then $f(R_i', \boldsymbol{R}_{-i}) \in A(R_i, R_i')$, and $f(\boldsymbol{R}) = f(R_i', \boldsymbol{R}_{-i})$ otherwise. *Swap-monotonicity* is logically between *strategy-proofness* and *bounded response*. Although it is not explicitly stated, it can be seen in Mishra (2016) that on the universal domain of preferences, *swap-monotonicity* and *unanimity* imply dictatorship. Under *bounded response*, the social choice can change even if it is in the part which is "irrelevant" from an agent's preferences changing, while this cannot happen under *swap-monotonicity*. As a result, *bounded response* and *swap-monotonicity* are distinct from each other in their normative meanings. Also, as Example 3.1 in Section 3.3 will show, *bounded response* and *unanimity* do not necessarily imply dictatorship.

Second, *set-monotonicity* introduced by Börgers (2015, Chapter 8) says that for each $\boldsymbol{R} \in \mathcal{L}^n$, $i \in N$, and $R_i' \in \mathcal{L}$ (which may not be adjacent to R_i), if there exists a subset of alternatives $Y \subset X$ such that $f(\boldsymbol{R}) \in Y$, and for each $(x, x') \in X^2$ with $x \neq x'$, $x \; R_i \; x'$ and $x' \; R_i' \; x$ imply $(x, x') \in Y^2$, then $f(R_i', \boldsymbol{R}_{-i}) \in Y$. It is readily shown that *set-monotonicity* is logically equivalent to *swap-monotonicity*. *Set-monotonicity* (a necessary condition of *strategy-proofness*) is used in the proof of the Gibbard–Satterthwaite theorem, but the proof relies on other properties of *strategy-proofness*.

First, suppose that $x \in U(R_i, R'_i)$ and $y \notin U(R_i, R'_i)$. By inequality (3), $y \in A(R_i, R'_i)$ and $\rho_{R_i}(y) - \rho_{R_i}(x) = 1$. Then, $\rho_{R'_i}(x) = \rho_{R_i}(x)$ and $\rho_{R'_i}(y) = \rho_{R_i}(y) + 1$, which contradicts inequality (4). This shows (a). Second, suppose that $x \in L(R_i, R'_i)$ and $y \notin L(R_i, R'_i)$. By inequality (3), $y \in A(R_i, R'_i)$ and $\rho_{R_i}(x) - \rho_{R_i}(y) = 1$. Then, $\rho_{R'_i}(x) = \rho_{R_i}(x)$ and $\rho_{R'_i}(y) = \rho_{R_i}(y) - 1$, which contradicts inequality (4). This shows (c). Finally, if $x \in A(R_i, R'_i)$ and $y \in U(R_i, R'_i)$, (a) is violated when the roles R_i and R'_i are exchanged. If $x \in A(R_i, R'_i)$ and $y \in L(R_i, R'_i)$, (c) is violated when the roles R_i and R'_i are exchanged. Therefore, (b) holds. □

Although *same-sidedness condition* might seem much weaker than *strategy-proofness*, it is in fact unexpectedly strong. For example, the Borda rule fails to satisfy *same-sidedness condition* on the universal domain of preferences. To see this, suppose that $n = 2$ and $m = 3$ for simplicity. Let $X = \{x, y, z\}$. Let $\boldsymbol{R} \in \mathcal{L}^2$ be such that $y \; R_1 \; x \; R_1 \; z$ and $x \; R_2 \; y \; R_2 \; z$. Two alternatives x and y have the same Borda count at this preference profile. Assume that the tie-breaking rule chooses x without loss of generality. If agent 1's preference changes from R_1 to R'_1 such that $y \; R'_1 \; z \; R'_1 \; x$, the new social choice under the Borda rule is y. Since R_1 and R'_1 are adjacent and $A(R_1, R'_1) = \{x, z\}$, *same-sidedness condition* is violated.

The following Lemma reveals the strength of *same-sidedness condition*.

Lemma 3.3. *If a SCF satisfies* same-sidedness condition *and* efficiency *then it is dictatorship.*

The proof of Lemma 3.3 is given in Section 4. Our main theorem is an immediate corollary of Lemmas 3.2 and 3.3:

Theorem 3.4. Bounded response *and* efficiency *imply dictatorship.*

Note that Theorem 3.4 (impossibility with *bounded response*) is logically weaker than Lemma 3.3 (impossibility with *same-sidedness condition*). Nevertheless, we present the impossibility with *bounded response* as our main result. This is because this axiom has a transparent normative meaning, while *same-sidedness condition* is just a technical property of SCFs.

3.2 Application

We consider a new condition related to incentives to misreport preferences. We assume that the options for misrepresentation are restricted to the adjacent preferences to the true one.

In investigating an opportunity of profitable misrepresentation, it is natural to pay particular attention to the alternatives whose ranks change by the misrepresentation. If the social choice moves across such an alternative in terms of their ranks, then the agent would find this change of the social choice significant or noticeable. Assume that agent i does not have a time to consider every possible candidate of misrepresentation, or he is reluctant to do so. Then, the agent misreports only when the misreport improves the social choice significantly in the above sense.

We do not argue that the agents always behave in this way. However, we believe that the above setting is plausible in some cases, and it is interesting to see whether we can construct a SCF which prevents such misrepresentation. The condition ensuring that each agent reports his true preference in such a setting is the following.

We say that a SCF f satisfies *weak AM-proofness* if for each $\boldsymbol{R} \in \mathcal{L}^n$, each $i \in N$, and each $R'_i \in \mathcal{L}$ which is adjacent to R_i, there exists no $x \in A(R_i, R'_i)$ such that $f(R'_i, \boldsymbol{R}_{-i})\ R_i\ x\ R_i\ f(\boldsymbol{R})$ and $f(\boldsymbol{R}) \neq f(R'_i, \boldsymbol{R}_{-i})$.[5]

Since it can be readily seen that *weak AM-proofness* implies *same-sidedness condition*, we have the following corollary of Lemma 3.3.

Corollary 3.5. *If a SCF satisfies* weak AM-proofness *and* efficiency*, then it is dictatorship.*

3.3 Unanimity

A SCF f satisfies *unanimity* if for each $\boldsymbol{R} \in \mathcal{L}^n$ and each $x \in X$ such that $r^1(R_i) = x$ for each $i \in N$, $f(\boldsymbol{R}) = x$. We note that *unanimity* follows from *efficiency*. Since the Gibbard–Satterthwaite theorem shows that *strategy-proofness* and *unanimity* imply dictatorship, it is of interest to ask whether *bounded response* and *unanimity* imply dictatorship. We have a negative answer to this question, as the following counterexample shows.

Example 3.1. Suppose $n = 3$ and $m = 4$. Consider the following SCF f. For each $\boldsymbol{R} \in \mathcal{L}^n$,

(a) if $\left|\{r^1(R_1), r^1(R_2), r^1(R_3)\}\right| = 1$, then $f(\boldsymbol{R}) = r^1(R_1)$.

(b) if $\left|\{r^1(R_1), r^1(R_2), r^1(R_3)\}\right| = 2$, then $f(\boldsymbol{R}) = r^1(R_i)$ where there exist $i, j, k \in N$ such that $\{i, j, k\} = N$ and $r^1(R_i) \neq r^1(R_j) = r^1(R_k)$.

(c) if $\left|\{r^1(R_1), r^1(R_2), r^1(R_3)\}\right| = 3$, then $f(\boldsymbol{R}) = w$ where w is the unique alternative in $X \setminus \{r^1(R_1), r^1(R_2), r^1(R_3)\}$.

