Prevalence of Truthtelling and Implementation

Olivier Bochet* Norovsambuu Tumennasan †

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Abstract

Saijo, Sjöström and Yamato (2007) introduce the protective criterion of secure implementation, requiring double implementation in dominant strategies and Nash equilibria. Most strategy-proof rules of interest fail to be secure implementable. We introduce and characterize a new notion coined coalitional secure implementation requiring implementation in dominant strategies and strong Nash equilibria. The difference with secure implementation is striking: when preferences are strict, the only conditions needed for coalitional secure implementation are strategy-proofness and non-bossiness. Next, we derive necessary and sufficient conditions for coalitional secure implementation and consider various domains and models of interest from the literature on mechanism design. We show that out of the three necessary conditions for secure implementation identified in Saijo et al. (2007), the outcome rectangular property can be completely dispensed with –the most stringent condition. In the Sprumont model (Sprumont, 1991), we in addition show that for any rule that is efficient, strategy-proof and satisfies a mild additional condition, any bad Nash equilibrium is Pareto dominated by a reversion to truthtelling. Our findings emphasize the prevalence of truthtelling. We link our findings to new conditions about the resilience of rules.

1 Introduction

Strategy-Proofness is a central condition in mechanism design. It states that truthtelling is a dominant strategy in the *direct revelation mechanism* associated with a given decision rule (henceforth SCF). A possible caveat with this notion is that an SCF can be strategy-proof while admitting many Nash equilibria in its direct revelation mechanism that differ from the outcome prescribed by the SCF. We call these *bad Nash equilibria.* Cason et al. (2006) identify a behavioral wedge among strategy-proof rules. Using an experiment,

*NYU-Abu Dhabi. *Email:* olivier.bochet@nyu.edu

†Department of Economics, Dalhousie University University. *Email:* norov@dal.ca

they compare two types of strategy-proof SCFs: one whose direct revelation mechanism admits bad Nash equilibria, and one that does not. They show that in the former case, dominant strategy rates of play barely reach 50% and subjects get stuck at bad Nash equilibria. In contrast, with the latter, dominant strategy rates are significantly boosted upward. In conjunction with these insights, Saijo et al. (2007) introduce the additional protective criterion of *secure implementation* which requires double implementation in dominant strategies and Nash equilibrium. They identify a new condition, the *rectangular property* which together with strategy-proofness is both necessary and sufficient for secure implementation. In a second characterization, they identify the outcome rectangular property (a variant of the rectangular property) and non-bossiness in welfare as the key conditions for a strategy-proof SCF to be secure implemented.[1] An advantage of the secure implementation criterion is that the quest for mechanisms is dramatically reduced. A strong version of the revelation principle is at play: any SCF that is securely implementable by a mechanism is also securely implementable via its direct revelation mechanism.

The news delivered by secure implementation is, perhaps not surprisingly, negative. Many strategy-proof SCFs encountered in the literature fail to satisfy the rectangular property. The reasons and consequences of this failure are not yet fully understood. A recent paper by Bochet and Sakai (2010) suggests that failures of the rectangular property may not always be so "severe". Specializing to a private-good economy with single-peaked preferences –the Sprumont model, (Sprumont, 1991)– they show that in the direct revelation of the *Uniform Rule*, bad Nash equilibria are not robust to coalitional deviations. Strikingly, each bad Nash equilibrium outcome is Pareto inefficient, and all profitable coalitional deviations involve reversions to truthtelling.

The findings of Bochet and Sakai (2010) is particularly interesting in environments in which agents communicate during or prior to submitting their reports. Indeed pre-play communication is usually not prohibited in many applications. For instance, many cities in the US allocate students to schools using some centralized allocation mechanism. Here, students or parents can discuss their preference reports before submitting to the relevant authorities. Patients can also freely discuss their organ preferences prior submitting them to donor-patient matching systems. In such environments, it is unreasonable to expect agents to be stuck at a Nash equilibrium that is vulnerable to coalitional deviations.[2] Thus, we require that each strong Nash equilibrium instead of Nash delivers the desirable outcome.

Our Contribution: The findings in Bochet and Sakai (2010) show the potential fragility and lack of credibility of both bad Nash equilibria and the failure of the rectangular prop-

[1]The rectangular property can thus be decomposed into the combination of the outcome rectangular property and non-bossiness in welfare.

[2]Indeed, this fact is established in experimental studies such as Moreno and Wooders (1998).

erty. We allow for coalitional moves and focus on the notion of *strong Nash equilibrium*. We introduce an extension of secure implementation to coalitional secure implementation. Once coalitional moves are excluded, the extension coincides with the standard secure implementation notion. Coalitional secure implementation requires that there exists a dominant strategy equilibrium delivering the right outcome while all strong Nash equilibria are good. When preferences are strict, we show that the only conditions needed for coalitional secure implementation are *strategy-proofness* and *non-bossiness*. Importantly, our proof techniques use simple deviations from bad Nash equilibria at which agents get stuck: a coalition just reverts to truthtelling and all of its members strictly gain from it. The conjunction of these two conditions is equivalent to *group strategy-proofness*, a coalitional strengthening of strategy-proofness. Among others, the full class of trading cycles rules (Pycia and Ünver, forthcoming) is coalitional secure implementable. This is in sharp contrast with the negative results on secure implementation in the house allocation model (Fujinaka and Wakayama, 2011).

For domains with possible indifferences, we derive necessary and sufficient conditions. The central condition identified is the *group reversal property*. Our main goal is to get a better understanding of the additional mileage one gets when going from secure implementation to its coalitional extension.

We generalize our result on domains with strict preferences in several directions. For this, we appeal to the second characterization of secure implementation identified in Saijo et al. (2007) using strategy-proofness, non-bossiness in welfare and the outcome rectangular property. Our results show that for many domains and models of interest, the only conditions needed for coalitional secure implementation via direct mechanisms are strategy-proofness and non-bossiness in welfare. The outcome rectangular property can be completely dispensed with, provided that the domain is sufficiently rich. Finally, specializing to the single-peaked model, we show that any rule satisfying efficiency, strategy-proofness and non-bossiness in welfare is coalitional secure.

There is a common interesting feature in the proof strategy of all our results: bad Nash equilibria are always broken by some coalitional reversion to truthtelling. This feature is stronger than the requirement of the group reversal property. In contrast, the group reversal property only requires that there exists some coalitional deviation. As such an interesting by-product of our results is the unexpected prevalence of truthtelling of strategy-proof SCFs that satisfy non-bossiness in welfare. Note that truthtelling is prevalent only when coalitional moves are possible. As shown with the failure of the rectangular property, unilateral deviations to truthtelling are typically not profitable when agents get stuck at joint misreport of preferences. On the other hand, in such situations coalitional reversions to truthtelling are always profitable for some group of agents provided that the SCF satisfies the additional condition of non-bossiness in welfare.

Pick a strategy-proof SCFs that satisfy non-bossiness in welfare. We have the following conclusion: then not only truthtelling is a weakly dominant strategy for each agent (strategy-proofness) but (i) in addition any joint misreport of preferences which makes the SCF to change can be signaled by a profitable coalitional reversion to truthtelling, and (ii) the combination of strategy-proofness and non-bossiness in welfare also precipitates group strategy-proofness, a more stringent condition on the non-manipulation of preferences. Hence non-bossiness in welfare precipitates an unexpected robustness (or prevalence, or resistance) of truthtelling for strategy-proof SCFs in many domains and models of interest.

We provide some discussion and extension of our results. First, we present a result that emphasizes the prevalence of truthtelling, in the sense discussed above. Specializing to the Sprumont model, we show that not only for any bad Nash equilibrium there exists a profitable coalitional reversion to truthtelling, but in fact every bad Nash equilibrium is Pareto dominated by truthtelling for any rule satisfying efficiency, strategy-proofness, non-bossiness and a mild additional requirement. The class of rules satisfying these properties is very large and contains, for instance, all fixed path rules (Moulin, 1999). In contrast, in the Sprumont model only the class of priority rules is secure implementable (Bochet and Sakai, 2010).

Next we show that, unlike with secure implementation, the revelation principle fails for coalitional secure implementation. Some rules that violate group strategy-proofness and the group reversal property may be coalitional secure implemented using indirect mechanisms. Finally, we close the gap between our implementation concept and full implementation in dominant strategies and strong Nash equilibrium. As it turns out, the only condition needed in addition to group strategy-proofness and the group reversal property is weak non-bossiness. This condition already appeared in the literature on dominant strategy implementation (Mizukami and Wakayama, 2007).

The paper proceeds as follows. We introduce the model and some of the necessary notations in Section 2. We provide some background on secure implementation in Section 3 and examples highlighting the intuition conveyed in Bochet and Sakai (2010). Section 4 contains our central results. Section 5 provides some discussion and extension of our results, while also discussing the prevalence of truthtelling in the Sprumont Model. We provide some concluding remarks in Section 6. Longer proofs are in the Appendix.

2 Setup

2.1 Model

Let $N = \{1, \ldots, n\}$ be a *set of agents.* Let $A = A_1 \times \ldots \times A_n$ be a *set of alternatives.* For $i \in N$, we call A_i *agent i's individual set of alternatives.* We assume that if $A_i \subseteq \mathbb{R}^m$

and $|A_i| = \infty$, then A_i is convex. Let $x = (x_1, \ldots, x_n) \in A$ be an alternative and $\mathbb{1} \equiv (1, \ldots, 1) \in \mathbb{R}^n$. If alternative x is such that for all $i, j \in N$, $x_i = x_j = \alpha$, then we denote $x = \alpha\mathbb{1}$. Next, let $F \subseteq A$ be the *set of feasible alternatives.* If for all $x \in F$ there exists α such that $x = \alpha\mathbb{1}$, then the set of feasible alternatives F *determines a public goods economy.* Otherwise, the set of feasible alternatives F *determines an economy with at least one private goods component.* Hence, our model encompasses public and private goods economies.

To fix ideas, let us give two examples. It will be clear from these examples that given the set A of alternatives, the set F of feasible alternatives fully determines whether we are working with a public or private goods model. Note that the Cartesian product notation we use for the set of alternatives is for notational convenience only; none of our results require it.

Example 2.1. *Let* $A = \{a_1, \ldots, a_n\} \times \ldots \times \{a_1, \ldots, a_n\}$.
Public Goods Model: Suppose that the agents have to choose one candidate out of the set $\{a_1, \ldots, a_n\}$ *of possible candidates. Then,* $F = \{x \in A : \textit{for all } i, j \in N,\ x_i = x_j\}$.
Private Goods Model: On the other hand, if agents have to allocate the set of indivisible objects or tasks $\{a_1, \ldots, a_n\}$ *among themselves, then* $F = \{x \in A : \textit{for all } i, j \in N,\ x_i \neq x_j\}$. ◇

Example 2.2. *Let* $A = [0, 1] \times \ldots \times [0, 1]$.
Public Goods Model: Suppose that the agents have to choose a single point in the interval [0,1] that everyone will consume without rivalry, e.g., a public facility on a street (see Moulin, 1980). Then, $F = \{x \in A : \textit{for all } i, j \in N,\ x_i = x_j\}$.
Private Goods Model: On the other hand, if agents have to choose a division of one unit of an infinitely divisible good among themselves (see Sprumont, 1991), then feasibility is determined by the size of the resource and $F = \{x \in A : \textit{for all } i \in N,\ x_i \geq 0 \textit{ and } \sum_{i \in N} x_i = 1\}$. ◇

For all $i \in N$, *preferences* are represented by a complete, reflexive, and transitive binary relation R_i over A_i. As usual, for all $a, b \in A_i$, $a\, R_i\, b$ is interpreted as "i weakly prefers a to b", $a\, P_i\, b$ as "i strictly prefers a to b", and $a\, I_i\, b$ as "i is indifferent between a and b". Preferences R_i over A_i are *strict* if for all $a, b \in A_i$, $a\, R_i\, a$ implies $a\, P_i\, b$ or $a = b$. Given $i \in N$, and preference relation R_i, $p(R_i) = \{a \in A_i : aR_ib \text{ for all } b \in A_i\}$. The set $p(R_i)$ is the set of elements of A_i that are top-ranked under R_i by agent i.

For public goods models, preferences R_i over the individual set of alternatives A_i can easily be extended to preferences over the set of alternatives A (since each agent consumes the same public alternative). Whenever our model captures a private goods component, we assume that agents only care about their own consumption. Then, for both public and private goods models, we can easily extend preferences R_i over the individual set of alternatives A_i to preferences over the set of alternatives A (both preference relations only depend on agent i's consumption in A_i). Therefore, from now on, we use R_i to describe agent i's preferences over A_i as well as over A, i.e., we use both notations $x\, R_i\, y$ and $x_i\, R_i\, y_i$. Note that for private goods models, strict preferences over A_i do not need to be strict over A.

For all $i \in N$, we call a set of preferences over A_i, denoted by $\mathcal{R}_i$, a *preference domain.* Let $\mathcal{R}^N \equiv \Pi_{i\in N}\mathcal{R}_i$ be the *domain of preference profiles.* A typical preference profile is $R = (R_i)_{i\in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}_i$. Profile $R \in \mathcal{R}^N$, is often written as (R_i, R_{-i}), where $R_{-i} = (R_1, ..., R_{i-1}, R_{i+1}, ..., R_n)$.

We now define several preference domains that some of our results will cover.

Strict preference domain: Preferences R_i on $A_i \subseteq \mathbb{R}$ are *strict* if for all $x_i, y_i \in A_i$, $x_i\, R_i\, y_i$ implies $x_i\, P_i\, y_i$ or $x_i = y_i$. We say that a domain of preference profiles $\mathcal{R}^N$ is the domain of strict preferences if for each $i \in N$, each $\mathcal{R}_i$ consists of all the possible strict preferences over A_i.

Single-peaked preference domain: Preferences R_i on $A_i \subseteq \mathbb{R}$ are *single-peaked* if there exists a point $p(R_i) \in A_i$ such that for all $x_i, y_i \in A_i$ satisfying either $y_i < x_i \leq p(R_i)$ or $p(R_i) \leq x_i < y_i$, $x_i\, P_i\, y_i$. We say a domain of preference profiles $\mathcal{R}^N \equiv \mathcal{R}^{sp}$ is the domain of single-peaked preferences if for each $i \in N$, each $\mathcal{R}_i$ consists of all the possible single-peaked preferences over A_i.

Let the set of alternatives A, the set of feasible alternatives F and the domain of preference profile $\mathcal{R}^N$ be given. Then a *social choice function, SCF*, f (or rule) is a function that assigns to every preference profile $R \in \mathcal{R}^N$ a feasible alternative $f(R) \in F$.

The planner designs a mechanism (game form) which is a pair $\Gamma = ((M_i)_{i\in N}, g)$ where M_i is agent i's message (strategy) set, and $g : \prod_{i\in N} M_i \to F$ is the outcome function mapping each message profile to an alternative. For each $R \in \mathcal{R}^N$, the pair (Γ, R) is a game in which the set of players is N, the set of strategy profiles is $M = \prod M_i$, and each player i's payoff is $g(m)$ where $m = (m_i)_{i\in N}$ is a message profile. We confine our attention to pure strategies.

For a given preference profile $R \in \mathcal{R}^N$, we use the usual notations that $R_S \equiv (R_i)_{i\in S}$ and $R_{-S} \equiv (R_j)_{j\notin S}$. Likewise, for each $S \subset N$, we let $\mathcal{R}^S$ be the domain of preference profiles for S. Similar notations are used for $\mathcal{R}$, m and M.

We now introduce several important definitions which are used repeatedly throughout the paper.

Nash Equilibrium: For a given mechanism $\Gamma = (M, g)$, a message profile m is a Nash equilibrium at profile $R \in \mathcal{R}^N$ if for each $i \in N$, $g(m)R_i g(m'_i, m_{-i})$ for each $m'_i \in M_i$. For each $R \in \mathcal{R}^N$, let $NE(\Gamma, R)$ be the set of Nash equilibria of (Γ, R).

