Dividing goods or bads under additive utilities*

Anna Bogomolnaia★♠, Hervé Moulin★♠,
Fedor Sandomirskiy♠, and Elena Yanovskaia♠.
★ *University of Glasgow*
♠ *Higher School of Economics, St Petersburg*

July 2016

Abstract

The *Competitive Equilibrium with Equal Incomes* is an especially appealing efficient and envy-free division of private goods when utilities are additive: it maximizes the Nash product of utilities and is single-valued and continuous in the marginal rates of substitution. The CEEI to divide *bads* captures similarly the critical points of the Nash product in the efficient frontier. But it is far from resolute, allowing routinely many divisions with sharply different welfare consequences.

Even the much more permissive *No Envy* property is profoundly ambiguous in the division of bads: the set of efficient and envy-free allocations can have many connected components, and has no single-valued selection continuous in the marginal rates.

The CEEI to divide goods is *Resource Monotonic* (RM): everyone (weakly) benefits when the manna increases. But when we divide bads efficiently, RM is incompatible with Fair Share Guarantee, a much weaker property than No Envy.

1 Introduction

The No Envy test ([9], [35]) captures the powerful fairness principle that participants of a given allocation problem should be given "equal opportunities". When we divide heterogenous private commodities in the common property regime it eschews interpersonal comparison of utilities and is logically related to the concept of competitive exchange. If the agents have Arrow-Debreu (in particular, convex) preferences over the goods to divide, there is at least one

*Support from the Basic Research Program of the National Research University Higher School of Economics is gratefully acknowledged. Moulin thanks the Simmons institute for Theoretical Computing for its hospitality during the Fall 2015.Sandomirskiy is partially supported by the grant 16-01-00269 of the Russian Foundation for Basic Research. The comments of William Thomson on an earlier version have been especially useful.

Competitive Equilibrium allocation where all agents have *Equal Incomes* (for short CEEI, or simply *Competitive*, allocation), and this allocation is both efficient and envy-free.

The implementation of the elegant Competitive division rule in user-friendly platforms like SPLIDDIT or ADJUSTED WINNER[1] asks visitors to report simple preferences. This has two advantages, one practical and the other conceptual.

On the practical side, we divide for instance the family heirlooms by asking each participant to distribute 100 points over the different items, and interpret these "bids" as fixed marginal rates of substitution; in other words we force the report of *linear preferences.* Eliciting complex complementarities between the different objects is a complex task with 6 objects, an outright impossible one with 10 or more' so we deliberately ignore that aspect of preferences.[2]. The proof of the pudding is in the eating: hundreds of visitors use these sites every month, fully aware of the interpretation of their bids ([12]).

On the conceptual side, the Competitive allocation of goods under additive utilities is remarkably (almost unreasonably so, say [5]) well behaved: it maximizes the (Nash) product of individual utilities over all feasible allocations, hence it is unique utility-wise, easily computed, and continuous in the marginal utility rates. It is natural to expect that the Competitive approach to fair division is equally successful for the division of "bads" and of "goods": just like labor as time not spent on leisure, a simple change of viewpoint turns 'bads' into 'goods'. Surprisingly, this is not the case at all: we show here that even the implementation of the much more permissive No Envy test is a formidable challenge when we divide bads.

This is important because many practical fair division problems involve *bads* (non disposable items generating disutility): workers distributing tasks (house chores, job shifts among substitutable workers ([3]) like teaching loads, babysitting, etc.), cities sharing noxious facilities, managers allocating cuts in the company's workforce between their respective units, and so on. Of course the division of *goods* has many applications as well: the family heirlooms ([24]), the assets of divorcing partners ([2]), office space between the colleagues, seats in overdemanded business school courses ([27], [4]), computing resources in peer-to-peer platforms ([11]), and so on.

The sharp difference between goods and bads is especially clear under linear preferences, because the two models are formally identical: preferences are described by a list of marginal rates interpreted as the marginal *utilities* for goods, or *disutilities* for bads.

The Competitive allocations (efficient and envy-free as usual) exist when we share *bads*, however there are routinely several of them with different welfare consequences. Here is why: eating x units of bad a is the same as consuming $1-x$ units of the good $\dashv a$, "not eating a"; however each agent can consume at most 1 unit of the good $\dashv a$, of which we have $n-1$ units to distribute among

[1] www.spliddit.org/ ; www.nyu.edu/projects/adjustedwinner/

[2] Similarly practical combinatorial auctions never ask buyers to report a ranking of all subsets of objects, ([1], [36], [7]).

n agents. It is well known that such constraints on individual consumption (sometimes interpreted as satiation of preferences) can enlarge considerably the set of competitive equilibria (**????**).

In the following example with two agents and three bads:

bad	a	b	c
u_1	3	2	8
u_2	6	3	2

efficiency means that if agent 1 eats (consumes a positive amount of) bad x and agent 2 eats some bad y, the letter x is alphabetically before y or $x = y$. We have no less than *five* Competitive allocations, reaching both ends of the envy-free and efficient set. At the worst competitive allocation z^1 for agent 1, she gets her *Fair Share* disutility 6.5 (as if she consumes half of each bad) and $z^1 =$

bad	a	b	c
z^1_1	1	1	3/16
z^1_2	0	0	13/16

is competitive for the prices $p = (\frac{6}{13}, \frac{4}{13}, \frac{16}{13})$ (normalized so that total cost of all objects is 2 and each agent has a budget of 1). Symmetrically the worst efficient and envy-free allocation z^5 for agent 2 gives him his Fair Share disutility 5.5. In two more competitive allocations the bads are not split: in z^2 (resp. z^4) agent 1 gets a, b (resp. a) and agent 2 gets c (resp. b, c); in the fifth allocation z^3 they split b.[3] Note that this pattern is robust to small changes in the disutility matrix.[4]

In general the number of Competitive disutility profiles can grow exponentially in the smallest of the number of bads and of agents (Proposition 3). There is no obvious way to deal with this embarrassing multiplicity.

The news is worse when we turn to No Envy, a much weaker fairness test than the competitive one. It turns out that with three or more agents, the set of efficient and envy-free allocations of bads, and the corresponding disutility profiles, can have up to roughly $\frac{2}{3}n$ connected components among n agents: see Proposition 4 and a four-agent two-bad example with three such components in Section 5. It follows that any single-valued, efficient and envy-free division rule, in particular any selection of the Competitive allocations, must be discontinuous in the disutility rates (Theorem 1). The No Envy property is the central concept of the equal opportunity approach to fair division: its implementation for the division of bads requires either to allow inefficient allocations, or tolerate significant jumps in individual welfare from minute differences in the disutility matrix.

Our second impossibility result pits two staples of the fair division literature against each other. *Resource Monotonicity* (RM) is a solidarity requirement in the common property regime: when the pile of goods (resp. bads) to divide

---

[3] So $z^5 =$

bad	a	b	c
z^5_1	11/12	0	0
z^5_2	1/12	1	1

with prices $p = (\frac{12}{11}, \frac{6}{11}, \frac{4}{11})$ and $z^3 =$

bad	a	b	c
z^3_1	1	1/12	0
z^3_2	0	11/12	1

with prices $p = (\frac{18}{19}, \frac{12}{19}, \frac{8}{19})$.

[4] The pattern is identical in problem

u_1	u_{1a}	u_{1b}	u_{1c}
u_2	u_{2a}	u_{2b}	u_{2c}

if $\frac{u_{1a}}{u_{2a}} > \frac{u_{1b}}{ub} > \frac{u_{1c}}{u_{2c}}$, and $u_{1a} > u_{1b} + u_{1c}$, $u_{2c} > u_{2a} + u_{2b}$, and $|\frac{u_{2a}}{u_{2b}} - \frac{u_{1c}}{u_{1b}}| < 1$.

increases (resp. decreases), everyone should benefit at least weakly. Beyond the fair division of private goods, RM has been applied to a broad range of resource allocation problems with production and/or indivisibilities.[5]

Fair Share Guarantee (FSG) says that nobody should end up worse off than by eating $1/n$-th of each good, or of each bad. This simple welfare lower bound, much weaker than No Envy in our model, is probably the least controversial fairness requirement in the literature; like RM it can be adapted to many other fair division contexts involving production and/or indivisibilities ([18], [19]).

Our second surprising contrast "goods versus bads" is that the Competitive division of goods meets RM (and FSG), but no efficient single-valued division of bads meets both RM and FSG: Theorem 2. Resource Monotonicity is one more appealing feature of the Competitive rule to divide *goods*: the linearity of preferences avoids the choice between RM and FSG. However if we divide bads, the incompatibility is back with a vengeance.

We conclude that, even in the benchmark domain of additive utilities, identifying a reasonable efficient division rule is as difficult for bads as it is easy for goods.

Contents After the literature review in Section 2, the model is defined in Section 3, and Competitive allocations in Section 4. They are characterized by systems of first order conditions identical up to a sign switch (Lemma 2), and are both related to the Nash product of utilities or disutilities (Proposition 1); they are both characterized by a single property that we call *Independence of Lost Bids* (ILB): Proposition 2. Proposition 3 gives some estimates on the number of competitive allocations of bads different welfare-wise, even precise ones for problems with two agents and/or two bads. Section 5 turns to efficient and envy-free allocations of bads: their set can have as many as $\lfloor(2n+1)/3\rfloor$ connected components (Proposition 4), which implies Theorem 1, our main impossibility result. Our second negative result, Theorem 2, is the object of Section 6. Most proofs are in Section 7.

2 Related literature

1. Our main motivation is the recent stream of work in algorithmic mechanism design (and its "computational social choice" subfield) on fair division of *goods*. It recognizes the practical convenience of additive utilities and the conceptual advantages of the Competitive solution. For instance in the same model as here, Megiddo and Vazirani ([16]) show that the Competitive utility profile depends continuously upon the rates of substitution and the total endowment; Jain and Vazirani ([14]) that it is can be computed in time polynomial in the dimension $n+p$ of the problem.

Steinhaus' 1948 "cake-division" model ([30]), assumes linear preferences represented by atomless measures over, typically, a compact euclidian set. It contains our model for goods as the special case where the measures are piecewise

[5] See subsections 7.1, 9.4, 10.3 and 11.3 in the recent survey [31].

constant. Sziklai and Segal-Halevi ([29]) show that it preserves the equivalence of the Competitive rule and the Nash product maximizer, and that this rule is Resource Monotonic (see the Remark in Section 6).

The fair division of *indivisible goods* with additive utilities is a variant of the standard model where the maximization of the Nash product still defines a very attractive solution: it loses its competitive interpretation and becomes hard to compute ([15]), however it is envy-free "up to at most one object" (Caragianis et al. [5]) and can be efficiently approximated ([6]). Finally Budish ([3]) approximates the Competitive allocation in problems with a large number of copies of several good-types by means of a little flexibility in the number of available copies.

2. Four decades earlier the microeconomic literature on the fair division of private goods worked in the much larger domain of Arrow-Debreu preferences, where the relation between the Nash product of utilities and the Competitive rule is lost, and provided several axiomatic characterizations of the latter. The most popular result appears first in Hurwicz ([13]) and Gevers ([10]), and is refined by Thomson ([32]) and Nagahisa ([21]): any efficient and Pareto indifferent rule meeting (some variants of) Maskin Monotonicity (MM) must contain the Competitive rule.[6] Our Independence of Lost Bids is in fact a weak variant of MM for the linear domain, so that the proof of our Proposition 2 follows the standard argument. The point is that the classic characterization extends to the model with bads.

3. In the Arrow-Debreu preference domain the Competitive allocation of goods is not Resource Monotonic. In fact no efficient single-valued division of *goods* can meet both FSG and RM: [20] (see also [34]). The incompatibility is easy to check for two agents with Leontief preferences over two goods. In view of our Theorem 2, much remains to understand about domain of preferences where the incompatibility disappears.

3 Division problems and division rules

The finite set of agents is N with generic element i, and $|N| = n \geq 2$. The finite set of divisible items is A with generic element a and $|A| = p \geq 2$. The manna consists of one unit of each item. In one interpretation of the model all items are goods, in the other they are all bads.