[5] In Sato (2013), a SCF f satisfies *AM-proofness* if for each $\boldsymbol{R} \in \mathcal{L}^n$, each $i \in N$, and each $R'_i \in \mathcal{L}$ which is adjacent to R_i, it is not the case that $f(R'_i, \boldsymbol{R}_{-i})\ R_i\ f(\boldsymbol{R})$ and $f(\boldsymbol{R}) \neq f(R'_i, \boldsymbol{R}_{-i})$. Here, "AM" stands for Adjacent Manipulation.

We explain this SCF in words. If the three agents agree on the best alternative, that alternative is chosen. If exactly two of them agree on the best alternative, the best alternative for the remaining agent is chosen. If the best alternatives for the three agents are distinct from each other, the alternative which is not the best for any of them is chosen.

By (a), f satisfies *unanimity*. Let us show that f satisfies *bounded response*.

For each preference profile $\boldsymbol{R} \in \mathcal{L}^3$, let $T(\boldsymbol{R}) = \{r^1(R_1), r^1(R_2), r^1(R_3)\} \subset X$ be the set of top alternatives. Let $\boldsymbol{R} \in \mathcal{L}^3$ and $i \in N$. Since f depends only on the top alternatives, it suffices to consider the flip between the top alternative and the second-best one. Let $R_i' \in \mathcal{L}$ be the preference adjacent to R_i obtained by flipping $r^1(R_i)$ and $r^2(R_i)$. We consider three cases in order.

CASE a: Suppose that $|T(\boldsymbol{R})| = 1$. Then, $|T(R_i', \boldsymbol{R}_{-i})| = 2$, and $f(R_i', \boldsymbol{R}_{-i}) = r^1(R_i') = r^2(R_i)$, as required by *bounded response*.

CASE b: Suppose that $|T(\boldsymbol{R})| = 2$.

SUBCASE b.1: If $|T(R_i', \boldsymbol{R}_{-i})| = 1$, *bounded response* holds by Case a.

SUBCASE b.2: Suppose that $|T(R_i', \boldsymbol{R}_{-i})| = 2$, that is, there exist $j, k \in N \setminus \{i\}$ such that $r^1(R_i) = r^1(R_j) \neq r^1(R_k) = r^1(R_i')$. Then, $f(\boldsymbol{R}) = r^1(R_k) = r^1(R_i') = r^2(R_i)$, and $f(R_i', \boldsymbol{R}_{-i}) = r^1(R_j) = r^1(R_i)$, as required by *bounded response*.

SUBCASE b.3: Suppose that $|T(R_i', \boldsymbol{R}_{-i})| = 3$, that is, there exist $j, k \in N \setminus \{i\}$ such that $r^1(R_i) = r^1(R_j) \neq r^1(R_k)$ and $r^1(R_i') = r^2(R_i) \in X \setminus T(\boldsymbol{R})$. Then, $f(\boldsymbol{R}) = r^1(R_k) \in X \setminus \{r^1(R_i), r^2(R_i)\}$, and $f(R_i', \boldsymbol{R}_{-i}) \in X \setminus \{r^1(R_i'), r^1(R_j), r^1(R_k)\} \subset X \setminus \{r^1(R_i), r^2(R_i)\}$. Thus, $\{f(\boldsymbol{R}), f(R_i', \boldsymbol{R}_{-i})\} \subseteq \{r^3(R_i), r^4(R_i)\}$, as required by *bounded response*.

CASE c: Suppose that $|T(\boldsymbol{R})| = 3$. Then, $|T(R_i', \boldsymbol{R}_{-i})| \geq 2$.

SUBCASE c.1: If $|T(R_i', \boldsymbol{R}_{-i})| = 2$. *bounded response* holds by Subcase b.3.

SUBCASE c.2: Suppose that $|T(R_i', \boldsymbol{R}_{-i})| = 3$, that is, $r^1(R_i') = r^2(R_i) \in X \setminus T(\boldsymbol{R})$. Then, $f(\boldsymbol{R}) = r^2(R_i)$ and $f(R_i', \boldsymbol{R}_{-i}) = r^2(R_i') = r^1(R_i)$, as required by *bounded response*.

Therefore, f satisfies *bounded response*.

Let $X = \{x, y, z, w\}$. In Example 3.1, if $\boldsymbol{R} = (R_1, R_2, R_3)$ is such that $r^1(R_1) = x$, $r^1(R_2) = y$, $r^1(R_3) = z$, and $r^4(R_1) = r^4(R_2) = r^4(R_3) = w$, then $f(\boldsymbol{R}) = w$. This is somewhat curious in that the worst alternative w is chosen even if the agents unanimously agree that the best three alternatives are x, y, and z. In fact, we can show that a strengthened version of *unanimity*, which excludes such cases, is enough to obtain the impossibility result.

We say that a SCF f satisfies *strong unanimity* if f satisfies *unanimity*, and for each $\boldsymbol{R} \in \mathcal{L}^n$ and each $x, y, z \in X$ such that $\{r^1(R_i), r^2(R_i), r^3(R_i)\} = \{x, y, z\}$ for each $i \in N$,

$f(\boldsymbol{R}) \in \{x, y, z\}$. *Strong unanimity* follows from *efficiency*.[6]

Proposition 3.6. *If a SCF satisfies* same-sidedness condition *and* strong unanimity*, then it is dictatorship.*

Proof. See the supplementary note Muto and Sato (2016c). □

By the definition of *strong unanimity*, if $m = 3$, *strong unanimity* is trivially equivalent to *unanimity*. Also, if $n = 2$, *same-sidedness condition* and *unanimity* imply *strong unanimity*. Hence, we have the following corollary.

Corollary 3.7. *Suppose that $n = 2$ or $m = 3$. If a SCF satisfies* same-sidedness condition *and* unanimity*, then it is dictatorship.*

Proof. It suffices to show that if $n = 2$, *same-sidedness condition* and *unanimity* imply *strong unanimity*.

Suppose that $n = 2$. Let f be a SCF satisfying *same-sidedness condition* and *unanimity*. Let $(R_1, R_2) \in \mathcal{L}^2$ be such that $\{r^1(R_1), r^2(R_1), r^3(R_1)\} = \{r^1(R_2), r^2(R_2), r^3(R_2)\}$, which implies $\rho_{R_1}(r^1(R_2)) \leq 3$. Let $R'_1 \in \mathcal{L}$ be the preference such that $r^1(R'_1) = r^1(R_2)$, and $x \; R'_1 \; y$ if and only if $x \; R_1 \; y$ for each pair $x, y \in X \setminus \{r^1(R_2)\}$. By *unanimity*, $f(R'_1, R_2) = r^1(R_2)$. By Definition 3.1 (a) and (b), $\rho_{R_1}(r^1(R_2)) = \rho_{R_1}(f(R'_1, R_2)) \geq \rho_{R_1}(f(R_1, R_2))$. Since $\rho_{R_1}(r^1(R_2)) \leq 3$, we have $\rho_{R_1}(f(R_1, R_2)) \leq 3$. Hence, f satisfies *strong unanimity*. □

3.4 Restricted domains

So far, we considered the universal domain of preferences $\mathcal{L}$. It is natural to ask if the impossibility result of Theorem 3.4 holds on restricted domains. We provide two examples of restricted domains on which the possibility result holds when $n = 3$ and $m = 4$.

The first example is a domain on which *unanimity* and *strategy-proofness* imply dictatorship. Thus, the possibility on this domain suggests a distance between *strategy-proofness* and *bounded response*.

Example 3.2. Suppose that X is indexed as $\{x_1, x_2, \ldots, x_m\}$. For each pair of integers ℓ, ℓ', let $x_{\ell'} = x_\ell$ if $\ell' \equiv \ell \mod m$.[7] Let $\mathcal{D} \subset \mathcal{L}$ be the restricted domain of preferences

[6] Thus, Theorem 3.4 is a corollary of Proposition 3.6. We nevertheless place Theorem 3.4 as the main theorem because *efficiency* is the standard axiom while *strong unanimity* is not. Moreover, the proof with *strong unanimity* is more complicated than the proof with *efficiency*.

