All-Pay Quality-Bids in Score Procurement Auctions*

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Abstract

We study scoring procurement auctions with risk-neutral parties, quasi-linear payoffs, and quality costs partly sunk regardless of winning. Under the difference-score rule, we characterize in closed form the unique symmetric pure-strategy equilibrium with scores decreasing in type and bounded price bids. We then derive the procurer-optimal scoring rule. With power-form costs, a higher all-pay share lowers quality, total surplus, and supplier payoffs but ambiguously affects procurer payoff. Reimbursing losers' all-pay quality costs increases quality and supplier payoffs but reduces total surplus and procurer payoff. The optimal score subtracts price from a quality index that underweights quality relative to the procurer's payoff function. Under a weakly convex inverse reversed hazard rate, its marginal quality weight weakly increases with the all-pay share and number of suppliers. Our equilibrium approach extends to minimum-score constraints, multidimensional quality bids, and general scoring rules.

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1 Introduction

Procurement is widely used to acquire goods, services, and public works. On average, about 15% of the yearly global gross domestic product is spent on public procurement alone, including military acquisitions.[1] Procurements involving multi-dimensional bids are ubiquitous. Typically, suppliers bid on both quality and price, which are combined into a single-dimensional score used to select the most economically advantageous offer.

Since the seminal work of Che [6], score auctions with winner-pay quality bids have been studied in many contexts, including Branco [5], Asker and Cantillon [1, 2], Wang and Liu [16], Nishimura [13], and Hanazono, Nakabayashi, Sano, and Tsuruoka [9].

In many settings, quality bidding is more naturally modeled as having a partially all-pay component rather than being entirely winner-pay.[2] For example, all-pay quality bids are often features of procurements in defense contracting and other military design competitions, architectural designs, government construction projects with a design component, and business-to-business customized sales. To the best of our knowledge, Che and Gale [7] are the first to study score procurement with all-pay quality bids in a complete information environment. Their focus is to demonstrate the optimality of shortlisting and handicapping in such environments. Koh [11] also studies shortlisting when suppliers' quality provision is uncertain.

To the best of our knowledge, our study is the first to introduce partially all-pay quality bids in score procurement auctions with incomplete information. Specifically, we study a multi-supplier environment in which each supplier's score is the difference between the supplier's quality and price bids. The procurer and suppliers are risk neutral. The supplier with the highest score wins and receives the submitted price; the other suppliers receive no payment, and all suppliers bear the sunk all-pay component of their quality costs. A supplier's payoff equals the payment received, if any, less the quality cost incurred. The procurer's payoff equals the winner's quality less the procurer's payments to the suppliers.

We first highlight how the all-pay nature of quality bidding changes equilibrium bidding behavior in score procurement, relative to the winner-pay paradigm of Che [6]. As in Che [6], the equilibrium quality bid can be identified directly from the observation

[1] See "Supplement to the 2013 Annual Statistical Report on United Nations Procurement: Procurement and Innovation," https://www.unsystem.org/content/supplement-2013-annual-statistical-report-united-nations-procurement (accessed Jan. 15, 2019).

[2] See Baye, Kovenock and de Vries [3] and Kaplan, Luski, Sela and Wettstein [10] among many others. Gershkov, Moldovanu, Strack, and Zhang [8] study an environment where bidders incur costs to increase their values.

that a supplier chooses quality and price bids optimally conditional on submitting the equilibrium score. The core departure is as follows. In the winner-pay benchmark, quality choice is separable from the competition: each supplier determines the quality bid independently of the probability of winning. With an all-pay component, quality costs are incurred even upon losing, so the winning probability enters the quality condition directly: a supplier optimally scales down quality provision as the probability of winning falls, and the more so the larger the all-pay share.

To characterize equilibrium, we view the quality-price bid pair as an incentive-compatible direct mechanism, with the quality bid playing the role of the allocation rule and the price bid that of the payment rule. This observation pins down the price schedule as a function of the quality schedule. Unlike in Che's winner-pay environment, however, the resulting quality strategy is not dominant: a deviation in score changes both quality and price. We call this property "non-separability" of the supplier's quality and price decisions.[3] It makes global incentive-compatibility verification necessary.[4]

The equilibrium analysis enables us to study the effect of the partially all-pay component of quality bids on quality provision, supplier payoffs, the procurer's payoff, and total surplus. When the effort cost function is convex and takes a power form, we find that a higher proportion of the all-pay component reduces the quality provision, the total surplus, and suppliers' payoffs. The effect on the procurer's payoff, however, is ambiguous. In particular, we illustrate that all-pay quality bids can indeed generate either a higher or a lower procurer payoff than winner-pay quality bids. The deeper reason is a virtual-surplus effect: under a fixed difference-score rule, the procurer is exposed to high-cost suppliers whose virtual contribution to her payoff can be negative under certain conditions, so inducing positive quality provision from them reduces her payoff. A higher all-pay share depresses quality provision most strongly for exactly these suppliers, and the resulting mitigation of losses can outweigh the quality reduction on efficient types.

We next examine reimbursement of the losing suppliers' all-pay quality costs and its effects on quality, payoffs, and total surplus.[5] With power-form costs, reimbursement increases quality provision and suppliers' payoffs but reduces total surplus and the procurer's payoff. Reimbursement raises quality and supplier payoffs because it lowers suppliers' perceived marginal cost; indeed, reimbursing a proportion of losers' all-pay

[3]We thank the advisory editor in charge for suggesting this term.

[4]Section 6.3 explains how the same procedure applies to nonlinear score rules, for which conditional bid optimization alone generally does not identify the quality schedule.

[5]These exercises thus resemble those of Baye, Kovenock, and de Vries [4] and Siegel [15] who study equilibria in auctions with contingent investment. Baye, Kovenock, and de Vries [4] study auctions with incomplete information, and Siegel [15] studies settings with complete information.

costs makes suppliers behave exactly as if the all-pay share were lower. It does not, however, reduce the social cost of quality provision; it shifts part of that cost to the procurer and induces socially excessive quality. Because supplier payoffs rise while total surplus falls, the procurer's payoff also falls.

The all-pay component changes the incentive-relevant allocation rule and, consequently, the procurer's payoff bound: both must account for suppliers' all-pay-adjusted cost exposure. Viewing the symmetric equilibrium quality bid as the allocation rule allows us to establish payoff equivalence in the all-pay environment: the procurer's and suppliers' expected payoffs are determined by the quality schedule and the least efficient type's payoff. Pointwise maximization of the procurer's payoff yields the optimal quality schedule, which we implement with an explicit quasi-linear score rule; the associated price schedule is also explicit. The optimal rule induces scores that are strictly decreasing in type. We find that the optimal score rule places less weight on quality than the procurer's payoff function does, echoing Che's [6] result for the winner-pay environment. Relative to Che's winner-pay optimum, the optimal quality schedule in our environment is modified by a single all-pay wedge that lies below one. Moreover, under the optimal score rule the procurer's payoff is strictly decreasing in the all-pay proportion: once the score rule internalizes the screening and quality-distortion tradeoff, the procurer no longer benefits from a higher all-pay share as a crude instrument for curbing quality provision by inefficient suppliers. Finally, when the suppliers' type distribution has a weakly convex inverse reversed hazard rate,[6] the optimal score rule places weakly greater marginal weight on quality when the all-pay share rises or when more suppliers participate.

As extensions, we show that our procedure for identifying a monotone symmetric pure-strategy equilibrium without first solving for equilibrium scores also applies to minimum-score requirements, multi-dimensional quality bids, and nonlinear score rules. For a fixed score rule, a properly chosen minimum score can improve the procurer's payoff. We also derive an equilibrium under a ratio score rule to illustrate how the procedure accommodates nonlinear scores. Our method is an alternative to, rather than a replacement for, the conventional approach that first solves for equilibrium score bids.

The rest of the paper is organized as follows. Section 2 sets up the model with partially all-pay quality bids. Section 3 develops the equilibrium analysis for the continuum from winner-pay to all-pay quality bids and derives comparative statics with respect to the all-pay share. Section 4 examines the impact of the procurer's reimbursement of

[6]Throughout the paper, the inverse reversed hazard rate refers to the ratio of the distribution function to its density; the reciprocal of this ratio is conventionally called the reversed hazard rate.

suppliers' all-pay quality bidding costs on quality provision, supplier payoffs, the procurer's payoff, and total surplus. Section 5 derives the procurer-optimal score rule and the associated bidding equilibrium. We also investigate the properties of the optimal score rule. Section 6 demonstrates that our procedure for identifying the monotone symmetric pure-strategy equilibrium can be extended to more general environments, including minimum-score requirements, multi-dimensional quality bids, and nonlinear score rules. Section 7 provides some concluding remarks. Technical proofs are relegated to the appendices.