Agent i's allocation (or share) is $z_i \in [0,1]^A$; the profile $z = (z_i)_{i \in N}$ is a feasible allocation if $\sum_N z_i = e^A$, the vector in $\mathbb{R}^A_+$ with all coordinates equal to 1. The set of feasible allocations is $\Phi(N, A)$.

Each agent is endowed with linear preferences over $[0,1]^A$, represented for convenience by a vector $u_i \in \mathbb{R}^A_+$: a utility function in the case of goods, a disutility function in that of bads. We keep in mind that only the underlying

[6] Another, logically unrelated characterization combines Consistency and Replication Invariance ([33]) or Consistency and Converse Consistency ([22]).

preferences matter, i. e., for any $\lambda > 0$, u_i and λu_i carry the same information. This restriction is formally included in Definition 1 below.

Given an allocation z we write i's corresponding utility/disutility (*util/disutil* for short) as $U_i = u_i \cdot z_i = \sum_A u_{ia} z_{ia}$.

Note that a "null agent" ($\forall a : u_{ia} = 0$) when we divide goods can clearly be ignored. When we divide bads, the presence of a null agent makes the optimal disutility profile $U = 0$ feasible (and uniquely efficient). Thus we only look at problems where all agents are non null. In fact the vector $U = 0$ is feasible if (and only if) each bad is harmless to at least one agent ($\forall a \exists i : u_{ia} = 0$): we call such problems (for bads) *trivial* and will rule them out in the Definitions below.

Similarly if item a gives $u_{ia} = 0$ for all i, it is a "useless good" or a "harmless bad" that can be ignored as well. But our Competitive rule to divide bads goes one step further: it will also ignore a bad a harmless to *some* agents, and give no credit to these agents for eating a. See Definition 3.

Definition 0

A division problem is a triple $\mathcal{Q} = (N, A, u)$ where $u \in \mathbb{R}_+^{N \times A}$ is such that the $N \times A$ matrix $[u_{ia}]$ has no null row, and in the case of bads there is at least one column with no null entry.

The corresponding set of feasible util/disutil profiles U is $\Psi(\mathcal{Q})$, and $\Psi^{eff}(\mathcal{Q})$ is its subset of efficient util/disutil profiles at $\mathcal{Q}$.[7]

The structure of efficient allocations in the linear domain is key to several of our results: in particular most entries of the allocation matrix $[z_{ia}]$ are nil. For any $z \in \Phi(N, A)$ define the bipartite $N \times A$ *consumption graph* $\Gamma(z) = \{(i, a) | z_{ia} > 0\}$.

Lemma 1

a) Fix a problem $\mathcal{Q} = (N, A, u)$. If $U \in \Psi^{eff}(\mathcal{Q})$ then there is some $z \in \Phi(N, A)$ representing U such that $\Gamma(z)$ is a forest (an acyclic graph). For such allocation z the matrix $[z_{ia}]$ has at least $(n-1)(p-1)$ zeros.

b) Fixing N, A, on an open dense subset $\mathcal{U}^*(N, A)$ of matrices $u \in \mathbb{R}_+^{N \times A}$, every efficient util/disutil profile $U \in \Psi^{eff}(N, A, u)$ is achieved by a single allocation z. At such profiles an efficient division rule f is single-valued.

The definition of $\mathcal{U}^*(N, A)$ and the proof of Lemma 1 is in Section 7.1.

We use two equivalent definitions of a division rule, in terms of *util/disutil* profiles or of feasible allocations. When we rescale each u_i as $\lambda_i u_i$ the new profile is written $\lambda * u$.

Definition 1

i) A division rule F associates to every problem $\mathcal{Q} = (N, A, u)$ a set of util/disutil profiles $F(\mathcal{Q}) \subset \Psi(\mathcal{Q})$. Moreover $F(N, A, \lambda * u) = \lambda * F(N, A, \lambda * u)$ for any rescaling λ with $\lambda_i > 0$ for all i.

ii) A division rule f associates to every problem $\mathcal{Q} = (N, A, u)$ a subset $f(\mathcal{Q})$ of $\Phi(N, A)$ such that for any $z, z' \in \Phi(N, A)$:

$$\{z \in f(\mathcal{Q}) \text{ and } u_i \cdot z_i = u_i \cdot z'_i \text{ for all } i \in N\} \Longrightarrow z' \in f(\mathcal{Q})$$

[7] So $U \in \Psi^{eff}(\mathcal{Q})$ means: there is no $U' \in \Psi(\mathcal{Q})$ such that $U' \gneq U$ (if we divide goods) or $U' \lneq U$ (if bads).

Moreover $f(N, A, \lambda * u) = f(\mathcal{Q})$ *for any rescaling* λ *where* $\lambda_i > 0$ *for all* i.

The one-to-one mapping from F to f is clear. Definition 1 makes no distinction between two allocations with identical welfare consequences, a property often called *Pareto-Indifference*. We speak of a *single-valued* division rule if $F(\mathcal{Q})$ is a singleton for all $\mathcal{Q}$, otherwise the rule is *multi-valued*.

Single-valued rules are of much greater value to the mechanism designer, as they eschew the further negotiation required to converge on a single division. The simplest efficient single-valued rule to divide goods or bads, is the *Egalitarian* one, equalizing as much as possible the relative util/disutil profiles. Pazner and Schmeidler ([23]) introduced it as a welfarist alternative to the competitive approach. Given a problem $\mathcal{Q}$ we look for an efficient util/disutil profile $U \in \Psi^{eff}(\mathcal{Q})$ such that[8]

$$\frac{U_i}{u_i \cdot e^A} = \frac{U_j}{u_j \cdot e^A} \text{ for all } i, j$$

This rule provides a useful contrast to the Competitive one.

4 Competitive rules: goods and bads

4.1 Definition and basic characterizations

Definition 3 *Fix a problem* $\mathcal{Q} = (N, A, u)$.
i) Goods: the feasible allocation $z \in \Phi(N, A)$ *is a Competitive allocation if there is a price* $p \in \mathbb{R}_+^A$ *such that* $\sum_A p_a = n$ *and*

$$z_i \in \arg \max_{y_i \in \mathbb{R}_+^A} \{u_i \cdot y_i | p \cdot y_i \leq 1\} \text{ for all } i \tag{1}$$

ii) Bads: we say that $z \in \Phi(N, A)$ *is a Competitive allocation if there is a price* $p \in \mathbb{R}_+^A$ *such that* $\sum_A p_a = n$ *and*

$$z_i \in \arg \min_{y_i \in \mathbb{R}_+^A} \{u_i \cdot y_i | p \cdot y_i \geq 1\} \text{ for all } i \tag{2}$$

and for all $a \in A$

$$p_a = 0 \text{ if } u_{ia} = 0 \text{ for some } i \in N \tag{3}$$

In the case of goods this Definition implies $U \gg 0$ because each row u_i is non null and agent i can afford *some share* of a good he likes. Inequality $U \gg 0$ holds for bads as well because agent i must buy some bads with a positive price, and by (3) he dislikes such bads.

In the absence of property (3), if some entries u_{ia} are zero we may find inefficient allocations meeting system (2). Here is a two-agent, two-bad example

[8] If we divide bads such a profile can always be found, but in the case of goods this may not be possible if some entries u_{ia} are zero. Then we must refine the definition and select a profile U such that $(\frac{U_i}{u_i \cdot e^A})_{i \in N}$ maximizes the leximin ordering.

	a	b
u_1	2	1
u_2	0	1

; at the prices $p_a = \frac{4}{3}, p_b = \frac{2}{3}$ the inefficient allocation

	a	b
z_1	1/4	1
z_2	3/4	0

meets (2). There are two more solutions of (2):

	a	b
z_1	0	1/2
z_2	1	1/2

for prices $p_a = 0, p_b = 2$, and

	a	b
z_1	0	1
z_2	1	0

for prices $p_a = p_b = 1$. Both are efficient and (3) rules out the latter.

Clearly we could expand the definition of competitive divisions of bads to include all efficient allocations meeting (2). This would bring additional technical issues but no new light on the limits of the competitive approach.

We write $f^c(\mathcal{Q}), F^c(\mathcal{Q})$ for the set of Competitive allocations and corresponding util/disutil profiles; we use the same notation for goods or bads as this will cause no confusion. Existence of a Competitive allocation in the goods case is well known, and in the case of bads we derive it from Lemma 2 and Proposition 1 below.

A Competitive division of goods or bads meets the **No Envy** test: for all $\mathcal{Q}$ and all $i, j \in N$

$$u^i \cdot z^i \geq u^i \cdot z^j \text{ (goods); } u^i \cdot z^i \leq u^i \cdot z^j \text{ (bads)}$$

and *Core stability from Equal Split*: no coalition S of agents can find a Pareto improvement by reallocating its initial endowment of $|S|/n$ units of each good or bad.

The joint characterization of our two Competitive rules by a simple system of inequalities, essential to most of our results, is also quite intuitive so we do not postpone its proof to the Appendix.

Lemma 2 *Fix a problem $\mathcal{Q} = (N, A, u)$ as in Definition 0. Then $U \in F^c(\mathcal{Q})$ if and only if:*
i) Case of goods: $U \gg 0$ and $U = (u_i \cdot z_i)_{i \in N}$ *for some $z \in \Phi(N, A)$ such that for all $i \in N$*

$$\text{for all } a \in A\text{: } z_{ia} > 0 \Longrightarrow \{\frac{u_{ia}}{U_i} \geq \frac{u_{ja}}{U_j} \text{ for all } j \in N\} \tag{4}$$

ii) Case of bads: $U \gg 0$ and $U = (u_i \cdot z_i)_{i \in N}$ *for some $z \in \Phi(N, A)$ such that for all $i \in N$*

$$\text{for all } a \in A\text{: } z_{ia} > 0 \Longrightarrow \{\frac{u_{ia}}{U_i} \leq \frac{u_{ja}}{U_j} \text{ for all } j \in N\} \tag{5}$$

Proof: *Case of goods*. Fix $\mathcal{Q}, U, z$ meeting (4) and $U \gg 0$. We set $p_a = \frac{u_{ia}}{U_i}$ for all i such that $z_{ia} > 0$ and note that $p \cdot z_i = 1$ for all i. For all a such that $z_{ia} = 0$ we have $\frac{u_{ia}}{U_i} \leq p_a$: therefore z_i is agent i's Walrasian demand at price p, and z is a competitive allocation.

Conversely let $z \in \Phi(N, A)$ and p meet (1). Recall $U \gg 0$ because each agent i likes at least one good. If $p_a = 0$ then nobody likes good a (if $u_{ia} > 0$ then i's demand is infinite, a contradiction of (1)) and system (4) holds for a. Consider now the support A^* of p_a. Because z_i is i's demand at price p the ratio $\frac{u_{ia}}{p_a}$ is a constant π_i over the support of z_i, and we have: $\sum_{A^*} u_{ia} z_{ia} = \pi_i (\sum_{A^*} p_a z_{ia}) \Longrightarrow \pi_i = U_i$. So $\frac{u_{ia}}{U_i} = p_a$ whenever i consumes a and $\frac{u_{ib}}{U_i} \leq p_b$ if i does not eat any b, as required by system (4). Note that this argument implies that the competitive price p is unique.

Case of bads. Fix $\mathcal{Q}, U, z$ meeting (5) and $U \gg 0$. Define $A^0 = \{a \in A | u_{ia} = 0$ for some $i \in N\}$ and set $p_a = 0$ for those bads. By (5) the bads in A^0 can only be eaten by agents who don't mind them: $z_{ia} > 0 \Longrightarrow u_{ia} = 0$. Next in the restriction of $\mathcal{Q}$ to $A \diagdown A^0$ the same utility profile U is still feasible, strictly positive, and meets (5). Like in the above argument we set $p_a = \frac{u_{ia}}{U_i}$ for all i who eat some a, and check that we have constructed a competitive price in the sense of Definition 3.