[7] For each pair of integers k, k' and each positive integer K, $k' \equiv k \mod K$ if and only if $k' - k$ is a multiple of K.

$R \in \mathcal{L}$ such that there exists an integer ℓ satisfying $r^1(R) = x_\ell$ and $r^2(R) \in \{x_{\ell-1}, x_{\ell+1}\}$. This domain $\mathcal{D}$ is a circular domain (Sato, 2010), and on a circular domain, *unanimity* and *strategy-proofness* imply dictatorship.

Suppose that $n = 3$ and $m = 4$. Consider the SCF f defined as follows. For each $\boldsymbol{R} = (R_1, R_2, R_3) \in \mathcal{D}^3$,

(a) if there exist $i, j \in N$ such that $i \neq j$ and $r^1(R_i) = r^1(R_j)$, then $f(\boldsymbol{R}) = r^1(R_i)$, and

(b) otherwise, there exists an integer ℓ such that $\{r^1(R_1), r^1(R_2), r^1(R_3)\} = \{x_{\ell-1}, x_\ell, x_{\ell+1}\}$. We define $f(\boldsymbol{R}) = x_\ell$ in this case.

This SCF f depends only on the profile of top alternatives $(r^1(R_1), r^1(R_2), r^1(R_3))$. If at most two alternatives appear in this profile, then $f(\boldsymbol{R})$ is defined by the plurality rule. If not, $f(\boldsymbol{R})$ is determined by the tie-breaking rule which picks the "middle" one among three. Since for each $\boldsymbol{R} \in \mathcal{D}^3$ there exists $i \in N$ such that $f(\boldsymbol{R}) = r^1(R_i)$, the SCF f satisfies *efficiency*. Let us observe that f satisfies *bounded response*.

For each preference profile $\boldsymbol{R} \in \mathcal{D}^3$, let $T(\boldsymbol{R}) = \{r^1(R_1), r^1(R_2), r^1(R_3)\} \subset X$ be the set of top alternatives. Let $\boldsymbol{R} \in \mathcal{D}^3$ and $i \in N$. Since f depends only on the top alternatives, it suffices to consider for each agent a flip between the top alternative and the second-best one. Let $R'_i \in \mathcal{D}$ be the preference adjacent to R_i given by flipping $r^1(R_i)$ and $r^2(R_i)$. We consider two cases.

CASE a: Suppose that $|T(\boldsymbol{R})| \leq 2$.

SUBCASE a.1: If $|T(\boldsymbol{R})| = 1$ or $|T(R'_i, \boldsymbol{R}_{-i})| = 1$, then $f(\boldsymbol{R}) = f(R'_i, \boldsymbol{R}_{-i})$. *Bounded response* is trivial in this case.

SUBCASE a.2: Suppose that $|T(\boldsymbol{R})| = |T(R'_i, \boldsymbol{R}_{-i})| = 2$ and $f(\boldsymbol{R}) \neq f(R'_i, \boldsymbol{R}_{-i})$, that is, there exist $j, k \in N \setminus \{i\}$ such that $r^1(R_i) = r^1(R_j) \neq r^1(R_k) = r^1(R'_i)$. Then, $f(\boldsymbol{R}) = r^1(R_i)$, and $f(R'_i, \boldsymbol{R}_{-i}) = r^1(R'_i) = r^2(R_i)$. *Bounded response* holds.

SUBCASE a.3: Suppose that $|T(\boldsymbol{R})| = 2$ and $|T(R'_i, \boldsymbol{R}_{-i})| = 3$, that is, there exist $j, k \in N \setminus \{i\}$ such that $r^1(R_i) = r^1(R_j) \neq r^1(R_k)$ and $r^1(R'_i) = r^2(R_i) \in X \setminus T(\boldsymbol{R})$. Then, $f(\boldsymbol{R}) = r^1(R_i)$.

Let $r^1(R_i) = x_\ell$. By the definition of $\mathcal{D}$, $r^1(R'_i) = r^2(R_i) \in \{x_{\ell-1}, x_{\ell+1}\}$. First, suppose that $r^1(R'_i) = x_{\ell-1}$. Then, $r^1(R_k) = x_{\ell+1}$ or $x_{\ell-2}$, and $f(R'_i, \boldsymbol{R}_{-i}) = x_\ell$ or $x_{\ell-1}$. This implies that $f(R'_i, \boldsymbol{R}_{-i}) = r^1(R_i)$, or $f(R'_i, \boldsymbol{R}_{-i}) = r^1(R'_i) = r^2(R_i)$. *Bounded response* holds in either case. Next, suppose that $r^1(R'_i) = x_{\ell+1}$. Then, $r^1(R_k) = x_{\ell+2}$ or $x_{\ell-1}$, and $f(R'_i, \boldsymbol{R}_{-i}) = x_{\ell+1}$ or x_ℓ. This implies that $f(R'_i, \boldsymbol{R}_{-i}) = r^1(R'_i) = r^2(R_i)$, or $f(R'_i, \boldsymbol{R}_{-i}) = r^1(R_i)$. *Bounded response* holds in either case.

CASE b: Suppose that $|T(\boldsymbol{R})| = 3$. Then, $|T(R'_i, \boldsymbol{R}_{-i})| \geq 2$.

SUBCASE b.1: If $\left|T(R_i', \boldsymbol{R}_{-i})\right| = 2$, then *bounded response* holds by Subcase a.3.

SUBCASE b.2: Suppose that $\left|T(R_i', \boldsymbol{R}_{-i})\right| = 3$, that is, $r^1(R_i') = r^2(R_i) \in X \setminus T(\boldsymbol{R})$. Let $r^1(R_i) = x_\ell$. By the definition of $\mathcal{D}$, $r^1(R_i') = r^2(R_i) \in \{x_{\ell-1}, x_{\ell+1}\}$. First, suppose that $r^1(R_i') = x_{\ell-1}$. Then, $\{r^1(R_j), r^1(R_k)\} = \{x_{\ell+1}, x_{\ell+2}\}$. We have $f(\boldsymbol{R}) = x_{\ell+1}$ and $f(R_i', \boldsymbol{R}_{-i}) = x_{\ell+2}$. Thus, $\{f(\boldsymbol{R}), f(R_i', \boldsymbol{R}_{-i})\} \subseteq \{r^3(R_i), r^4(R_i)\}$. *Bounded response* holds in this case. Next, suppose that $r^1(R_i') = x_{\ell+1}$. Then, $\{r^1(R_j), r^1(R_k)\} = \{x_{\ell-1}, x_{\ell-2}\}$. We have $f(\boldsymbol{R}) = x_{\ell-1}$ and $f(R_i', \boldsymbol{R}_{-i}) = x_{\ell-2}$. Thus, $\{f(\boldsymbol{R}), f(R_i', \boldsymbol{R}_{-i})\} \subseteq \{r^3(R_i), r^4(R_i)\}$. *Bounded response* holds in this case.

Therefore, f satisfies *bounded response* in all cases.

The second example is the single-peaked domain. On this domain, we provide a nondictatorial SCF which satisfies *bounded response* and *efficiency* but violates *strategy-proofness*. This also suggests a distance between *strategy-proofness* and *bounded response*.

Example 3.3. Suppose that n is odd, and $m = 4$. Let $X = \{x_1, x_2, x_3, x_4\}$, and $\mathcal{D} \subset \mathcal{L}$ be the single-peaked domain with respect to the above indexes, that is, $\mathcal{D}$ is the set of all preferences R such that there exists $k \in \{1,2,3,4\}$ such that if $4 \geq k > k' > k'' \geq 1$ or $1 \leq k < k' < k'' \leq 4$, then $x_{k'}\ R\ x_{k''}$. Consider the following SCF f. For each $\boldsymbol{R} \in \mathcal{D}^n$, if there exists $y \in X$ such that $r^1(R_i) = y$ for each $i \in N$, then $f(\boldsymbol{R}) = y$. Otherwise,

(a) if $\left|\{i \in N \mid x_1\ R_i\ x_4\}\right| \geq (n+1)/2$, then

  (i) if x_2 is Pareto efficient at $\boldsymbol{R}$, then $f(\boldsymbol{R}) = x_2$,

  (ii) otherwise, x_3 must be Pareto efficient at $\boldsymbol{R}$,[8] and $f(\boldsymbol{R}) = x_3$.