2 Model

We consider a score procurement auction with a risk-neutral procurer and $N \geq 2$ risk-neutral suppliers. The suppliers, $i = 1, \ldots, N$, simultaneously choose their nonnegative bids of quality q_i and price p_i to compete for a contract. Only the winner receives the submitted price p, while every bidder bears the all-pay component of the quality cost. Specifically, a type c_i of supplier i must incur a cost of $\rho C_i(q; c_i)$ to bid a quality q whether he wins the contract or not. Here, $C_i(q; c_i)$ is the total provision cost that a type c_i of supplier i must incur to provide quality q if he is selected as the winner. As in Baye, Kovenock, and de Vries [4] and Siegel [15], we introduce a parameter $\rho \in [0, 1]$ that represents the proportion of the all-pay component in the total provision cost. The case $\rho = 0$ corresponds to the winner-pay benchmark of Che [6] and is retained throughout as a limiting reference; unless otherwise noted, our interest is in $\rho > 0$. We assume $C_i(q; c_i) = c_i\zeta(q)$, where ζ is continuously differentiable on $[0, +\infty)$, twice continuously differentiable on $(0, +\infty)$ with $\zeta''(q) > 0$ for every $q > 0$, and satisfies $\zeta(0) = \zeta'(0) = 0$, $\zeta'(q) > 0$ for $q > 0$, and $\lim_{q\to+\infty} \zeta'(q) = +\infty$. Here $\zeta'(0)$ denotes the right derivative. For example, $\zeta(q) = q^\gamma$ satisfies these conditions for every $\gamma > 1$. c_i is supplier i's private type; a higher c_i implies a higher marginal cost of quality. We discuss how to accommodate the case of $\zeta'(0) > 0$ in Section 5 (following Proposition 8) and in Section 6.3 together with Appendix B.

We assume that suppliers' types c_i are independently and identically distributed according to a cumulative distribution $F(\cdot)$ on $[\underline{c}, \bar{c}]$. On $(\underline{c}, \bar{c}]$, the distribution has a continuous, strictly positive density $f(\cdot)$, and $F(c)/f(c)$ admits a continuous extension to $[\underline{c}, \bar{c}]$, with its value at $\underline{c}$ interpreted as zero. We assume $0 < \underline{c} < \bar{c} < +\infty$.

The higher score wins, and ties are broken uniformly at random. The score function

is

$$s(q,p) = q - p. \tag{1}$$

For later use, define

$$H(c) \equiv [1 - F(c)]^{N-1},$$

the probability that a type-c supplier wins when scores are strictly decreasing in type. We use $H(\cdot)$ throughout for this rank-order winning probability. Note that $H(c) > 0$ for $c < \bar{c}$ and $H(\bar{c}) = 0$; expressions with $H(c)$ in a denominator are evaluated for $c < \bar{c}$ and extended to the endpoint by continuity when needed.

The procurer's payoff is $q - p$ for the winning bid. Supplier i's payoff is $p_i - C_i(q_i; c_i)$ upon winning and $-\rho C_i(q_i; c_i)$ upon losing.

The equilibrium analysis extends to cases in which the procurer's payoff takes the form of $u(q) - p$, with $u(\cdot)$ increasing and concave, and the score rule takes the quasi-linear form $s = v(q) - p$, with $v(\cdot)$ continuously differentiable and increasing on the relevant quality range. Concavity of $v(\cdot)$ is not required. See Section 5 for details.

3 Equilibrium analysis and comparative statics

3.1 Equilibrium analysis

We now derive the equilibrium within the class of symmetric pure-strategy equilibria with scores decreasing in type and bounded price bids, that is, equilibrium price schedules $p(\cdot)$ with $\sup_c p(c) < +\infty$. The restriction to bounded price bids reflects limited liability of the procurer, who cannot commit to pay an arbitrarily large price. Due to the boundedness of price bids and the participation constraint, the limit equilibrium payoff must clearly be zero as $c \uparrow \bar{c}$. Note that the limit equilibrium winning chance is zero. Moreover, for $\rho > 0$, the limit equilibrium quality must be zero as $c \uparrow \bar{c}$.

There are two key observations that facilitate identifying the equilibrium. Suppose the score of type c_i is $s(c_i)$ in the equilibrium. First, if we view the equilibrium strategy $(q(c_i), p(c_i))$ as an incentive-compatible direct mechanism — in which the quality bid function $q(\cdot)$ plays the role of the allocation rule and the price bid function $p(\cdot)$ that of the payment rule — then the Myerson [12] approach, together with a zero-payoff condition for type $\bar{c}$, allows us to pin down a payment rule $p(c_i; q(\cdot))$ for any given allocation rule $q(\cdot)$ decreasing in type.[7] Second, define the score by $s(c_i) = q(c_i) - p(c_i; q(\cdot))$

[7] See Riley and Samuelson [14] for an alternative treatment of optimal auctions in the early literature.

for every c_i. Thus, $s(\cdot)$ is solely determined by the allocation rule $q(\cdot)$. Given that a type-c_i supplier submits score $s(c_i)$ in equilibrium, the supplier must choose $(q, p) = (q(c_i), p(c_i))$ to maximize expected payoff conditional on that score. This further gives us conditions on the equilibrium bidding functions $(q(\cdot), p(\cdot))$.[8] With a quasi-linear score, as in Che [6], the equilibrium quality bid can be identified directly from the second insight. If the resultant $s(\cdot)$ is decreasing in type and no type has an incentive to deviate to a different score, then the identified $(q(\cdot), p(\cdot))$ following the above procedure is the symmetric pure-strategy equilibrium. Uniqueness follows from the procedure itself: the two observations, together with the boundary conditions above, are necessary conditions that every equilibrium in the maintained class satisfies, and they determine the candidate uniquely. If the candidate fails either verification, it is not an equilibrium; when the preceding conditions uniquely determine the candidate, no equilibrium exists within the maintained class.

Unlike in Che [6], the quality bid identified from the second observation is not a dominant strategy when $\rho > 0$,[9] because quality and price decisions are not separable (Section 3.1.2); the candidate equilibrium must therefore be verified for global incentive compatibility.

3.1.1 Implications of the first observation (Myerson incentive compatibility)

We first consider the implications of the first observation. Fix any $q(c_i)$. We can view any equilibrium strategy $(q(c_i), p(c_i))$ as an incentive-compatible direct mechanism that leads to decreasing scores in type. The quality bid function $q(c_i)$ can be viewed as an allocation rule, and the price function $p(c_i)$ can be viewed as a payment rule. Then the Myerson approach allows us to pin down the payment rule $p(c_i)$ as a function of the allocation rule $q(\cdot)$. The details follow.

Given a $(q(c_i), p(c_i))$ pair that leads to scores decreasing in type, consider a type-c_i supplier's problem of choosing $\tilde{c}_i$ when the other suppliers report truthfully:

$$\begin{aligned} &\max_{\tilde{c}_i} \quad \pi_i(\tilde{c}_i, c_i), \\ &\pi_i(\tilde{c}_i, c_i) \equiv H(\tilde{c}_i)\big[p(\tilde{c}_i) - (1-\rho)c_i\zeta(q(\tilde{c}_i))\big] - \rho c_i \zeta(q(\tilde{c}_i)). \end{aligned} \tag{2}$$

[8] With a general score rule and $\rho > 0$, the restrictions on $(q(\cdot), p(\cdot))$ implied by the above two observations, together with the boundary condition $q(\bar{c}) = 0$, pin down a candidate equilibrium quality schedule $q(\cdot)$. Please refer to Section 6.3 for details. The quality bid typically cannot be identified by solely utilizing the second observation when the score rule is not quasi-linear.

[9] Lemma 1 of Che [6] shows that with quasi-linear score rules, the quality bidding strategy in (5) is a dominant strategy when $\rho = 0$.

If $(q(\cdot), p(\cdot))$ is a pure-strategy equilibrium, then $\tilde{c}_i^*$, the solution to (2), satisfies $\tilde{c}_i^* = c_i$ for every c_i. Equivalently, incentive compatibility requires that the objective in (2) be maximized at $\tilde{c}_i = c_i$ for each type.