For the converse statement, recall that (2), (3) together imply $U \gg 0$, and mimic the argument in the case of goods. ■

Lemma 2 implies a geometric characterization of the competitive util/disutil profiles. Let $\mathcal{N}(U) = \Pi_{i \in N} U_i$ be the Nash product of util/disutils. At $U \in \mathbb{R}^N_{++}$ the hyperplane $\mathcal{H}(U) = \{W \in \mathbb{R}^N_+ | \sum_N \frac{W_i}{U_i} = n\}$ supports the upper contour $\mathcal{UC}^{\mathcal{N}}(U)$ of $\mathcal{N}$. We say that the feasible util/disutil vector $U \in \Psi(\mathcal{Q})$ is a *critical profile* of $\mathcal{N}$ if the hyperplane $\mathcal{H}(U)$ supports $\Psi(\mathcal{Q})$ at U.

Assume first we divide goods and pick $U \in F^c(\mathcal{Q})$ with price p. Choose an arbitrary $U' \in \Psi(\mathcal{Q})$ with $U'_i = u_i \cdot z'_i$, then use (4) and $p_a = \frac{u_{ia}}{U_i}$ to derive the variational inequality

$$\sum_N \frac{U'_i}{U_i} = \sum_{N \times A} \frac{u_{ia} z'_{ia}}{U_i} \leq \sum_{N \times A} p_a z'_{ia} = \sum_A p_a = n$$

implying that $\mathcal{H}(U)$ supports $\Psi(\mathcal{Q})$ *from above* at U. But $\mathcal{H}(U)$ supports $\mathcal{UC}^{\mathcal{N}}(U)$ from below, therefore U maximizes the strictly quasi-concave function $\mathcal{N}$ in $\Psi(\mathcal{Q})$, and we conclude that $F^c(\mathcal{Q})$ is a singleton. This convex maximization problem, known as the Eisenberg-Gale optimization problem ([8]) can be efficiently solved by the standard numerical methods.

Next if we divide bads and pick $U \in F^c(\mathcal{Q})$ with price p, the system (5) gives $\sum_N \frac{U'_i}{U_i} \geq n$ for any $U' \in \Psi(\mathcal{Q})$ and $\mathcal{H}(U)$ supports $\Psi(\mathcal{Q})$ *from below* (see Figures 1,2); it is easy to see that this inequality is not only necessary but also sufficient for U to be competitive. Therefore U is still a critical profile of $\mathcal{N}$ but is no longer related to its global maximum or minimum in $\Psi(\mathcal{Q})$. Note that the variational characterization (5) of competitive allocations implies that they always exist: if the profile U maximizes $\mathcal{N}$ on the efficient frontier $\Psi^{eff}(\mathcal{Q})$, the hyperplane $\mathcal{H}(U)$ supports $\Psi(\mathcal{Q})$ *from below* so it is competitive. But finding competitive allocation of bads is not a convex optimization problem and with more than two agents we do not know of any efficient algorithms discovering $F^c(\mathcal{Q})$.

After summarizing this discussion, we illustrate it with two examples.

Proposition 1 *Fix a problem* $\mathcal{Q} = (N, A, u)$.
i) Goods: the Competitive utility profile $F^c(\mathcal{Q})$ *is the unique maximizer of the Nash product in* $\Psi(\mathcal{Q})$.
ii) Bads: the Competitive disutility profiles in $F^c(\mathcal{Q})$ *are precisely all the critical points of the Nash product in* $\Psi^{eff}(\mathcal{Q})$.

Statement *i)* is well known and goes back to Eisenberg and Gale ([8]). Statement *ii)* is new.

Example 1: Two agents and three items: $u = \begin{array}{cccc} & a & b & c \\ u_1 & 2 & 1 & 4 \\ u_2 & 1 & 1 & 5 \end{array}$. The unique competitive allocations are depicted on Figure 1:

$$\text{goods: } \begin{array}{cccc} & a & b & c \\ z_1 & 1 & 1 & 1/8 \\ z_2 & 0 & 0 & 7/8 \end{array} \qquad \text{bads: } \begin{array}{cccc} & a & b & c \\ z_1 & 0 & 0 & 7/10 \\ z_2 & 1 & 1 & 3/10 \end{array}$$

Example 2: Two agents and two items: $u = \begin{array}{ccc} & a & b \\ u_1 & 1 & 2 \\ u_2 & 3 & 1 \end{array}$. The Competitive allocations of goods simply gives b to 1 and a to 2. We have three competitive divisions of bads, depicted in Figure 2: in two of them one agent gets exactly his or her Fair Share, as in the example discussed in the Introduction.

4.2 Independence of lost bids

Think of u_{ia} as agent i's bid for item a, which is "lost" if she ends up not consuming any. The new axiom we introduce now means that when we revise a losing bid and the bid remains losing, the allocation selected by the rule does not change. It is an incentive property inasmuch as a misreport on an item that we do not end up consuming is "cheap": it is presumably harder to verify *ex post* my marginal utility/disutility for that item than for an item I am actually eating. This axiom yields a simple joint characterization of both Competitive rules.

Under the Egalitarian rule if an agent i knows that upon reporting her true preference she will not consume a certain good a, $z_{ia} = 0$, she has a transparent strategic manipulation by raising her *losing bid* u_{ia} to u'_{ia} while making sure that this new bid remains losing, $z'_{ia} = 0$: she needs then to be compensated for the resulting loss in the relative value of her "true" share. Similarly in the case of bads, she benefits by lowering a losing bid as long as it remains losing.

That no such manipulation is possible under the Competitive rule is clear from systems (4), (5). If $z \in f^c(\mathcal{Q})$ in the problem $\mathcal{Q} = (N, A, u)$ with goods A and $z_{ia} = 0$, then z is still a solution of (4) when we (only) lower u_{ia} to u'_{ia}. Similarly if we divide bads, an allocation z such that $z_{ia} = 0$ remains in $f^c(\mathcal{Q})$ when we increase u_{ia}.

Definition 4 *The rule* f *is Independent of Lost Bids (ILB) if for any two problems* $\mathcal{Q}, \mathcal{Q}'$ on N, A *where* u, u' *differ only in the entry* ia*, and such that*

$u_{ia} > u'_{ia}$ (goods) or $u_{ia} < u'_{ia}$ (bads), *we have*

$$\forall z \in f(\mathcal{Q}) : z_{ia} = 0 \Longrightarrow z \in f(\mathcal{Q}') \tag{6}$$

We combine ILB with either one of two uncontroversial fairness properties. **Equal Treatment of Equals** (ETE) is the universal requirements that the rule should not discriminate between two agents with identical characteristics, in this case identical preferences: for all $\mathcal{Q}$ and $i, j \in N$

$$u_i = u_j \Longrightarrow U_i = U_j \text{ for all } U \in F(\mathcal{Q})$$

Fair Share Guarantee (FSG) : the default option of consuming a fair share of each item sets a lower bound on individual welfare; for all $\mathcal{Q}$, $i \in N$ and $U \in F(\mathcal{Q})$

$$U_i \geq u_i \cdot (\frac{1}{n} e^A) \text{ (goods); } U_i \leq u_i \cdot (\frac{1}{n} e^A) \text{ (bads)}$$

Proposition 2 *(goods or bads)*
If a division rule meets Efficiency, Equal Treatment of Equals and/or Fair Share Guaranteed, and Independence of Lost Bids, it contains the Competitive rule.

The Competitive rule for goods is characterized by adding single-valuedness to the above requirements.

The statement is tight. The Egalitarian rule only fails ILB. The *Equal Division* rule ($F_i(\mathcal{Q}) = u_i \cdot (\frac{1}{n} e^A)$ for all $\mathcal{Q}$) only fails Efficiency. Finally the *Competitive Equilibrium with Fixed Income Shares* (replacing in Definition 1 the common budget of 1 by fixed budgets β_i, independent of preferences) only fails both ETE and FSG when the shares are not equal.

We show in Section 7.2 that ILB is a strictly weaker requirement than Maskin Monotonicity in our preference domain, thus connecting Proposition 2 to earlier results mentioned in Section 2.

4.3 The multiplicity issue

We start with two examples, one with goods and the other with bads, where the contrast between Egalitarian and Competitive rules is especially stark. In the case of bads the number of Competitive allocations is particularly large.

The canonical example with goods We have n agents and $(n-1)$ goods. The first $(n-1)$ agents are *single-minded*: agent $i, 1 \leq i \leq n-1$, likes only good a_i. Agent n is *flexible*, he likes all goods equally:

good	a_1	a_2	$\cdots$	a_{n-1}
u_1	1	0	0	0
u_2	0	1	0	0
$\cdots$	0	0	1	0
u_{n-1}	0	0	0	1
u_n	1	1	1	1

The competitive price is $\frac{n}{n-1}$ for every good: each single-minded agent buys $\frac{n-1}{n}$ units of "his" good while the flexible agent gets a $\frac{1}{n}$-th share of each good

$$z_i = \frac{n-1}{n} e^{a_i} \text{ for } 1 \leq i \leq n-1 \text{ ; } z_n = \frac{1}{n} e^A \tag{7}$$

This is tough on the flexible agent who gets his fair share and no more, while everybody else gets $(n-1)$ times more! The reason is that in this example the Competitive allocation is the only one in the core from equal split: the coalition of all single-minded agents does not need agent n to achieve its competitive surplus.

The Egalitarian rule splits each good i equally between agent i and agent n so that everyone ends up with a share worth one half of the entire manna, much above her Fair Share.

$$z_i = \frac{1}{2} e^{a_i} \text{ for } 1 \leq i \leq n-1 \text{ ; } z_n = \frac{1}{2} e^A$$

Here we submit that the Egalitarian allocation gives *too much* to agent n, who gets (much) more than his fair share of *every* good. By contrast at a Competitive allocation, *here and always*, everyone gets *at most* a $\frac{1}{n}$-th share of *at least* one good: for all i we have $\min_{a \in A} z_{ia} \leq \frac{1}{n}$.[9]

For fixed sizes of N and A it would be interesting to understand at which problems the ℓ_∞ or ℓ_1 distance between the profiles of normalized utilities at the competitive and egalitarian allocations, is the largest possible. The canonical example may be a step toward the answer.

The canonical example with bads Now we turn the above example upside down as

bad	a_1	a_2	$\cdots$	a_{n-1}
u_1	1	γ	γ	γ
u_2	γ	1	γ	γ
$\cdots$	γ	γ	1	γ
u_{n-1}	γ	γ	γ	1
u_n	1	1	1	1

where $1 < \gamma < \infty$

If γ is large each agent i, $1 \leq i \leq n-1$ is single-minded in her preference for bad a_i. Even if γ is barely above 1, the allocation (7) where the flexible agent gets no relief from eating $\frac{1}{n}$-th of each bad is competitive, for the same uniform price $\frac{n}{n-1}$.

However in this example there is in total $2^{n-1}-1$ Competitive allocations, all different welfare-wise. Recall the notation e^S for the vector in $\mathbb{R}^A$ with $e_i^S = 1$

[9] If $z_{ia} > \frac{1}{n}$ for all a the competitive price must be parallel to u_i (or eating of each good would not be a competitive demand) and the equal budget condition $p \cdot z_i = p \cdot (\frac{1}{n} e^A)$ gives $u_i \cdot z_i = u_i \cdot (\frac{1}{n} e^A)$, contradiction. If we divide bads every Competitive allocation satisfies similarly $\max_{a \in A} z_{ia} \geq \frac{1}{n}$.

if $i \in S$ and zero otherwise. Pick an integer $q, 1 \leq q \leq n-1$ and check that the allocation

$$z_i = \frac{q}{q+1} e^{a_i} \text{ for } 1 \leq i \leq q \text{ ; } z_j = e^{a_j} \text{ for } q+1 \leq j \leq n-1 \text{ ; } z_n = \frac{1}{q+1} e^{\{a_1, \cdots, a_q\}} \tag{8}$$

is Competitive for the prices $p_{a_i} = \frac{q+1}{q}$ for $1 \leq i \leq q$ and $p_{a_j} = 1$ for $q+1 \leq j \leq n-1$. In particular agent n's disutility varies from $\frac{1}{2}$ to $\frac{n-1}{n}$. We could have chosen any subset S of bads with size q, which proves the claim.