(b) if $\left|\{i \in N \mid x_1\ R_i\ x_4\}\right| \leq (n-1)/2$, then

  (i) if x_3 is Pareto efficient at $\boldsymbol{R}$, then $f(\boldsymbol{R}) = x_3$,

  (ii) otherwise, x_2 must be Pareto efficient at $\boldsymbol{R}$, and $f(\boldsymbol{R}) = x_2$.

This SCF f satisfies *unanimity*. Suppose that at a preference profile $\boldsymbol{R}$, some agents disagree with the most-preferred alternative. In this case, $f(\boldsymbol{R})$ is defined by two steps. Either x_1 or x_4 is the worst alternative at every preference in the single-peaked domain. In the first step, agents determine the socially worst alternative by the plurality rule between x_1 and x_4. In the second step, the social alternative is chosen from $\{x_2, x_3\}$ by the rule which chooses the one "more distant" from the worst as long as it is efficient.

[8] If $\boldsymbol{R} \in \mathcal{D}^n$, either x_2 or x_3 is Pareto efficient at $\boldsymbol{R}$. Suppose that neither x_2 nor x_3 is Pareto efficient. Then, no agent ranks x_2 or x_3 at the top of his preference. Thus, for each $i \in N$, $r^1(R_i) = x_1$ or x_4. Since the agents do not agree on the best alternative, $x_2\ R_i\ x_3\ R_i\ x_4$ for some $i \in N$, and $x_2\ R_j\ x_1$ for some $j \in N$. Thus, x_2 is Pareto efficient, which is a contradiction.

The SCF f satisfies *efficiency* by definition. f violates *strategy-proofness* because when $n = 3$ and a preference profile $\boldsymbol{R} \in \mathcal{L}^n$ satisfies $x_2\ R_1\ x_3\ R_1\ x_4\ R_1\ x_1$, $x_2\ R_2\ x_3\ R_2\ x_1\ R_2\ x_4$, and $x_3\ R_3\ x_2\ R_3\ x_4\ R_3\ x_1$, agent 1 may change the reported preference to $R'_1 = R_2$ and can manipulate the social choice from $f(\boldsymbol{R}) = x_3$ to $f(R'_1, \boldsymbol{R}_{-1}) = x_2$.

Let us show that f satisfies *bounded response*. By symmetry, we can focus on the cases in which $f(\boldsymbol{R}) \in \{x_1, x_2\}$. First, suppose that $f(\boldsymbol{R}) = x_1$. By definition, $r^1(R_i) = x_1$ for each $i \in N$. By the assumption of the single-peaked domain, $x_1\ R_i\ x_2\ R_i\ x_3\ R_i\ x_4$ for each $i \in N$. The only flip available in $\mathcal{D}$ is exchanging x_1 and x_2. This flip changes the social choice to x_2. Therefore, *bounded response* holds in this case.

Next, suppose that $f(\boldsymbol{R}) = x_2$. Let $i \in N$ and $R'_i \in \mathcal{L}$ be adjacent to R_i. If $f(R'_i, \boldsymbol{R}_{-i}) = x_1$, then the flip between R_i and R'_i exchanges x_1 and x_2. Thus, *bounded response* holds in this case. If $f(R'_i, \boldsymbol{R}_{-i}) = x_2 = f(\boldsymbol{R})$, then *bounded response* is trivial. If $f(R'_i, \boldsymbol{R}_{-i}) = x_4$, then $x_4\ R'_i\ x_3\ R'_i\ x_2\ R'_i\ x_1$ and $x_4\ R_j\ x_3\ R_j\ x_2\ R_j\ x_1$ for each $j \in N \setminus \{i\}$. The only flip between R_i and R'_i available in $\mathcal{D}$ consists of exchanging x_3 and x_4, and thus $f(\boldsymbol{R}) = x_3$. This contradicts the assumption $f(\boldsymbol{R}) = x_2$. Thus, we assume $f(R'_i, \boldsymbol{R}_{-i}) = x_3$. We consider three cases.

CASE 1: Suppose that either [$x_1\ R_i\ x_4$ and $x_4\ R'_i\ x_1$] or [$x_4\ R_i\ x_1$ and $x_1\ R'_i\ x_4$]. By the assumption of the single-peaked domain, x_1 and x_4 are the bottom two alternatives at R_i and R'_i. This implies that $\{r^1(R_i), r^2(R_i)\} = \{r^1(R'_i), r^2(R'_i)\} = \{x_2, x_3\}$. Thus, *bounded response* holds. Therefore, in the following cases, we assume that either [$x_1\ R_i\ x_4$ and $x_1\ R'_i\ x_4$] or [$x_4\ R_i\ x_1$ and $x_4\ R'_i\ x_1$].

CASE 2: Suppose that $\left|\{j \in N \mid x_1\ R_j\ x_4\}\right| \geq (n+1)/2$, x_3 is inefficient at $\boldsymbol{R}$, and x_3 is efficient at $(R'_i, \boldsymbol{R}_{-i})$. If x_4 Pareto dominates x_3 at $\boldsymbol{R}$, then $x_4\ R_j\ x_3\ R_j\ x_2\ R_j\ x_1$ for each $j \in N$. This contradicts $f(\boldsymbol{R}) = x_2$. If x_1 Pareto dominates x_3 at $\boldsymbol{R}$, then x_2 also Pareto dominates x_3 at $\boldsymbol{R}$ by the assumption of the single-peaked domain. Therefore, we assume that x_2 Pareto dominates x_3 at $\boldsymbol{R}$. Since x_3 is efficient at $(R'_i, \boldsymbol{R}_{-i})$, x_2 and x_3 are exchanged between R_i and R'_i, that is, x_2 and x_3 are consecutively ranked at R_i and R'_i. Thus, *bounded response* holds.

CASE 3: Suppose that $\left|\{j \in N \mid x_1\ R_j\ x_4\}\right| \leq (n-1)/2$, x_2 is efficient at $\boldsymbol{R}$, and x_2 is inefficient at $(R'_i, \boldsymbol{R}_{-i})$. If x_1 Pareto dominates x_2 at $(R'_i, \boldsymbol{R}_{-i})$, then $x_1\ R'_i\ x_2\ R'_i\ x_3\ R'_i\ x_4$ and $x_1\ R_j\ x_2\ R_j\ x_3\ R_j\ x_4$ for each $j \in N \setminus \{i\}$. This contradicts $f(R'_i, \boldsymbol{R}_{-i}) = x_3$. If x_4 Pareto dominates x_2 at $(R'_i, \boldsymbol{R}_{-i})$, then x_3 also Pareto dominates x_2 at $(R'_i, \boldsymbol{R}_{-i})$ by the assumption of the single-peaked domain. Therefore, we assume that x_3 Pareto dominates x_2 at $(R'_i, \boldsymbol{R}_{-i})$. Since x_2 is efficient at $\boldsymbol{R}$, x_2 and x_3 are exchanged between R_i and R'_i, that is, x_2 and x_3 are consecutively ranked at R_i and R'_i. Thus, *bounded response* holds.

4 Proof

In this section, we prove Lemma 3.3 which immediately implies Theorem 3.4. We divide the proof into several steps. Namely, we prove three Lemmas 4.1, 4.2, and 4.3 as milestones of the proof, and then show Lemma 3.3. Each of Lemmas 4.1, 4.2, and 4.3 states that there exists a dictator i^* in a certain special situation.