The standard single-crossing argument implies that an allocation rule can be supported by a type-contingent payment rule. Appendix A shows that when $q'(c_i) < 0$, single crossing holds and the quality schedule $q(\cdot)$ is supported by the following price rule:

$$p(c_i) = \frac{\int_{c_i}^{\bar{c}} \{H(t)(1-\rho)\zeta(q(t)) + \rho\zeta(q(t))\}dt + \rho c_i \zeta(q(c_i))}{H(c_i)} + (1-\rho)c_i\zeta(q(c_i)) \geq 0. \quad (3)$$

3.1.2 Implications of the second observation (optimal bids for a given score)

We now turn to the implications of the second observation. Suppose $(q(c_i), p(c_i))$ is a symmetric pure-strategy equilibrium. By the first observation, we must have that (3) holds. Define

$$s(c_i) = s(q(c_i), p(c_i)). \quad (4)$$

The second observation says that any equilibrium $(q(c_i), p(c_i))$ must solve the following optimization problem:

$$\begin{aligned} (q(c_i), p(c_i)) &= \arg\max_{(q,p)} \{H(c_i)[p - (1-\rho)c_i\zeta(q)] - \rho c_i \zeta(q)\} \\ s.t. \quad &: \quad s = q - p = s(c_i). \end{aligned}$$

Solving the Lagrangian first-order conditions, which are derived in Appendix A, pins down the candidate equilibrium quality

$$q(c_i; \rho) = (\zeta')^{-1}\left(\frac{H(c_i)}{c_i\{(1-\rho)H(c_i) + \rho\}}\right). \quad (5)$$

For $\rho > 0$, the argument of $(\zeta')^{-1}$ in (5) converges to zero as $c_i \uparrow \bar{c}$; thus, $q(\bar{c}) = 0$ under the assumption $\zeta'(0) = 0$. (In the winner-pay benchmark $\rho = 0$, the quality condition reduces to $\zeta'(q) = 1/c_i$, so that $q(\bar{c}) = (\zeta')^{-1}(1/\bar{c}) > 0$, as in Che [6].) The assumption $\zeta''(q) > 0$ implies that $q(c)$ is strictly decreasing for $c < \bar{c}$.

The marginal condition underlying (5), $\zeta'(q(c_i;\rho)) = 1/\{c_i[(1-\rho)+\rho/H(c_i)]\}$, makes precise the departure from Che [6] described in the Introduction: for $\rho = 0$ it collapses to $\zeta'(q) = 1/c_i$, independent of $H(c_i)$, whereas for $\rho > 0$ the winning probability enters it directly. Since $H(c_i)$ falls with c_i, this explains why equilibrium quality falls with ρ

(Proposition 1 below), and why the effect is especially strong for high-cost suppliers with low winning probabilities — a feature that drives the comparative statics of Section 3.2.

When supplier i deviates to score $s(c_i')$, his quality and price bids solve the following optimization problem:

$$\begin{aligned}(q^*, p^*) &= \arg\max_{(q,p)} \{H(c_i')[p - (1-\rho)c_i\zeta(q)] - \rho c_i\zeta(q)\} \\ s.t. &: \quad s = q - p = s(c_i').\end{aligned}$$

We thus have

$$q^* = (\zeta')^{-1}\left(\frac{1}{c_i\{(1-\rho) + \frac{\rho}{H(c_i')}\}}\right),$$

which depends on both c_i and c_i'.

This is the non-separability of quality and price decisions noted in the Introduction.

Since $q(c_i)$ is decreasing, the above $(q(\cdot), p(\cdot))$ characterized in (5) and (3) would constitute a symmetric pure-strategy equilibrium within the class we consider if and only if the corresponding score function $s(c_i) = q(c_i) - p(c_i)$ is decreasing in type and no type c_i deviates from bidding $s(c_i)$. This is indeed the case as verified by the following lemma, whose proof is relegated to Appendix A.

Lemma 1 *(i) $s'(c_i) = q'(c_i) - p'(c_i) < 0$. (ii) No type c_i has an incentive to deviate from generating score $s(c_i)$.*

We thus have the following theorem.

Theorem 1 *The symmetric bidding strategy $(q(\cdot), p(\cdot))$ identified in (5) and (3) constitutes the unique symmetric pure-strategy equilibrium with scores decreasing in type and bounded price bids, up to the behavior of the zero-probability endpoint type. Moreover, $q(\cdot)$ is strictly decreasing on $[\underline{c}, \bar{c})$.*[10]

Note that the quality bid $q(c_i)$ is identified by only using the second observation. This is the case for the difference-form score function that we consider, but it is not the case in general. Please refer to Section 6.3 for the procedure for identifying $q(c_i)$ without solving for the equilibrium score bidding function for a general score function. In particular, the case of a ratio score function is studied there as an example.

[10]The price schedule in (3) is bounded. For $\rho > 0$, convexity of ζ and $\zeta(0) = 0$ give $\zeta(q) \leq q\zeta'(q)$, so (5) yields $\zeta(q(c)) \leq q(c)H(c)/(\rho c)$. Since $q(\cdot)$ is decreasing and $(1-\rho)H(t) + \rho \leq 1$, the integral in (3) is at most $(\bar{c} - c)\zeta(q(c))$, and hence $p(c) \leq [1 + (\bar{c} - c)/(\rho c)]q(c) + (1-\rho)c\zeta(q(c))$, which tends to zero as $c \uparrow \bar{c}$. For $\rho = 0$, since $H(t) \leq H(c)$ for $t \geq c$, the integral in (3) is at most $(\bar{c} - c)H(c)\zeta(q(c))$, and hence $p(c) \leq \bar{c}\zeta(q(c)) \leq \bar{c}\zeta(q(\underline{c}))$. Analogous bounds hold for the price bids in Proposition 8 and Section 6.1.

3.2 Comparative statics

We now examine comparative statics with respect to ρ, the proportion of the all-pay component. We will establish that a higher ρ leads to lower quality provision in general. For the class of cost functions $\zeta(q) = q^\gamma, \gamma > 1$, it leads to lower total surplus and suppliers' payoffs. The procurer's payoff is in general neither monotonically increasing nor decreasing in ρ. Nevertheless, a higher ρ always leads to a lower procurer payoff when $(1 - \frac{1}{\gamma}) - \frac{1}{\gamma c}\frac{F(c)}{f(c)} \geq 0, \forall c \in [\underline{c}, \bar{c}]$. Conversely, when γ approaches 1 from above, so that $\zeta(q)$ is close to linear, the procurer's payoff can increase with ρ near zero, as the sufficient conditions below demonstrate.

First, by (5), it is clear that the equilibrium quality bid decreases in ρ.

Proposition 1 *A higher ρ weakly lowers quality for every type and strictly lowers it for $c \in (\underline{c}, \bar{c})$.*

Applying the envelope theorem to the incentive-compatibility problem in (2) yields

$$\frac{d\pi_i(c_i, c_i)}{dc_i} = -\{H(c_i)(1-\rho) + \rho\}\zeta(q(c_i; \rho)). \tag{6}$$

Let $\varphi = \zeta \circ (\zeta')^{-1}$. Substituting (5) into (6) gives

$$\frac{d\pi_i(c_i, c_i)}{dc_i} = -\{H(c_i)(1-\rho) + \rho\}\varphi\left(\frac{H(c_i)}{c_i\{(1-\rho)H(c_i) + \rho\}}\right).$$

Note that $\pi_i(\bar{c}, \bar{c}) = 0$ for every ρ. If $|\frac{d\pi_i(c_i,c_i)}{dc_i}|$ decreases with ρ, then a higher ρ leads to lower supplier payoffs. This is the case when φ is convex, as will be verified in the proof of the following proposition. The power-cost specification $\zeta(q) = q^\gamma$, $\gamma > 1$, satisfies this condition.

Proposition 2 *If $\varphi = \zeta \circ (\zeta')^{-1}$ is convex, then a higher ρ weakly lowers every supplier type's payoff.*

We next turn to total surplus and the procurer's payoff. For the power-cost specification $\zeta(q) = q^\gamma$, $\gamma > 1$, the following lemma gives convenient reduced-form expressions; the definitions and derivations are relegated to its proof.

Lemma 2 *Assume $\zeta(q) = q^\gamma$, $\gamma > 1$. Let $TS(\rho)$, $\pi_S(\rho)$, and $\pi_P(\rho)$ denote expected total surplus, a representative supplier's expected payoff, and the procurer's expected payoff,*

respectively. Then

$$TS(\rho) = NE_c\left\{q(c;\rho)H(c)\left(1-\frac{1}{\gamma}\right)\right\},$$

$$N\pi_S(\rho) = NE_c\left\{q(c;\rho)H(c)\frac{1}{\gamma c}\frac{F(c)}{f(c)}\right\},$$

$$\pi_P(\rho) = NE_c\left\{q(c;\rho)H(c)\left[\left(1-\frac{1}{\gamma}\right)-\frac{1}{\gamma c}\frac{F(c)}{f(c)}\right]\right\}.$$

Lemma 2 immediately yields the following results.