Note that the Egalitarian allocation depends heavily upon γ, the extent to which the first $n-1$ agents prefer "their" bad:

$$z_i = \frac{1+(n-2)\gamma}{2+(n-2)\gamma + \frac{1}{n-1}} e^{a_i} \text{ for } 1 \leq i \leq n-1 \text{ ; } z_n = \frac{n}{(n-1)(2+(n-2)\gamma + \frac{1}{n-1})} e^A$$

As γ grows agent n eats a vanishingly small fraction of every bad.

Proposition 3 *For any problem $\mathcal{Q}$ with bads*:
i) The number of Competitive allocations distinct welfare-wise is finite.
ii) For general $n, m = |A|$ it can be as high as $2^{\min\{n,m\}} - 1$ if $n \neq m$, and $2^{n-1} - 1$ if $n = m$.
iii) For $n = 2$ t*he upper bound on this number is* $2m - 1$.
iv) For $m = 2$ t*he upper bound on this number is* $2n - 1$.

We offer no guess about the upper bound on the number of distinct Competitive allocations for general n, m.

Proof of statement *i*). Recall that (in a non trivial problem) the disutility profile U of each Competitive allocation is strictly positive, and achieved by some allocation z such that $\Gamma(z)$ is a forest. There are finitely many (bipartite) forests in $N \times A$ therefore it is enough to check that to each forest Γ corresponds at most one Competitive allocation disutility-wise. Consider a tree T in Γ with vertices N_0, A_0. If agents $i, j \in N_0$ are both linked to $a \in A_0$, system (5) implies that U_i, U_j are proportional to u_{ia}, u_{ja} . Repeating this observation along the paths of T we see that the profile $(U_i)_{i \in N_0}$ is determined up to a multiplicative constant. Now in total the agents in N_0 consume exactly A_0 so by efficiency we cannot have two distinct $(U_i)_{i \in N_0}$ meeting (5).[10]

Proof of statement *ii*). In the case $n > m$ we adapt the canonical example as follows. For agent $i, 1 \leq i \leq m$, set as before $u_{ia_i} = 1, u_{ia_j} = \gamma$ for $j \neq i$, and for agents $m+1$ to n pick $u_{ia} = 1$ for all a. Then for any $q, 1 \leq q \leq n-1$, the allocation

$$z_i = \frac{m}{n} e^{a_i} \text{ for } 1 \leq i \leq q \text{ ; } z_j = e^{a_j} \text{ for } q+1 \leq j \leq m \text{ ; } z_j = \frac{1}{n} e^{\{a_1, \cdots, a_q\}} \text{ for } m+1 \leq j \leq n$$

[10] Note that the finiteness result holds even if we drop requirement (3) in Definition 3 but still insist that a competitive allocation be efficient. If A^0 is the set of bads a such that $u_{ia} = 0$ for some i, then some items in A^0 can have a positive price, and be eaten by agents who do not mind them, eat only in A^0, and enjoy a disutility of zero; while the other bads in A^0 have zero price, are also eaten by agents who do not mind them but those agents eat also some real bads in $A \diagdown A^0$. For each such partition of A^0 there are finitely many competitive disutility profiles.

generalizing (8), is a Competitive allocation as before.

For statement *ii*) in the case $m = n + 1$ we use the following example:

bad	a_1	$\cdots$	$\cdots$	a_n	a_{n+1}
u_1	1	γ	γ	γ	1
$\cdots$	γ	1	γ	γ	1
$\cdots$	γ	γ	1	γ	1
u_n	γ	γ	γ	1	1

where $1 < \gamma < \infty$

For any subset of agents $N^* \subseteq N$ the allocation where those agents share equally the bad $n+1$, while bad $a_i, 1 \leq i \leq n$ goes to agent i, is a Competitive allocation with prices $p_{a_{n+1}} = p_{a_i} = \frac{n^*}{n^*+1}$ for $i \in N^*$, $p_{a_j} = 1$ for $j \in N \diagdown N^*$.

For the case $m > n$ we take a disutility matrix with $m - n$ copies of the last column. We omit the details as well as the easy argument for the case $n = m$. ■

The longer proofs of statements *iii*) and *iv*) are in the Appendix. They rely on the fact that for $n = 2$ a problem is entirely described by the sequence of ratios $\frac{u_{1a}}{u_{2a}}$, and for $m = 2$ by the sequence of ratios $\frac{u_{ia}}{u_{ib}}$: this allows a closed form description of all Competitive allocations.

A by-product of these proofs is that, on an open dense subset of the problems where $n = 2$ and/or $m = 2$, the number of different competitive allocations is odd.[11] A very plausible conjecture is that this is true as well for any n, m.

5 Envy-free division of bads: discontinuity

We derive our main impossibility result from a careful analysis of the set $\mathcal{A}$ of efficient and envy-free allocations in problems with two bads a, b, and any number of agents. This is tractable because as we just noted, a problem is entirely described by the sequence of ratios $\frac{u_{ia}}{u_{ib}}$.

We give a simple numerical example where $\mathcal{A}$ "jumps" from one to three connected components:

bads	a	b
u_1	1	4
u_2	1	$4 - \varepsilon$
u_3	$4 - \varepsilon$	1
u_4	4	1

. Start with the case $\varepsilon = 0$. No Envy implies $U_1 = U_2$ and $U_3 = U_4$, and we have essentially a two-person problem. There is a symmetric Competitive allocation z^2 where agents $1, 2$ share a while $3, 4$ share b, so that $U^2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$; and two asymmetric ones z^1 and z^3. At z^1 the price is $p^1 = (\frac{2}{5}, \frac{8}{5})$ agents $1, 2$ get their Fair Share disutility while $3, 4$ keep all the surplus, and we get $U^1 = (\frac{5}{4}, \frac{5}{4}, \frac{5}{16}, \frac{5}{16})$; and the third allocation z^3 by exchanging the roles of $1, 2$ and $3, 4$. Here $\mathcal{A}$ is connected, and the corresponding disutility profiles cover the union of intervals $[U^1, U^2] \cup [U^2, U^3]$.

[11] Moreover the Competitive allocations are naturally ordered, and the median allocation is a natural selection from $F^c(\mathcal{Q})$. However this selection has no particular normative justification.

Now if ε is positive and small, it is easy to check that the Competitive allocations are unchanged; however $\mathcal{A}$ is now made of three connected components. Note that efficiency implies that at most one agent eats both bads. If agent 1 eats both, an envy-free allocation is $z_1 = (0, y), z_2 = z_3 = z_4 = (0, \frac{1-y}{3})$ where (in the first order) $y \in [\frac{1}{16}(1 - \frac{3\varepsilon}{4}), \frac{1}{16}]$, so this interval $I^1$of $\mathcal{A}$ is very close to the competitive allocation z^1. If agent 2 eats both, an envy-free allocation is $z_1 = (1-x, 0), z_2 = (x, y), z_3 = z_4 = (0, \frac{1-y}{2})$. The corresponding subset of $\mathcal{A}$ is now a triangle in (x, y) that is disjoint from I^1 because $2x + 4y \geq 1$ (1 does not envy 2), and $8x + 3y \geq 1$ (3 does not envy 2), so that $x \in [\frac{1}{26}, \frac{1}{2}]$; one end of this quasi-interval I^2 of $\mathcal{A}$ is the competitive allocation z^2. The other cases are symmetric and yield I^3 connected to I^2 and the third component I^4.

Proposition 4 *If we divide at least two bads between at least three agents, there are problems $\mathcal{Q}$ where the set $\mathcal{A}$ of efficient and envy-free allocations, and the corresponding set of disutility profiles, have $\lfloor \frac{2n+1}{3} \rfloor$ connected components.*

If we divide goods or bads between *two* agents, No Envy coincides with Fair Share Guarantee, so the set $\mathcal{A}$ is clearly connected.

If we divide two *goods,* one can easily check that $\mathcal{A}$ is connected, whereas in the case of two bads, the maximal number of components is indeed $\lfloor \frac{2n+1}{3} \rfloor$ (this is clear from the proof of the Proposition in Section 7.5).

Beyond these simple cases we do not know if $\mathcal{A}$ remains connected for any number of goods, and we risk no guess about the maximal number of components in the case of bads.

Definition 5

We call the division rule f Continuous if for each choice of N, A, and $i \in N$, the disutility $U_i = u_i \cdot f_i(N, A, u)$ is a continuous function of $u \in \mathbb{R}_+^{N \times A}$. We call the rule f Efficient (resp. Envy-Free) if it selects an efficient (resp. an envy-free) allocation for every problem $\mathcal{Q}$.

Theorem 1 *If we divide at least two bads between at least four agents, no single-valued division rule can be Efficient, Envy-Free and Continuous.*

This incompatibility result is tight. The equal division of all the bads, irrespective of disutility functions, is Envy-Free and Continuous. A single-valued selection of the Competitive correspondence is Efficient and Envy-free. The Egalitarian rule is Efficient and Continuous (by Berge Theorem).

We submit that understanding the optimal trade-offs between Efficiency and No Envy in our model is the most important step toward turning the idea of equal opportunities into concrete division rules.

6 Resource Monotonicity

More goods or fewer bads to divide should not be bad news to anyone: all agents "own" the goods/bads equally and welfare should be comonotonic to ownership. This simple normative property has played a major role in the modern fair division literature (see Section 2). When it fails someone has an incentive to sabotage the discovery of additional goods, or bring new bads to the table.

It is easy to check that the Egalitarian rule, for goods or for bads, is not Resource Monotonic among three or more agents. Compare the following two problems with $A^1 = \{a, b, c\}$and $A^2 = \{a, b, c, d\}$ respectively:

$$\begin{array}{cccc} & a & b & c \\ u_1 & 3 & 1 & 1 \\ u_2 & 1 & 3 & 1 \\ u_3 & 1 & 1 & 3 \end{array} \quad \text{and} \quad \begin{array}{ccccc} & a & b & c & d \\ u_1 & 3 & 1 & 1 & 0 \\ u_2 & 1 & 3 & 1 & 4 \\ u_3 & 1 & 1 & 3 & 4 \end{array}$$

The A^1-problem is symmetric. Any efficient and symmetric rule allocates goods "diagonally": agent 1 gets all of a and so on; normalized utilities are $\frac{3}{5}$. In the A^2-problem the natural idea is to keep the same allocation of a, b, c and divide d equally between agents 2 and 3, because agent 1 does not care for d. This is what the Competitive rule recommends (prices are $(1, \frac{3}{5}, \frac{3}{5}, \frac{4}{5})$). But the normalized utilities at this allocation are $(\frac{3}{5}, \frac{5}{9}, \frac{5}{9})$, so the Egalitarian rule must compensate agents $2, 3$ for the *loss* in normalized utilities caused by the *gain* of some new good! Equality is restored at the allocation

$$z^e = \begin{array}{cccc} a & b & c & d \\ 55/59 & 0 & 0 & 0 \\ 2/59 & 1 & 0 & 1/2 \\ 2/59 & 0 & 1 & 1/2 \end{array}$$

where agent 1's welfare has decreased.

In the following definition we write $u_{[B]}$ for the restriction to $\mathbb{R}_+^{N \times B}$ of the util/disutil matrix $u \in \mathbb{R}_+^{N \times A}$:

Resource Monotonicity (RM): for all $\mathcal{Q} = (N, A, u)$ and all $B \subset A$

$$F(N, B, u_{[B]}) \leq F(\mathcal{Q}) \text{ (goods or bads)} \tag{9}$$

(going from A to B is bad news if we deal with goods, and good news if with bads).

Theorem 2
i) The Competitive rule to divide goods is Resource Monotonic.
ii) With three or more agents and two or more bads, no efficient single-valued rule to divide bads can meet Resource Monotonicity and Fair Share Guarantee.

Recall from [20] that in any domain containing the Leontief preferences, we cannot divide **goods** efficiently while ensuring FSG and RM (this is true even with two agents and two goods). This makes the goods versus bads contrast in the case of linear preferences all the more intriguing.