Weak Domination: Fix A, F and $\mathcal{R}^N$. For all $x, y \in A$, x weakly dominates y at preference profile $R \in \mathcal{R}^N$ via coalition $S \subseteq N$ if $x_i R_i y_i$ for each $i \in S$, and $x_j \; P_j \; y_j$ for at least one j in S. We write $x \,\text{wdom}[R, S] \; y$ to denote that x weakly dominates y via coalition $S \subseteq N$ at preference profile R.

Dominant strategies: For a given mechanism $\Gamma = (M, g)$, a strategy profile m is (weakly) dominant at profile $R \in \mathcal{R}^N$, if for each $i \in N$ and $\tilde{m} \in M$, $g(m_i, \tilde{m}_{-i}) R_i g(m_i, \tilde{m}_{-i})$. For each $R \in \mathcal{R}^N$, let $DS(\Gamma, R)$ be the set of dominant strategies of (Γ, R).

Secure Implementation: An SCF f is secure implementable if there exists a mechanism Γ such that for each $R \in \mathcal{R}^N$,
(i) there exists a strategy profile $m \in DS(\Gamma, R)$ such that $g(m) = f(R)$
(ii) for each $m \in NE(\Gamma, R)$, $g(m) = f(R)$.

The above definition is the one introduced in Saijo et al. (2007). Note that secure implementation is nothing more than the requirement of double implementation in dominant strategies and Nash equilibrium. The idea behind secure implementation is very appealing for the following reasons: (a) When a mechanism has a dominant strategy, agents are likely to play it. Thus, the first requirement of secure implementation says that in each state, the "right" outcome must be delivered by at least one dominant strategy. (b) At the same time, secure implementation recognizes the fact that agents may get stuck at any Nash equilibrium. Thus, the second requirement asks that each Nash equilibrium delivers the "right" outcome.

As we mentioned in the Introduction, preplay communication among participating agents is allowed in many practical mechanisms. Once pre-play communication is allowed, agents will not get stuck at any Nash equilibrium which is susceptible to coalitional deviations. In this paper, we assume that agents coordinate at a strong Nash equilibrium – a strategy that is immune to coalitional deviations (Aumann, 1959). With this motivation in mind, we depart from secure implementation by requiring that all the strong Nash equilibria deliver the "right" outcome. Let us now define strong Nash equilibria and our implementation notion formally below.

Strong Nash Equilibrium: For a given mechanism $\Gamma = (M, g)$, a message profile m is a strong Nash equilibrium at profile $R \in \mathcal{R}^N$, if there exists no $\tilde{m}_S$ such that $g(\tilde{m}_S, m_{-S}) \,\text{wdom}[R, S] g(m)$. For each $R \in \mathcal{R}^N$, let $SNE(\Gamma, R)$ be the set of strong Nash equilibria of (Γ, R).

Coalitional Secure Implementation: An SCF f is coalitional secure implementable if there exists a mechanism Γ such that for each $R \in \mathcal{R}^N$,
(i) there exists a strategy profile $m \in DS(\Gamma, R) \cap SNE(\Gamma, R)$ such that $g(m) = f(R)$
(ii) for each $m \in SNE(\Gamma, R)$, $g(m) = f(R)$.

Our extension of secure implementation allows for coalitional deviations to take place. When only coalition of size 1 are possible, this definition coincides with the standard definition of secure implementation. Note however that item (i) does not impose that all dominant strategy equilibria deliver the right outcome. Instead it only requires that there exists a dominant strategy equilibrium that is also a strong Nash equilibrium. Item (ii) then makes sure that all strong Nash equilibria coincide with rule f at the true preference profile. If a dominant strategy equilibrium m delivers the wrong outcome, item (ii) then guarantees that m cannot be a strong Nash equilibrium. We will comment on this definition in Section 6.

For the most part, we concentrate on direct mechanisms. Given an SCF f, the direct mechanism associated to f is $\Gamma^* = (\mathcal{R}^N, f)$.

Direct Coalitional Secure Implementation: An SCF f is coalitional secure implementable via its direct mechanism Γ^* if for each $R \in \mathcal{R}^N$,
(i) $R \in DS(\Gamma^*, R) \cap SNE(\Gamma^*, R)$
(ii) for each $\tilde{R} \in SNE(\Gamma^*, R)$, $f(\tilde{R}) = f(R)$.

We now introduce several properties of an SCF which are central for the paper.

Strategy-Proofness: An SCF f satisfies strategy-proofness if for each $R \in \mathcal{R}^N$, each agent $i \in N$ and $R'_i \in \mathcal{R}_i$, $f(R) R_i f(R'_i, R_{-i})$.

Strategy-proofness of an SCF is a strong incentive compatibility notion. It guarantees that truthtelling is a (weakly) dominant strategy in the direct revelation mechanism associated to f. We introduce a strengthening of strategy-proofness once coalitions can form.

Group Strategy-Proofness: An SCF f is group strategy-proof if for each $R \in \mathcal{R}^N$ and coalition $S \subseteq N$, there does not exist $R'_S \in \mathcal{R}^S$ such that $f(R'_S, R_{-S})\,\text{wdom}[R, S] f(R)$.

Non-Bossiness: An SCF f satisfies non-bossiness if whenever $f_i(R) = f_i(\tilde{R}_i, R_{-i})$ for some $i \in N$, $\tilde{R}_i \in \mathcal{R}$ and $R \in \mathcal{R}^N$, then $f(R) = f(\tilde{R}_i, R_{-i})$.

We close this section with a condition that is central for the standard notion of secure implementation.

Rectangular Property: An SCF f satisfies the rectangular property if for each $R, \tilde{R} \in \mathcal{R}^N$ with $f(R) \neq f(\tilde{R})$, there exists $i \in N$ such that $f(R_i, \tilde{R}_{-i}) P_i f(\tilde{R})$.

The rectangular property imposes a strong invariance on an SCF across preference profiles, as we explain next.

3 A Difficulty with Secure Implementation

As emphasized in Saijo et al. (2007), many strategy-proof rules of interest fail to be secure implementable. Indeed many rules of interest violate the rectangular property, a necessary condition for secure implementation. We recall first the intuition behind the

	R_2	$\tilde{R}_2$
R_1	$f_1(R_1, R_2),\ f_2(R_1, R_2)$	$f_1(R_1, \tilde{R}_2),\ f_2(R_1, \tilde{R}_2)$
$\tilde{R}_1$	$f_1(\tilde{R}_1, R_2),\ f_2(\tilde{R}_1, R_2)$	$f_1(\tilde{R}_1, \tilde{R}_2),\ f_2(\tilde{R}_1, \tilde{R}_2)$

Table 1: The Rectangular Property

necessity of the rectangular property as the result in Saijo et al. (2007) may be not be as well-known as it deserves.

Our discussion is based on table 1 for which we have fixed an SCF f. At the true preference profile (R_1, R_2), suppose that $f_1(R_1, \tilde{R}_2) I_1 f_1(\tilde{R}_1, \tilde{R}_2)$. By strategy-proofness, this implies that reporting $\tilde{R}_1$ is a best-response for agent 1 when agent 2 reports $\tilde{R}_2$. Assume next that $f_2(\tilde{R}_1, R_2) I_2 f_2(\tilde{R}_1, \tilde{R}_2)$. By the same token, $\tilde{R}_2$ is a best-response when agent 1 reports $\tilde{R}_1$. Hence $(\tilde{R}_1, \tilde{R}_2)$ is a Nash equilibrium at (R_1, R_2) with $f(\tilde{R}_1, \tilde{R}_2)$ as a Nash equilibrium outcome. By secure implementability, since all Nash equilibria are "good", we must have that $f(R_1, R_2) = f(\tilde{R}_1, \tilde{R}_2)$. One can see right away the possibility of a tension between strategy-proofness and the rectangular property. Strategy-proofness imposes "rigidities" on f so that unilateral change in preference reports have a limited impact on the outcome delivered by the rule. This in turn may create problems since joint untruthful reports may be unbreakable through unilateral deviations: agents may get stuck at bad Nash equilibria.

We briefly present here some problems stemming from the failure of the rectangular property. We focus on four different examples. The first one is taken from Bochet and Sakai (2010).

Example 3.1. *Structure and coalitional instability of bad Nash equilibria I*
Let $F = \{x \in [0, \Omega]^n : \text{for all } i \in N,\ x_i \geq 0 \text{ and } \sum_{i \in N} x_i = \Omega\}$ where $\Omega \in \mathbb{R}_{++}$. The preference domain is the one of single-peaked preferences, $\mathcal{R}^{sp}$, over $[0, \Omega]$. Then $(\mathcal{R}^{sp}, F)$ determine the Sprumont Model (Sprumont, 1991) of division under single-peaked preferences. A rule that is central in this model is the so-called *Uniform rule* f^U, defined for each $R \in \mathcal{R}^N$ and each $i \in N$ as,

$$f^U(R) = \begin{cases} min\{p(R_i), \lambda\} \text{ if } \sum\limits_{i \in N} P(R_i) \geq \Omega \\ max\{p(R_i), \lambda\} \text{ if } \sum\limits_{i \in N} P(R_i) \leq \Omega \end{cases}$$

where λ solves $\sum\limits_{i \in N} f^U(R) = \Omega$.

The uniform rule has many desirable properties. In particular it is group strategy-proof. However, it fails to be securely implementable as shown below.

Let $n = 3$, $\Omega = 6$ and pick $R \in \mathcal{R}^N$ with peak profile $p(R) = (1, 2, 4)$. Consider

$\Gamma = (\mathcal{R}^{sp}, f^U)$, the direct revelation game of the uniform rule. Observe that $\tilde{R}$ with $p(\tilde{R}) = (2, 2, 2)$ is a Nash equilibrium. Indeed, $f^U(\tilde{R}) = (2, 2, 2)$ and by definition of the uniform rule, no one can change this allocation by any unilateral deviation from $\tilde{R}$. Notice that $f^U(R) = (1, 2, 3)$ Pareto dominates $f^U(\tilde{R})$ at R. Agents 1 and 3 have the joint profitable deviation of simply reporting their true preferences so that the true uniform allocation is obtained. By the same token, observe that the report R' with $p(R') = (1.5, 2, 2.5)$ and $f^U(R') = (1.5, 2, 2.5)$ is also a Nash equilibrium at R. One can verify that $NE(\Gamma^*, R)$ contains in fact an infinity of bad Nash equilibria described by the following sets,

$$\{\tilde{R} \in \mathcal{R}^N : 1 < p(\tilde{R}_1) \leq 2 = p(\tilde{R}_2) \leq p(\tilde{R}_3) < 4, \sum_i p(\tilde{R}_i) = 6\}$$

$$\{\tilde{R} \in \mathcal{R}^N : p(\tilde{R}_1) = 2, p(\tilde{R}_2) = 2, p(\tilde{R}_3) \leq 2\}$$

$$\{\tilde{R} \in \mathcal{R}^N : p(\tilde{R}_1) \geq 2, p(\tilde{R}_2) = 2, p(\tilde{R}_3) = 2\}$$

$\diamond$

Example 3.2. *Structure and coalitional instability of bad Nash equilibria II*
Let $A_i = \{h_1, ..., h_n\}$ for each $i \in N$ and $F = \{x = (x_1, ..., x_n) \in A : x_i \neq x_j$ for each $i \neq j\}$, i.e., F determines a private good economy with indivisible goods, as in example 2.1. Let the preference domain $\mathcal{R}^N$ be the set of strict preferences over $\{h_1, ..., h_n\}$. Pick f^{TTC} to be the top-trading cycle SCF in which each agent i is endowed with object h_i.[3]
Let $N = \{1, 2, 3\}$ and $A = \{h_1, h_2, h_3\}^3$. Let the set of preferences be as follows:

R_1	R_2	R_3	$\tilde{R}_1$	$\tilde{R}_2$
h_2	h_1	h_2	h_1	h_3
h_1	h_3	$\vdots$	$\vdots$	$\vdots$
h_3	h_2			

[3] $f^{TTC}(R)$ is determined according to the following process:

Step m: Each agent who have not been allocated an object in the previous steps points to the agent who owns her/his most preferred object among those which are not assigned to any agent yet. There exist at least one cycle of agents $\{i_1, \cdot, i_k\}$ such that each i_l where $l < k$ points to i_{l+1} while i_k points to i_1. Under the TTC rule, each agent in a cycle is allocated the object of the agent to whom she points.

The above process is terminated once every agent is allocated an object.

Observe here that the TTC rule f^{TTC} gives the following allocations,

$$
\begin{aligned}
f^{TTC}(R_1, R_2, R_3) &= (h_2, h_1, h_3) \\
f^{TTC}(\tilde{R}_1, R_2, R_3) &= (h_1, h_3, h_2) \\
f^{TTC}(R_1, \tilde{R}_2, R_3) &= (h_1, h_3, h_2) \\
f^{TTC}(\tilde{R}_1, \tilde{R}_2, R_3) &= (h_1, h_3, h_2).
\end{aligned}
$$

Observe that when the state is $R = (R_1, R_2, R_3)$, the dominant strategy equilibrium is (R_1, R_2, R_3), i.e., truthtelling. Consider now another strategy profile $(\tilde{R}_1, \tilde{R}_2, R_3)$ which results in (h_1, h_3, h_2). Clearly, agent 3 has no incentive to unilaterally deviate as she obtains her top choice under this preference report. If either agent 1 or 2 unilaterally deviates from $(\tilde{R}_1, \tilde{R}_2, R_3)$ to truth telling, then the outcome remains (h_1, h_3, h_2). Combining this with the result that truthtelling is a weakly dominant strategy for each agent, none of agents 1 and 2 have an incentive to deviate unilaterally from $(\tilde{R}_1, \tilde{R}_2, R_3)$. Thus, $(\tilde{R}_1, \tilde{R}_2, R_3)$ is a Nash equilibrium. Notice that it is not a strong Nash equilibrium. Agents 1 and 2 jointly deviating from $(\tilde{R}_1, \tilde{R}_2)$ to (R_1, R_2) (while agent 3 reports R_3) leads to allocation (h_2, h_1, h_3). This is a profitable deviation for both agents. An important difference compared to the previous example is that the allocation under $(\tilde{R}_1, \tilde{R}_2, R_3)$ is not Pareto comparable with the one obtained under the report (R_1, R_2, R_3). Hence following the coalitional deviation by agents 1 and 2, agent 3 is worse-off since he was getting his top choice h_2 under the report $(\tilde{R}_1, \tilde{R}_2, R_3)$. ◇

From the previous two examples, two conclusions may be drawn. The first and most important one is that, for many models and rules of interest, at a bad Nash equilibrium, there may exists a coalition of agents which can benefit by changing their preference report in the direct revelation mechanism. Hence, in many models, the requirement of having no bad Nash equilibria is too strong if pre-play communication is allowed. The deviation is simple since it is a coalitional reversion to truthtelling. Theorem 4.2 and Theorem 4.9 introduced in the next section will show under which conditions the above conclusion is true. The second conclusion is that the observation that bad Nash equilibria are Pareto inefficient is not a general observation, as shown in Example 3.2.

Our third example considers a public decision model.

Example 3.3. *Structure and coalitional instability of bad Nash equilibria III*
Let $F = \{x \in [0, 1]^n : \text{ for all } i, j \in N, \ x_i = x_j\}$. Let the common preference domain be the set of single-peaked preferences $\mathcal{R}^N$ over $[0, 1]$. Consider the median rule $f^{med} = med_{i \in N}(p(R_i))$ where med is the median operator. The median rule has many desirable properties. In particular it is group strategy-proof. However, it fails to be securely implementable as shown below.

Let $n = 3$ and pick $R \in \mathcal{R}^N$ with peak profile $p(R) = (\frac{1}{10}, \frac{1}{5}, \frac{2}{5})$. By the median rule,

$f^{med}(R) = \frac{1}{5}$. Consider $\Gamma^* = (\mathcal{R}^N, f^{med})$, the direct revelation game of the median rule. Observe that $\tilde{R}$ with $p(\tilde{R}) = (\frac{1}{10}, \frac{1}{10}, \frac{1}{10})$ is a Nash equilibrium. Indeed, $f^{med}(\tilde{R}) = (\frac{1}{10})$ and by definition of the median rule, no one can change this allocation by any unilateral deviation from $\tilde{R}$. Hence $\tilde{R} \in NE(\Gamma^*, R)$ and f^{med} cannot be securely implemented. But agents 2 and 3 have the joint profitable deviation of simply reporting their true preference relation. Notice, like in the previous example, that truthtelling does not Pareto dominate all bad Nash equilibria.