Proposition 3 *If $\zeta(q) = q^\gamma$, $\gamma > 1$, then a higher ρ (i) strictly lowers expected total surplus and aggregate supplier payoffs; and (ii) strictly lowers the procurer's payoff when $F(c)/f(c) \leq (\gamma-1)c$ for every $c \in [\underline{c}, \bar{c}]$.*

The intuition behind Proposition 3 is immediate from Lemma 2. With $\zeta(q) = q^\gamma$, $\gamma > 1$, equation (5) gives $q(c;\rho) = [H(c)/(\gamma c\{(1-\rho)H(c)+\rho\})]^{1/(\gamma-1)}$; hence $q(c;\rho)H(c)$ is weakly decreasing in ρ for every $c < \bar{c}$ and strictly decreasing for every $c \in (\underline{c}, \bar{c})$. In the first two expressions in Lemma 2, this is the only factor that depends on ρ, while the remaining weights, $1-1/\gamma$ and $[F(c)/f(c)]/(\gamma c)$, are nonnegative and positive on a set of interior types of positive measure. It follows directly that both total surplus and suppliers' payoffs strictly decrease in ρ. The third expression shows that the procurer's payoff weakly decreases in ρ whenever $F(c)/f(c) \leq (\gamma-1)c$ for every $c \in [\underline{c}, \bar{c}]$, because the bracketed virtual-contribution term is then nonnegative for every type. The decrease is also strict: since $F(\underline{c})/f(\underline{c})$ is interpreted as zero and $\underline{c} > 0$, continuity of the extended ratio $F(c)/f(c)$ implies that the virtual-contribution term is strictly positive for all types in a right neighborhood of $\underline{c}$, on whose interior $q(c;\rho)H(c)$ is strictly decreasing in ρ. The factor $F(c)/f(c)$ is the inverse reversed hazard rate and captures suppliers' information rents in the procurement environment.

Consider the class $F(c) = (\frac{c-\underline{c}}{\bar{c}-\underline{c}})^\eta$ on $[\underline{c}, \bar{c}]$, with $\underline{c} > 0$ and $\eta > 0$. For this class, $F(c)/f(c) = (c-\underline{c})/\eta$ on the interior, with the continuous extension equal to zero at $\underline{c}$. Hence, if $1 + 1/\eta < \gamma$, then $F(c)/f(c) \leq (\gamma-1)c$ for every $c \in [\underline{c}, \bar{c}]$. We thus have the following corollary.

Corollary 1 *Suppose $\zeta(q) = q^\gamma, \gamma > 1$, and $F(c) = (\frac{c-\underline{c}}{\bar{c}-\underline{c}})^\eta$ on $[\underline{c}, \bar{c}]$ with $\underline{c} > 0$ and $\eta > 0$. If $1 + \frac{1}{\eta} < \gamma$, then a higher ρ strictly lowers the procurer's payoff.*

Although Proposition 3 identifies conditions under which a higher ρ hurts the procurer, the question remains whether a higher all-pay share can benefit the procurer. For the two propositions below, we say that a continuous function $D(c)$ single-crosses zero from above if there exists $c_0 \in (\underline{c}, \bar{c})$ such that $D(c) > 0$ for $c < c_0$, $D(c_0) = 0$, and $D(c) < 0$ for $c > c_0$. For a given $\gamma > 1$, let

$$D(c) = (\gamma - 1) - \frac{1}{c}\frac{F(c)}{f(c)}.$$

Whenever $D(c)$ single-crosses zero from above, let c_0 denote its crossing point and define

$$g(c) = (\gamma c)^{-\frac{1}{\gamma-1}} H(c) D(c), \qquad r(c) = \frac{1}{H(c)} - 1.$$

The upper endpoint of the interval imposed in Proposition 4 below is strictly positive under the maintained single-crossing condition, as shown in the proof of the proposition.

Proposition 4 *Suppose $\zeta(q) = q^\gamma$ with $\gamma > 1$, and $D(c)$ single-crosses zero from above. If*

$$\pi_P(0) \in \left(0,\ \frac{N}{\gamma} \cdot \frac{E_c\left[g(c)\left(r(c_0) - r(c)\right)\right]}{r(c_0)}\right), \tag{7}$$

then there exists $\bar{\rho} \in (0,1)$ such that $\pi_P'(\rho) > 0$ for every $\rho \in [0, \bar{\rho}]$.

Note that if $F(c)$ is a generalized uniform distribution as in Corollary 1, then $(\gamma - 1) - \frac{1}{c}\frac{F(c)}{f(c)}$ is strictly decreasing, and it single-crosses zero from above if and only if it is negative at $\bar{c}$, i.e., if and only if $\gamma < 1 + \frac{\bar{c}-\underline{c}}{\eta\bar{c}}$. The interval condition in (7) is satisfiable, for example, for F uniform on $[1,2]$ with $N = 2$ and $\gamma = 1.1$. The expression $(\gamma-1)-\frac{1}{c}\frac{F(c)}{f(c)}$ single-crosses zero from above at $c_0 = 10/9$, and we verify numerically that $\pi_P(0) \approx 0.00096$ lies below the upper endpoint ≈ 0.0080 in (7), with $\pi_P'(0) \approx 0.0089 > 0$.

The intuition behind Proposition 4 follows from the third expression in Lemma 2. The bracketed term $(1 - 1/\gamma) - [F(c)/f(c)]/(\gamma c)$ is the normalized virtual contribution of a type-c supplier to the procurer per unit of quality-weighted winning probability. The required single-crossing property means that it is positive for low-cost types and negative for high-cost types: when c is low, the total surplus is shared between the procurer and the suppliers, whereas when c is high, supplier information rents exceed the total surplus generated at the margin, so procuring positive quality from these types reduces the procurer's payoff — and the more quality they provide, the larger the loss. Under the fixed difference-score rule with no minimum score requirement, the procurer cannot screen out these negative-virtual-surplus suppliers. A higher ρ depresses quality provision across the board, but — as the quality formula (5) shows — proportionally most strongly for

high-cost suppliers with low winning probabilities, precisely the negative-virtual-surplus types; the reduction in their quality provision mitigates the procurer's losses on them. When $\pi_P(0)$ is small, this mitigation of losses dominates at the margin, so the procurer's payoff rises with ρ near zero; the upper endpoint in (7) makes "small" precise.

Proposition 4 gives a sufficient condition under which the procurer's payoff rises with ρ over an initial interval. Next, we show that even if $\pi_P'(\rho)$ is not positive for every ρ, it is still possible that $\pi_P(1) > \pi_P(0)$. Using the same c_0 and $g(c)$ defined above, define

$$\widetilde{r}(c) = 1 - H(c)^{\frac{1}{\gamma-1}}.$$

The upper endpoint of the interval imposed in Proposition 5 below is likewise strictly positive under the maintained single-crossing condition, as shown in the proof of the proposition.

Proposition 5 *Suppose $\zeta(q) = q^\gamma$ with $\gamma > 1$, and $D(c)$ single-crosses zero from above. If*

$$\pi_P(0) \in \left(0,\ \frac{N}{\gamma} \cdot \frac{E_c\left[g(c)\left(\widetilde{r}(c_0) - \widetilde{r}(c)\right)\right]}{\widetilde{r}(c_0)}\right), \tag{8}$$

then $\pi_P(1) > \pi_P(0)$.

The intuition behind Proposition 5 is the same virtual-surplus logic as that of Proposition 4: moving from $\rho = 0$ to $\rho = 1$ reduces quality provision proportionally most strongly for the negative-virtual-surplus high-cost types, thereby mitigating the procurer's losses on them. When $\pi_P(0)$ is small, in the precise sense of the threshold in (8), this mitigation dominates. In the numerical example following Proposition 4 (F uniform on $[1, 2]$, $N = 2$, $\gamma = 1.1$), this condition is also satisfied: we verify numerically that $\pi_P(0) \approx 0.00096$ lies below the threshold value ≈ 0.0023 in (8), and $\pi_P(1) - \pi_P(0) \approx 0.00094 > 0$.

Propositions 4 and 5 therefore identify circumstances in which the quality reduction caused by a higher all-pay share improves the procurer's payoff under the fixed difference-score rule. Section 5 will further show that this effect disappears when the score rule is chosen optimally: once the rule internalizes the screening and quality-distortion tradeoff, a higher all-pay share is no longer useful as a crude instrument for curbing quality provision by negative-virtual-surplus suppliers.

4 Reimbursing suppliers' all-pay bidding costs

The preceding analysis raises a natural policy question: should the procurer reimburse suppliers' all-pay bidding costs fully or partially? We study reimbursement of the losing suppliers' all-pay costs.