Lemma 4.1. *Suppose that a SCF f satisfies* same-sidedness condition *and* efficiency*. For each $\bar{R} \in \mathcal{L}$, there exists an agent $i^* \in N$ such that for each $\bar{\boldsymbol{R}}_{-i^*} \in \mathcal{L}^{n-1}$ satisfying $r^1(\bar{R}_j) = r^2(\bar{R})$ and $r^m(\bar{R}_j) = r^1(\bar{R})$ for each $j \in N \setminus \{i^*\}$, we have $f(\bar{R}, \bar{\boldsymbol{R}}_{-i^*}) = r^1(\bar{R})$.*

We use figures to illustrate preference profiles. For example, the situation considered in the statement of Lemma 4.1 is illustrated by Figure 2, which is interpreted as follows. For each $\bar{R} \in \mathcal{L}$, let $x = r^1(\bar{R})$, $y = r^2(\bar{R})$, and the cells with vertical dots represent arbitrary alternatives. Then, Lemma 4.1 says that for each preference $\bar{R} \in \mathcal{L}$, there exists a dictator $i^* \in N$ when the top alternative in every other agent's preference is y, and the bottom alternative in every other agent's preference is x. Since i^* is the dictator in this situation, the social choice is x. In Figure 2 and those in the subsequent proofs, the square brackets indicate the social choice at the preference profile specified by the figure.

R_1	$\cdots$	R_{i^*-1}	R_{i^*}	R_{i^*+1}	$\cdots$	R_n
y	$\cdots$	y	$[x]$	y	$\cdots$	y
$\vdots$	$\cdots$	$\vdots$	y	$\vdots$	$\cdots$	$\vdots$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$[x]$	$\cdots$	$[x]$	$\vdots$	$[x]$	$\cdots$	$[x]$

Figure 2:

To show Lemma 4.1, we basically follow the proof strategy of Steps 1–4 in Reny (2001) who proved the Gibbard–Satterthwaite theorem. Since some manipulation in Reny (2001) is not applicable under *bounded response*, we focus on the top three alternatives in steps 1–3, and then consider every alternative in X. In the following proof, the numbers of the steps correspond to those in Reny (2001).

Proof of Lemma 4.1. Fix a preference $\bar{R} \in \mathcal{L}$ arbitrary. Let $x \equiv r^1(\bar{R})$, $y \equiv r^2(\bar{R})$, and $z \equiv r^3(\bar{R})$.

STEP 1: We start with a preference profile in which every agent's preference is $R \in \mathcal{L}$ such that $r^1(R) = x$, $r^2(R) = z$, $r^3(R) = y$, and $r^k(R) = r^k(\bar{R})$ for each $k \geq 4$. By *efficiency*, the social choice is x. This setting is shown in Figure 3. Then, exchange x and z in agent

R	$\cdots$	R	R	R	$\cdots$	R
$[x]$	$\cdots$	$[x]$	$[x]$	$[x]$	$\cdots$	$[x]$
z	$\cdots$	z	z	z	$\cdots$	z
y	$\cdots$	y	y	y	$\cdots$	y
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$

Figure 3:

1's preference. By *efficiency*, the social choice is x or z. If it is x, exchange x and z in agent 2's preference. If it is x, repeat the same procedure until for some $i^* \in N$, the social choice becomes z. We eventually obtain Figures 4 and 5.

R_1	$\cdots$	R_{i^*-1}	R_{i^*}	R_{i^*+1}	$\cdots$	R_n
z	$\cdots$	z	$[x]$	$[x]$	$\cdots$	$[x]$
$[x]$	$\cdots$	$[x]$	z	z	$\cdots$	z
y	$\cdots$	y	y	y	$\cdots$	y
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$

Figure 4:

R_1	$\cdots$	R_{i^*-1}	R_{i^*}	R_{i^*+1}	$\cdots$	R_n
$[z]$	$\cdots$	$[z]$	$[z]$	x	$\cdots$	x
x	$\cdots$	x	x	$[z]$	$\cdots$	$[z]$
y	$\cdots$	y	y	y	$\cdots$	y
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$

Figure 5:

STEP 2: In Figure 5, exchange z and y in the preferences of agents $i^* + 1$ to n. By Definition 3.1 (b), the social choice is z or y, and by *efficiency*, the social choice is z. We have Figure 6. In Figure 6, exchange x and y in the preferences of agents 1 to $i^* - 1$, and

R_1	$\cdots$	R_{i^*-1}	R_{i^*}	R_{i^*+1}	$\cdots$	R_n
$[z]$	$\cdots$	$[z]$	$[z]$	x	$\cdots$	x
x	$\cdots$	x	x	y	$\cdots$	y
y	$\cdots$	y	y	$[z]$	$\cdots$	$[z]$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$

Figure 6:

R_1	$\cdots$	R_{i^*-1}	R_{i^*}	R_{i^*+1}	$\cdots$	R_n
$[z]$	$\cdots$	$[z]$	$[z]$	y	$\cdots$	y
y	$\cdots$	y	x	x	$\cdots$	x
x	$\cdots$	x	y	$[z]$	$\cdots$	$[z]$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$

Figure 7:

also exchange x and y in the preferences of agents $i^* + 1$ to n. By Definition 3.1 (a) and (c), the social choice is neither x nor y, and by *efficiency*, the social choice is z. We have Figure 7.

In Figure 7, exchange z and x in agent i^*'s preference. By Definition 3.1 (b), the social choice is x or z. We can show that it is x: If it is z, exchange y and x in the preferences of agents i^*+1 to n, exchange y and x in the preferences of agents 1 to i^*-1, and exchange y and z in the preferences of agents i^*+1 to n. Because of *efficiency* and Definition 3.1, the social choice remains z. Since it returns to Figure 4 in which the social choice is x, this is a contradiction. Therefore, we have Figure 8.

R_1	$\cdots$	R_{i^*-1}	R_{i^*}	R_{i^*+1}	$\cdots$	R_n
z	$\cdots$	z	$[x]$	y	$\cdots$	y
y	$\cdots$	y	z	$[x]$	$\cdots$	$[x]$
$[x]$	$\cdots$	$[x]$	y	z	$\cdots$	z
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$

Figure 8:

STEP 3: In Figure 8, exchange z and y in the preferences of agents 1 to i^*-1, and also i^*. The social choice is neither z nor y by Definition 3.1 (a) and (c), and by *efficiency*, the social choice remains x. We have Figure 9.

R_1	$\cdots$	R_{i^*-1}	R_{i^*}	R_{i^*+1}	$\cdots$	R_n
y	$\cdots$	y	$[x]$	y	$\cdots$	y
z	$\cdots$	z	y	$[x]$	$\cdots$	$[x]$
$[x]$	$\cdots$	$[x]$	z	z	$\cdots$	z
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$

Figure 9:

R_1	$\cdots$	R_{i^*-1}	R_{i^*}	R_{i^*+1}	$\cdots$	R_n
y	$\cdots$	y	$[x]$	y	$\cdots$	y
z	$\cdots$	z	y	z	$\cdots$	z
$\vdots$	$\cdots$	$\vdots$	z	$\vdots$	$\cdots$	$\vdots$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$[x]$	$\cdots$	$[x]$	$\vdots$	$[x]$	$\cdots$	$[x]$

Figure 10:

STEP 4: In Figure 9, lower the positions of x to the bottom in the preferences of the agents except for i^*. By Definition 3.1 (b), the social choice cannot be y and by *efficiency*, the social choice remains x. We have Figure 10.