Statement *i*) is an easy consequence of Lemmas 1 and 2: see Section 7.7 for details. The proof of statement *ii*) hinges on a simple two-person, two-bad example. Suppose the rule F meets all properties listed there and consider the problem $\mathcal{Q}$: $\begin{array}{ccc} \text{bads} & a & b \\ u_1 & 1 & 4 \\ u_2 & 4 & 1 \end{array}$. Set $U = F(\mathcal{Q})$ and observe that $(1, 1)$ is an efficient disutility profile so one U_i is ≤ 1, say $U_1 \leq 1$.

Consider $\mathcal{Q}'$:

bads	$\frac{1}{9}a$	b
u_1	1/9	4
u_2	4/9	1

(where we treat $\frac{1}{9}a$ as a whole bad) and pick $z' \in f(\mathcal{Q}')$. By FSG and feasibility:

$$z'_{2b} \leq u_2 \cdot z'_2 \leq u_2 \cdot (\frac{1}{2}e^{A'}) = \frac{13}{18}$$

$$\Longrightarrow z'_{1b} \geq \frac{5}{18} \Longrightarrow u_1 \cdot z'_1 = U'_1 \geq \frac{10}{9} > U_1$$

contradicting RM. Details of the proof for general $n \geq 3, m \geq 2$ are in Section 7.7.

Remark As mentioned in Section 2 Sziklai and Segal-Halevi ([29]) prove that the Competitive solution is Resource Monotonic in the general cake-division problem, which implies statement i) in Theorem 2. They show that as the cake increases the (normalized) price of the old cake goes down, a different proof technique than ours.

7 Appendix: Proofs

7.1 Lemma 1

7.1.1 $a)$ the consumption forest at efficient allocations

Fix a problem $\mathcal{Q}$. We can clearly assume that the matrix u has no null row or column. To fix ideas we think of the items as goods but the proof is identical for bads. Pick z representing $U \in \Psi^{eff}(\mathcal{Q})$ and assume there is a K-cycle in $\Gamma(z)$: $z_{ka_k}, z_{ka_{k-1}} > 0$ for $k = 1, \cdots, K$, where $a_0 = a_K$. Then $u_{ka_k}, u_{ka_{k-1}}$ are positive for all k: if $u_{ka_k} = 0$ efficiency and $\sum_{i \in N} u_{ia_k} > 0$ imply $z_{ka_k} = 0$. Assume now

$$\frac{u_{1a_1}}{u_{2a_1}} \cdot \frac{u_{2a_2}}{u_{3a_2}} \cdot \cdots \cdot \frac{u_{(K-1)a_{K-1}}}{u_{Ka_{K-1}}} \cdot \frac{u_{Ka_K}}{u_{1a_K}} > 1 \tag{10}$$

Then we can pick arbitrarily small positive numbers ε_k such that

$$\frac{u_{1a_1} \cdot \varepsilon_1}{u_{1a_K} \cdot \varepsilon_K} > 1, \frac{u_{2a_2} \cdot \varepsilon_2}{u_{2a_1} \cdot \varepsilon_1} > 1, \cdots, \frac{u_{Ka_K} \cdot \varepsilon_K}{u_{Ka_{K-1}} \cdot \varepsilon_{K-1}} > 1 \tag{11}$$

and the corresponding transfer to each agent k of ε_k units of good k against ε_{k-1} units of good $k-1$ is a Pareto improvement, contradiction. Therefore (10) is impossible; the opposite strict inequality is similarly ruled out so we conclude

$$\frac{u_{1a_1}}{u_{2a_1}} \cdot \frac{u_{2a_2}}{u_{3a_2}} \cdot \cdots \cdot \frac{u_{(K-1)a_{K-1}}}{u_{Ka_{K-1}}} \cdot \frac{u_{Ka_K}}{u_{1a_K}} = 1 \tag{12}$$

Now if we perform a transfer as above where

$$\frac{u_{1a_1} \cdot \varepsilon_1}{u_{1a_K} \cdot \varepsilon_K} = \frac{u_{2a_2} \cdot \varepsilon_2}{u_{2a_1} \cdot \varepsilon_1} = \cdots = \frac{u_{Ka_K} \cdot \varepsilon_K}{u_{Ka_{K-1}} \cdot \varepsilon_{K-1}} = 1$$

the utility profile U is unchanged. If we choose the numbers ε_k as large as possible for feasibility, this will bring at least one entry (k, a_k) or (k, a_{k-1}) to zero, so in our new representation z' of U the graph $\Gamma(z')$ has fewer edges. We can clearly repeat this operation until we eliminate all cycles of $\Gamma(z)$.

The last statement follows at once from the fact that a forest with $n+m$ vertices contains at most $n+m-1$ edges.

7.1.2 *b*) at almost all profiles each efficient utility profile is achieved by a single allocation

We let $\mathcal{U}^*(N, A)$ be the open and dense subset of $\mathbb{R}_+^{N\times A}$ such that for any cycle $\mathcal{C} = \{1, a_1, 2, a_2, \cdots, a_K, 1\}$ in the complete bipartite graph $N \times A$ we have $\pi(\mathcal{C}) = \frac{u_{1a_1}}{u_{2a_1}} \cdot \frac{u_{2a_2}}{u_{3a_2}} \cdot \cdots \cdot \frac{u_{(K-1)a_{K-1}}}{u_{Ka_{K-1}}} \cdot \frac{u_{Ka_K}}{u_{1a_K}} \neq 1$ (property (12) fails) and moreover $u_{ia} > 0$ for all i, a. It is clearly an open dense subset of $\mathbb{R}_+^{N\times A}$.

We pick a problem $\mathcal{Q}$ with $u \in \mathbb{R}_+^{N\times A}$, fix $U \in \Psi^{eff}(\mathcal{Q})$ and assume there is two different $z, z' \in \Phi(N, A)$ such that $u \cdot z = u \cdot z' = U$. There must be some pair $1, a_1$ such that $z_{1a_1} > z'_{1a_1}$. Because a_1 is eaten in full there is some agent 2 such that $z_{2a_1} < z'_{2a_1}$ and because $u_2 \cdot z_2 = u_2 \cdot z'_2$ there is some good a_2 such that $z_{2a_2} > z'_{2a_2}$. Continuing in this fashion we build a sequence $1, a_1, 2, a_2, 3, a_3, \cdots$, such that $\{z_{ka_{k-1}} < z'_{ka_{k-1}}$ and $z_{ka_k} > z'_{ka_k}\}$ for all $k \geq 2$. This sequence must cycle, i. e., we must reach K, a_K such that $z_{Ka_K} > z'_{Ka_K}$ and $z_{\widetilde{k}a_K} < z'_{\widetilde{k}a_K}$ for some $\widetilde{k}, 1 \leq \widetilde{k} \leq K-1$. Without loss we label $\widetilde{k}$ as 1, and the corresponding cycle as $\mathcal{C}$.

From the reasoning above it follows that for $z'' = (z + z\prime)/2$, an efficient allocation, there is a cycle in $\Gamma(z'')$. Then the argument in Section a). implies that u is not in $\mathcal{U}^*(N, A)$, as was to be proved.

7.2 Proposition 2

We already checked that the rule f^c meets ILB; also ETE and FSG are clear. Conversely we fix f meeting EFF, ETE or FSG, and ILB and an arbitrary problem $\mathcal{Q} = (N, A, u)$, where items are goods or bads. Without loss of generality we assume that the matrix u has no null row or column. In the proof we consider several problems (N, A, v) where v varies in $\mathbb{R}_+^{N\times A}$, and for simplicity we write $f(v)$ in lieu of $f(N, A, v)$.

We pick $z \in f^c(u)$ and check that $z \in f(u)$ as well. Set $U_i = u_i \cdot z_i$ and let p be the competitive price at z. In the proof of Lemma 2 we saw that $p_a = \frac{u_{ia}}{U_i}$ for all i such that $z_{ia} > 0$, and for all j we have $p_a \geq \frac{u_{ja}}{U_j}$ (resp. $p_a \leq \frac{u_{ja}}{U_j}$) if we divide goods (resp. bads). Moreover $p \cdot z_i = 1$ for all i, and $p \cdot e^A = n$.

Consider the problem $\mathcal{Q}^* = (N, A, w)$ where $w_i = p$ for all i. The equal split allocation is efficient in $\mathcal{Q}^*$ therefore ETE implies $F(w) = e^N$ and so does FSG, because $p \cdot (\frac{1}{n}e^A) = 1$. Now if we set $\widetilde{w}_i = U_i p$ the scale invariance property of F (Definition 1) gives $F(\widetilde{w}) = U$; moreover $z \in f(\widetilde{w})$ because $\widetilde{w}_i \cdot z = U_i$ for all i. If $z_{ia} > 0$ we have $u_{ia} = U_i p_a = \widetilde{w}_{ia}$; if $z_{ia} = 0$ we have similarly $u_{ia} \leq \widetilde{w}_{ia}$

for the goods case, or $u_{ia} \geq \widetilde{w}_{ia}$ if bads. Apply finally ILB: after lowering (for goods) or raising (for bads) every lost bid $\widetilde{w}_{ia}$ to u_{ia}, the allocation z is still in $f(u)$, as desired. ■

Finally we check that ILB is in fact a weaker form of Maskin Monotonicity (MM). We do this in the case of bads only, as both cases are similar. Individual allocations z_i vary in $[0,1]^A$ and utilities in $\mathbb{R}_+^A$, so the MM axiom for rule f means that for any two problems $\mathcal{Q}, \mathcal{Q}'$ on N, A and $z \in f(\mathcal{Q})$ we have

$$\{\forall i \in N, \forall w \in [0,1]^A \quad u_i \cdot z_i \leq u_i \cdot w \Longrightarrow u_i' \cdot z_i \leq u_i' \cdot w\} \Longrightarrow z \in f(\mathcal{Q}') \quad (13)$$

We fix $\mathcal{Q}, i \in N$ and $z \in f(\mathcal{Q})$, and for $\varepsilon = 0,1$ we write $A^\varepsilon = \{a | z_{ia} = \varepsilon\}$ and $A^+ = A \diagdown (A^0 \cup A^1)$. The implication in the premises of (13) reads

$$\forall w \in [0,1]^A \ u_i \cdot (w - z_i) \geq 0 \Longrightarrow u_i' \cdot (w - z_i) \geq 0$$

The cone generated by the vectors $w - z_i$ when w covers $[0,1]^A$ is $C = \{\delta \in \mathbb{R}^A | \delta_a \geq 0 \text{ for } a \in A^0,\ \delta_a \leq 0 \text{ for } a \in A^1\}$. By Farkas Lemma the implication $\{\forall \delta \in C : u_i \cdot \delta \geq 0 \Longrightarrow u_i' \cdot \delta \geq 0\}$ means that, up to a scaling factor,

$$u_{ia}' = u_{ia} \text{ on } A^+ \ ; \ u_{ia}' \geq u_{ia} \text{ on } A^0 \ ; \ u_{ia}' \leq u_{ia} \text{ on } A^1$$

Thus MM says that after lowering a lost bid, or increasing one that gets the whole bad, the initial allocation will remain in the selected set. Now ILB only considers raising a lost bid, so it is only "half" of MM. Naturally both Competitive rules meet both parts of MM.

7.3 Proposition 3, *Statement* iii) and oddness of $|F^c(\mathcal{Q})|$

We fix a problem $(\{1,2\}, A, u)$. We label the bads $k \in \{1, \cdots, m\}$ so that the ratios $\frac{u_{1k}}{u_{2k}}$ increase weakly in k. A harmless bad k ($u_{ik} = 0$ for $i = 1,2$) plays no role, so the ratios are well defined with the convention $\frac{1}{0} = \infty$.
Step 1. Suppose $\frac{u_{1k}}{u_{2k}} = \frac{u_{1(k+1)}}{u_{2(k+1)}}$. Then in any Competitive allocation with price p we have

$$\frac{p_k}{p_{k+1}} = \frac{u_{ik}}{u_{i(k+1)}} \text{ for } i = 1,2 \quad (14)$$

Indeed if one of $i = 1,2$ eats both k and $k+1$, (14) follows by the linearity of preferences, If on the contrary i eats bad k and j eats bad $k+1$, then (5) gives $\frac{u_{ik}}{p_k} \geq \frac{u_{i(k+1)}}{p_{k+1}}$ and $\frac{u_{j(k+1)}}{p_{k+1}} \geq \frac{u_{jk}}{p_k}$.