Let $\alpha \in [0,1]$ denote the proportion of all-pay costs to be reimbursed to losers. Consider a type-c_i supplier's problem of choosing $\tilde{c}_i$ to maximize his expected payoff when the other suppliers report truthfully:

$$\max_{\tilde{c}_i} \pi_i(\tilde{c}_i, c_i) = H(\tilde{c}_i)[p(\tilde{c}_i) - c_i\zeta(q(\tilde{c}_i))] + \{1 - H(\tilde{c}_i)\}(-(1-\alpha)\rho c_i\zeta(q(\tilde{c}_i))). \quad (9)$$

Let $\tilde{\rho} = (1-\alpha)\rho$. Note that $\pi_i(\tilde{c}_i, c_i)$ can be rewritten as

$$\pi_i(\tilde{c}_i, c_i) = H(\tilde{c}_i)[p(\tilde{c}_i) - (1-\tilde{\rho})c_i\zeta(q(\tilde{c}_i))] - \tilde{\rho}c_i\zeta(q(\tilde{c}_i)). \quad (10)$$

Comparing equations (10) and (2) gives the following result.

Lemma 3 *With reimbursement to losers of a proportion α of all-pay costs, suppliers' bidding behavior is identical to that in our model with $\tilde{\rho} = (1-\alpha)\rho$.*

Lemma 3 isolates the economics of reimbursement. Reimbursing a proportion α of the losers' all-pay costs makes suppliers behave exactly as if the all-pay share were lower, $\tilde{\rho} = (1-\alpha)\rho$; however, it does not reduce the social cost of quality provision — it only shifts part of that cost from the suppliers to the procurer, and thereby induces higher quality provision. Since the social marginal cost of quality is unchanged, the additional quality provision is socially excessive. As the results below formalize, supplier payoffs rise, while total surplus falls and — since the procurer's payoff is total surplus net of supplier payoffs — so does the procurer's payoff.

Lemma 3 and the comparative statics of Section 3.2 yield the following result.

Proposition 6 *Suppose $\rho > 0$ and the procurer reimburses losers for a proportion α of their all-pay costs. A higher reimbursement proportion weakly raises quality for every type (strictly for $c \in (\underline{c}, \bar{c})$) and weakly raises every type's payoff when $\varphi = \zeta \circ (\zeta')^{-1}$ is convex, as when $\zeta(q) = q^\gamma$ with $\gamma > 1$.*

Proposition 6 follows from Lemma 3: a higher α lowers the effective all-pay share $\tilde{\rho} = (1-\alpha)\rho$, so Propositions 1 and 2 apply with ρ replaced by $\tilde{\rho}$. A higher reimbursement

rate lowers suppliers' effective marginal cost and therefore raises quality and supplier payoffs. We next examine total surplus and the procurer's payoff.

For the sake of tractability, we investigate these issues under the assumption that $\zeta(q) = q^\gamma, \gamma > 1$. Note that under this specification, by (5), we have that

$$q(c;\tilde{\rho}) = \left[\frac{1}{\gamma c[(1-\tilde{\rho}) + \frac{\tilde{\rho}}{H(c)}]}\right]^{\frac{1}{\gamma-1}}.$$

We first look at total surplus denoted by $TS(\alpha)$. Let $c_N = \min\{c_i\}$ and $F_N(c_N) = 1 - [1 - F(c_N)]^N$ be the cumulative distribution function of c_N. To simplify notation, write $q_\alpha(c) \equiv q(c;\tilde{\rho})$. Then

$$TS(\alpha) = E_{c_N}[q_\alpha(c_N) - (1-\rho)c_N\zeta(q_\alpha(c_N))] - N\rho E_c[c\zeta(q_\alpha(c))]\,. \tag{11}$$

Lemma 4 *If $\rho > 0$, $TS(\alpha)$ is weakly decreasing in α and is strictly lower for every $\alpha > 0$ than at $\alpha = 0$.*

Proposition 6 and Lemma 4 lead to the following results.

Proposition 7 *Suppose $\rho > 0$ and $\zeta(q) = q^\gamma$ with $\gamma > 1$. Any positive reimbursement of losers' all-pay costs lowers total surplus and the procurer's payoff relative to no reimbursement; both are weakly decreasing in α.*

The distinction between reimbursement and a genuine reduction in ρ is important. Reimbursement changes suppliers' behavior through the effective share $\tilde{\rho} = (1-\alpha)\rho$, but it neither reduces the real resource cost of quality nor eliminates the procurer's reimbursement payment. It therefore raises supplier payoffs while lowering total surplus, and both effects reduce the procurer's payoff.

5 Procurer-optimal score rule

We now turn to deriving the procurer-optimal score rule. In this section, the procurer's payoff is generalized to be $u(q) - p$, where $\{q, p\}$ is the bid of the winning bidder. Here $u(\cdot)$ is continuously differentiable, increasing, and concave, with $u(0) = 0$ and $u'(q) > 0$ for every $q \geq 0$. Supplier i's payoff is the same as specified before. In this section, we assume that the virtual cost function $c + \frac{F(c)}{f(c)}$ is strictly increasing in c. Our analysis proceeds in two steps. First, we derive the optimal quality schedule $q^*(\cdot)$

among symmetric incentive-compatible direct mechanisms with the rank-order winning rule; Proposition 8 states the class precisely, and Remark 1 delimits its scope. Second, we show that $q^*(\cdot)$ is implemented by an explicit quasi-linear score rule.

Based on the revelation principle, we consider incentive-compatible symmetric direct mechanisms denoted by $(q(\cdot), p(\cdot))$. A report c_i' selects the quality bid $q(c_i')$; if it is the lowest report, supplier i wins and receives $p(c_i')$.

Under the all-pay cost structure, the object that governs incentive compatibility is neither the quality schedule $q(\cdot)$ nor the winning probability $H(c)$ in isolation, but the all-pay-adjusted cost exposure

$$K(c) = [(1-\rho)H(c) + \rho]\,\zeta(q(c)), \tag{12}$$

the expected quality cost borne by a truthfully reporting type-c supplier per unit of his cost parameter. Writing $U(c)$ for type c's equilibrium expected payoff, the envelope condition takes the form

$$U'(c) = -K(c), \tag{13}$$

which specializes (6) to the direct-mechanism formulation. Because the winning rule is fixed, $H(\cdot)$ is exogenous; the procurer optimizes over the quality schedule $q(\cdot)$ — equivalently, over the exposure profile $K(\cdot)$ — with payments pinned down by (13) and the boundary condition $U(\bar{c}) = 0$. It is precisely in this sense that the quality bid function $q(\cdot)$ plays the role of the allocation rule.

We first write the procurer's expected payoff as a function of the quality schedule $q(\cdot)$.

Lemma 5 *Given $q(\cdot)$, the procurer's expected payoff is*

$$PS(q(\cdot)) = NE_c\left\{u(q(c))H(c) - [(1-\rho)H(c)+\rho]\left(c + \frac{F(c)}{f(c)}\right)\zeta(q(c))\right\}. \tag{14}$$

Pointwise maximization of $PS(q(\cdot))$ leads to the following $q^*(c)$:

$$\frac{\zeta'(q^*(c))}{u'(q^*(c))} = \frac{H(c)}{[(1-\rho)H(c)+\rho]} \cdot \frac{1}{c + \frac{F(c)}{f(c)}}. \tag{15}$$

Note that both $\frac{H(c)}{[(1-\rho)H(c)+\rho]}$ and $\frac{1}{c+\frac{F(c)}{f(c)}}$ (under the assumption that the virtual cost $c + \frac{F(c)}{f(c)}$ is strictly increasing) are nonincreasing in c, and $\frac{\zeta'(q)}{u'(q)}$ increases with q. We thus have

$$q^*(c) = \left(\frac{\zeta'}{u'}\right)^{-1}\left(\frac{H(c)}{[(1-\rho)H(c)+\rho]} \cdot \frac{1}{c + \frac{F(c)}{f(c)}}\right), \tag{16}$$

which is decreasing in c. For $\rho > 0$, $q^*(\bar{c}) = 0$ because $\zeta'(0)/u'(0) = 0$. In the winner-pay benchmark $\rho = 0$, the right-hand side of (15) remains positive at $\bar{c}$, so $q^*(\bar{c})$ need not be zero. For every $c < \bar{c}$, $q^*(c)$ is weakly decreasing in ρ, and the decrease is strict for $c > \underline{c}$. Moreover, when $q = q^*$, the all-pay-adjusted cost exposure $K(c)$ in (12) is nonincreasing in c, because both $(1-\rho)H(c) + \rho$ and $q^*(c)$ are nonincreasing and ζ is increasing. Hence the pointwise maximizer satisfies the monotonicity condition required by incentive compatibility.

Equation (15) makes the marginal contribution of the all-pay environment precise, relative to Che [6]. The standard virtual-cost term $c + \frac{F(c)}{f(c)}$ is unchanged; the all-pay feature introduces exactly one new factor, the *all-pay wedge*[11]

$$\frac{H(c)}{(1-\rho)H(c) + \rho}.$$

In the winner-pay case ($\rho = 0$), the wedge equals one and (15) reduces to Che's optimality condition $\zeta'(q)/u'(q) = 1/[c + F(c)/f(c)]$. For $\rho > 0$, the wedge lies below one (strictly, for $c > \underline{c}$) and decreases in c, so the all-pay component depresses the optimal quality schedule, and does so most strongly for high-cost types.