In Figure 10, shuffle the alternatives in $X \setminus \{x, y\}$ in the preferences except for that of agent i^*, so that for each $j \in N \setminus \{i^*\}$, the preference of agent j becomes $\bar{R}_j$. By *efficiency*, the social choice is either x or y in the entire process of shuffling, and by Definition 3.1 (c), the social choice cannot be y. Hence, the resulting social choice is $f(\bar{R}, \bar{\boldsymbol{R}}_{-i^*}) = x = r^1(\bar{R})$. □

The above proof of Lemma 4.1 has followed the proof strategy of Steps 1–4 in Reny (2001). The proof of Reny (2001) proceeds to his last step, which cannot be directly applied to the setting with *bounded response*. We instead prove the next lemma which states that for each preference $\bar{R} \in \mathcal{L}$, there exists a dictator $i^* \in N$ under an assumption that the bottom alternative in every other agent's preference is the top in i^*'s.

Lemma 4.2. *Suppose that a SCF f satisfies* same-sidedness condition *and* efficiency. *For each $\bar{R} \in \mathcal{L}$, there exists an agent $i^* \in N$ such that for each $\bar{\boldsymbol{R}}_{-i^*} \in \mathcal{L}^{n-1}$ satisfying $r^m(\bar{R}_j) = r^1(\bar{R})$ for each $j \in N \setminus \{i^*\}$, we have $f(\bar{R}, \bar{\boldsymbol{R}}_{-i^*}) = r^1(\bar{R})$.*

Given Lemma 4.1, the above Lemma 4.2 says that in the situation of Figure 2, the social choice remains the same if the position of y in the preference of agent $j \in N \setminus \{i^*\}$ is lowered while x stays at the bottom in the preference of j. If *monotonicity* is assumed as in Reny (2001) (or *swap-monotonicity* in Mishra (2016)), Lemma 4.2 is immediate because the upper contour set of the social choice is unchanged by such a manipulation. Under *bounded response*, however, Lemma 4.2 is fairly nontrivial.

proof of Lemma 4.2. Fix $\bar{R} \in \mathcal{L}$ arbitrarily. Let $x \equiv r^1(\bar{R})$ and $y \equiv r^2(\bar{R})$. Let $i^* \in N$ be the agent given in Lemma 4.1. For each preference profile $\boldsymbol{R}_{-i^*} \in \mathcal{L}^{n-1}$, let $\tau(\boldsymbol{R}_{-i^*}) = \sum_{j \in N \setminus \{i^*\}} \rho_{R_j}(y)$.

We prove the lemma by induction. The following induction base is given by Lemma 4.1:

THE INDUCTION BASE: For each $\bar{\boldsymbol{R}}_{-i^*} \in \mathcal{L}^{n-1}$, if $r^m(\bar{R}_j) = x$ for each $j \in N \setminus \{i^*\}$, and $\tau(\bar{\boldsymbol{R}}_{-i^*}) = n - 1$, then $f(\bar{R}, \bar{\boldsymbol{R}}_{-i^*}) = x$.

The induction proceeds with the following hypothesis and step.

THE INDUCTION HYPOTHESIS: For each $\bar{\boldsymbol{R}}_{-i^*} \in \mathcal{L}^{n-1}$, if $r^m(\bar{R}_j) = x$ for each $j \in N \setminus \{i^*\}$, and $\tau(\bar{\boldsymbol{R}}_{-i^*}) = t$ (where $n - 1 \leq t \leq (m-1)(n-1) - 1$), then $f(\bar{R}, \bar{\boldsymbol{R}}_{-i^*}) = x$.

THE INDUCTION STEP: For each $\bar{\boldsymbol{R}}_{-i^*} \in \mathcal{L}^{n-1}$, if $r^m(\bar{R}_j) = x$ for each $j \in N \setminus \{i^*\}$, and $\tau(\bar{\boldsymbol{R}}_{-i^*}) = t + 1$, then $f(\bar{R}, \bar{\boldsymbol{R}}_{-i^*}) = x$.

Fix $\bar{\boldsymbol{R}}_{-i^*} \in \mathcal{L}^{n-1}$ such that $\tau(\bar{\boldsymbol{R}}_{-i^*}) = t + 1$, arbitrarily. We assume that $f(\bar{R}, \bar{\boldsymbol{R}}_{-i^*}) \neq x$, and derive a contradiction.

STEP 1: We show that $f(\bar{R}, \bar{\boldsymbol{R}}_{-i^*}) \neq y$.

Assume $f(\bar{R}, \bar{\boldsymbol{R}}_{-i^*}) = y$. Since $t + 1 \geq (n-1) + 1$, there exist $j \in N \setminus \{i^*\}$ and $k \geq 2$ such that $y = r^k(\bar{R}_j)$. Let $R_j \in \mathcal{L}$ be the preference obtained by exchanging the ranks of $r^{k-1}(\bar{R}_j)$ and $y = r^k(\bar{R}_j)$ in $\bar{R}_j$. Since $x = r^m(\bar{R}_j) \neq r^{k-1}(\bar{R}_j)$, by Definition 3.1 (b), we have $f(\bar{R}, R_j, \bar{\boldsymbol{R}}_{-(i^*,j)}) \neq x$. Because $\tau(R_j, \bar{\boldsymbol{R}}_{-(i^*,j)}) = t$, this contradicts the induction hypothesis.

STEP 2: We show that for each $j \in N \setminus \{i^*\}$, $\rho_{\bar{R}_j}(y) < \rho_{\bar{R}_j}\big(f(\bar{R}, \bar{\boldsymbol{R}}_{-i^*})\big)$.

By Step 1, this inequality is immediate if $\rho_{\bar{R}_j}(y) = 1$. Assume that there exists $j \in N \setminus \{i^*\}$ such that $\rho_{\bar{R}_j}(y) = k \geq 2$ and $k \geq \rho_{\bar{R}_j}\big(f(\bar{R}, \bar{\boldsymbol{R}}_{-i^*})\big)$. Let $R_j \in \mathcal{L}$ be the preference obtained by exchanging the ranks of $r^{k-1}(\bar{R}_j)$ and $y = r^k(\bar{R}_j)$ in $\bar{R}_j$. Then by Definition 3.1 (a) and (b), $f(\bar{R}, R_j, \bar{\boldsymbol{R}}_{-(i^*,j)}) \neq x$ $(= r^m(\bar{R}_j))$. Because $\tau(R_j, \bar{\boldsymbol{R}}_{-(i^*,j)}) = t$, this contradicts the induction hypothesis.

STEP 3: We derive a contradiction.

Since $r^m(\bar{R}_j) = x$ for each $j \in N \setminus \{i^*\}$, $\rho_{\bar{R}_j}(y) \leq m - 1$. By Step 2, $\rho_{\bar{R}_j}(y) < \rho_{\bar{R}_j}\big(f(\bar{R}, \bar{\boldsymbol{R}}_{-i^*})\big)$ for all $j \in N \setminus \{i^*\}$. Since we assumed $f(\bar{R}, \bar{\boldsymbol{R}}_{-i^*}) \neq x$, we also have $\rho_{\bar{R}}(y) < \rho_{\bar{R}}\big(f(\bar{R}, \bar{\boldsymbol{R}}_{i^*})\big)$. These inequalities contradict *efficiency*.

Therefore, the induction step is shown. This completes the proof. □

Next, we show the following lemma, which states that for each preference $\bar{R} \in \mathcal{L}$, agent i^* given in Lemma 4.2 is the dictator when i^*'s preference is $\bar{R}$.