One can verify that $NE(\Gamma^*, R)$ contains in fact an infinity of bad Nash equilibria at preference profile R described by the following sets,

$$\{\tilde{R} \in \mathcal{R}^N : p(\tilde{R}_i) = p(\tilde{R}_j) \neq \frac{1}{5} \text{ for each } i, j \in N\}$$

$$\{\tilde{R} \in \mathcal{R}^N : 1 > p(\tilde{R}_1) = p(\tilde{R}_2) = k > \frac{1}{5} \text{ and } p(\tilde{R}_3) > k\}$$

$$\{\tilde{R} \in \mathcal{R}^N : 0 < p(\tilde{R}_2) = p(\tilde{R}_3) = k < \frac{1}{5} \text{ and } p(\tilde{R}_1) < k\}$$

◇

We introduce one last example which shows that some rules of interest may remain beyond reach even if coalitional deviations are possible.

Example 3.4. *Survival of bad Nash equilibria*
Let $F = \{(x, t) \in \{0, 1\}^n \times \mathbb{R}^n : \sum_{i \in N} x_i = 1\}$. The feasible set F stands for a model in which there is one object to be given to one out of the n agents, and monetary transfers are possible. Let SCF f^V be the Vickrey rule, i.e. the second price auction. Let $\mathcal{R}^N = \mathbb{R}_+$. For each $i \in N$, each preference relation R_i is indexed by a real number that stands for the valuation agent i attaches to the object. With a slight abuse of notation, each R_i is such that for (x, t), (x, t') with $t > t'$, then $(x_i, t_i) P_i (x_i, t'_i)$. Hence, preferences are said to be quasi-linear. For each $R \in \mathcal{R}^N$, $f(R) = (x, t) \in F$ with (i) $x_i = 1$ if $R_i = \max_{j \in N} R_j$, and $x_i = 0$ otherwise, (ii) $t_i = max_{j \neq i} R_j$ if $x_i = 1$, and $t_i = 0$ otherwise. While the Vickrey rule is strategy-proof, it fails to be securely implementable as shown below.

Let $n = 2$ and fix a preference profile R with $R_1 > R_2$. The joint report (R_1, R_2) is a weakly dominant strategy: agent 1 receives the object and pays $t_2 = R_2$ while agent 2 pays nothing. However, there is an infinity of bad Nash equilibria in the direct revelation mechanism of f at profile R, as described by the following set,

$$\{(\tilde{R}_1, \tilde{R}_2) \in \mathcal{R}^N : 0 \leq \tilde{R}_1 \leq R_2 \text{ and } R_1 \leq \tilde{R}_2\}$$

Because agent 2 gets a non-negative payoff at R when the report is $(\tilde{R}_1, \tilde{R}_2)$, this joint lie is a Nash equilibrium at R. Within the above set, $(\tilde{R}_1, \tilde{R}_2) = (0, \tilde{R}_2)$ is a strong Nash equilibrium for any $\tilde{R}_2 \geq R_1$. Notice also that truthtelling is not a strong Nash equilibrium since $(R_1, \tilde{R}_2)$ is a profitable coalitional deviation for any $\tilde{R}_2 < R_2$. The Vickrey rule violates group strategy-proofness but also the non-bosiness condition. As shown next, group strategy-proofness is a central condition for coalitional secure implementation by direct mechanisms. ◇

4 Recovering Positive Results

We show how positive results on secure implementation can be recovered if agents can communicate with one another before participating in the mechanism. In such cases, agents will not get stuck at a "bad" Nash equilibrium as long as some coalition has a profitable deviation from it. Consequently, requirement (ii) for coalitional secure implementation is weaker than the one for secure implementation. On the other hand, requirement (i) of coalitional secure implementation demands that the alternative in the SCF is reached by a preference report which is both weakly dominant and a strong Nash under coalitional secure implementation. Thus, requirement (i) of our implementation notion is more demanding than the one under secure implementation. We start first with results for the domain of strict preferences.

4.1 Strict Preferences domain

Our first result is striking in contrast to secure implementation. When preferences are strict, the only conditions needed for coalitional secure implementation are strategy-proofness and non-bossiness. Because these two conditions are equivalent to group strategy-proofness for the strict preference domain, under this restriction the only condition needed for coalitional secure implementation is group strategy-proofness.[4]

Lemma 4.1. *Let $\mathcal{R}^N$ be the strict preference domain and let F determine a private good economy. Rule f satisfies group strategy-proofness if and only if it satisfies strategy-proofness and non-bossiness.*

Proof. See for instance Pápai (2000) for a proof of this result. □

Theorem 4.2. *Let $\mathcal{R}^N$ be the strict preference domain and let F determine a private good economy. Pick f and consider $\Gamma^* = (\mathcal{R}^N, f)$. For each $R, \tilde{R} \in \mathcal{R}^N$ with $f(R) \neq f(\tilde{R})$, there exists $S \subseteq N$ and $R_S \in \mathcal{R}^S$ such that $f(R_S, \tilde{R}_{-S}) P_i\, f(\tilde{R})$ for all $i \in S$ if f satisfies strategy-proofness and non-bossiness. Furthermore, f is coalitional secure implementable via the direct mechanism if and only if f satisfies strategy-proofness and non-bossiness.*

Proof. See Appendix. □

[4]Group strategy-proofness is an obvious necessary condition for coalitional secure implementation via a direct revelation mechanism.

Corollary 4.3. *Let $\mathcal{R}^N$ be the strict preference domain and let F determine a public good economy. Pick f and consider $\Gamma^* = (\mathcal{R}^N, f)$. Then f is coalitional secure implementable if and only if f satisfies strategy-proofness.*

Given what we know regarding public good economies the above corollary is of limited interest. But Theorem 4.2 is very instructive. We obtain a direct corollary for several models of interest.

Corollary 4.4. *Let $\mathcal{R}^N$ be the strict preference domain, and let $F = \{x \in \{1, ..., h\}^N : h \geq n, |x_i| = 1 \ \forall i \in N, x_i \neq x_j, i \neq j\}$. If an SCF satisfies strategy-proofness and non-bossiness, then it satisfies group strategy proofness. Hence any trading cycle rule (Pycia and Ünver, 2011) is coalitional secure implementable.*

Fujinaka and Wakayama (2011) show that the only secure implementable efficient rules in the so-called housing model are the *priority rules* which allocate objects based on a fixed ordering of the set of agents.[5] Here, any trading cycle rule identified in Pycia and Ünver (forthcoming) is coalitional secure implementable and efficient: any bad Nash equilibrium can be broken by a coalitional deviation.

Before we move on, we make three remarks based on the proof of Theorem 4.2.

Remark 4.5. In the proof observe that any undesirable Nash equilibria is broken by a coalition which reverts to truthtelling. Given that truthtelling is a dominant strategy for any agent and arguably the most focal point, the coordination issue for the deviating coalition from a undesirable Nash equilibrium is very mild. In fact, it turns out that in some domains truthtelling Pareto dominates any other Nash equilibria. We will investigate this apparent *superiority of truthtelling* in Section 5.3.

Remark 4.6. The second most important observation in the proof is that every member of the coalition which deviates from a undesirable Nash equilibrium gets strictly better off. This means that every member of the deviating coalition has a very strong incentive to block a "bad" Nash equilibrium. In this sense, any strategy-proof and non-bossy rule would be implementable in the environment of Theorem 4.2 even if we strengthen our implementation notion by requiring that (i) truthtelling is a dominant strategy which no coalition can block even if we use the "weak" blocking and (ii) any undesirable outcome can be blocked by some coalition in the strong sense.

Remark 4.7. Any report that is blocked by some coalition through a joint deviation is not a strong Nash equilibrium. However, this deviation could be itself not self-enforcing in the sense that some subcoalition may benefit by further deviating from the initial deviation. Indeed in such cases some members would opt not to block the original report. Hence, the original report may arise at an equilibrium. This motivation led to the concept

[5]Priority rules are also known as serial dictatorships.

of coalition-proof Nash equilibria (Bernheim et al., 1987) which requires that no coalition has a self-enforcing deviation. If we replace strong Nash equilibria by coalition-proof Nash equilibria in our definition of implementation, any strategy-proof and non-bossy rule would be implementable in private good economies with strict preferences. Indeed any bad outcome is blocked by some coalition which revert to truthtelling. However, no coalition can benefit by further deviating because of the group-strategy proofness of f.

The preceding result identifies group strategy-proofness as the only required condition for coalitional secure implementation via direct mechanisms for a specific preference domain. We now turn our attention to identifying necessary and sufficient conditions for coalitional secure implementation in more general domains.

When the preference domain admits indifferences, group strategy-proofness is no longer sufficient. We introduce below a condition dubbed the *group reversal property*, a necessary preference reversal requirement across preference profiles when the SCF varies.

Group Reversal Property: An SCF f satisfies the *Group Reversal Property* if for each $R, \tilde{R} \in \mathcal{R}^N$ with $f(R) \neq f(\tilde{R})$, there exist $S \subseteq N$ and $R'_S \in \mathcal{R}^S$ such that $f(R'_S, \tilde{R}_{-S}) \,\mathrm{wdom}[R, S] f(\tilde{R})$.

When, say, $\tilde{R}$ is bad Nash equilibrium at R in the direct revelation mechanism of f, notice that the group reversal property only requires a deviation to an alternative preference profile for some $S \subseteq N$, without pinning down the type of alternative preference profile that S must revert to. Our previous result showed that when preferences are strict, not only there exists such an alternative preference profile, but it is in fact the true profile R_S for some coalition S. Going back to Table 1, we see that despite the fact that $(\bar{R}_1, \bar{R}_2)$ is a Nash equilibrium at (R_1, R_2), all we need is to identify some profitable coalitional deviation to break the undesirable Nash equilibrium $(\bar{R}_1, \bar{R}_2)$. We first show the independence of group strategy-proofness and the group reversal property.

Example 4.8. *Independence of group strategy proofness and the group reversal property* Suppose that there are two agents $\{1, 2\}$ and let the set of preferences for each agent be $\mathcal{R}_i = \{R_i, \tilde{R}_i\}$. The set of social alternatives is $A = \{a, b, c, d\}$.

The agents' preferences are given as follows:

R_1	R_2	$\tilde{R}_1$	$\tilde{R}_2$
$b \sim d$	a	c	b
a	$c \sim d$	d	d
c	b	$a \sim b$	$a \sim c.$

The SCF f is found in the following table:

	R_2	$\tilde{R}_2$
R_1	a	b
$\tilde{R}_1$	c	d

In this example, group strategy-proofness is satisfied, but the group reversal property is violated. Specifically, $f(R) \neq f(\tilde{R})$ and there is no coalition $S \subseteq N$ and $R'_S \in \mathcal{R}^S$ such that $f(R'_S, \tilde{R}_{-S}) \,\text{wdom}[R,S] f(\tilde{R})$. In the direct revelation mechanism of f, $\tilde{R}$ is a strong Nash equilibrium at R and $f(\tilde{R}) = d$ is not in the SCF. Thus, f is not coalitional secure implementable by the direct mechanism of f. ◇

We now show that both group strategy-proofness and group reversal properties are necessary and sufficient for coalitional secure implementation.

Theorem 4.9. *An SCF f is coalitional secure implementable by the direct mechanism $\Gamma^* = (\mathcal{R}^N, f)$ if and only if*

- *f is group-strategy-proof*
- *f satisfies the group reversal property.*

Proof. If f is not group-strategy proof, then there exist $R \in \mathcal{R}^N$, $S \subseteq N$ and $\tilde{R}_S \in \mathcal{R}^S$ such that $f(\tilde{R}_S, R_{-S}) \,\text{wdom}[R,S] f(R)$. Clearly, $R \notin SNE(\Gamma^*, R)$. Thus, group strategy-proofness is a necessary condition for coalitional secure implementation via the direct mechanism of f.

If f does not satisfy the group reversal property, then there exist $R, \tilde{R} \in \mathcal{R}^N$ such that (i) $f(R) \neq f(\tilde{R})$ and (ii) for no $S \subseteq N$ and no $R'_S \in \mathcal{R}^S$, $f(R'_S, \tilde{R}_{-S}) \,\text{wdom}[R,S] f(\tilde{R})$. Thus, $f(\tilde{R}) \in SNE(\Gamma^*, R)$, but $f(\tilde{R}) \neq f(R)$ which is a contradiction. Thus, the group reversal property is a necessary condition for coalitional secure implementation via the direct mechanism of f.

The sufficiency part follows immediately from the definitions. □

We have already pointed out that the set of rules that are coalitional secure implemented is significantly larger than the secure implementable rules in private good economies with the strict preference domain. We now further analyze how the requirements for coalitional secure implementation compare to the ones for secure implementation in either more permissive or restrictive preference domains.

4.2 A Direct Comparison with Secure Implementation

In order to make a more meaningful comparison between secure implementation and its coalitional counterpart, we need to compare the rectangular property to the combination of both group strategy-proofness and the group reversal property. For this purpose, we find useful to invoke the result from Saijo et al. (2007) that shows the equivalence between the rectangular property and the combination of the outcome rectangular property and non-bossiness in welfare. We define these conditions below.

Non-bossiness in welfare: An SCF f satisfies non-bossiness in welfare if whenever $f_i(R) I_i f_i(\tilde{R}_i, R_{-i})$ for some $i \in N$, $\tilde{R}_i \in \mathcal{R}_i$ and $R \in \mathcal{R}^N$, then $f(R) = f(\tilde{R}_i, R_{-i})$.[6]

Outcome rectangular property: An SCF f satisfies the outcome rectangular property if for each $R, R' \in \mathcal{R}^N$, if $f(R_i, R'_{-i}) = f(R')$ for each $i \in N$, then $f(R) = f(R')$.

Equipped with these definitions we now show that in many preference domains, the outcome rectangular property can be completely dispensed with for coalitional secure implementation. Our results are at times general, and at times preference domain-specific. In particular, if the preference domain is too narrow, coalitional secure implementation may be more demanding than secure implementation. However, an intuitive rich domain condition is satisfied, this negative conclusion is typically false and that our implementation notion expands the set of secure implementable rules.

We first start with an unequivocal result: the combination of strategy-proofness and non-bossiness in welfare yield the group reversal property, independently of the preference domain at hand.

Theorem 4.10. *If an SCF f satisfies strategy-proofness and non-bossiness in welfare then f satisfies the group reversal property.*

Proof. See Appendix. □

The theorem above shows that group reversal property is satisfied for SCFs which are both strategy-proof and non-bossy in welfare. However, these two conditions do not necessarily imply group strategy-proofness. We illustrate this point below.

Example 4.11. *Narrow domain: strategy-proofness and non-bossiness in welfare*
Let $N = \{1, 2\}$ and $F = \{w, x, y, z\}$. Suppose that the set of preferences for each agent is $R_i \cup \tilde{R}_i$ where

$$c \; P_1 \; d \; P_1 \; a \; P_1 \; b \qquad\qquad b \; \tilde{P}_1 \; d \; \tilde{P}_1 \; a \; \tilde{P}_1 \; c$$

$$b \; P_2 \; d \; P_2 \; a \; P_2 \; c \qquad\qquad c \; \tilde{P}_2 \; d \; \tilde{P}_2 \; a \; \tilde{P}_2 \; b.$$

Now consider the following SCF f:

	R_2	$\tilde{R}_2$
R_1	a	c
$\tilde{R}_1$	b	d

[6] Saijo et.al. (2007) labels this condition simply as non-bossiness.