We now look at how to induce $q^*(c)$ using a properly set score rule. Specifically, we choose $v(q)$ in the following quasi-linear specification $s = v(q) - p$ to induce $q^*(c)$, which generates an upper bound $PS(q^*(\cdot))$ for the procurer's expected payoff.

With score function $s = v(q) - p$, any equilibrium $(q(c_i), p(c_i))$ must solve the following optimization problem:

$$\begin{aligned} (q(c_i), p(c_i)) &= \arg\max_{(q,p)} \{H(c_i)[p - (1-\rho)c_i\zeta(q)] - \rho c_i \zeta(q)\} \\ s.t. \quad &: \quad s = v(q) - p = s(c_i), \end{aligned}$$

where $s(c_i)$ denotes the equilibrium score for type c_i.

The corresponding Lagrangian for the above problem is

$$\max_{(q,p,\lambda)} L(q, p, \lambda) = H(c_i)[p - (1-\rho)c_i\zeta(q)] - \rho c_i\zeta(q) + \lambda[(v(q) - p) - s(c_i)].$$

The first-order conditions give

$$\frac{\zeta'(q)}{v'(q)} = \frac{H(c_i)}{c_i\{H(c_i)(1-\rho) + \rho\}}. \tag{17}$$

[11] We thank the advisory editor for proposing this term, which is informative.

On the range of $q^*(\cdot)$, let $c(q;\rho)$ denote its inverse. From (15) and (17), to induce $q^*(c)$, set $v^*(\cdot)$ such that

$$v^{*\prime}(q) = \frac{c(q;\rho)}{(c(q;\rho) + \frac{F(c(q;\rho))}{f(c(q;\rho))})} u'(q) > 0. \tag{18}$$

For $\rho > 0$, set $v^*(0) = 0$. For $\rho = 0$, where the range of q^* begins at the positive value $q^*(\bar{c})$, extend the rule below that value by setting $v^{*\prime}(q) = \bar{c}\,\zeta'(q)$ and normalize $v^*(0) = 0$. This extension makes $q^*(\bar{c})$ optimal for the endpoint type and leaves the incentives of all types $c < \bar{c}$ unchanged.

In Appendix A, Lemma A1 shows that $\zeta'(q)/v^{*\prime}(q)$ is nondecreasing for $q \leq q^*(\underline{c})$ and strictly increasing on the equilibrium-quality range for types $c < \bar{c}$. When $\rho = 0$, the ratio equals $1/\bar{c}$ below $q^*(\bar{c})$, whereas the right-hand side of (17) equals $1/c > 1/\bar{c}$ for $c < \bar{c}$. Thus, (17) uniquely pins down $q^*(c)$ for every $c < \bar{c}$; at $\rho = 0$, the zero-probability endpoint is completed as described above.

Consider a procurement with the score rule $s^*(q,p) = v^*(q) - p$, for $q \leq q^*(\underline{c})$; $s^*(q,p) = v^*(q^*(\underline{c})) - p$, for $q > q^*(\underline{c})$. Applying the procedure in Section 3.1 yields the following result.

Proposition 8 *Consider the class of symmetric incentive-compatible direct procurement mechanisms in which each supplier's own report determines quality through a common schedule $q(\cdot)$, the supplier with the lowest reported cost wins, only the winner receives a transfer, and quality expenditures retain the maintained partially all-pay structure. If the virtual cost function $c + \frac{F(c)}{f(c)}$ is strictly increasing, then within this class a first-price procurement with quasi-linear score rule $s^*(q,p)$ maximizes the procurer's expected payoff. It induces equilibrium quality bid $q^*(c)$ and, for $c < \bar{c}$, the following price bid:*

$$p^*(c) = \frac{\int_c^{\bar{c}} \{H(t)(1-\rho)\zeta(q^*(t)) + \rho\zeta(q^*(t))\}dt + \rho c\zeta(q^*(c))}{H(c)} + (1-\rho)c\zeta(q^*(c)) \geq 0. \tag{19}$$

The equilibrium score is strictly decreasing for $c < \bar{c}$. Under the maintained assumption $\zeta'(0) = 0$, every type $c < \bar{c}$ is active.

Under $s^*(q,p)$, the schedules $(q^*(c), p^*(c))$ form the candidate equilibrium. The following lemma verifies score monotonicity and global incentive compatibility. Its proof uses only monotonicity and integrated envelope identities, so it does not require $q^*(\cdot)$ to be differentiable.

Lemma 6 *Define the equilibrium score by $s^*(c) = v^*(q^*(c)) - p^*(c)$. Then (i) $s^*(c)$ is strictly decreasing for $c < \bar{c}$; and (ii) no type c_i has an incentive to deviate from submitting $s^*(c_i)$.*

Lemma 6 establishes that $(q^*(c), p^*(c))$ is a symmetric equilibrium under s^*. Because it induces $q^*(c)$ and satisfies $\pi_i(\bar{c}, \bar{c}) = 0$, it attains the bound $PS(q^*(\cdot))$ and therefore maximizes the procurer's expected payoff within the class stated in Proposition 8.

As in Myerson [12], a version of payoff equivalence holds in our setting: the procurer's and suppliers' expected payoffs are determined by the quality schedule $q(\cdot)$ and the least efficient type's payoff $\pi_i(\bar{c}, \bar{c})$. Allowing endogenous entry within this class does not change $q^*(\cdot)$; because $q^*(c) > 0$ for every $c < \bar{c}$, the optimum does not exclude any positive-probability type.

If $\zeta'(0) > 0$, the pointwise problem in (16) may instead yield $q^*(c) = 0$ above a cutoff, so the all-participation conclusion need not hold. A complete implementation of that corner case lies outside the maintained assumptions of Proposition 8. Section 6.3 and Appendix B illustrate how our equilibrium-characterization procedure accommodates $\zeta'(0) > 0$ under a ratio score rule.

Remark 1 *Proposition 8 is an optimality result within the class stated there, in which payments are pinned down by incentive compatibility via (13). In our one-dimensional private-cost environment, incentive compatibility itself imposes the relevant monotonicity restriction, so the binding restrictions are symmetry and the rank-order winning rule rather than monotonicity per se. We do not claim optimality over broader classes of mechanisms: asymmetric mechanisms, mechanisms with arbitrary loser transfers or entry subsidies, mechanisms that alter the all-pay nature of quality expenditures, and mechanisms that optimize jointly over a more general winning-probability rule all lie outside the scope of our analysis.*

Corollary 2 *Relative to the procurer's payoff function, the optimal score rule places less weight on quality: $v^*(q) < u(q)$ for $q > 0$, while $v^{*\prime}(q) \leq u'(q)$ on the relevant quality range, with strict inequality except at its upper endpoint.*

On the equilibrium-quality range, (18) gives $v^{*\prime}(q) \leq u'(q)$, with equality only at the upper endpoint, where $c(q; \rho) = \underline{c}$. When $\rho = 0$, the extension below $q^*(\bar{c})$ satisfies $v^{*\prime}(q) = \bar{c}\zeta'(q) < u'(q)$, while above $q^*(\underline{c})$ the score is capped. Since $u(0) = v^*(0) = 0$, these derivative comparisons imply $v^*(q) < u(q)$ for every $q > 0$. Because u is concave while quality costs are convex, stronger quality incentives increase required payments faster than the procurer's marginal benefit from quality. Higher quality also raises information rents. The procurer therefore optimally places less marginal weight on quality than does the payoff function u. Our finding echoes that of Che [6], who establishes the same result in his Section 4 for winner-pay auctions.

We further have the following result.

Corollary 3 *If the inverse reversed hazard rate $F(c)/f(c)$ is weakly convex, then, at every quality in the common domain, $v^{*\prime}(q)$ is weakly higher when ρ is higher. Thus, the optimal score rule places weakly more marginal weight on quality as the all-pay share rises.*

Equation (18) can be written as $v^{*\prime}(q) = u'(q)/[1 + (F(c(q;\rho))/f(c(q;\rho)))/c(q;\rho)]$. A higher ρ lowers $c(q;\rho)$. When $F(c)/f(c)$ is weakly convex and $\underline{c} > 0$, the ratio $[F(c)/f(c)]/c$ is nondecreasing in c; hence $v^{*\prime}(q)$ is weakly higher at the larger all-pay share. Intuitively, the procurer responds to the stronger all-pay distortion by making the score more responsive to quality.

The same argument applies to the number of suppliers. For fixed $\rho > 0$, a higher N lowers $q^*(c)$ and therefore lowers its inverse $c(q;N)$. Under weak convexity of $F(c)/f(c)$, the ratio $[F(c(q;N))/f(c(q;N))]/c(q;N)$ weakly falls with N.