So for a given amount of money spent by i on bads k and $k+1$, she gets the same disutility no matter how she splits this expense between the two bads. Hence there is an interval of Competitive allocations obtained by shifting the consumption of k and $k+1$ while keeping the total expense on these two bads fixed for each agent. They all give the same disutility profile and use the same price. So if we merge k and $k+1$ into a bad k^* with disutilities $u_{ik^*} = u_{ik} + u_{i(k+1)}$, all the Competitive allocations of the initial problem become a single Competitive allocation for the new price $p_{k^*} = p_k + p_{k+1}$, with p unchanged elsewhere. By successively merging all the bads sharing the same ratio $\frac{u_{1k}}{u_{2k}}$, we do not change the number of Competitive allocations distinct disutility-wise, and reach a problem with fewer bads where the ratios $\frac{u_{1k}}{u_{2k}}$ increase strictly in k. So we only need to prove the statement in this case.

Step 2. Efficiency means that if 1 eats some k and 2 some k', then $k \leq k'$. In particular the agents split at most one bad, and efficient allocations z are of two types. In a $k/k+1$-*cut* 1 eats all bads $\ell \in \{1, \cdots, k\}$ while 2 eats all $\ell \in \{k+1, \cdots, m\}$; in a k-*split* 1 and 2 share bad k, while bads $\ell \in \{1, \cdots, k-1\}$ go to 1, and $\ell \in \{k+1, \cdots, m\}$ to 2.

By Definition 3 if the $k/k+1$-*cut* is Competitive, the corresponding (normalized) price is

$$p_\ell = \frac{u_{1\ell}}{U_1(k)} \text{ for } \ell \leq k \; ; \; p_\ell = \frac{u_{2\ell}}{U_2(k+1)} \text{ for } \ell \geq k+1$$

where we use the notation $U_1(k) = \sum_1^k u_{1\ell}$, $U_2(k) = \sum_k^m u_{2\ell}$. System (5) has two parts: agent 1's demand at this price is $e^{\{1,\cdots,k\}}$:

$$\{\frac{u_{1\ell}}{p_\ell} \leq \frac{u_{1\ell'}}{p_{\ell'}} \text{ for all } \ell \leq k < k+1 \leq \ell'\} \Longleftrightarrow \frac{U_1(k)}{U_2(k+1)} \leq \frac{u_{1(k+1)}}{u_{2(k+1)}}$$

and agent 2's demand is $e^{\{k+1,\cdots,m\}}$:

$$\{\frac{u_{2\ell'}}{p_{\ell'}} \leq \frac{u_{2\ell}}{p_\ell} \text{ for all } \ell \leq k < k+1 \leq \ell'\} \Longleftrightarrow \frac{u_{1k}}{u_{2k}} \leq \frac{U_1(k)}{U_2(k+1)}$$

Thus the $k/k+1$-cut is competitive if and only

$$\frac{u_{1k}}{u_{2k}} \leq \frac{U_1(k)}{U_2(k+1)} \leq \frac{u_{1(k+1)}}{u_{2(k+1)}} \tag{15}$$

We turn to a k-*split* allocation. If it is competitive by Definition 3 the corresponding *non normalized* price is

$$p_\ell = \frac{u_{1\ell}}{u_{1k}} \text{ for } \ell \leq k-1 \; ; \; p_k = 1 \; ; \; p_\ell = \frac{u_{2\ell}}{u_{2k}} \text{ for } \ell \geq k+1$$

and the inequalities $\frac{u_{1\ell}}{p_\ell} \leq \frac{u_{1\ell'}}{p_{\ell'}}$ and $\frac{u_{2\ell'}}{p_{\ell'}} \leq \frac{u_{2\ell}}{p_\ell}$ hold by construction for $\ell \leq k \leq \ell'$. To reach a Competitive allocation it suffices to find a split of bad k for which z_1 and z_2 have the same cost:

$$\{\sum_1^{k-1} p_\ell + x = 1 - x + \sum_{k+1}^{m} p_{\ell'} \text{ for some } x \in]0,1[\} \Longleftrightarrow |\frac{U_1(k-1)}{u_{1k}} - \frac{U_2(k+1)}{u_{2k}}| < 1 \tag{16}$$

Note that for $k = 1$ inequalities (16) reduce to $U_2(2) < u_{21}$, and for $k = m$ to $U_1(m-1) < u_{1m}$.

In particular there is at most one k-split Competitive allocation for each k, so the largest conceivable number of Competitive allocations is $2m-1$, because the maximal number of $k/k+1$-cuts and k-split allocations is respectively $m-1$ and m.

Step 3. We use the notation $(x)_+ = \max\{x, 0\}$ to give an example where this bound is achieved:

$$u_{1k} = 2^{(k-2)_+} \text{ for } 1 \leq k \leq m-1 \text{ ; } u_{1m} = 2^{m-2} + 1$$

$$u_{21} = 2^{m-2} + 1 \text{ ; } u_{2k} = 2^{(m-1-k)_+} \text{ for } 2 \leq k \leq m$$

Check first $U_1(k-1) = u_{1k}$ and $U_2(k+1) = u_{2k}$ for $2 \leq k \leq m-1$; also $U_2(2) = U_1(m-1) = 2^{m-2} < u_{21} = u_{1m}$ so (16) holds for all k. Next $\frac{U_1(k)}{U_2(k+1)} = \frac{u_{1(k+1)}}{u_{2k}}$ for $2 \leq k \leq m-2$, so that (15) is clear for such k. And (15) holds as well for $k = 1, m-1$.

This example is clearly robust: small perturbations of the disutility matrix preserve the number of competitive allocations.

Step 4. Oddness of $|F^c(\mathcal{Q})|$. Recall that $F^c(\mathcal{Q})$ is the set of critical points of the Nash product in $\Psi^{eff}(\mathcal{Q})$.The $k/k+1$-cut allocations are the extreme points of the set of feasible allocations $\Phi(N, A)$, and their utility profiles are the extreme points[12] of $\Psi^{eff}(\mathcal{Q})$ if the ratios $\frac{u_{1k}}{u_{2k}}$ increase strictly in k. Excluding the set of utility profiles u such that $|\frac{U_1(k-1)}{u_{1k}} - \frac{U_2(k+1)}{u_{2k}}| = 1$ (see (16)), it follows that the $k/k+1$ cut is competitive if and only if it is a local minimum of $\mathcal{N}$. On the other hand the utility profile of a k-split allocation is on a one-dimensional face of $\Psi^{eff}(\mathcal{Q})$, and is competitive if and only if it is a local maximum of $\mathcal{N}$. Then the statement follows from the fact that if a continuous non-negative function on the interval is zero at the end-points, the number of its local maxima exceeds the number of its local minima (different than the end-points) by one: the extrema alternate and the closest to the end-points are the maxima.

Note that the above argument implies that in a typical problem with two agents, if $|F^c(\mathcal{Q})| = 1$ then the competitive allocation is a $k/k+1$-split, and if $|F^c(\mathcal{Q})| \geq 2$, at least one k-cut allocation is competitive.

7.4 Proposition 3, *Statement* iv) and oddness of $|F^c(\mathcal{Q})|$

We fix a problem $(N, \{a, b\}, u)$ and label the agents $i \in \{1, \cdots, n\}$ in such a way that the ratios $\frac{u_{ia}}{u_{ib}}$ increase weakly in i.

Step 1. Assume that the sequence $\frac{u_{ia}}{u_{ib}}$ *increases strictly.* If z is an efficient allocation, then for all i, j $\{z_{ia} > 0$ and $z_{jb} > 0\}$ implies $i \leq j$. In particular at most one agent is eating both goods, and we have two types of efficient and envy-free allocations. The $i/i+1$-*cut* $z^{i/i+1}$ is defined for $1 \leq i \leq n-1$ by: $z_j^{i/i+1} = (\frac{1}{i}, 0)$ for $j \leq i$, and $z_j^{i/i+1} = (0, \frac{1}{n-i})$ for $j \geq i+1$. Next for $2 \leq i \leq n-1$ the allocation z is an *i-split* if there are numbers x, y such that

$$z_j = (\frac{1-x}{i-1}, 0) \text{ for } j \leq i-1 \text{ ; } z_j = (0, \frac{1-y}{n-i}) \text{ for } j \geq i+1 \tag{17}$$

[12] An $U \in \Psi^{eff}(\mathcal{Q})$ is *extreme* if it is not between two other points of $\Psi^{eff}(\mathcal{Q})$.

$$z_i = (x, y) \text{ with } 0 \le x \le \frac{1}{i},\ 0 \le y \le \frac{1}{n-i+1} \tag{18}$$

Also, z is a 1-split if $z_1 = (1, y)$ and $z_j = (0, \frac{1-y}{n-1})$ for $j \ge 2$; and z is a n-split if $z_n = (x, 1)$ and $z_j = (\frac{1-x}{n-1}, 0)$ for $j \le n-1$. Note that the cut $z^{i/i+1}$ is both an i-split and an $i+1$-split.

If the sequence $\frac{u_{ia}}{u_{ib}}$ increases strictly, it is clear that an efficient and envy-free allocation must be an i-split. In the next Step we show that this is still true, welfare-wise, if that sequence increases only weakly, then we provide a full characterization in Step 3.

Step 2. Assume the sequence $\frac{u_{ia}}{u_{ib}}$ increases only weakly, for instance $\frac{u_{ia}}{u_{ib}} = \frac{u_{(i+1)a}}{u_{(i+1)b}}$. Then if z is efficient and envy-free we may have $z_{(i+1)a} > 0$ and $z_{ib} > 0$, however we can find z' delivering the same disutility profile and such that one of $z'_{(i+1)a}$ and z'_{ib} is zero. Indeed No Envy and the fact that u_i and u_{i+1} are parallel gives $u_i \cdot z_i = u_i \cdot z_{i+1}$ and $u_{i+1} \cdot z_{i+1} = u_{i+1} \cdot z_i$, from which the claim follows easily. We conclude that the i-split allocations contain, utility-wise, all efficient and envy-free allocations.

Step 3. If the cut $z^{i/i+1}$ is a Competitive allocation, the corresponding price is $p = (i, n-i)$, and the system (5) reads $\frac{u_{ja}}{i} \le \frac{u_{jb}}{n-i}$ for $j \le i$, $\frac{u_{jb}}{n-i} \le \frac{u_{ja}}{i}$ for $j \ge i+1$, which boils down to

$$\frac{u_{ia}}{u_{ib}} \le \frac{i}{n-i} \le \frac{u_{(i+1)a}}{u_{(i+1)b}} \text{ for } 1 \le i \le n-1 \tag{19}$$

Next for $2 \le i \le n-1$ if the i-split allocation z (17) is competitive, the (normalized) price must be $p = n(\frac{u_{ia}}{u_{ia}+u_{ib}}, \frac{u_{ib}}{u_{ia}+u_{ib}})$ and each agent must be spending exactly 1:

$$p_a \frac{1-x}{i-1} = p_b \frac{1-y}{n-i} = p_a x + p_b y = 1$$

which gives

$$x = \frac{1}{n u_{ia}}((n-i+1)u_{ia} - (i-1)u_{ib})\ ;\ y = \frac{1}{n u_{ib}}(i u_{ib} - (n-i)u_{ia}) \tag{20}$$

We let the reader check that these formulas are still valid when $i = 1$ or $i = n-1$.

An i-split allocation z is *strict* if it is not a cut, which happens if and only if both x, y in (17) are strictly positive. By (20), for any $i \in \{1, \cdots, n\}$ there is a strict i-split allocation that is competitive if and only if

$$\frac{i-1}{n-i+1} < \frac{u_{ia}}{u_{ib}} < \frac{i}{n-i} \tag{21}$$

(with the convention $\frac{1}{0} = \infty$).

Step 4. Counting competitive allocations. There are at most n competitive (strict) i-split allocations, and $n-1$ cuts $z^{i/i+1}$, hence the upper bound $2n-1$. An example where the bound is achieved uses any sequence $\frac{u_{ia}}{u_{ib}}$ meeting (21) for all $i \in \{1, \cdots, n\}$, as these inequalities imply (19) for all $i \in \{1, \cdots, n-1\}$.