Observe here that f satisfies strategy-proofness and non-bossiness in welfare. In fact, f satisfies the rectangular property and is thus secure implementable. However, f is not group strategy-proof since $f(\tilde{R}) = d$ wdom$[R, N]$ $a = f(R)$, and it is thus not coalitional secure implementable. We conclude that coalitional secure implementation can be more demanding than secure implementation in some cases. ◇

A key feature for the failure of coalitional secure implementation in the above example is the narrowness of the preference domain. When the preference domain is sufficiently "large", the combination of strategy-proofness and non-bossiness in welfare implies group strategy-proofness. We introduce below a richness condition on the preference domain which guarantees that the two conditions imply group strategy-proofness.

Rich domain: Domain $\mathcal{R}^N$ is *rich* if for each $i \in N$, $x_i \neq y_i \in A_i$, there exists $R_i \in \mathcal{R}_i$ such that $x_i P_i y_i P_i z_i$ for all $z_i \in A_i$ with $z_i \neq x_i$ and $z_i \neq y_i$.

If the domain $\mathcal{R}^N$ is rich, then for any given agent $i \in N$ and for any two alternatives $x_i, y_i \in A_i$ for this agent there must exist preferences for i which place these alternatives as the top two alternatives. For instance, in private good economies the strict preference domain is rich. Clearly, any preference domain containing the strict preference domain is also rich. Furthermore, some domains not containing the strict preference domain are rich. For instance, suppose that the agents only care about their top two alternatives. Such domains would be rich as long any two alternatives are the top two at some point. However, some important domains of interest such as the single peaked domain in the Sprumont setting are not rich. First we show that in rich domains, strategy-proofness and non-bossiness in welfare together imply group-strategy proofness.

Theorem 4.12. *Let $\mathcal{R}^N$ be a rich domain. If an SCF f satisfies strategy-proofness and non-bossiness in welfare then it satisfies group strategy-proofness. Hence, f is coalitional secure implementable by its direct revelation mechanism.*

Proof. See Appendix. □

We first point out that Remarks 4.6 and 4.7 apply for SCFs that are strategy-proof and satisfy non-bossiness in welfare. Theorem 4.10 uses a proof technique which differs from the one used in Theorem 4.2. However, by inspecting the if Part of the latter theorem, it is easy to see that replacing non-bossiness by non-bossiness in welfare takes care of cases where an indifference could occur. Hence, whenever for some $R, \tilde{R} \in \mathcal{R}^N$, $f(R) \neq f(\tilde{R})$ then there exists a strictly profitable deviation to truthtelling for some $S \subseteq N$. Importantly, if the domain $\mathcal{R}^N$ is rich, Theorem 4.12 then implies that this deviation is self-enforcing by the group strategy-proofness of f.

The combination of Theorems 4.10 and 4.12 imply that non-bossiness in welfare and strategy-proofness are sufficient for coalitional secure implementation in rich preference domains.[7] We have already pointed out that the domain of strict preferences in private good economies is rich. Furthermore, there non-bossiness in welfare is equivalent to non-bossiness. The very large set of trading cycles rules satisfy these two conditions along with efficiency –arguably the most desirable property in private good economies (Pycia and Ünver, forthcoming). However, this set reduces to the serial dictatorships once the additional requirement imposed by the outcome rectangularity is in play. This observation suggest that the outcome rectangular property is key for the set of secure implementable rules to be very narrow.

Out of the three necessary conditions for secure implementation identified by Saijo et al. (2007), our results show that the outcome rectangular property can be completely dispensed with. The other two conditions turn out to be far less demanding, and the class of coalitional secure implementation is relatively large compared to the class of secure implementable rules. A vexing aspect of our characterization is that we rely on two conditions which are together sufficient for our implementation notion, emphasizing that they are necessary for secure implementation. In general, non-bossiness in welfare is actually not necessary for coalitional secure implementation via direct mechanisms. Unfortunately, the combination of strategy-proofness and non-bossiness is not enough to obtain group strategy-proofness –on that ground recall also Example 4.11 which showed that the combination of stragegy-proofness and non-bossiness in welfare does not necessarily imply group strategy-proofness. We provide support for these observations in the next two examples.

Example 4.13. *Non-bossiness in welfare is not necessary*
Consider a house allocation problem with for alternatives, a, b, and c. Let there be two agents, 1 and 2. The set of preferences for agent 1 is unrestricted while the one for agent 2 consist of only one strict preference which ranks a ahead of c. The SCF in this case is as follows: $f(R) = (a, c)$ if $a\ P_1 b$ but $f(R) = (b, a)$ if $b\ R_1 a$. It is easy to see that f is strategy-proof. Next, fix a preference profile R in which aI_1b. In this case $f(R) = (b, a)$. Consider another preference $\tilde{R}_1$ of agent 1 in which $a\tilde{P}_1b$. Now $f(\tilde{R}_1, R_2) = (a, c) \neq f(R)$ but $f_1(R) = b\ I_1\ a = f_1(\tilde{R}_1, R_2)$. This shows that f violates non-bossiness in welfare. Consequently, f is not secure implementable.

However, f satisfies both group strategy-proofness and the group reversal property. Clearly, if agent 1 strictly prefers a to b, then any group deviation leads to either allocation

[7]Note that group strategy-proofness does not imply non-bossiness in welfare on a rich domain. To see this suppose that group strategy-proofness is satisfied but non-bossiness in welfare is violated, i.e. there exists $i \in N$, $R'_i \in \mathcal{R}_i$ such that $f_i(R'_i, R_{-i})I_if_i(R)$ and $f(R'_i, R_{-i}) \neq f(R)$. The cases where there exists $j \neq i$ such that either $f_j(R'_i, R_{-i})P_jf_j(R)$ or $f_j(R)P_jf_j(R'_i, R_{-i})$ imply a contradiction of group strategy-proofness. If none of these two cases apply, this means that $f_j(R'_i, R_{-i})I_jf_j(R)$ for each $j \neq i$. The only way to undo this is for the SCF to satisfy a kind of *no-total indifference condition* which would state that whenever $f_j(R'_i, R_{-i})I_jf_j(R)$ for each $j \in N$ then $f(R'_i, R_{-i}) = f(R)$. The latter condition precludes reshuffling of bundles or objects across agents that lead to no welfare change with respect to the original preference profile. Whether this is a desirable condition is obviously besides the point.

(a,c) or to (b,a). Thus, 1 cannot be a part of any deviating group in this situation. A similar argument holds if 1 strictly prefers b to a. If 1 is indifferent between a and b then any group deviation leads to either (a,c) or to (b,a). In both cases, agent 2 is weakly worse off. Thus, f is group strategy-proof. To show that f satisfies the group reversal property, let us consider any R with $f(R) = (a,c)$. By construction, it must be that $a\ P_1\ b$. For any $\tilde{R}$ with $f(\tilde{R}) = (b,a)$, agent 1 can deviate to R_1 and obtain a. Consider any R with $f(R) = (a,c)$. It must be then aP_1b. For any $\tilde{R}$ with $f(\tilde{R}) = (b,a)$, we know that $f(R)\,\text{wdom}[R,N]f(\tilde{R})$. Thus, f satisfies the group reversal property. ◇

Example 4.14. *Non-bossiness is not sufficient*
Consider a private-good economy with three agents, $\{1,2,3\}$, and three goods, $\{a,b,c\}$. The preference domain for each agent is unrestricted and the SCF f for any given preference profile R is as follows:

$$f(R) = \begin{cases} (a,b,c) & \text{if } aP_1b \text{ OR } aI_1bP_1c \\ (b,c,a) & \text{otherwise.} \end{cases}$$

Observe here that the SCF above depends only on agent 1's preferences. Clearly, in any case agent 1 cannot alter the allocation without affecting his own. Thus, f satisfies non-bossiness. In addition, agent 1 has no incentive to misreport her preferences since such a move could only make him worse-off. Consequently, f is strategy-proof. However, f is not group-strategy proof. To be precise consider a prefence profile R in which aI_1bP_1c and cP_2b. Here $\{1,2\}$ can deviate to $\tilde{R}_{\{1,2\}}$ where $b\tilde{P}_1a$ and $\tilde{R}_2 = R_2$. Observe that $f(\tilde{R}_{\{1,2\}}, R_3) = (b,c,a)$. Allocation (b,c,a) is better for coalition $\{1,2\}$. Hence, f is not group-strategy proof. ◇

4.3 Non-Rich Domain: Single-Peakedness

Let us now turn our attention to the Sprumont model introduced in Examples 2.2 and 3.1. We have pointed out that the domain of single peaked preferences is not rich. However, it turns out that non-bossiness in welfare and strategy-proofness imply group-strategy proofness even in this narrow domain. Recall that $\Omega \in \mathbb{R}_{++}$ is the resource stock. Here, the feasible set of allocations is $F = \{x \in A : \sum_i x_i = \Omega\}$.

Theorem 4.15. *Let $\mathcal{R}^N$ be the single-peaked preferences domain and let F determine the feasible set of the Sprumont model. If an SCF f satisfies strategy-proofness and non-bossiness in welfare, then it satisfies group strategy-proofness.*

Proof. See Appendix. □

Theorems 4.9, 4.10 and 4.15 imply that strategy-proofness and non-bossiness in welfare – two of the three necessary conditions for secure implementation – already guarantee coalitional secure implementation in the Sprumont setting. One may wonder how strong of a condition non-bossiness in welfare is in this setting. In the following result, we show that efficiency –arguably the most desirable criteria in any resource allocation problem– along with strategy proofness and non-bossiness imply non-bossiness in welfare.

Theorem 4.16. *Let $\mathcal{R}^N$ be the single-peaked preferences domain and let F determine the feasible set of the Sprumont model. If an SCF f satisfies efficiency, strategy-proofness and non-bossiness, then it satisfies non-bossiness in welfare.*

Proof. Suppose f does not satisfy non-bossiness in welfare. Therefore, there must exist $R \in \mathcal{R}^N$, $i \in N$ and $R'_i \in \mathcal{R}_i$ such that $f(R'_i, R_{-i}) \neq f_i(R)$ and $f_i(R'_i, R_{-i}) I_i f_i(R)$. In fact, because f satisfies non-bossiness, $f_i(R'_i, R_{-i}) \neq f_i(R)$. This means that $f_i(R'_i, R_{-i})$ and $f_i(R)$ are on the opposite sides of $p(R_i)$. Without loss of generality assume that $f_i(R) < p(R_i) < f_i(R'_i, R_{-i})$. Then by efficiency, $f_j(R) \leq p(R_j)$ for all $j \neq i$. Indeed, if $f_j(R) > p(R_j)$ for some j, we can improve i and j by taking tiny amount from j's allocation and by increasing i's by the the same amount. This implies that $\sum_j f_j(R) = \Omega < \sum_j p(R_j)$.

Fix a preference $\hat{R}_i \in \mathcal{R}_i$ such that $p(\hat{R}_i) = p(R_i)$ but $f_i(R)\hat{P}_i f_i(R'_i, R_{-i})$. By strategy-proofness, $f_i(\hat{R}_i, R_{-i}) = f_i(R'_i, R_{-i}) > p(\hat{R}_i)$. Then by efficiency, $\Omega > p(\hat{R}_i) + \sum_{j\neq i} p(R_j) = \sum_j p(R_j)$. This contradicts our earlier conclusion that $\Omega < \sum_j p(R_j)$. □

Theorems 4.9, 4.10, 4.15, and 4.16 imply that any SCF that satisfies efficiency, strategy-proofness and non-bossiness is coalitional secure implementable. This class contains the whole family of fixed path rules by Moulin (1999). Bochet and Sakai (2010) show that the only secure implementable rules within the fixed path rules are the priority rules, i.e., serial dictatorships. This suggests that the extra requirement imposed by the outcome rectangular property for secure implementation over coalitional secure implementation restricts the set of implementable rules significantly in the Sprumont model.

5 Discussion and extensions

5.1 On the prevalence of truthtelling

5.1.1 The truth-resistance properties

Our previous implementation results can be seen under the light of two new properties which capture the notion of prevalence of truthtelling. We introduce them in turn.

Resilience: An SCF f is resilient if it is strategy-proof and for each $R, \tilde{R} \in \mathcal{R}^N$ with $f(R) \neq f(\tilde{R})$, there exists $i \in N$ such that $f_i(R_i, \tilde{R_{-i}}) P_i f_i(\tilde{R})$.

An SCF is resilient if not only agents have an individual incentive to the truth, but they also have an incentive to revert to truthtelling if a joint misreport occurred. Do SCFs typically satisfy resilience? From secure implementation we already know the answer. The second part of the resilience is simply the requirement of the rectangular property provided that f is strategy-proof. Hence we know that most strategy-proof rules of interest fail to be resilient. Notice that the condition of non-bossiness in welfare is of no help here. Indeed Saijo et al. show that the combination of strategy-proofness, non-bossiness in welfare and the outcome rectangular property is both necessary and sufficient for secure implementation. But what about a group version of resilience?

Group Resilience: An SCF f is group resilient if it is group strategy-proof and for each $R, \tilde{R} \in \mathcal{R}^N$ with $f(R) \neq f(\tilde{R})$, there exists $S \subseteq N$ such that $f_S(R_S, \tilde{R}_{-S}) P_S f_S(\tilde{R})$.

While resilience is typically out of reach, our previous results show that group resilience is obtained for any SCF that satisfies strategy-proofness and non-bossiness in welfare. As such, there is an close connection between group strategy-proofness and strategy-proofness. But not only. Non-bossiness in welfare induces an unexpected robustness to manipulations which makes truthtelling particularly salient. An SCF that satisfies strategy-proofness and non-bossiness in welfare automatically satisfies group resilience provided of course that the domain of preferences is rich enough –see e.g. Example 4.11 which shows that group resilience can be more demanding than resilience when the domain is too narrow.

Hence group resilience, while looking like a very demanding condition is in fact not more demanding than the combination of two conditions which have been central in the mechanism design literature, strategy-proofness and non-bossiness in welfare. Our findings show an unexpected precipitation/robustness induced by non-bossiness in welfare for strategy-proof SCFs.

5.1.2 Pareto dominance of truthtelling

Contrary to the findings on secure implementation, many rules which violate the rectangular property are coalitional secure implementable. Importantly, several of our results show that bad Nash equilibria can be broken by a reversion to truthtelling for some coalition $S \subseteq N$. Our next result goes further and show the strong prevalence of truthtelling in the Sprumont model for any SCF that satisfies efficiency, strategy-proofness and a mild property –*DNEN*– that we introduce below. Not only for any bad Nash equilibrium there exists a profitable coalitional reversion to truthtelling, but in fact every bad Nash equilibrium is Pareto dominated by truthtelling. Before we go to the result, let us introduce some additional definitions.

In this subsection, we fix $\mathcal{R}^N$ to be the single-peaked preferences domains, $A = [0, \Omega]^N$ and $F = \{x \in A : \sum_{i \in N} x_i = \Omega\}$ to be the feasible set of the Sprumont model.

Definition 5.1 (Deviator's Negative Effect on Nondeviators)**.** We say SCF f satisfies the property of *deviator's negative effect on nondeviators* (DNEN) if whenever $f_i(R) \neq f_i(\tilde{R}_i, R_{-i})$ for $i \in N$, $R \in \mathcal{R}^N$ and $\tilde{R}_i \in \mathcal{R}_i$, we either have $f_i(R) < f_i(\tilde{R}_i, R_{-i})$ and $f_j(R) \geq f_j(\tilde{R}_i, R_{-i})$ for all $j \neq i$ OR $f_i(R) > f_i(\tilde{R}_i, R_{-i})$ and $f_j(R) \leq f_j(\tilde{R}_i, R_{-i})$ for all $j \neq i$.