Corollary 4 *If $F(c)/f(c)$ is weakly convex and $\rho > 0$, then, at every quality in the common domain, $v^{*\prime}(q)$ is weakly higher when N is higher. Thus, the optimal score rule places weakly more marginal weight on quality as competition increases.*

Propositions 4 and 5 show that, with a linear score rule, a higher ρ can raise the procurer's payoff. The next proposition shows that this ambiguity disappears under the optimal score rule.

Proposition 9 *The procurer's expected payoff under the optimal score rule is strictly decreasing in ρ.*

Although the optimal score rule itself varies with ρ, Lemma 5 and the envelope theorem give the derivative of the optimized payoff as

$$NE_c\left\{[H(c)-1]\left(c+\frac{F(c)}{f(c)}\right)\zeta(q^*(c))\right\}<0.$$

Proposition 9 connects directly to the virtual-surplus intuition behind Propositions 4 and 5. Under the fixed difference-score rule with no threshold, the procurer is exposed to high-cost suppliers with negative virtual surplus, and a higher all-pay share serves as a crude instrument that suppresses precisely those suppliers' quality provision. Once the score rule is chosen optimally, it already internalizes the screening and quality-distortion

tradeoff — the all-pay wedge in (15) scales down quality incentives for inefficient types — so the crude instrument is no longer needed. The envelope calculation above then shows that only the direct cost effect of ρ remains, and the optimized procurer payoff is strictly decreasing in ρ. The possible beneficial effect of a higher all-pay proportion discovered in Section 3.2 is therefore an artifact of limited flexibility in choosing the score rule, not a general property of all-pay procurement.

5.1 Restricted optimum

In situations where the procurer has limited flexibility in choosing the optimal score rule, our procedure can be adapted to identify the restricted optimum among a family of score rules.

For example, suppose the procurer is restricted to a family of linear score rules of the following form: $s = \mu q - p$ where $\mu > 0$. Assume $u(q) = q$ and $\zeta(q) = q^2$. From (17), we have

$$q(c;\mu) = \frac{\mu}{2}\frac{H(c)}{c\{H(c)(1-\rho)+\rho\}}.$$

Plugging $q(c;\mu)$ into $PS(q(\cdot))$, we have

$$PS(\mu) = N\int_{\underline{c}}^{\bar{c}} \left\{ \frac{H(c)^2}{c\{H(c)(1-\rho)+\rho\}} \left(\frac{\mu}{2} - (c + \frac{F(c)}{f(c)}) \left(\frac{\mu}{2}\right)^2 \frac{1}{c} \right) \right\} f(c)dc.$$

The unique optimum is

$$\mu^* = \frac{\int_{\underline{c}}^{\bar{c}} \frac{H(c)^2}{c\{H(c)(1-\rho)+\rho\}} f(c)dc}{\int_{\underline{c}}^{\bar{c}} \frac{H(c)^2}{c\{H(c)(1-\rho)+\rho\}} (1 + \frac{F(c)}{cf(c)}) f(c)dc} < 1.$$

6 Extensions

This section extends our equilibrium-characterization procedure to minimum-score requirements, multi-dimensional quality, and nonlinear score rules. We also characterize an all-pay equilibrium under a ratio score rule to illustrate the procedure.

6.1 Minimum score requirements

In this subsection, we show how the approach accommodates a minimum-score requirement. Throughout the subsection, $\rho > 0$. A minimum score $\hat{s} \geq 0$ may induce endogenous participation; a positive threshold excludes an upper tail of types. We focus on the

entry equilibrium where all suppliers adopt a symmetric entry threshold $\hat{c}$, and we look for a symmetric bidding equilibrium across participants. In our analysis, we assume that the number of participants is not disclosed when they make their bidding decisions.

To identify the entry and bidding equilibrium, start with an entry threshold $\hat{c} \in [\underline{c}, \bar{c}]$, determine the minimum score $\hat{s}$ that induces it, and then characterize bidding by the participating types.

The threshold type must earn zero expected payoff in equilibrium, i.e., $\pi_i(\hat{c}, \hat{c}) = 0$. First, the second observation in Section 3.1 remains valid. Therefore, the equilibrium quality bid $q(c_i)$ of any participating type remains the same as identified in (5). We next pin down their equilibrium price bids.

Using $\pi_i(\hat{c}, \hat{c}) = 0$ and the envelope condition (25) that is derived based on the first observation in Section 3.1, we have

$$\pi_i(c_i, c_i; \hat{c}) = \int_{c_i}^{\hat{c}} \{H(t)(1-\rho)\zeta(q(t)) + \rho\zeta(q(t))\}dt. \tag{20}$$

Combining the definition of $\pi_i(\tilde{c}_i, c_i)$ in (2) with (20) gives the equilibrium price rule

$$p(c_i; \hat{c}) = \frac{\int_{c_i}^{\hat{c}} \{H(t)(1-\rho)\zeta(q(t)) + \rho\zeta(q(t))\}dt + \rho c_i \zeta(q(c_i))}{H(c_i)} + (1-\rho)c_i\zeta(q(c_i)) \geq 0. \tag{21}$$

The price rule can equivalently be written as

$$p(c_i; \hat{c}) = p(c_i; \bar{c}) - \frac{\int_{\hat{c}}^{\bar{c}} \{H(t)(1-\rho)\zeta(q(t)) + \rho\zeta(q(t))\}dt}{H(c_i)}.$$

Let $s(c_i; \hat{c}) = q(c_i) - p(c_i; \hat{c})$ for $c_i \leq \hat{c}$. Thus,

$$s(c_i; \hat{c}) = s(c_i; \bar{c}) + \frac{\int_{\hat{c}}^{\bar{c}} \{H(t)(1-\rho)\zeta(q(t)) + \rho\zeta(q(t))\}dt}{H(c_i)}, \forall c_i \leq \hat{c}.$$

The score $s(c_i; \hat{c})$ decreases with $\hat{c}$. In Appendix A, we further verify (Lemma A2) that $s(c_i; \hat{c})$ also decreases with $c_i \leq \hat{c}$.

Because $s(c_i; \hat{c})$ decreases in both arguments, the diagonal score $s(\hat{c}; \hat{c})$ decreases in $\hat{c} \in [\underline{c}, \bar{c}]$. Note that $s(\bar{c}; \bar{c}) = 0$ as $q(\bar{c}) = 0$ and $p(\bar{c}; \bar{c}) = 0$. Let $\bar{s} = s(\underline{c}; \underline{c}) > 0$. Then, for every $\hat{s} \in [0, \bar{s})$, there is a unique $\hat{c} \in (\underline{c}, \bar{c}]$ satisfying $s(\hat{c}; \hat{c}) = \hat{s}$. At $\hat{s} = \bar{s}$, procurement occurs with probability zero. We thus have the following result.

Proposition 10 *For any $\hat{s} \in [0, \bar{s})$ satisfying $s(\hat{c}; \hat{c}) = \hat{s}$, the symmetric entry threshold is $\hat{c}$, and each participating type $c_i \leq \hat{c}$ bids $\{q(c_i), p(c_i; \hat{c})\}$.*[12] *At $\hat{s} = \bar{s}$, procurement*

[12] $q(c_i)$ is defined in (5), and $p(c_i; \hat{c})$ is defined in (21).

occurs only on a zero-probability event; for $\hat{s} > \bar{s}$, no type is willing to meet the minimum score.

The same single-crossing argument as in Lemma 1 verifies truthful score choice among participants. Scores below $\hat{s}$ are ineligible, while a score above the highest equilibrium score wins for sure and is weakly worse than the highest equilibrium score, which already wins with probability one. For an excluded type $c_i > \hat{c}$, the best eligible equilibrium score is the cutoff score $s(\hat{c}; \hat{c})$; its optimized payoff is below the cutoff type's zero payoff because the optimized payoff is strictly decreasing in the true cost parameter. Hence no excluded type enters. Finally, for $\hat{s} > \bar{s}$, observe that $\bar{s} = s(\underline{c}; \underline{c}) = \max_{q \geq 0}\{q - \underline{c}\zeta(q)\}$, because the cutoff type $\underline{c}$ wins with certainty, earns zero, and its quality satisfies $\underline{c}\zeta'(q(\underline{c})) = 1$ by (5) with $H(\underline{c}) = 1$. A type c_i whose score bid meets $\hat{s} > \bar{s}$ would thus earn $p - c_i\zeta(q) \leq \max_{q \geq 0}\{q - c_i\zeta(q)\} - \hat{s} \leq \bar{s} - \hat{s} < 0$ upon winning and a nonpositive payoff upon losing, so no type is willing to meet the minimum score.