Step 5. Oddness of $|F^c(\mathcal{Q})|$. For the utility profiles such that all the inequalities (19) and (21) are strict, we draw the two sequences $\frac{u_{ia}}{u_{ib}}$ and $\frac{i}{n-i}$ on the real line. Clearly the left-most and the right-most competitive allocations must be splits: if there is no competitive i-split allocation for $1 \leq i \leq i^*$ then (21) gives successively $\frac{u_{1a}}{u_{1b}} > \frac{1}{n-1}$, then $\frac{u_{2a}}{u_{2b}} > \frac{2}{n-2}, \cdots, \frac{u_{i^*a}}{u_{i^*b}} > \frac{i^*}{n-i^*}$, hence the i^*/i^*+1-cut is not competitive. Similarly one checks that between two adjacent competitive split allocations there is exactly one competitive cut allocation.

7.5 Proposition 4

Step 1 the case $m = 2$
As in the previous proof we fix a problem $(N, \{a, b\}, u)$ where the ratios $r_i = \frac{u_{ia}}{u_{ib}}$ increase strictly in $i \in \{1, \cdots, n\}$. We write S^i for the closed rectangle of i-split allocations (17), (18): we have $S^i \cap S^{i+1} = \{z^{i/i+1}\}$ for $i = 1, \cdots, n-1$, and $S^i \cap S^j = \varnothing$ if i and j are not adjacent. We saw that envy-free and efficient allocations must be in the connected union of rectangles $\mathcal{B} = \cup_{i=1}^n S^i$. Writing $\mathcal{EF}$ for the set of envy-free allocations, we describe now the connected components of $\mathcal{A} = \mathcal{B} \cap \mathcal{EF}$. Clearly the set of corresponding disutility profiles has the same number of connected components.

We let the reader check that the cut $z^{i/i+1}$ is EF (envy-free) if and only if it is competitive, i. e. inequalities (19) hold, that we rewrite as:

$$r_i \leq \frac{i}{n-i} \leq r_{i+1} \tag{22}$$

If $z^{i/i+1}$ is EF then both $S^i \cap \mathcal{EF}$ and $S^{i+1} \cap \mathcal{EF}$ are in the same component of $\mathcal{A}$ as $z^{i/i+1}$, because they are convex sets containing $z^{i/i+1}$. If both $z^{i-1/i}$ and $z^{i/i+1}$ are EF, so is the interval $[z^{i-1/i}, z^{i/i+1}]$; then these two cuts as well as $S^i \cap \mathcal{EF}$ are in the same component of $\mathcal{A}$. And if $z^{i/i+1}$ is EF but $z^{i-1/i}$ is not, then the component of $\mathcal{A}$ containing $z^{i/i+1}$ is disjoint from any component of $\mathcal{A}$ in $\cup_1^{i-1} S^j$ (if any), because $S^i \cap \cup_1^{i-1} S^j = \{z^{i-1/i}\}$; a symmetrical statement holds if $z^{i-1/i}$ is EF but $z^{i/i+1}$ is not.

Finally if $S^i \cap \mathcal{EF} \neq \varnothing$ while neither $z^{i-1/i}$ nor $z^{i/i+1}$ is in $\mathcal{EF}$, the convex set $S^i \cap \mathcal{EF}$ is a connected component of $\mathcal{A}$ because it is disjoint from $S^{i-1} \cap \mathcal{EF}$ and $S^{i+1} \cap \mathcal{EF}$, and all three sets are compact. In this case we speak of an interior component of $\mathcal{A}$. We claim that S^i contains an interior component if and only if

$$\frac{i-1}{n-i+1} < r_{i-1} < r_i < r_{i+1} < \frac{i}{n-i} \tag{23}$$

where for $i = 1$ this reduces to the two right-hand inequalities, and for $i = n$ to the two left-hand ones. The claim is proven in the next Step.

Now consider a problem with the following configuration:

$$r_1 < r_2 < \frac{1}{n-1} < \frac{3}{n-3} < r_3 < r_4 < r_5 < \frac{4}{n-4} <$$
$$< \frac{6}{n-6} < r_6 < r_7 < r_8 < \frac{7}{n-7} < \frac{9}{n-9} \cdots$$

By inequalities (22) we have $z^{i/i+1} \in \mathcal{EF}$ for $i = 3q-1$, and $1 \leq q \leq \lfloor \frac{n}{3} \rfloor$, and no two of those cuts are adjacent so they belong to distinct components. Moreover S^i contains an interior component of $\mathcal{A}$ for $i = 3q-2$, and $1 \leq q \leq \lfloor \frac{n+2}{3} \rfloor$, and only those. So the total number of components of $\mathcal{A}$ is $\lfloor \frac{n}{3} \rfloor + \lfloor \frac{n+2}{3} \rfloor = \lfloor \frac{2n+1}{3} \rfloor$ as desired.

We let the reader check that we cannot reach a larger number of components.

Step 2: $\{$*S^i contains an interior component*$\} \Longleftrightarrow \{$*inequalities (23) hold*$\}$

Pick $z \in S^i$ as in (17), (18) and note first that for $2 \leq i \leq n-1$, the envy-freeness inequalities reduce to just four inequalities: agents $i-1$ and i do not envy each other, and neither do agents i and $i+1$ (we omit the straightforward argument). Formally

$$\frac{1}{r_{i+1}}(\frac{1}{n-i} - \frac{n-i+1}{n-i}y) \leq x \leq \frac{1}{r_i}(\frac{1}{n-i} - \frac{n-i+1}{n-i}y) \tag{24}$$

$$r_{i-1}(\frac{1}{i-1} - \frac{i}{i-1}x) \leq y \leq r_i(\frac{1}{i-1} - \frac{i}{i-1}x) \tag{25}$$

In the (non negative) space (x, y) define the lines $\Delta(\lambda)$: $y = \lambda(\frac{1}{i-1} - \frac{i}{i-1}x)$ and $\Gamma(\mu)$: $x = \mu(\frac{1}{n-i} - \frac{n-i+1}{n-i}y)$. As shown on Figure 3 when λ varies $\Delta(\lambda)$ pivots around $\delta = (\frac{1}{i}, 0)$, corresponding to $z^{i/i+1}$, and similarly $\Gamma(\mu)$ pivots around $\gamma = (0, \frac{1}{n-i+1})$, corresponding to $z^{i-1/i}$. The above inequalities say that (x, y) is in the cone Δ^* of points below $\Delta(r_i)$ and above $\Delta(r_{i-1})$, and also in the cone Γ^* below $\Gamma(\frac{1}{r_i})$ and above $\Gamma(\frac{1}{r_i+1})$. Thus $\delta \in \Gamma^*$ if and only if $z^{i/i+1}$ is EF, and $\gamma \in \Delta^*$ if and only if $z^{i-1/i}$ is EF. If neither of these is true γ is above or below Δ^* on the vertical axis and δ is to the left or to the right of Γ^* the horizontal axis. But if γ is below Δ^* while δ is right of Γ^*, the two cones do not intersect and $S^i \cap \mathcal{EF} = \varnothing$; ditto if γ is above Δ^* while δ is left of Γ^* (see Figures 3A,3B,3C). Moreover γ above Δ^* and δ right of Γ^* is impossible as it would imply

$$\frac{1}{n-i+1} > \frac{r_i}{i-1} \text{ and } \frac{1}{i} > \frac{1}{r_i(n-i)}$$

a contradiction. We conclude that $\{S^i \cap \mathcal{EF} \neq \varnothing$ and $z^{i-1/i}, z^{i/i+1} \notin \mathcal{EF}\}$ holds if and only if γ is below Δ^* and δ is to the left of Γ^*, which is exactly the system (23).

In the case $i = 1$ the EF property of z reduces to (24) and the i-split allocation has $x = 1$. If $r_1 > \frac{1}{n-1}$ the right-hand inequality in (24) is impossible with $x = 1$, therefore $r_1 < \frac{1}{n-1}$; but then the fact that $z^{1/2}$ is not EF gives (see (22)) $r_2 < \frac{1}{n-1}$ as desired. A similar argument applies for the case $i = n$.

Step 3: general m

Fix a problem $(N, \{a, b\}, u)$ with $\lceil \frac{2n+1}{3} \rceil$ connected components as in Step 1. Given any $m \geq 3$, construct a problem $(N, \widetilde{A}, \widetilde{u})$ with $\widetilde{A} = \{a, b_1, \cdots, b_{m-1}\}$ and for all agents i

$$\widetilde{u}_{ia} = u_{ia} \text{ ; } \widetilde{u}_{ib_k} = \frac{1}{m-1}u_{ib} \text{ for all } 1 \leq k \leq m-1$$

The bads b_k are smaller size clones of b. If some $\widetilde{z}$ is efficient and EF in the new problem, then the following allocation z is efficient and EF in the initial problem:

$$z_{ib} = \sum_{1}^{m-1} \widetilde{z}_{ib_k} \ ; \ z_{ia} = \widetilde{z}_{ia}$$

and $z, \widetilde{z}$ deliver the same disutility profile. Therefore in the two problems the sets of efficient and EF allocations have the same number of components.

7.6 Theorem 1

Fix a rule f single-valued, Efficient and Envy-Free. Assume first $n = 4$, $m = 2$. Consider $\mathcal{Q}^1$ where, with the notation in the previous proof, we have

$$r_1 < r_2 < \frac{1}{3} < 1 < 3 < r_3 < r_4$$

(note that the numerical example at the beginning of Section 5 is of this type)

By (22) and (23) $\mathcal{A}$ has three components: one interior to S^1 (excluding the cut $z^{1/2}$), one around $z^{2/3}$ intersecting S^2 and S^3, and one interior to S^4 excluding $z^{3/4}$. Assume without loss that f selects an allocation in the second or third component just listed, and consider $\mathcal{Q}^2$ where r_1, r_2 are unchanged but the new ratios r'_3, r'_4 are

$$r_1 < r_2 < 3 < r'_3 < 1 < r'_4 < \frac{1}{3}$$

Here, again by (22) and (23), $\mathcal{A}$ has a single component interior to S^1, the same as in $\mathcal{Q}^1$: none of the cuts $z^{i/i+1}$ is in $\mathcal{A}$ anymore, and there is no component interior to another S^i. When we decrease continuously r_3, r_4 to r'_3, r'_4, the allocation $z^{1/2}$ remains outside $\mathcal{A}$ and the component interior to S^1 does not move. Therefore the allocation selected by f cannot vary continuously in the ratios r_i, or in the underlying disutility matrix u.

We can clearly construct a similar pair of problems to prove the statement when $n \geq 5$ and $m = 2$. And for the case $m \geq 3$ we use the cloning technique in Step 3 of the previous proof.

7.7 Theorem 2

Statement i) the Competitive rule for goods is Resource Monotonic

We first generalize the definition of F^c, f^c to problems where the endowment ω_a of each good is arbitrary, and let the reader check that the system (4) capturing the optimal allocations $f^c(N, A, \omega, u)$ is unchanged. Then we fix N, A, u, ω, ω' such that $\omega \leq \omega'$. We assume without loss of generality that u contains no null row or column (all agents are interested and all goods are useful).For $\lambda \in [0, 1]$ we write $\omega^\lambda = (1 - \lambda)\omega + \lambda\omega'$, and for every forest Γ in $N \times A$ we define

$$\mathcal{B}(\Gamma) = \{\lambda \in [0, 1] | \exists z \in f^c(N, A, \omega^\lambda, u) : \Gamma(z) = \Gamma\}$$

Note that $\mathcal{B}(\Gamma)$ can be empty or a singleton, but if it is not, then it is an interval. To see this take $z \in f^c(\omega^\lambda), z' \in f^c(\omega^{\lambda'})$ such that $\Gamma(z) = \Gamma(z')$. For any $\omega'' = (1-\mu)\omega^\lambda + \mu\omega^{\lambda'}$ the allocation $z'' = (1-\mu)z + \mu z'$ is feasible, $z^{''} \in \Phi(N, A, \omega'')$, the forest $\Gamma(z'')$ is unchanged, and the system (5), which holds at z and z', also holds at z''. Thus $z'' \in f^c(\omega'')$ and the claim is proven.