The property above says that the allocations of a deviator and the others move in opposite directions. This property can be derived from other well-known properties. For instance, consistency and resource monotonicity, which play a prominent role in the literature, imply DNEN. In our setting, both the agent pool and resource stock is fixed. However, to define consistency and resource monotonocity, one considers a collection of allocation problems that differ in the agent pool and resource stock, i.e., a collection of problems like ours. Then consistency and resource monotonicity provide a link between the allocation rules in individual allocation problems. Specifically, consistency says that one has to follow the same rule whenever the same amount of resources has to allocated among the same group of agents. For instance, suppose that S is a subset of the agent pool in two different allocation problems. If the agents in S report the same preferences in both problems and the allocation rules devote the same total quantity of resources to S, then the members of S must obtain the same allocation in both problems under consistency. Resource monotonicity says that if the resource stock increases while everything else remains constant, then no agent's allocation decreases.[8] An obvious consequence of consistency in a specific problem like ours is that if a deviating coalition secures the same total resource by reporting different preferences then the nondeviators' allocation does not change. Resource monotonicity in addition to consistency means that if a deviating coalition increases the allocation of its total resource by misreporting then the nondeviators' allocation cannot increase. Clearly, this is more demanding condition than our DNEN. Thus, the combination of consistency and resource monotonicity is more restrictive than DNEN. Furthermore, the rules in different problems can satisfy DNEN and be independent of each other. For instance, consider serial dictatorship rules that have different priority orders in different problems. For each problem, its corresponding serial dictatorship satisfies DNEN, strategy-proofness, efficiency and non-bossiness. However, the serial dictatorships across different problems do not necessarily satisfy consistency and resource monotonicity.

[8]We give here the definition of consistency and resource monotonicity in the settings in which these properties are usually defined. To do this, we need some new definitions: An allocation problem is a pair $\langle S, \omega \rangle$ where $S \subset N$ and $\omega \leq \Omega$. In addition, let $F^S(\omega) = \{x_S \in [0,\omega]^{|}S| : x_i \geq 0 \ \forall i \in S \ \& \ \sum_{i \in S} x_i = \omega\}$. An allocation mechanism for $\langle S, \omega \rangle$, $f^{\langle S, \omega \rangle}$, is a mapping that maps $\mathcal{R}^S$ to $F^S(\omega)$. The collection of allocation mechanisms for all the possible allocation problems is called a mechanism. A mechanism $(f^{\langle S, \omega \rangle})_{S \subset N \ \& \ \omega \in [0,\Omega]}$ is consistent if for all S, ω, and $T \subset S$, we have $f^{\langle S, \omega \rangle}_{S \setminus T}(R^S) = f^{\langle S \setminus T, \omega - \sum_{i \in T} f_i^{\langle S, \omega \rangle}(R^S) \rangle}(R^{S \setminus T})$. In addition, it is resource monotonic if each $f^{\langle S, \omega \rangle}$ is non-decreasing in ω.

DNEN along with efficiency, strategy-proofness and non-bossiness guarantees in the Sprument model that truthtelling dominates all other Nash equilibria.

Theorem 5.2. *If an SCF f satisfies efficiency, strategy-proofness, non-bossiness and DNEN, then truthtelling Pareto dominates all the Nash equilibria in the direct revelation game $\Gamma^* = (\mathcal{R}^N, f)$, i.e., if $\tilde{R} \in NE(\Gamma^*, R)$ and $f(R) \neq f(\tilde{R})$ for some R and $\tilde{R}$, then $f(R)$ wdom$[R, N]$ $f(\tilde{R})$*

Proof. See Appendix. □

As we discussed earlier, consistency and resource monotonicity together imply DNEN. In addition, it is easy to see that consistency alone guarantees non-bossiness. Moulin (1999) shows that consistency, resource monotonicity, efficiency and strategy-proofness characterizes the class of fixed path rules. Consequently, the class of rules that satisfies efficiency, strategy-proofness, non-bossiness and DNEN is contains all the fixed path rules. This means that in the Sprument setting, the class of rules in which truthtelling Pareto dominates all the Nash equilibria is very large.

We demonstrate in the following example that one cannot prove Theorem 5.2 without assuming DNEN (or some weaker version of DNEN).

Example 5.3. *Indispensable DNEN*
Let $N = \{1, 2, 3\}$ and $\Omega = 6$. The SCR f works as follows: if both agents 1 and 2 report preferences with peaks strictly below 2, then agents 1, 2 and 3 select their allocation sequentially in that order. However, if at least one of the first two agents has preferences with peak at 2 or above, then f is the uniform rule.

In this case, f is either a serial dictatorship or uniform rule depending on the first two agents' peaks. Both rules are efficient implying that so is f. Furthermore, observe that whenever both agents 1 and 2 report their peaks strictly below 2, both obtains their reported peak. However, whenever one (or both) of them reports her peak at 2 or above, she obtains at least 2 units of resource. Thus, no agent can force f to switch from or to the serial dictatorship without changing her own allocation. In addition, both the serial dictatorships and uniform rules are nonbossy. Thus, f is nonbossy.

We now argue that f is strategy-proof. For agent 3 this is obvious because his reported preferences alone can not force f to switch to or from the serial dictatorship (and the uniform rule). Given that both rules are strategy-proof, truthtelling is a dominant strategy for agent 3. Now let us consider agent 1. Suppose agent 2's peak is 2 or more. Then f is the uniform rule regardless of 1's report. Given the uniform rule is strategy-proof, truthtelling is a dominant strategy. Suppose agent 2's peak is strictly below 2. If agent 1's peak is strictly below 2, agent 1 has no incentive to misreport her preferences because she obtains her peak by reporting truthfully. If agent 1's peak is 2 or above, she should not have any incentive to misreport her preferences above 2 because in all such cases f is the uniform rule, which is strategy-proof. By reporting her peak strictly below 2, agent 1 obtains below 2. However, by reporting truthfully, agent 2 gets at least 2 but never more than her peak. Thus, agent 1 has no incentive to lie. Similar arguments prove that agent to has no incentive to misrepresent her preferences.

However, f does not satisfy DNEN. To see this consider an preference profile R with peaks at $(1, 2, 2)$. Because the peak of agent 2 is at 2, f allocates according to the uniform rule. Thus, $f(R) = (2, 2, 2)$. However, if agent 2 deviates and report her peak at 1, f allocated according to the serial dictatorship and the allocation is $(1, 1, 4)$. Clearly, the allocations of 1 and 3, who do not deviate, move in different directions.

Finally, let us show that truthtelling does not pareto dominate all the Nash equilibria in some cases. Let R be a profile with peaks at $(1, 1, 2)$. In this case, $f(R) = (1, 1, 4)$. Suppose that $\tilde{R}$ be a profile with peaks at $(2, 2, 2)$. It is easy to see that $f(\tilde{R}) = (2, 2, 2)$. In addition, $\tilde{R}$ is a Nash equilibrium at profile R. However, the allocation under truthtelling does not Pareto dominate the one under $\tilde{R}$.

◇

We conclude this section with the following example demonstrating that DNEN is not a necessary condition for Theorem 5.2.

Example 5.4. *Sequential Dictatorship: DNEN is not a necessary condition*
Let $N = \{1, 2, 3\}$ and $\Omega = 6$. The SCR f works as follows: agent 1 selects her allocation first and who selects next depends on agent 1's allocation. Specifically, agent 2 selects second if agent 1 is allocated 2 or less and agent 3 in all other cases. This rule is known as the sequential dictatorship in the literature and is strategy-proof, non-bossy and efficient. Furthermore, any Nash equilibrium yields the allocation at any preference profile. Specifically, at any Nash equilibrium agent 1 must obtain her favorite allocation. Consequently, who chooses after agent 1 is the same at all Nash equilibria. This agent's allocation thus must be the same at all Nash equilibria. As a result, the allocation for all agents at any Nash is the same. However, let us note here that f does not satisfy DNEN. For instance, consider a profile R with peaks at $(1, 3, 3)$. In this case, $f(R) = (1, 3, 2)$. However, if agent 1 reports $\tilde{R}_1$ with her peak at 2.5, then $f(\tilde{R}_1, R_{-1}) = (2.5, 0.5, 3)$. Clearly, f violates DNEN.

◇

5.2 A failure of the revelation principle

Our results so far are centered around the use of direct mechanisms. Our choice is made on the one hand for simplicity, but on the other hand, because direct mechanisms are commonly used in practice. In Saijo et al. (2007)., the rectangular property turns out to be so demanding that any rule that can be secure implemented by an indirect mechanism can also be (fully) secure implemented by its direct mechanism. While the quest for implementing mechanism is thus drastically reduced, the cost is obvious: the class of secure implementable rules is small. No such result is available in our context: more rules can be coalitional secure implemented using indirect mechanisms. We illustrate this below.

Example 5.5. *A failure of the revelation principle*
Let $n = 3$. Each agent i's preferences are of two types, R_i and $\tilde{R}_i$. The set of social alternatives is $\{a, b, c, d, e, f, g, h, k, l, m, n\}$.

R_1	R_2	R_3	$\tilde{R}_1$	$\tilde{R}_2$	$\tilde{R}_3$
b	c	$a \sim l$	$g \sim h$	$f \sim h$	$e \sim f$
m	n	$b \sim m$	$e \sim f$	$e \sim g$	$g \sim h$
d	d	$c \sim n$	c	b	$l \sim m$
a	a	$k \sim d$	$n \sim d$	$m \sim d$	$n \sim d$
c	b	e	m	n	$a \sim b$
k	k	f	l	l	$c \sim k$
l	l	g	k	k	
n	m	h	$a \sim b$	$a \sim c$	
e	e				
f	g				
g	f				
h	h				

The SCF f is as follows:

R_3:

	R_2	$\tilde{R}_2$
R_1	a	b
$\tilde{R}_1$	c	d

$\tilde{R}_3$:

	R_2	$\tilde{R}_2$
R_1	e	f
$\tilde{R}_1$	g	h

Clearly, f does not satisfy both group-strategy proofness and the group reversal property. Thus, f is not coalitional secure implementable via its direct mechanism
However, f is coalitional secure implementable by the following indirect mechanism, $\Gamma = (M, g)$, in which the respective set of messages for the agents are $M_1 = \{m_1, \tilde{m}_1\}$, $M_2 = \{m_2, \tilde{m}_2\}$ and $M_3 = \{m_3, m'_3, \tilde{m}_3\}$, and the outcome function $g : M \to A$ is described as follows:

m_3

	m_2	$\tilde{m}_2$
m_1	a	b
$\tilde{m}_1$	c	k

m_3'

	m_2	$\tilde{m}_2$
m_1	l	m
$\tilde{m}_1$	n	d

$\tilde{m}_3$

	m_2	$\tilde{m}_2$
m_1	e	f
$\tilde{m}_1$	g	h

It is straightforward to see that agent 1's dominant strategy is to play m_1 ($\tilde{m}_1$) when her preferences are R_1 ($\tilde{R}_1$). The same is true for agent 2. For agent 3 it is dominant to play m_3 or m_3' when her preferences are R_3. On the other hand, playing $\tilde{m}_3$ is dominant when her preferences are $\tilde{R}_3$. We need to show that in each state only the dominant strategy profile that yields the alternative in the SCF of that state is a strong Nash equilibrium in that state.

When the preference profile is R strategy profile (m_1, m_2, m_3) is both a dominant strategy and a strong Nash equilibrium. In addition, $g(m) = a = f(R)$. No other strategy is a strong Nash equilibrium. Indeed strategy profile (m_1, m_2, m_3') is weakly dominated by $(\tilde{m}_1, \tilde{m}_2, m_3')$ for agents 1 and 2.

When the preference profile is $(\tilde{R}_1, R_2, R_3)$ strategy profile $(\tilde{m}_1, m_2, m_3)$, which yields c, is both a dominant strategy and a strong Nash equilibrium. No other strategy is strong Nash equilibrium. Indeed strategy profile $(\tilde{m}_1, m_2, m_3')$ is dominated by $(\tilde{m}_1, m_2, m_3)$ for agents 1 and 3.

When the preference profile is $(R_1, \tilde{R}_2, R_3)$ strategy profile $(m_1, \tilde{m}_2, m_3)$, which yields b, is both a dominant strategy and a strong Nash equilibrium. No other strategy is strong Nash equilibrium. Indeed strategy profile $(m_1, \tilde{m}_2, m_3')$ is dominated by $(m_1, \tilde{m}_2, m_3)$ for agents 2 and 3.

When the preference profile is $(\tilde{R}_1, \tilde{R}_2, R_3)$ strategy profile $(\tilde{m}_1, \tilde{m}_2, m_3')$, which yields d, is both a dominant strategy and a strong Nash equilibrium. No other strategy is strong Nash equilibrium. Indeed strategy profile $(\tilde{m}_1, \tilde{m}_2, m_3)$ is dominated by $(\tilde{m}_1, \tilde{m}_2, m_3')$ for agents 1,2 and 3.

In the remaining cases (i.e., the cases in which agent 3's preferences are $\tilde{R}_3$) it is straightforward to see that agent 3 would always be worse-off if she is a part of another coalition in which she does not play $\tilde{m}_3$. Without agent 3's involvement, the other two agents cannot improve simultaneously by deviating from their dominant strategies. $\diamond$

Our characterization in Section 4 holds only for direct mechanisms. From the above example, we see that neither group strategy-proofness nor the group reversal property are necessary for coalitional secure implementation when using indirect mechanisms. We unfortunately do not have a characterization of coalitional secure implementation using indirect mechanisms to offer. Next we discuss a strengthening of our implementation concept, a direct extension of Saijo et al. (2007) once coalitional moves are possible. We argue that the failure of the revelation principle identified in this subsection extends to a

	R_2	R'_2	R''_2
R_1	$2, 1.5$	$2, 0$	$1, 0$
R'_1	$1, 3$	$2, 1.4$	$1, 1$
R''_1	$2, 1$	$2, 0.5$	$1, 1$

Table 2: Strict Coalitional Secure Implementation and non-bossiness in welfare

stricter implementation notion that excludes dominant strategy equilibria not delivering the right outcome.

5.3 A stronger version of coalitional secure implementation

The notion of coalitional secure implementation we introduced in Section 2 does not rule out that some dominant strategy equilibria may also deliver the wrong outcome for some preference profiles. In Section 4 we argued that this seems like a minor departure from a stronger notion that would require all dominant strategy equilibria to be "good". Indeed, given our implementation notion, any dominant strategy equilibria which delivers the wrong outcome is destroyed by some coalitional deviation since for any SCF f and for any preference profile R, $SNE(\Gamma^*, R) \subseteq DS(\Gamma^*, R) \subseteq NE(\Gamma^*, R)$. An advantage of coalitional secure implementation is also the possibility to implement a larger class of SCFs using indirect mechanisms –as shown in Example 5.5. Of course, strategic uncertainty is not entirely absent given the previous implementation concept. Also, the enlargement of the class of coalitional secure SCFs with the use of indirect mechanisms requires to rely more on the complete information assumption inherent behind the strong Nash equilibrium concept. However, the following strengthening of our implementation retains the same failure of the revelation principle.

Let us now define and characterize the stronger notion of coalitional secure implementation. As said above, it rules out strategic uncertainty and also extends the secure implementation concept as initially defined in Saijo et al. (2007).

Definition 5.6. Strict Coalitional Secure Implementation An SCF f is *strict coalitional secure implementable* if there exists a mechanism Γ such that

1. for each $R \in \mathcal{R}^N$ there exists $m \in DS(\Gamma, R) \cap SNE(\Gamma, R)$ such that $g(m) = f(R)$
2. for each $R \in \mathcal{R}^N$, for each $m \in DS(\Gamma, R)$, $g(m) = f(R)$
3. for each $R \in \mathcal{R}^N$, for each $m \in SNE(\Gamma, R)$, $g(m) = f(R)$.

This stronger notion of coalitional secure implementation is of course the direct extension of the secure implementation criterion introduced in Saijo et al. (2007) when coalitional deviations can take place, as discussed in Bochet and Sakai (2010). Not surprisingly, if we focus on the use of direct mechanisms, group strategy-proofness and the group reversal property are no longer sufficient for this more stringent implementation concept. We illustrate this using Table 2. There we have constructed a direct mechanism for an SCF and we have indicated directly the utility numbers for agents at the true preference profile $R = (R_1, R_2)$. Notice first that the SCF f defined is not only strategy-proof but in fact group strategy-proof –it is easy to extend our construction so that it satisfies the property at all preference profiles. Next, it is easy to see that the rectangular property is violated so that f is not secure implementable. At R_1, agent 1 has both R_1 and $R_1^{''}$ as dominant strategies while agent 2 has only R_2 at R_2. Because f violates the rectangular property at (R_1, R_2), $(R_1^{''}, R_2^{''})$ is a bad Nash equilibrium. The latter can be broken by a coalitional deviation to (R_1', R_2'), and (R_1', R_2') is itself not a Nash equilibrium. However $(R_1^{''}, R_2)$ is a bad dominant strategy equilibrium. Yet the group reversal property is satisfied since (R_1'', R_2) is not a strong Nash as it is broken again by a coalitional deviation to (R_1', R_2'). Hence the conjunction of group strategy-proofness and the group reversal property is no longer sufficient. A property that the SCF constructed in Table 2 obviously fails is non-bossiness in welfare.