Lemma 4.3. *Suppose that a SCF f satisfies* same-sidedness condition *and* efficiency*. For each $\bar{R} \in \mathcal{L}$, there exists an agent $i^* \in N$ such that for each $\bar{\boldsymbol{R}}_{-i^*} \in \mathcal{L}^{n-1}$, we have $f(\bar{R}, \bar{\boldsymbol{R}}_{-i^*}) = r^1(\bar{R})$.*

Given Lemma 4.2, the above Lemma 4.3 says that the social choice remains the same if the position of the bottom alternative, which equals the social choice, in the preference of agent $j \in N \setminus \{i^*\}$ is raised. If *monotonicity* is assumed as in Reny (2001), Lemma 4.3 is immediate because the upper contour set of the social choice is reduced by such a change. Under *bounded response*, however, Lemma 4.3 needs an elaborate proof.

proof of Lemma 4.3. Fix a preference $\bar{R} \in \mathcal{L}$ arbitrarily. Let $x = r^1(\bar{R})$. Let $i^* \in N$ be the agent given in Lemma 4.2. For each $\boldsymbol{R}_{-i^*} \in \mathcal{L}^{n-1}$, let $\sigma(\boldsymbol{R}_{-i^*}) = \sum_{j \in N \setminus \{i^*\}} \rho_{R_j}(x)$.

We prove the theorem by induction. The following induction base is given by Lemma 4.2:

THE INDUCTION BASE: For each $\bar{\boldsymbol{R}}_{-i^*} \in \mathcal{L}^{n-1}$, if $\sigma(\bar{\boldsymbol{R}}_{-i^*}) = (n-1)m$, then $f(\bar{R}, \bar{\boldsymbol{R}}_{-i^*}) = x$.

The induction proceeds with the following hypothesis and step.

THE INDUCTION HYPOTHESIS: For each $\bar{\boldsymbol{R}}_{-i^*} \in \mathcal{L}^{n-1}$, if $\sigma(\bar{\boldsymbol{R}}_{-i^*}) = t$ (where $n \leq t \leq (n-1)m$), then $f(\bar{R}, \bar{\boldsymbol{R}}_{-i^*}) = x$.

THE INDUCTION STEP: For each $\bar{\boldsymbol{R}}_{-i^*} \in \mathcal{L}^{n-1}$, if $\sigma(\bar{\boldsymbol{R}}_{-i^*}) = t - 1$, then $f(\bar{R}, \bar{\boldsymbol{R}}_{-i^*}) = x$.

Fix $\bar{\boldsymbol{R}}_{-i^*} \in \mathcal{L}^{n-1}$ such that $\sigma(\bar{\boldsymbol{R}}_{-i^*}) = t-1$ arbitrarily. Let $y = f(\bar{R}, \bar{\boldsymbol{R}}_{-i^*})$. Let $J_1 = \{j \in N \setminus \{i^*\} \mid \rho_{\bar{R}_j}(x) \leq m-2\}$, $J_2 = \{j \in N \setminus \{i^*\} \mid \rho_{\bar{R}_j}(x) = m-1\}$, and $J_3 = \{j \in N \setminus \{i^*\} \mid \rho_{\bar{R}_j}(x) = m\}$. Since $\sigma(\bar{\boldsymbol{R}}_{-i^*}) = t-1 < m(n-1)$, $J_1 \cup J_2 \neq \emptyset$.

We assume that $y \neq x$ and derive a contradiction.

STEP 1: We show that for each $j \in J_1 \cup J_2$, if $x = r^k(\bar{R}_j)$, then $y = r^{k+1}(\bar{R}_j)$.

Assume not. Then, there exist $j \in J_1 \cup J_2$ and $z \neq y$ such that $x = r^k(\bar{R}_j)$, and $z = r^{k+1}(\bar{R}_j)$. Let R_j be the preference obtained by exchanging the ranks of x and z in $\bar{R}_j$. Then, because of Definition 3.1 (a) and (c), $f(\bar{R}, R_j, \bar{\boldsymbol{R}}_{-(i^*,j)}) \notin \{x, z\}$. This contradicts the induction hypothesis Because $\sigma(R_j, \bar{\boldsymbol{R}}_{-(i^*,j)}) = t$, this contradicts the induction hypothesis.

Therefore, we have Figure 11. Since the choice of $\bar{\boldsymbol{R}}_{-i^*}$ was arbitrary, we have shown that for each $j \in N \setminus \{i^*\}$ and each $\boldsymbol{R}_{-i^*} \in \mathcal{L}^{n-1}$ such that $f(\bar{R}, \boldsymbol{R}_{-i^*}) \neq x$ and $\sigma(\boldsymbol{R}_{-i^*}) = t-1$, if there exists $k \leq m-1$ such that $r^k(R_j) = x$, then $r^{k+1}(R_j) = f(\bar{R}, \boldsymbol{R}_{-i^*})$.

i^*	J_1			J_2			J_3		
x	⋮	⋯	⋮	⋮	⋯	⋮	⋮	⋯	⋮
⋮	x	⋯	⋮	⋮	⋯	⋮	⋮	⋯	⋮
⋮	$[y]$	⋯	x	⋮	⋯	⋮	⋮	⋯	⋮
⋮	⋮	⋯	$[y]$	⋮	⋯	⋮	⋮	⋯	⋮
⋮	⋮	⋯	⋮	⋮	⋯	⋮	⋮	⋯	⋮
⋮	⋮	⋯	⋮	x	⋯	x	⋮	⋯	⋮
⋮	⋮	⋯	⋮	$[y]$	⋯	$[y]$	x	⋯	x

Figure 11:

i^*	J_2			J_3		
x	⋮	⋯	⋮	⋮	⋯	⋮
⋮	⋮	⋯	⋮	⋮	⋯	⋮
⋮	x	⋯	x	⋮	⋯	⋮
⋮	$[y]$	⋯	$[y]$	x	⋯	x

Figure 12:

STEP 2: We show that $J_1 \neq \emptyset$.

Assume $J_1 = \emptyset$. Since $J_1 \cup J_2 \neq \emptyset$, then $J_2 \neq \emptyset$. We have Figure 12. For each $j \in J_3$, lower the rank of y to the penultimate position in agent j's preference. By Definition 3.1 (a) and (b), the resulting social choice cannot be x. Since $J_2 \neq \emptyset$, Step 1 shows that the social choice remains y. By *efficiency*, $r^2(\bar{R}) = y$. Letting $z = r^3(\bar{R})$, we have Figure 13. In Figure 13, exchange the ranks of y and z in the preference of agent i^*. By Definition 3.1 (b), the social choice is y or z, and by *efficiency*, it is z. We have Figure 14.

In Figure 14, exchange the ranks of x and y in $\bar{R}_j$ for some $j \in J_2$. By Definition 3.1 (a), the resulting social choice cannot be x. Next, exchange the ranks of z and y in the preference of agent i^*. By Definition 3.1 (b) and (c), the resulting social choice cannot be x.

i^*	J_2			J_3		
x	⋮	⋯	⋮	⋮	⋯	⋮
$[y]$	⋮	⋯	⋮	⋮	⋯	⋮
z	x	⋯	x	$[y]$	⋯	$[y]$
⋮	$[y]$	⋯	$[y]$	x	⋯	x

Figure 13:

i^*	J_2			J_3		
x	⋮	⋯	⋮	⋮	⋯	⋮
$[z]$	⋮	⋯	⋮	⋮	⋯	⋮
y	x	⋯	x	y	⋯	y
⋮	y	⋯	y	x	⋯	x

Figure 14:

Because the value of σ is t after the these manipulations, this contradicts to the induction hypothesis. Therefore, $J_1 \neq \emptyset$.

STEP 3: We show that $J_2 = \emptyset$ and there exists $j^* \in N \setminus \{i^*\}$ such that $J_1 = \{j^*\}$ arbitrarily.