Next we check that inside an interval $\mathcal{B}(\Gamma)$ the rule F^c is resource monotonic. The forest Γ is a union of trees. If a tree contains a single agent i, she eats (in full) the same subset of goods for any λ in $\mathcal{B}(\Gamma)$, hence her utility increases weakly in λ. If a sub-tree of Γ connects the subset S of agents, then system (5) fixes the direction of the utility profile $(U_i)_{i \in S}$, because along a path of Γ the equalities $\frac{u_{ia}}{U_i} = \frac{u_{ja}}{U_j}$ ensure that all ratios $\frac{U_i}{U_j}$ are independent of λ in $\mathcal{B}(\Gamma)$. As λ increases in $\mathcal{B}(\Gamma)$ the agents in S together eat the same subset of goods, therefore the U_i-s increase weakly by efficiency.

Finally Lemma 1 implies that the finite set of intervals $\mathcal{B}(\Gamma)$ cover $[0, 1]$. On each true interval (not a singleton) the utility profile $U^\lambda = F(N, A, \omega^\lambda, u)$ and there is at most a finite set of isolated points not contained in any true interval. Moreover the mapping $\lambda \to U^\lambda$ is continuous because $\omega \to U(\omega)$ is (an easy consequence of Berge Theorem). The desired conclusion $U(\omega) \leq U(\omega')$ follows.

Statement *ii*) Here we generalize the example after Theorem 2, first to the case where $n = 2n'$ even, $n' \geq 2$. Fix two bads a, b. At $\mathcal{Q}$ we have n' agents with $u_i = (1, 5n')$, $i \in N_1$, and n' agents with $u_j = (5n', 1)$, $j \in N_2$. The profile $U = \frac{1}{n'} e^N$ is feasible. Also, at an efficient profile if at least one in N_1 eats some b, then no one in N_2 eats any a, and vice versa. Thus at $U = F(\mathcal{Q})$ at least one of $U_{N_1} \leq 1$ or $U_{N_2} \leq 1$ is true, say $U_{N_1} \leq 1$.

Then consider $\mathcal{Q}' = (\frac{1}{10n'} a, b)$ and use again FSG and feasibility:

$$\text{for } j \in N_2\text{: } z'_{jb} \leq u_j \cdot z'_j \leq u_j \cdot (\frac{1}{2n'} e^{A'}) = \frac{3}{4n'}$$

$$\Longrightarrow \sum_{i \in N_1} z'_{ib} \geq \frac{1}{4} \Longrightarrow U'_{N_1} \geq \frac{5}{4} n' > U_{N_1}$$

contradicting RM.

The case $n = 2n' + 1$ odd is very similar, except that the two groups are of size n' and $n' + 1$, with the same utilities as above. If we have more than two bads, say $c, d, \ldots$, we can either assume they are harmless, $u_{ic} = u_{id} = 0$ for all i, or these disutilities are very small with respect those for a, b.

References

[1] S. Bouveret and J. Lang. 2008. Efficiency and envy-freeness in fair division of indivisible goods: logical representation and complexity. *Journal of Artificial Intelligence Research* 32, 525–564.

[2] S. J. Brams and A. D. Taylor. 1996. *Fair Division: From Cake-Cutting to Dispute Resolution.* Cambridge University Press.

[3] E. Budish. 2011. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy* 119, 6, 1061–1103.

[4] E. Budish and E. Cantillon. 2010. The Multi-unit Assignment Problem: Theory and Evidence from Course Allocation at Harvard, Manuscript.

[5] I. Caragiannis, D. Kurokawa, H. Moulin, A. Procaccia, N. Shah, J. Wang, 2016. The Unreasonable Fairness of Maximum Nash Welfare, *Proceedings of the 17th ACM Conference on Electronic Commerce*, Maastricht, June 24-27.

[6] R. Cole and V. Gkatzelis. 2015. Approximating the Nash social welfare with indivisible items. In *Proceedings of the 47th STOC Conference*. 371–380.

[7] Peter Cramton, Y. Shoham, and R. Steinberg, eds. 2006. *Combinatorial auctions*. Vol. 475. Cambridge: MIT press.

[8] E. Eisenberg and D. Gale. 1959. Consensus of subjective probabilities: The pari-mutuel method. *The Annals of Mathematical Statistics* 30, 1, 165–168.

[9] D. Foley. 1967. Resource allocation and the public sector. *Yale Economics Essays* 7, 45–98.

[10] L. Gevers.1986. Walrasian social choice: some simple axiomatic approaches. In W. P. Heller, R. M. Starr& D. A. Starrett (Eds.), *Social choice and public decision making*, Essays in honor of K. J. Arrow,97-114. Cambridge University Press

[11] A. Ghodsi, M. Zaharia, B. Hindman, A. Konwinski, S. Shenker, and I. Stoica. 2011. Dominant resource fairness: Fair allocation of multiple resource types. In *Proceedings of the 8th NSDI Conference*. 24–37.

[12] J. Goldman and A. D. Procaccia. 2014. Spliddit: Unleashing fair division algorithms. *SIGecom Exchanges* 13, 2, 41–46.

[13] L. Hurwicz. 1979. On allocations attainable through Nash equilibria. *Journal of Economic Theory,* 21, 1, 140-165.

[14] K. Jain and V. Vazirani. 2010. Eisenberg–Gale markets: Algorithms and game-theoretic properties. *Games and Economic Behavior*, 70, 1, 84-106.E. Lee. 2015. APX-hardness of maximizing Nash social welfare with indivisible items. arXiv:1507.01159.

[15] E. Lee. 2015. APX-hardness of maximizing Nash social welfare with indivisible items. arXiv:1507.01159.

[16] N. Megiddo and V. Vazirani. 2007. Continuity Properties of Equilibrium Prices and Allocations in Linear Fisher Markets. *Internet and Network Economics*, vol 4858 in Lecture Notes in Computer Science, 362-367, Springer.

[17] H. Moulin. 2003. *Fair Division and Collective Welfare*. MIT Press. Volume 54, Issue 2, August 1991, Pages 321-337

[18] H. Moulin. 1990. Uniform externalities: Two axioms for fair allocation. *Journal of Public Economics,* 43, 3, 305-326.

[19] H. Moulin. 1991. Welfare bounds in the fair division problem. *Journal of Economic Theory,* 54, 2, 321-337.

[20] H. Moulin and W. Thomson. 1988. Can everyone benefit from growth?: Two difficulties. *Journal of Mathematical Economics*, 17, 4, 339-345.

[21] R. Nagahisa, 1991. A Local independence Condition for Characterization of Walrasian Allocations Rule, *Journal of Economic Theory* 54, 106-123.

[22] A. van den Nouweland, B. Peleg and S. Tijs, 1996. Axiomatic characterizations of the Walras correspondence for generalized economies, *Journal of Mathematical Economics*, 25, 3, 355-372.

[23] E. Pazner and D. Schmeidler. 1978. Egalitarian equivalent allocations: A new concept of economic equity. *Quarterly Journal of Economics* 92, 4, 671–687.

[24] J. Pratt and R. Zeckhauser. 1990. The Fair and Efficient Division of the Winsor Family Silver. *Management Science*, 36, 11, 1293 - 1301.

[25] S. Ramezani and U. Endriss. 2010. Nash social welfare in multiagent resource allocation. In *Proceedings of the 12th AMEC Conference*. 117–131.

[26] J. Schummer. 1997. Strategy-proofness versus efficiency on restricted domains of exchange economies. *Social Choice and Welfare* 14, 47–56.

[27] T. Sönmez and U. Ünver. 2010. Course Bidding at Business Schools. *International Economic Review*, 51, 1, 99-123.

[28] W. Shafer and H. Sonnenschein. 1993. Market demand and excess demand functions. Chapter 14 in *Handbook of Mathematical Economics*, vol.2, 671-693, Elsevier.

[29] E. Segal-Halevi and B. Sziklai. 2015. Resource-monotonicity and Population-monotonicity in Cake-cutting, arXiv:1510.05229 [cs.GT].

[30] H. Steinhaus. 1948. The problem of fair division. *Econometrica*, 16, 1, 101-104.

[31] W. Thomson. 2010. Fair Allocation Rules. Chapter 21 in the *Handbook of Social Choice and Welfare*, vol 2, 393-492, Elsevier.

[32] W. Thomson.1987. The vulnerability to manipulative behavior of economic mechanisms designed to select equitable and efficient outcomes. In T. Groves, R. Radner & S. Reiter (Eds.), Chapter 14 of *Information, Incentives and Economic Mechanisms*, p. 375–396. University of Minnesota Press.

[33] W. Thomson.1988. A study of choice correspondences in economies with a variable number of agents. *Journal of Economic Theory*, 46, 237–254.

[34] W. Thomson and C Kayi. 2005 Monotonicity properties of the Walrasian correspondence in linear exchange economies, University of Rochester.

[35] H. Varian. 1974. Equity, envy and efficiency. *Journal of Economic Theory* 9, 63–91.

[36] S. de Vries and R. Vohra. 2003. Combinatorial Auctions: A Survey. *Journal on Computing*, 15,3, 284 - 309.

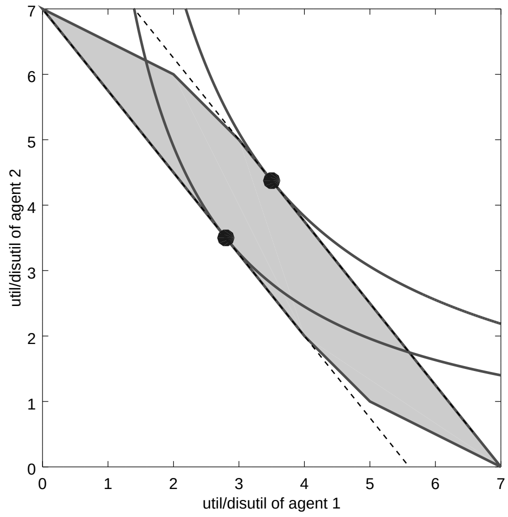


Figure 1: Competitive util/disutil profiles for example 1

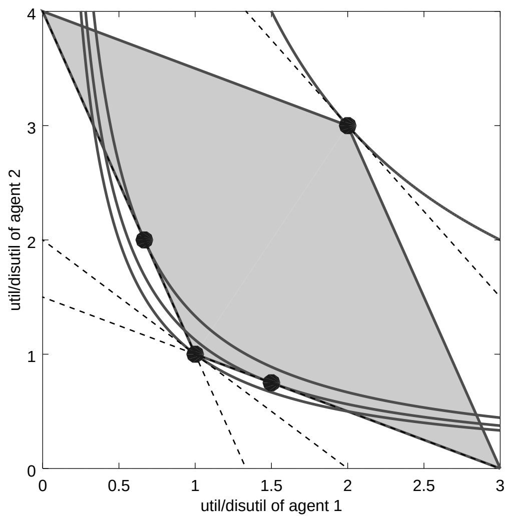
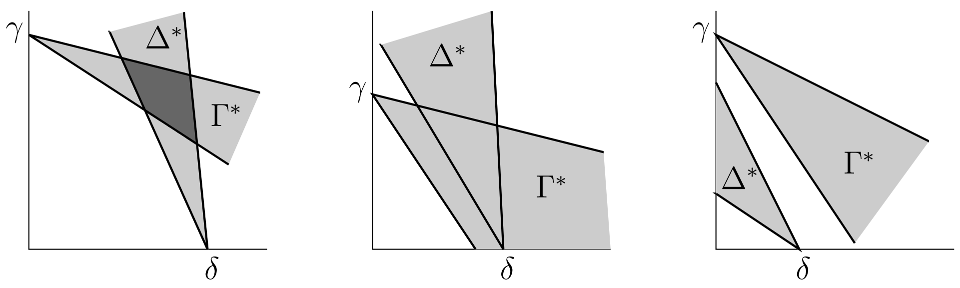


Figure 2: Competitive util/disutil profiles for examples 2



Figures 3A, 3B, 3C