Strategy profile (R_1'', R_2) is a dominant strategy at (R_1, R_2) and we know it delivers the wrong outcome. Note that the premise of non-bossiness in welfare is met and hence (R_1'', R_2) should deliver the same outcome as (R_1, R_2). This condition is however too strong. Indeed if $f(R_1'', R_2) = f(R_1, R_2)$, then f would satisfy the rectangular property at R. As it turns out the only thing that is needed is the following weaker condition.

Weak non-bossiness: An SCF f satisfies weak non-bossiness if whenever $f_i(R) \neq f_i(\tilde{R}_i, R_{-i})$ for some $i \in N$, $\tilde{R}_i \in \mathcal{R}_i$ and $R \in \mathcal{R}^N$, then there exists $\hat{R}_{-i}$ such that $f_i(R_i, \hat{R}_{-i}) P_i f_i(\tilde{R}_i, \hat{R}_{-i})$.

Weak non-bossiness is a sufficient condition for a strategy-proof SCF to be dominant strategy implemented by its direct revelation in mechanism. In Saijo et al. (2007) it is shown that weak non-bossiness is not sufficient to guarantee secure implementation of an SCF that is dominant strategy implemented. It is also not sufficient to guarantee that an SCF that satisfies both strategy-proofness and the outcome rectangular property is secure implementable. We show below that weak non-bossiness is in fact enough to guarantee that an SCF that satisfies both group strategy-proofness and the group reversal property is strict coalitional secure implemented by its direct revelation mechanism.

Theorem 5.7. *Let f be an SCF and consider $\Gamma^* = (\mathcal{R}^N, f)$. Then f is strict coalitional secure implemented by Γ^* if and only if f satisfies group strategy-proofness, the group reversal property and weak non-bossiness.*

Proof. See Appendix. □

6 Conclusion

Our paper takes a significant step towards understanding of the limitations of secure implementation. Our results confirm some of the intuition developed in Bochet and Sakai (2010). Many of the observed failures of secure implementation are not severe, and positive results can be recovered once pre-play communication is allowed. Recall that the main reason for secure implementation to fail is that, given an SCF, some bad Nash equilibria cannot be broken. Indeed the rectangular property imposes strong invariance of an SCF across preference profiles. Three features of our results are important as take aways. First, bad Nash equilibria can often be broken by simple coalitional deviations to truthtelling, with which all members of the deviating coalition are made strictly better off. This goes beyond the necessary condition –group reversal property– that we identified. For practical purposes, the prevalence of truthtelling is important as we expect it to play some focal role. Our findings for the Sprumont model reinforces this finding since there truthtelling in fact Pareto dominates any bad Nash equilibria. The conditions for which this result is true are rather mild. In addition to strategy-proofness, an obvious necessary condition for coalitional secure implementation, only efficiency and condition DNEN are needed. Finally, out of the three necessary conditions identified in Saijo et al., the outcome rectangular property (a variant of the rectangular property) can be completely dispensed with in many settings of interest. The latter shows the permissiveness of coalitional secure implementation via direct revelation mechanisms.

Several questions remain open at this stage. We list a few. First, we lack a complete characterization of domains/rules for which truthtelling Pareto dominates bad Nash equilibria in the direct revelation mechanism of a strategy-proof SCF. Our result on the Sprumont model are promising but it would be of interest to investigate if this conclusion can hold with more generality. Next, several of our results rely on non-bossiness in welfare, a condition which we have shown not to be necessary for coalitional secure implementation. At the same time, we have also shown that an SCF that satisfies strategy-proofness and non-bossiness may not satisfy group strategy-proofness, and this is true for many domains and models of interest. A condition which lies between non-bossiness and its counterpart in welfare would provide us with a complete characterization based on necessary and sufficient conditions. Finally, we have confined our attention to pure strategies only. Saijo et al. made some steps toward a characterization of secure implementation in mixed strategies, but their analysis is indirect as it relies on correlated equilibria. Incorporating mixed strategies would be an important step as necessary and sufficient conditions can be affected (see for instance Mezzetti and Renou (2012)'s paper on Nash implementation in mixed strategies). We leave these questions open for future research.

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Appendix

Proof of Theorem 4.2. We first prove part 1 of the theorem. Let $R, \tilde{R} \in \mathcal{R}^N$ be such that $f(R) \neq f(\tilde{R})$. If $\tilde{R} \notin NE(\Gamma^*, R)$, then there must exist some agent $i \in N$ and $\bar{R}_i \in \mathcal{R}_i$ such that $f(\bar{R}_i, \tilde{R}_{-i}) P_i f(\tilde{R})$. By combining this with the strategy-proofness of f, we find that $f(R_i, \tilde{R}_{-i}) R_i f(\bar{R}_i, \tilde{R}_{-i}) P_i f(\tilde{R})$. Thus, we have proved part 1 if $\tilde{R} \notin NE(\Gamma^*, R)$. For the remainder of the proof of part 1, we assume that $\tilde{R} \in NE(\Gamma^*, R)$. Let $T \subset N$ with $|T| = t$ be the subset of agents who lie at $\tilde{R}$, i.e. $(\tilde{R}_T, R_{-T}) = (\tilde{R}_T, \tilde{R}_{-T}) = \tilde{R}$. The following claim is crucial for our proof.

Claim: Fix any nonnegative integer $s < t$. If $f(R_{\bar{S}}, \tilde{R}_{-\bar{S}}) = f(\tilde{R})$ for all $\bar{S} \subset T$ with $|\bar{S}| \leq s$, then for all $S \subseteq T$ with $|S| = s + 1$, it must be either (i) $f(R_S, \tilde{R}_{-S}) P_i f(\tilde{R})$ for all $i \in S$ or (ii) $f(R_S, \tilde{R}_{-S}) = f(\tilde{R})$.

The claim above, which we prove below, immediately yields that either (a) there exists some $S \subseteq T$ with $f(R_S, \tilde{R}_{-S}) P_i f(\tilde{R})$ or (b) $f(R_T, \tilde{R}_{-T}) = f(\tilde{R})$. In the latter case, $f(R) = f(\tilde{R})$ (recall $(R_T, \tilde{R}_{-T}) = (R_T, R_{-T})$). This would contradict our assumption that $f(R) \neq f(\tilde{R})$. Thus, the only possible case is (a) which concludes the proof of part 1.

Proof of the Claim. Let us prove the claim when $s = 0$. Fix any $S \subset T$ with $|S| = 1$. By construction, $S = \{i\}$ for some $i \in T$. By the Nash equilibrium assumption of $\tilde{R} = (\tilde{R}_S, R_{-S})$ at R, we have

$$f_i(\tilde{R}) \; R_i \; f_i(R_i, \tilde{R}_{-i}) = f_i(R_S, \tilde{R}_{-S}). \tag{1}$$

By strategy-proofness,

$$f_i(R_i, \tilde{R}_{-i}) \; R_i \; f_i(\tilde{R}). \tag{2}$$

By combining the two relations above, we obtain that

$$f_i(R_i, \tilde{R}_{-i}) \; I_i \; f_i(\tilde{R}).$$

By the strict preference assumption and non-bossiness of f, we have

$$f_i(R_i, \tilde{R}_{-i}) = f(R_S, \tilde{R}_{-S}) = f_i(\tilde{R}). \tag{3}$$

This means that (ii) in the claim is always satisfied if $s = 0$.

Now fix any s with $0 < s < t$. Pick any $S \subseteq T$ with $|S| = s + 1$. Because of the strategy-proofness of f, we know that $f(R_S, \tilde{R}_{-S}) \; R_i \; f(R_{S \setminus i}, \tilde{R}_i, \tilde{R}_{-S})$ for all $i \in S$. By construction, $|S \setminus i| = s$. By the assumption used in the claim, $f(R_{S \setminus i}, \tilde{R}_i, \tilde{R}_{-S}) = f(\tilde{R})$.

Consequently,

$$f(R_S, \tilde{R}_{-S})\ R_i\ f(R_{S\setminus i}, \tilde{R}_i, \tilde{R}_{-S}) = f(\tilde{R}) \text{ for all } i \in S. \tag{4}$$

If the relation above holds for everyone with a strict one, then we are in (i) of the claim. If not, there must be at least one agent i for whom $f(R_S, \tilde{R}_{-S})\ I_i\ f(R_{S\setminus i}, \tilde{R}_i, \tilde{R}_{-S}) = f(\tilde{R})$. Then by the strict preference assumption and non-bossiness of f, we obtain $f(R_S, \tilde{R}_{-S}) = f(R_{S\setminus i}, \tilde{R}_i, \tilde{R}_{-S}) = f(\tilde{R})$ which is (ii) of the claim. This completes the proof of the claim.

Now let us prove that f is coalitional secure implementable if f satisfies strategy-proofness and non-bossiness. The definition of coalitional secure implementation immediately gives that group strategy-proofness is a necessary condition for coalitional secure implementation via a direct mechanism. By combining this with Lemma 4.1 we prove the only if part of theorem. On the other hand, the if part is easily proved because (i) $R \in DS(\Gamma^*, R) \cap SNE(\Gamma^*, R)$ by the group strategy-proofness of f and (ii) any $\tilde{R} \notin NE(\Gamma^*, R)$ with $f(\tilde{R}) \neq f(R)$ is blocked by some coalition S by reverting to truthtelling (the first part of the theorem). Thus, strategy-proofness and non-bossiness together are sufficient for coalitional secure implementation via a direct mechanism. □

Proof of Theorem 4.10 . In contradiction, suppose that f does not satisfy the group reversal property. Then there exists R and $\tilde{R}$ such that (i) $f(R) \neq f(\tilde{R})$ and (ii) no S and R'_S such that $f(R'_S, \tilde{R}_{-S})\,\text{wdom}[R,S] f(\tilde{R})$. Fix such R and $\tilde{R}$. We now argue that (i) and (ii) are incompatible.

Consider any coalition S with only one agent. Let $S = \{i\}$. By (ii), it must be that $f_i(\tilde{R}) R_i f_i(R_i, \tilde{R}_{-i})$. On the other hand, f is strategy-proof implying that $f_i(R_i, \tilde{R}_{-S}) R_i f_i(\tilde{R})$. Hence $f_i(R_i, \tilde{R}_{-S}) I_i f_i(\tilde{R})$. By non-bossiness in welfare $f(R_i, \tilde{R}_{-S}) = f(\tilde{R})$. Since i was chosen arbitrarily, the latter conclusion holds for all $i \in N$.

To complete the proof we argue by induction.

Suppose that for any coalition S with $p-1$ agents (where $2 \leq p \leq n$) , $f(R_S, \tilde{R}_{-S}) = f(\tilde{R})$. We have shown this already for the $p = 2$ case. Now we show that for any coalition S consisting of p agents, it must be that $f(R_S, \tilde{R}_{-S}) = f(\tilde{R})$. Fix any S with p agents. Take any $i \in S$. By the induction assumption, $f(R_{S\setminus\{i\}}, \tilde{R}_{-S\setminus\{i\}}) = f(\tilde{R})$. By strategy proofness, $f_i(R_S, \tilde{R}_{-S}) R_i f_i(R_{S\setminus\{i\}}, \tilde{R}_{-(S\setminus\{i\})}) = f_i(\tilde{R})$. Given that we chose i arbitrarily, $f_i(R_S, \tilde{R}_{-S}) R_i f_i(R_{S\setminus\{i\}}, \tilde{R}_{-(S\setminus\{i\})}) = f_i(\tilde{R})$ for each $i \in S$. Given our assumption (by contradiction) of no domination, it must be the case that $f_i(R_S, \tilde{R}_{-S}) I_i f_i(R_{S\setminus\{i\}}, \tilde{R}_{-(S\setminus\{i\})}) = f_i(\tilde{R})$ for each $i \in S$. By non-bossiness in welfare, $f(R_S, \tilde{R}_{-S}) = f(R_{S\setminus\{i\}}, \tilde{R}_{-(S\setminus\{i\})}) = f(\tilde{R})$ for any $i \in S$.

Since S was chosen arbitrarily, we finally obtain that $f(R) = f(\tilde{R})$, a contradiction with our initial assumption. Items (i) and (ii) are therefore incompatible. □

Proof of Theorem 4.12. Pick $R \in \mathcal{R}^N$ and suppose that f violates group strategy-proofness at R. Hence, there exists $S \subseteq N$, $R'_S \in \mathcal{R}^S$ such that $f(R'_S, R_{-S})\,\text{wdom}[R,S]\ f(R)$. Let $\hat{R}_S \in \mathcal{R}^S$ be the preference for S such that

(i) for any $i \in S$ for whom $f_i(R) = f_i(R'_S, R_{-S})$, $f_i(R'_S, R_{-S})$ is the most preferred alternative for agent i under $\hat{R}_i$

(ii) for any $i \in S$ for whom $f_i(R) \neq f_i(R'_S, R_{-S})$, $f_i(R'_S, R_{-S})\hat{P}_i f_i(R)\hat{P}_i z_i$ for $z_i \neq f_i(R'_S, R_{-S})$ and $z_i \neq f_i(R)$.

The existence of such a profile of preferences for S is guaranteed because our domain is rich. We now change R to $(\hat{R}_S, R_{-S})$, one agent's preference at a time. We show that the initial selection operated by f, $f(R)$, does not change at any step of this process. Pick any $i \in S$. If $f_i(R'_S, R_{-S}) = f_i(R)$, then by strategy-proofness we must have $f_i(\hat{R}_i, R_{-i}) = f_i(R)$. Otherwise, agent i would have a profitable deviation at $(\hat{R}_i, R_{-i})$ because $f_i(R)$ is the most preferred alternative for i at $\hat{R}_i$ (by construction). Suppose that $f_i(R'_S, R_{-S}) \neq f_i(R)$. Then strategy-proofness of f implies that $f_i(\hat{R}_i, R_{-i})$ is either $f_i(R'_S, R_{-S})$ or $f_i(R)$. Otherwise, agent i would have a profitable deviation at $(\hat{R}_i, R_{-i})$ because by construction, $f_i(R'_S, R_{-S})$ and $f_i(R)$ are the two most preferred alternatives for i at $\hat{R}_i$. Because $f(R'_S, R_{-S})\,\text{wdom}[R,S]\ f(R)$ and $i \in S$, we must have that $f_i(R'_S, R_{-S})R_i f_i(R)$. If $f_i(R'_S, R_{-S})P_i f_i(R)$, then strategy-proofness implies that $f_i(\hat{R}_i, R_{-i}) \neq f_i(R'_S, R_{-S})$. If $f_i(R'_S, R_{-S})I_i f_i(R)$, then non-bossines in welfare implies that $f_i(\hat{R}_i, R_{-i}) \neq f_i(R'_S, R_{-S})$. Thus, in all cases $f_i(\hat{R}_i, R_{-i}) = f_i(R)$. Then by non-bossiness, we obtain that $f(\hat{R}_i, R_{-i}) = f(R)$. Now pick any $j \neq i \in S$. By applying the same arguments as above we obtain that $f(\hat{R}_{\{i,j\}}, R_{-\{i,j\}}) = f(\hat{R}_i, R_{-i}) = f(R)$. The same reasoning applies for the remaining agents in S. Hence, we obtain that $f(\hat{R}_S, R_{-S}) = f(R)$.