By Step 2, $J_1 \neq \emptyset$. Fix $j^* \in J_1$. Let $r^k(\bar{R}_{j^*}) = x$ and $w = r^{k+2}(\bar{R}_{j^*})$. We have Figure 15, in which the left column in J_1 represents agent j^*'s preference. In Figure 15, exchange the ranks of y and w in $\bar{R}_{j^*}$. By Definition 3.1 (b), the social choice is y or w, and by Step 1, the social choice is w. Assume that $(J_1 \cup J_2) \setminus \{j^*\} \neq \emptyset$, and fix $j \in (J_1 \cup J_2) \setminus \{j^*\}$. Since the social choice is not y, this contradicts Step 1. Therefore, $J_1 = \{j^*\}$ and $J_2 = \emptyset$. We have Figure 16.

i^*	J_1			J_2			J_3		
x	⋮	⋯	⋮	⋮	⋯	⋮	⋮	⋯	⋮
⋮	x	⋯	⋮	⋮	⋯	⋮	⋮	⋯	⋮
⋮	$[y]$	⋯	x	⋮	⋯	⋮	⋮	⋯	⋮
⋮	w	⋯	$[y]$	⋮	⋯	⋮	⋮	⋯	⋮
⋮	⋮	⋯	⋮	x	⋯	x	⋮	⋯	⋮
⋮	⋮	⋯	⋮	$[y]$	⋯	$[y]$	x	⋯	x

Figure 15:

i^*	j^*	J_3		
x	⋮	⋮	⋯	⋮
⋮	x	⋮	⋯	⋮
⋮	$[y]$	⋮	⋯	⋮
⋮	w	⋮	⋯	⋮
⋮	⋮	⋮	⋯	⋮
⋮	⋮	x	⋯	x

Figure 16:

STEP 4: We derive a contradiction.

In Figure 16, raise the rank of y until the rank of y exceeds the rank of $w = r^{k+2}(\bar{R}_{j^*})$ in the preference of each $j \in J_3$. (If y's rank exceeds w's rank in the initial preference, then do nothing.) By Step 1, the resulting social choice is y or x, and by Definition 3.1 (a) and (b), the social choice is y. Next, lower the rank of w to the penultimate position in the preference of each $j \in J_3$. By Definition 3.1 (a), the resulting social choice cannot be x. Step 1 implies that it remains y. As a result, we have Figure 17.

i^*	j^*	J_3		
x	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\vdots$	x	$\vdots$	$\cdots$	$\vdots$
$\vdots$	$[y]$	$\vdots$	$\cdots$	$\vdots$
$\vdots$	w	$\vdots$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	w	$\cdots$	w
$\vdots$	$\vdots$	x	$\cdots$	x

Figure 17:

i^*	j^*	J_3		
x	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\vdots$	x	$\vdots$	$\cdots$	$\vdots$
$\vdots$	$[y]$	$\vdots$	$\cdots$	$\vdots$
$\vdots$	w	$\vdots$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	x	$\cdots$	x
$\vdots$	$\vdots$	w	$\cdots$	w

Figure 18:

i^*	j^*	J_3		
x	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\vdots$	x	$\vdots$	$\cdots$	$\vdots$
$\vdots$	w	$\vdots$	$\cdots$	$\vdots$
$\vdots$	$[y]$	$\vdots$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	x	$\cdots$	x
$\vdots$	$\vdots$	w	$\cdots$	w

Figure 19:

i^*	j^*	J_3		
x	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\vdots$	x	$\vdots$	$\cdots$	$\vdots$
$\vdots$	w	$\vdots$	$\cdots$	$\vdots$
$\vdots$	y	$\vdots$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	w	$\cdots$	w
$\vdots$	$\vdots$	x	$\cdots$	x

Figure 20:

In Figure 17, for each $j \in J_3$, exchange the ranks of w and x in the preference of j. By Definition 3.1 (a), the social choice cannot be x or w. We can show that it is y: Suppose that the social choice changes to some alternative distinct from y. Then, exchange the ranks of x and y in $\bar{R}_{j^*}$. By Definition 3.1 (a) and (c), the social choice cannot be x, and by *efficiency*, it cannot be w. Exchange the ranks of w and x in the preference of each $j \in J_3$. By Definition 3.1 (a), the social choice cannot be x. This contradicts the induction hypothesis. Thus, the social choice must be y after the above changes. We have Figure 18.

In Figure 18, exchange the ranks of y and w in $\bar{R}_{j^*}$. By Definition 3.1 (b), the social choice is y or w, and by *efficiency*, the social choice y. We have Figure 19. In Figure 19, for each $j \in J_3$, exchange the ranks of x and w in the preference of j. By Definition 3.1 (b), the resulting social choice should not be x or w. We have Figure 20. Since the value of σ in the preference profile presented in Figure 20 is $t-1$, this contradicts Step 1.

Hence, we have $y = x$. □

Finally, we prove Lemma 3.3.

Proof of Lemma 3.3. By Lemma 4.3, for each $\bar{R} \in \mathcal{L}$, there exists a dictator $i^* \in N$ at $\bar{R}$, i.e., there exists an agent $i^* \in N$ such that $f(\bar{R}, \boldsymbol{R}_{-i^*}) = r^1(\bar{R})$ for each $\boldsymbol{R}_{-i^*} \in \mathcal{L}^{n-1}$. We show that such an agent i^* is determined independent of the choice of $\bar{R}$.

Suppose that $i^* \in N$ is the dictator at $\bar{R} \in \mathcal{L}$, and $j^* \in N$ is the dictator at $R \in \mathcal{L}$. Assume $i^* \neq j^*$. Then for each $\boldsymbol{R}_{-(i^*,j^*)} \in \mathcal{L}^{n-2}$, $f(\bar{R}, R, \boldsymbol{R}_{-(i^*,j^*)}) = r^1(\bar{R})$ because i^* is the dictator, and also $f(\bar{R}, R, \boldsymbol{R}_{-(i^*,j^*)}) = r^1(R)$ because j^* is the dictator. Thus, $r^1(\bar{R}) = r^1(R)$. Take a preference $R' \in \mathcal{L}$ such that $r^1(R') \neq r^1(\bar{R})$, and suppose that agent $k^* \in N$ is the dictator at R'. Since $r^1(R') \neq r^1(\bar{R})$, then $k^* = i^*$, and also because $r^1(R') \neq r^1(R)$, then $k^* = j^*$. This contradicts the assumption $i^* \neq j^*$.

Therefore, f is dictatorship. □

5 Concluding remarks

We have introduced a new axiom called *bounded response*, and proved that *bounded response* and *efficiency* imply dictatorship. Since *bounded response* follows from *strategy-proofness*, the Gibbard–Satterthwaite theorem is shown as a corollary of our impossibility result. This result also suggests that even if profitable misrepresentation is permitted, the impossibility is inevitable as long as the degree of the improvement due to the misrepresentation is restricted.

On the universal domain, *strategy-proofness* is not a useful condition of nonmanipulability in the sense that no plausible SCF satisfies it. As we mentioned in the Introduction, there are recent researches investigating the result of weakening *strategy-proofness* in some natural or interesting ways. Our result shows that as long as we want a deterministic SCF on the universal domain, unfortunately, it is hard to find a useful nonmanipulability condition except for some extreme ones.[9] On the one hand, this might imply that we have to be satisfied with SCFs satisfying necessary conditions for *strategy-proofeness* which are not usually considered as nonmanipulability conditions. Examples of such conditions are *unanimity*, *efficiency*, and *weak monotonicity*. On the other hand, this might imply the limit of the classical social choice framework, and invite us to consider other models in which the possibility of constructing nonmanipulable SCFs is not investigated very much. For example, let us assume that agents have rankings over alternatives *and* evaluations, either "acceptable" or "unacceptable". This is the preference-approval model by Brams and Sanver (2009). Among few papers considering nonmanipulability in the preference-approval model, Erdamar et al. (2016) find some plausible rules satisfying an axiom called *evaluationwise strategy-proofness*.

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[9]For example, Muto and Sato (2016b) present a possibility result by employing *top-restricted strategy-proofness*, an axiom which requires that each agent cannot change the social choice from the second preferred one to the most preferred one. This axiom is a severe restriction of *strategy-proofness* in the sense that it considers only the top two alternatives.

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