We now reach $(\hat{R}_S, R_{-S})$ from (R'_S, R_{-S}) by sequentially changing preferences of agents in S, one at a time. We claim that the initial selection operated by f, $f(R'_S, R_{-S})$, does not change at any step of this process. Pick any $i \in S$. By construction of $\hat{R}_S$, $f_i(R'_S, R_{-S})$ is the most preferred alternative for agent i at $\hat{R}_i$. Then strategy-proofness of f yields that $f_i(\hat{R}_i, R'_{S\setminus\{i\}}, R_{-S}) = f_i(R'_S, R_{-S})$. Now because f satisfies non-bossiness we get that $f(\hat{R}_i, R'_{S\setminus\{i\}}, R_{-S}) = f(R'_S, R_{-S})$. Similar arguments apply for the remaining agents in S. Consequently, we have that $f(\hat{R}_S, R_{-S}) = f(R'_S, R_{-S})$. Recall that earlier we showed that $f(\hat{R}_S, R_{-S}) = f(R)$. Thus, $f(R) = f(R'_S, R_{-S})$ which contradicts $f(R'_S, R_{-S})\,\text{wdom}[R,S]\ f(R)$. □

We need the following two lemmas for the proof of Theorem 4.15.

Lemma 6.1. *Let $\mathcal{R}^N$ be the single-peaked preferences domain and let F determine the feasible set of the Sprumont model. If an SCF f satisfies strategy-proofness and non-*

bossiness in welfare, then it satisfies peak-onliness, i.e., for any $R, \tilde{R} \in \mathcal{R}^N$ *with* $p(R_i) = p(\tilde{R}_i)$*, it must be that* $f(R) = f(\tilde{R})$

Proof. We first show that for all $R \in \mathcal{R}^N$, $i \in N$ and $R'_i \in \mathcal{R}_i$ with $p(R_i) = p(R'_i)$, we have $f_i(R) = f_i(R'_i, R_{-i})$. Suppose otherwise. Let us denote $p(R'_i) = p(R_i) = \bar{p}$. Then because f is strategy-proof and the preferences are single peaked we have $f_i(R) \neq \bar{p}$ and $f_i(R'_i, R_i) \neq \bar{p}$. Without loss of generality let us assume that $f_i(R) < \bar{p}$. If $f_i(R'_i, R_i) < \bar{p}$, then f cannot be strategy-proof because R_i and R'_i are single-peaked and $f_i(R'_i, R_i) \neq f_i(R)$ (by assumption). Thus, we have that $f_i(R'_i, R_i) > \bar{p} > f_i(R)$. Then we can find a single peaked preference $\tilde{R}_i \in \mathcal{R}_i$ such that $p(\tilde{R}_i) = \bar{p}$ and $f_i(R'_i, R_i)\tilde{I}_i f_i(R)$. Again strategy-proofness and the single-peakedness of preferences imply that either $f_i(\tilde{R}_i, R_i) = f_i(R)$ or $f_i(\tilde{R}_i, R_i) = f_i(R'_i, R_i)$. Without loss of generality assume that $f_i(\tilde{R}_i, R_i) = f_i(R)$. As a result when agent i's preference is $\tilde{R}_i$, he can deviate to R'_i and obtain $f_i(R'_i, R_{-i})$. By construction, $f_i(R'_i, R_i)\tilde{I}_i f_i(R) = f_i(\tilde{R}_i, R_i)$. Then non-bossiness in welfare implies that $f(R'_i, R_i) = f(\tilde{R}_i, R_{-i})$. This contradicts $f_i(\tilde{R}_i, R_i) = f_i(R) \neq f_i(R'_i, R_i)$. Therefore, $f_i(R) = f_i(R'_i, R_{-i})$. The rest of the proof is a simple consequence of non-bossiness in welfare. □

Lemma 6.2. *Let* $\mathcal{R}^N$ *be the single-peaked preferences domain and let* F *determine the feasible set of the Sprumont model. Let SCF* f *satisfy strategy-proofness and peak-onliness. For any* $R \in \mathcal{R}^N$ *and* $\tilde{R}_i \in \mathcal{R}_i$ *with* $p(R_i) < p(\tilde{R}_i)$*, one of the following cases must occur:*

(i) $f_i(R) = f_i(\tilde{R}_i, R_{-i}) \leq p(R_i) < p(\tilde{R}_i)$.

(ii) $p(R_i) \leq f_i(R) \leq f_i(\tilde{R}_i, R_{-i}) \leq p(\tilde{R}_i)$, $p(R_i) < f_i(\tilde{R}_i, R_{-i})$ *and* $f_i(R) < p(\tilde{R}_i)$.

(iii) $p(R_i) < p(\tilde{R}_i) \leq f_i(R) = f_i(\tilde{R}_i, R_{-i})$.

Proof. The lemma is a direct consequence of the following results which we prove next.

(a) $f_i(R) \leq f_i(\tilde{R}_i, R_{-i})$.

(b) If either $f_i(R) < p(R_i)$ or $f_i(\tilde{R}_i, R_{-i}) > p(\tilde{R}_i)$, then $f_i(\tilde{R}_i, R_{-i}) = f_i(R)$.

(c) If either $p(\tilde{R}_i) \leq f_i(R)$ or $p(R_i) \geq f_i(\tilde{R}_i, R_{-i})$, then $f_i(\tilde{R}_i, R_{-i}) = f_i(R)$.

(a) On the contrary, assume $f_i(R) > f_i(\tilde{R}_i, R_{-i})$. If $f_i(\tilde{R}_i, R_{-i}) \geq p(R_i)$, then by the single-peakedness of f, i gains by reporting $\tilde{R}_i$ at R, a contradiction with the strategy-proofness of f. Hence, $f_i(\tilde{R}_i, R_{-i}) < p(R_i)$. A similar argument gives $p(\tilde{R}_i) < f_i(R)$. Consequently, $f_i(\tilde{R}_i, R_{-i}) < p(R_i) < p(\tilde{R}_i) < f_i(R)$. We next show that strategy-proofness is violated if $f_i(\tilde{R}_i, R_{-i})$ and $f_i(R)$ in on the opposite sides of $p(R_i)$. Fix any $\bar{R}_i$ with $p(\bar{R}_i) = p(R_i)$ such that i prefers $f_i(\tilde{R}_i, R_{-i})$ to $f_i(R)$ under $\bar{R}_i$. By peak-onliness, $f_i(R) = f_i(\bar{R}_i, R_{-i})$. Thus, i prefers $f_i(\tilde{R}_i, R_{-i})$ to $f_i(\bar{R}_i, R_{-i})$ under $(\bar{R}_i, R_{-i})$, a contradiction with the strategy-proofness of f. This completes the proof of (a).

(b) We concentrate on the $f_i(R) < p(R_i)$ case because the proof of the other case is a mirror image of the current case. By (a), $f_i(R) \leq f_i(\tilde{R}_i, R_{-i})$. Thus, in contradiction to (b), let $f_i(R) < f_i(\tilde{R}_i, R_{-i})$. If $f_i(\tilde{R}_i, R_{-i}) \leq p(R_i)$, then by the single-peakedness of f, i manipulates f at $(\tilde{R}_i, R_{-i})$ by reporting R_i. Consequently, $f_i(R) < p(R_i) < f_i(\tilde{R}_i, R_{-i})$. Because $f_i(R)$ and $f_i(\tilde{R}_i, R_{-i})$ are on the opposite sides of $p(R_i)$, as in the proof of (a), we reach a contradiction with strategy-proofness. Therefore, $f_i(R) = f_i(\tilde{R}_i, R_{-i})$.

(c) As with case (b) we only concentrate on the $p(\tilde{R}_i) \leq f_i(R)$ case. By (a), $f_i(R) \leq f_i(\tilde{R}_i, R_{-i})$. Hence, in contradiction to (c), let $f_i(R) < f_i(\tilde{R}_i, R_{-i})$. Then $p(\tilde{R}_i) \leq f_i(R) < f_i(\tilde{R}_i, R_{-i})$. By the single-peakedness of f, i manipulates f at $(\tilde{R}_i, R_{-i})$ by reporting R_i, a contradiction to the strategy-proofness of f. □

We are now ready to prove Theorem 4.15.

Proof of Theorem 4.15. In contradiction to the theorem, let f satisfy both strategy-proofness and non-bossiness in welfare but not group strategy-proofness. Consequently, there exist $R \in \mathcal{R}^N$, $S \subset N$ and $\tilde{R}_S \in \mathcal{R}^S$ such that $f(\tilde{R}_S, R_{-S}) \,\text{wdom}[R, S]\, f(R)$. By Lemma 6.1, f is peak-only. Let $\hat{R} \in \mathcal{R}^N$ be a preference profile such that $p(\hat{R}_i) = f_i(\tilde{R}_S, R_{-S})$. Pick any $i \in S$. We will now show that $f(\hat{R}_i, R_{-i}) = f(R)$. Suppose otherwise. Non-bossiness[9] would yield $f(\hat{R}_i, R_{-i}) = f(R)$ if $f_i(\hat{R}_i, R_{-i}) = f_i(R)$. Thus, $f_i(\hat{R}_i, R_{-i}) \neq f_i(R)$. If $p(\hat{R}_i) = p(R_i)$, then by peak-onliness, $f_i(\hat{R}_i, R_{-i}) = f_i(R)$. Hence, we must have that $p(\hat{R}_i) \neq p(R_i)$. The proofs for the $p(\hat{R}_i) < p(R_i)$ and $p(\hat{R}_i) > p(R_i)$ cases are similar. Subsequently, let us only consider the $p(\hat{R}_i) > p(R_i)$ case. By Lemma 6.2, the only possibility in which $f_i(\hat{R}_i, R_{-i}) \neq f_i(R)$ occurs if $p(R_i) \leq f_i(R) < f_i(\hat{R}_i, R_{-i}) \leq p(\hat{R}_i)$. However, this implies that $f_i(R) P_i p(\hat{R}_i) = f_i(\tilde{R}_S, R_{-S})$ because R_i is single-peaked. This contradicts that $i \in S$ and $f(\tilde{R}_S, R_{-S}) \,\text{wdom}[R, S] f(R)$. Consequently, we have that $f(\hat{R}_i, R_{-i}) = f(R)$.

Now pick any $j \in S$ and $j \neq i$. Because $f(\hat{R}_i, R_{-i}) = f(R)$, by following the same steps as above, we obtain that $f(\hat{R}_{\{i,j\}}, R_{-\{i,j\}}) = f(\hat{R}_i, R_{-i}) = f(R)$. By continuing with the same logic, it must be that $f(\hat{R}_S, R_{-S}) = f(R)$.

We now move from $(\tilde{R}_S, R_{-S})$ to $(\hat{R}_S, R_{-S})$ by changing the preferences of agents in S, one at a time. We claim that at each step of this process, the allocation prescribed by f remains unaffected. To see this select any $i \in S$ and consider $(\hat{R}_i, \tilde{R}_{S\setminus\{i\}}, R_{-S})$. By the strategy-proofness of f, we must have that $f_i(\hat{R}_i, \tilde{R}_{S\setminus\{i\}}, R_{-S}) = f_i(\tilde{R}_S, R_{-S})$ because $f_i(\tilde{R}_S, R_{-S}) = p(\hat{R}_i)$. Thus, by non-bossiness we have that $f(\hat{R}_i, \tilde{R}_{S\setminus\{i\}}, R_{-S}) = f(\tilde{R}_S, R_{-S})$. By employing similar arguments for the remaining steps of the process, we find that $f(\hat{R}_S, R_{-S}) = f(\tilde{R}_S, R_{-S})$. This contradicts our earlier conclusion $f(\hat{R}_S, R_{-S}) = f(R)$ because $f(\tilde{R}_S, R_{-S}) \neq f(R)$. □

[9]The stronger version of non-bossiness – non-bossiness in welfare – is not needed here.

Proof of Theorem 5.2. Pick $R, \tilde{R} \in \mathcal{R}^N$. Suppose that $\tilde{R} \in NE(\Gamma^*, R)$. If $\sum_{i\in N} p(R_i) = \Omega$, then the theorem is a simple consequence of efficiency. Therefore, we will only concentrate on the $\sum_{i\in N} p(R_i) \neq \Omega$ case. In addition, because the proof for the $\sum_{i\in N} p(\tilde{R}_i) > \Omega$ case is similar to the one for the $\sum_{i\in N} p(\tilde{R}_i) < \Omega$ case, we will consider only two cases: (1) $\sum_{i\in N} p(\tilde{R}_i) = \Omega$ and (2) $\sum_{i\in N} p(\tilde{R}_i) < \Omega$. Note that efficiency in these cases yield that $p(\tilde{R}_i) \leq f_i(\tilde{R})$, for all $i \in N$.

Case (1). To keep the proof of this case applicable to some subcases of (2), we will not use any result specific to Case (1). Let $S_1 = \{i \in N : p(R_i) \leq f_i(\tilde{R})\}$ and $S_2 = \{i \in N : p(R_i) > f_i(\tilde{R})\}$. Recall that we are focusing on the $\sum_{i\in N} p(R_i) \neq \Omega$ case. Without loss of generality we assume that $\sum_{i\in N} p(R_i) < \Omega$.

Step1. It must be that $f(R_{S_1}, \tilde{R}_{-S_1}) = f(\tilde{R})$.
Proof of Step1. We first argue that for all $i \in S_1$, $f(R_i, \tilde{R}_{-i}) = f(\tilde{R})$. Fix any $i \in S_1$. If $p(\tilde{R}_i) = p(R_i)$, then by peak-onliness, $f_i(\tilde{R}) = f_i(R_i, \tilde{R}_i)$. Suppose now $p(\tilde{R}_i) \neq p(R_i)$. Because $p(\tilde{R}_i) \leq f_i(\tilde{R})$ and $p(R_i) \leq f_i(\tilde{R})$ (because $i \in S_1$), due to Lemma 6.2, we have $f_i(\tilde{R}) = f_i(R_i, \tilde{R}_i)$ unless $p(R_i) \leq f_i(R_i, \tilde{R}_{-i}) < f_i(\tilde{R}) = p(\tilde{R}_i)$. Suppose that $p(R_i) \leq f_i(R_i, \tilde{R}_{-i}) \leq f_i(\tilde{R}) = p(\tilde{R}_i)$. Since $\tilde{R} \in NE(\Gamma^*, R)$ we have that $f_i(\tilde{R}) R_i f_i(R_i, \tilde{R}_{-i})$. Because f is strategy-proof, we also have that $f_i(R_i, \tilde{R}_{-i}) R_i f_i(\tilde{R})$. Hence, $f_i(R_i, \tilde{R}_{-i}) I_i f_i(\tilde{R})$. Because $p(R_i) \leq f_i(R_i, \tilde{R}_{-i}) \leq f_i(\tilde{R}) = p(\tilde{R}_i)$, by the single-peakedness assumption, it must be that $f_i(R_i, \tilde{R}_{-i}) = f_i(\tilde{R})$. Consequently, $f_i(R_i, \tilde{R}_{-i}) = f_i(\tilde{R})$ in all cases. Because f satisfies non-bossiness we have that $f(R_i, \tilde{R}_{-i}) = f(\tilde{R})$.

Observe here that we have completed the proof of Step 1 if $|S_1| = 1$. Thus, let us now assume $|S_1| > 1$ and we prove Step 1 by induction.
The induction assumption: Fix any k such that $1 \leq k \leq |S_1| - 1$. For all $T' \subset S_1$ with $|T'| \leq k$, we have that $f(R_{T'}, \tilde{R}_{-T'}) = f(\tilde{R}) = f(\tilde{R})$.

We know that the induction assumption is true if $k = 1$. We now show that for all $T \subseteq S_1$ with $|T| = k + 1$ we have $f(R_T, \tilde{R}_{-T}) = f(\tilde{R})$. In contrast, suppose that there exists T with $|T| = k + 1$ such that $f(R_T, \tilde{R}_{-T}) \neq f(\tilde{R})$. We first show that $f_i(R_T, \tilde{R}_{-T}) < f_i(\tilde{R})$ for each $i \in T$. Pick any $i \in T$, and set $T' = T \setminus \{i\}$. Because $|T'| = k$, by the induction assumption, $f(R_{T'}, \tilde{R}_{-T'}) = f(\tilde{R})$. If $p(\tilde{R}_i) = p(R_i)$, then by peak-onliness, $f_i(R_T, \tilde{R}_{-T}) = f_i(R_{T'}, \tilde{R}_{-T'})$ which along with non-bossiness leads to $f(R_{T'}, \tilde{R}_{-T'}) = f(R_T, \tilde{R}_{-T}) = f(\tilde{R})$, a contradiction. In fact, we would reach the same contradiction if $f_i(R_T, \tilde{R}_{-T}) = f_i(R_{T'}, \tilde{R}_{-T'})$. Subsequently, $p(\tilde{R}_i) \neq p(R_i)$ and $f_i(R_T, \tilde{R}_{-T}) \neq f_i(R_{T'}, \tilde{R}_{-T'})$. Recall that for the cases we are focusing on, $p(\tilde{R}_j) \leq f_j(\tilde{R}) = f_j(R_{T'}, \tilde{R}_{-T'})$ for all $j \in S_1$. By Lemma 6.2, we have $f_i(R_T, \tilde{R}_{-T}) = f_i(R_{T'}, \tilde{R}_{-T'})$ unless $p(R_i) \leq f_i(R_T, \tilde{R}_{-T}) < f_i(R_{T'}, \tilde{R}_{-T'}) = p(\tilde{R}_i)$. Consequently, we have that $p(R_i) \leq f_i(R_T, \tilde{R}_{-T}) < f_i(R_{T'}, \tilde{R}_{-T'}) = p(\tilde{R}_i)$. Now because $f_i(R_T, \tilde{R}_{-T}) < f_i(R_{T'}, \tilde{R}_{-T'})$, we have $f_j(R_T, \tilde{R}_{-T}) \geq f_j(R_{T'}, \tilde{R}_{-T'})$ for all $j \neq i$ by DNEN. Because $f(\tilde{R}) = f(R_{T'}, \tilde{R}_{-T'})$, we obtain that $f_i(R_T, \tilde{R}_{-T}) < f_i(\tilde{R})$ and

$f_j(R_T, \tilde{R}_{-T}) \geq f_j(\tilde{R})$ for all $j \neq i$. However, we picked i randomly from T, which means that $f_j(R_T, \tilde{R}_{-T}) < f_j(\tilde{R})$ for all $j \in T$. This is a contradiction which concludes our proof of Step 1.

Observe here that the proof of the theorem would have been complete if $S_2 = \emptyset$. Let $S_2 \neq \emptyset$. By the definition of S_2, we have $p(R_i) > f_i(\tilde{R}) = f(R_{S_1}, \tilde{R}_{-S_1})$ for any $i \in S_2$. Since $f_i(R_{S_1}, \tilde{R}_{-S_1}) \geq p(\tilde{R}_i)$ in the cases we are considering, we have $p(R_i) > p(\tilde{R}_i)$. Thus, because $\sum p(R_i) < \Omega$ (by assumption), we have $\sum_{i \in S} p(R_i) + \sum_{i \in N \setminus S} p(\tilde{R}_i) < \Omega$ for all $S \supseteq S_1$.

Denote by $S^* = \{i \in S_1 : p(R_i) = f_i(\tilde{R})\}$. Pick any $i \in S_2$. Let $T = S_1 \cup \{i\}$. We claim that

$$\begin{aligned} &f_j(R_T, \tilde{R}_{-T}) = p(R_j) && \text{for all } j \in S^* \cup \{i\} \\ &f_j(R_T, \tilde{R}_{-T}) = p(\tilde{R}_j) && \text{for all } j \in S_2 \setminus \{i\} \\ &f_j(R_T, \tilde{R}_{-T}) \in [p(R_j), f_j(R_{S_1}, \tilde{R}_{-S_1})] && \text{for all } j \in S_1 \setminus S^*. \end{aligned}$$

To prove the claim consider the preference profiles $(R_{S_1}, \tilde{R}_{-S_1})$ and $(R_T, \tilde{R}_{-T})$ which differ only in the preferences of i. As $T \supseteq S_1$, we have $\sum_{j \in T} p(R_j) + \sum_{j \in N \setminus T} p(\tilde{R}_j) < \Omega$ (as discussed earlier). Now efficiency yields that

$$\begin{aligned} &f_j(R_T, \tilde{R}_{-T}) \geq p(R_j) && \text{for all } j \in S^* \cup \{i\} \\ &f_j(R_T, \tilde{R}_{-T}) \geq p(R_j) && \text{for all } j \in S_1 \setminus S^* \\ &f_j(R_T, \tilde{R}_{-T}) \geq p(\tilde{R}_j) && \text{for all } j \in S_2 \setminus \{i\} \end{aligned}$$

Because $p(\tilde{R}_i) < p(R_i)$ (as we pointed out earlier) and $f_i(\tilde{R}) = f_i(R_{S_1}, \tilde{R}_{-S_1}) < p(R_i)$ ($i \in S_2$), Lemma 6.2 gives that $p(\tilde{R}_i) \leq f_i(R_{S_1}, \tilde{R}_{-S_1}) \leq f_i(R_T, \tilde{R}_{-T}) \leq p(R_i)$. Combining this with the above conditions, we obtain that $f_i(R_T, \tilde{R}_{-T}) = p(R_i)$. Subsequently, $f_i(R_{S_1}, \tilde{R}_{-S_1}) < f_i(R_T, \tilde{R}_{-T})$. Then DNEN yields that $f_j(R_T, \tilde{R}_{-T}) \leq f_j(R_{S_1}, \tilde{R}_{-S_1})$ for all $j \neq i$. Combining this with the three conditions above, we prove the claim.

Now by sequentially selecting agents from S_2 and proving similar claims as above, we prove the theorem for case 1.

Case 2. If $\sum_i p(R_i) < \Omega$, then the proof of this case is identical to the proof of case 1. Consequently, we turn our attention to the $\sum_i p(R_i) > \Omega$ case. By efficiency, $p(\tilde{R}_i) \leq f_i(\tilde{R})$ for all $i \in N$. If $p(R_i) \leq f_i(\tilde{R})$ for all $i \in N$, then $\sum_i p(R_i) \leq \Omega$ which is not compatible with the case we are focusing. Hence, suppose there exists some i with $p(R_i) > f_i(\tilde{R})$.

Fix an agent i^* with $p(R_{i^*}) > f_{i^*}(\tilde{R})$. We claim that there is no agent $i' \neq i^*$ with $p(\tilde{R}_{i'}) < f_{i'}(\tilde{R})$. Suppose otherwise. Then there must exist a preference relation R'_{i^*} such that $p(R'_{i^*}) \in (f_{i^*}(\tilde{R}), p(R_{i^*})]$ and $p(R'_{i^*}) \leq \Omega - \sum_{j \neq i^*, i'} f_j(\tilde{R}) - p(\tilde{R}_{i'})$. Observe that $p(R_{i^*}) \geq p(R'_{i^*}) > f_{i^*}(\tilde{R}) \geq p(\tilde{R}_{i^*})$. Because $p(R'_{i^*}) > p(\tilde{R}_{i^*})$ and $p(R'_{i^*}) > f_{i^*}(\tilde{R})$, by Lemma 6.2, $p(\tilde{R}_{i^*}) \leq f_{i^*}(\tilde{R}) \leq f_{i^*}(R'_{i^*}, \tilde{R}_{-i^*}) \leq p(R'_{i^*})$. If $f_i(R'_{i^*}, \tilde{R}_{-i^*}) > f_{i^*}(\tilde{R})$, then by single peakedness, i^* would improve by deviating from $\tilde{R}_{i^*}$ to R'_{i^*} at profile $\tilde{R}$, a contradiction to $\tilde{R} \in NE(\Gamma^*, R)$. Thus, to complete the proof of the claim, we need to dispose of the $f_{i^*}(R'_{i^*}, \tilde{R}_{-i^*}) = f_{i^*}(\tilde{R})$ case. If this was the case, by non-bossiness, we obtain $f(R'_{i^*}, \tilde{R}_{-i^*}) = f(\tilde{R})$. But then $f_{i^*}(R'_{i^*}, \tilde{R}_{-i^*}) < p(R'_{i^*})$ and $f_{i'}(R'_{i^*}, \tilde{R}_{-i^*}) > p(\tilde{R}_{i'})$. This would contradict the efficiency of f. As a result, for all $j \neq i^*$, $p(\tilde{R}_j) = f_j(\tilde{R})$. Then $p(\tilde{R}_{i^*}) < f_{i^*}(\tilde{R}) < p(R_{i^*})$ because $f_{i^*}(\tilde{R}) + \sum_{j \neq i^*} f_j(\tilde{R}) = \Omega$ and $p(\tilde{R}_{i^*}) + \sum_{j \neq i^*} p(\tilde{R}_j) < \Omega$. In addition, i^* is the only agent with $p(R_{i^*}) > f_{i^*}(\tilde{R})$. Otherwise, similar arguments lead to the conclusion that $p(\tilde{R}_{i^*}) = f_{i^*}(\tilde{R})$ which contradicts that $p(\tilde{R}_{i^*}) < f_{i^*}(\tilde{R})$. As a result, for all $j \neq i^*$, $p(R_j) \leq p(\tilde{R}_j)$ because $p(R_j) \leq f_j(\tilde{R})$.

Recall that we are concentrating on the $p(R_{i^*}) + \sum_{i \neq i^*} p(R_i) > \Omega$ case. Let $S_1 = \{i \in N : p(R_i) \leq f_i(\tilde{R})\}$. We know that every $i \neq i^*$ is in S_1. In fact, we have already shown that $p(R_i) \leq p(\tilde{R}_i) = f_i(\tilde{R})$ for every $i \in S_1$ and $f_{i^*}(\tilde{R}) < p(R_{i^*})$.

Because $\tilde{R} \in NE(\Gamma^*, R)$, it must be that $f_{i^*}(\tilde{R}) R_{i^*} f_{i^*}(R_{i^*}, \tilde{R}_{-i^*})$. On the other hand, by strategy-proofness, $f_{i^*}(R_{i^*}, \tilde{R}_{-i^*}) R_{i^*} f_{i^*}(\tilde{R})$. Consequently, $f_{i^*}(\tilde{R}) I_{i^*} f_{i^*}(R_{i^*}, \tilde{R}_{-i^*})$. In addition, because $p(\tilde{R}_{i^*}) < f_{i^*}(\tilde{R}) < p(R_{i^*})$, by applying Lemma 6.2, we obtain

$$p(\tilde{R}_{i^*}) < f_{i^*}(\tilde{R}) \leq f_{i^*}(R_{i^*}, \tilde{R}_{-i^*}) \leq p(R_{i^*}).$$

Subsequently, the single-peakedness of f and the relation $f_{i^*}(\tilde{R}) I_{i^*} f_{i^*}(R_{i^*}, \tilde{R}_{-i^*})$ give that $f_{i^*}(\tilde{R}) = f_{i^*}(R_{i^*}, \tilde{R}_{-i^*})$. Then by non-bossiness, $f(\tilde{R}) = f(R_{i^*}, \tilde{R}_{-i^*})$. Set $S^* = \{i \in S_1 : p(R_i) = f_i(\tilde{R})\}$.

Pick any $i \in S_1$. Let $T = \{i^*, i\}$. We claim that

$$\begin{aligned} f_j(R_T, \tilde{R}_{-T}) &= p(R_j) && \text{for all } j \in S^* \cup \{i\} \\ f_j(R_T, \tilde{R}_{-T}) &= p(R_j) && \text{for all } j \in S_1 \setminus \{i\} \\ f_{i^*}(R_T, \tilde{R}_{-T}) &= \Omega - p(R_i) - \sum_{j \neq i^*} p(\tilde{R}_i) \end{aligned}$$

Let us prove the claim. Note that $(R_{i^*}, \tilde{R}_{-i^*})$ and $(R_T, \tilde{R}_{-T})$ differ only in the preferences of i. Recall that for all $j \in N \setminus T \subset S_1$, $p(R_j) \leq p(\tilde{R}_j)$. Because $\sum_{j \in N} p(R_j) > \Omega$, it must be that $p(R_{i^*}) + p(R_i) + \sum_{j \in N \setminus T} p(\tilde{R}_j) > \Omega$. Now efficiency yields that

$$\begin{aligned} f_j(R_T, \tilde{R}_{-T}) &\leq p(R_j) && \text{for all } j \in S^* \cup \{i\} \\ f_j(R_T, \tilde{R}_{-T}) &\leq p(R_j) && \text{for all } j \in S_1 \setminus \{i\} \\ f_{i^*}(R_T, \tilde{R}_{-T}) &\leq p(\tilde{R}_{i^*}) \end{aligned}$$

If $p(R_i) = p(\tilde{R}_i)$ then by peak-onliness, $f_i(R_{i^*}, \tilde{R}_{-i^*}) = f_i(R_T, \tilde{R}_{-T})$. This along with non-bossiness gives $f(R_{i^*}, \tilde{R}_{-i^*}) = f(R_T, \tilde{R}_{-T})$. If $p(R_i) < p(\tilde{R}_i)$, then Lemma 6.2 yields $p(R_i) \leq f_i(R_T, \tilde{R}_{-T}) \leq f_i(R_{i^*}, \tilde{R}_{-i^*}) = p(\tilde{R}_i)$. Combining this with the three conditions above, we obtain that $f_i(R_T, \tilde{R}_{-T}) = p(R_i)$. Subsequently, $f_i(R_T, \tilde{R}_{-T}) < f_i(R_{i^*}, \tilde{R}_{-i^*})$. Now DNEN yields that $f_j(R_T, \tilde{R}_{-T}) \geq f_j(R_{i^*}, \tilde{R}_{-i^*}) = p(\tilde{R}_j)$ for all $j \neq i$. Combining this with the three conditions above, we prove the claim. Finally, by sequentially changing the preferences of each agent $j \neq i \in S_1$ from $\tilde{R}_j$ to R_j and proving similar claims as above we complete the proof of the Theorem. □

Proof of Theorem 5.7. Let f satisfy group strategy-proofness, the group reversal property and weak non-bossiness in welfare. Consider Γ^* the direct revelation mechanism of f, and let $R \in \mathcal{R}^N$ be the true preference profile. By group strategy-proofness, profile R is a both a dominant strategy and a strong Nash equilibrium of (Γ^*, R). Suppose there is $i \in N$, and $R'_i \in \mathcal{R}_i$ that is also a dominant strategy for i at R_i. If $f(R'_i, R_{-i}) \neq f(R)$, the group reversal property ensures that (R'_i, R_{-i}) is not a strong Nash equilibrium at R. By weak non-bossiness, there exists $\hat{R}_{-i} \in \mathcal{R}^{N\setminus -i}$ such that $f_i(R_i, \hat{R}_{-i}) P_i f_i(R'_i, R_{-i})$. Hence, R'_i cannot in fact be dominant after all, a contradiction, so $f(R'_i, R_{-i}) = f(R)$. Finally if there is $\tilde{R} \in \mathcal{R}^N$ such that $f(R) \neq f(\tilde{R})$, the group reversal property ensures that $\tilde{R}$ cannot be a strong Nash equilibrium. We have established that (i) $R \in DS(\Gamma^*, R) \cap SNE(\Gamma^*, R)$, (ii) $DS(\Gamma^*, R) = f(R)$ and (iii) $SNE(\Gamma^*, R) = f(R)$. Hence f is strict coalitional secure implemented by Γ^*.

Next suppose that f is strict coalitional secure implemented by Γ^*. By Theorem 4.9, f must satisfy group strategy-proofness and the group reversal property. Let us show that f must also satisfy weak non-bossiness. Pick $i \in N$, $R \in \mathcal{R}^N$, $R'_i \in \mathcal{R}_i$, and assume that $f(R) \neq f(R'_i, R_{-i})$. By the strict coalitional secure implementation requirement, we know that $(R'_i, R_{-i}) \notin SNE(\Gamma^*, R)$ and $(R'_i, R_{-i}) \notin DS(\Gamma^*, R)$. The former is ensured by the group reversal property at R. For the latter, R'_i must be dominated by R_i for agent i at preference R_i so that there exists $\hat{R}_{-i} \in \mathcal{R}^{N\setminus -i}$ such that $f_i(R_i, \hat{R}_{-i}) P_i f_i(R'_i, R_{-i})$. Hence f satisfies weak non-bossiness. □