6.1.1 Optimal minimum score

We next ask whether a minimum-score requirement can improve the procurer's payoff under the difference-score rule. Let $q(\cdot)$ denote the equilibrium quality schedule in (5).

Suppose the entry threshold is $\hat{c}$. In Appendix A, we show (Lemma A3) that the procurer's expected payoff $PS(\hat{c})$ is

$$N \int_{\underline{c}}^{\hat{c}} \left\{ q(c)H(c) - [(1-\rho)H(c) + \rho] \left(c + \frac{F(c)}{f(c)} \right) \zeta(q(c)) \right\} f(c)dc. \tag{22}$$

Differentiating (22) yields

$$\frac{dPS(\hat{c})}{d\hat{c}} = N \left\{ q(\hat{c})H(\hat{c}) - [(1-\rho)H(\hat{c}) + \rho] \left(\hat{c} + \frac{F(\hat{c})}{f(\hat{c})} \right) \zeta(q(\hat{c})) \right\} f(\hat{c}). \tag{23}$$

Consider $\zeta(q) = q^\gamma$, $\gamma > 1$. The equilibrium first-order condition implies

$$\gamma c\big[(1-\rho)H(c) + \rho\big] q(c)^{\gamma - 1} = H(c).$$

Using this identity in (23) yields

$$\frac{dPS(\hat{c})}{d\hat{c}} = N\big[(1-\rho)H(\hat{c}) + \rho\big] q(\hat{c})^\gamma \left[(\gamma - 1)\hat{c} - \frac{F(\hat{c})}{f(\hat{c})} \right] f(\hat{c}). \tag{24}$$

The derivative is positive for $\hat{c} > \underline{c}$ sufficiently close to $\underline{c}$. It is negative for $\hat{c}$ sufficiently close to $\bar{c}$ whenever $(\gamma - 1)\bar{c} < 1/f(\bar{c})$. Hence the zero minimum score, which corresponds to $\hat{c} = \bar{c}$, is then suboptimal.

Proposition 11 *Assume $\rho > 0$ and $\zeta(q) = q^\gamma$. If*

$$1 < \gamma < 1 + \frac{1}{\bar{c} f(\bar{c})},$$

then the optimal minimum score is strictly positive.

Proposition 11 shows that a minimum score can improve the procurer's payoff under a fixed score rule. Any interior optimal cutoff sets (23) to zero, and the corresponding minimum score is $\hat{s}^* = s(\hat{c}^*; \hat{c}^*)$. Under its sufficient condition, the derivative is positive just above $\underline{c}$ and negative near $\bar{c}$, so a global maximizer lies in $(\underline{c}, \bar{c})$.

The virtual-surplus logic behind Propositions 4 and 5 also suggests how a minimum-score requirement interacts with the all-pay share. The potential fixed-score benefit from a higher ρ identified in Section 3.2 stems from the procurer's exposure to negative-virtual-surplus suppliers, whom she cannot screen out absent an entry threshold. An optimally chosen minimum score can screen out an inefficient upper tail of suppliers. One would therefore expect the potential benefit from a higher all-pay share to be reduced, and possibly eliminated, once the threshold is chosen optimally — just as it disappears under the optimal score rule (Proposition 9). We do not pursue a formal analysis of this conjecture here.

6.2 Multi-dimensional quality

We next extend the analysis to certain multidimensional-quality environments. Let quality be $M \geq 2$ dimensional, $\mathbf{q} = (q_1, \ldots, q_M)$. We consider a given score function $S(\mathbf{q}, p) = v(\mathbf{q}) - p$. A representative supplier i's cost function is $C_i(\mathbf{q}; c_i) = c_i \zeta(\mathbf{q})$. This problem can be reduced to one with a single-dimensional quality bid.

For each feasible value v, consider the following problem:

$$\min_{\mathbf{q}} \zeta(\mathbf{q})$$

$$s.t. \quad : \quad v(\mathbf{q}) = v.$$

Assume this problem has a solution for each feasible v, and let $\mathbf{q}^*(v)$ denote a measurable cost-minimizing selection. Define $\tilde{C}_i(v; c_i) = c_i \zeta(\mathbf{q}^*(v))$, and $\tilde{S}(v, p) = v - p$. The equilibrium analysis under the score rule $\tilde{S}(v, p) = v - p$ and cost function $\tilde{C}_i(v; c_i)$ while treating v as a single-dimensional quality choice can be carried out following our procedure. Denote the resulting equilibrium, if it exists, by $\{v(c), p(c)\}$. It follows that the equilibrium bidding strategy in the original multi-dimensional quality setting is $\{\mathbf{q}^*(v(c)), p(c)\}$.

6.3 Nonlinear score rules

Our two-step procedure for identifying a symmetric pure-strategy equilibrium, when one exists, extends to a general score rule $s(q,p)$. However, the quality bid typically cannot be identified by solely utilizing the second observation in Section 3.1 when the score rule is not quasi-linear. With a general score rule, the restrictions on $(q(\cdot),p(\cdot))$, which are implied by the two observations in Section 3.1, together with the boundary conditions that the limit equilibrium payoff is zero as $c \uparrow \bar{c}$ and, for $\rho > 0$, the limit equilibrium quality is zero as $c \uparrow \bar{c}$, which follow from bounded price bids (Section 3.1), can pin down a candidate equilibrium quality schedule $q(\cdot)$ without first solving for the equilibrium score. The details are included in Appendix B.

In an application presented in Appendix B, we study a procurement with all-pay quality bids (i.e., $\rho = 1$) under the ratio score rule $s = s(q,p) = q/p$, defined for $p > 0$. Recall from Section 3.1 that, since price bids are bounded, we look for equilibria in which the limit quality and payoff are zero as $c \uparrow \bar{c}$. We show that if $\zeta'(0) = 0$, then no symmetric pure-strategy equilibrium with strictly decreasing scores exists in the maintained class for all-pay quality bids. This nonexistence result relies on the boundedness of price bids: without it, equilibria with strictly decreasing scores can exist, in which price bids diverge as $c_i \uparrow \bar{c}$ and the limit payoff can be positive. Appendix B gives an example. If $\zeta'(0) > 0$, the candidate equilibrium is, for $c_i < \bar{c}$,

$$q(c_i) = (\zeta')^{-1}\left(\frac{\bar{c}\zeta'(0)}{c_i}\right),\ p(c_i) = \frac{\int_{c_i}^{\bar{c}} \zeta(q(t))dt + c_i\zeta(q(c_i))}{H(c_i)}.$$

Whenever this price bid is bounded (for example, when $N = 2$ and the right limit $\zeta''(0)$ is finite and positive), it is the unique symmetric pure-strategy equilibrium with strictly decreasing scores in the maintained class; as in Section 3.1, uniqueness follows because the conditions that pin down the candidate are necessary for every equilibrium in the class. The equilibrium is weak, because every type is indifferent among all scores in the equilibrium support.

Hanazono, Nakabayashi, Sano, and Tsuruoka [9] provide a general treatment of winner-pay quality-bid score auctions (i.e., $\rho = 0$) under a general score rule. They adopt an approach that first identifies the equilibrium score bids. In particular, they explicitly obtain the bidding equilibrium under the ratio score rule. The key to identifying equilibrium bidding strategies is to reduce the multidimensional bidding problem to a one-dimensional problem in which suppliers bid only on a score. This reduction relies on the observation that a supplier can optimally choose quality and price to maximize his payoff conditional on winning with any given score. For each score s, the supplier's

optimal quality–price choice induces a value function that depends solely on the supplier's type and the score bid s. Using this value function, the original multidimensional bidding problem can be reformulated as a standard single-dimensional auction in terms of score bids.

The approach of Hanazono, Nakabayashi, Sano, and Tsuruoka [9] also applies to our environment with all-pay quality bids.[13] Our approach should be viewed as an alternative to, not a replacement for, their method.

7 Concluding remarks

In this paper, we study score procurement auctions in which suppliers' quality bids carry an all-pay component. Under the standard difference-score rule, we characterize the unique symmetric pure-strategy equilibrium with scores decreasing in type and bounded price bids, and derive policy-relevant comparative statics. We then characterize the procurer-optimal scoring rule. With power-form costs, increasing the proportion of the all-pay component of quality cost reduces equilibrium quality, total surplus, and suppliers' payoffs, while its effect on the procurer's payoff is ambiguous. Policies that reimburse losing suppliers for the all-pay component of their quality costs increase quality provision and raise suppliers' payoffs, but reduce total surplus and the procurer's payoff. Moreover, the possible benefit to the procurer from a higher all-pay proportion under a fixed score rule is not robust: it disappears once the score rule is chosen optimally, reflecting that it is an artifact of limited flexibility in the score rule.

The optimal scoring rule takes the form $s^*(q,p) = v^*(q) - p$: it subtracts the price from a quality index $v^*(\cdot)$ that places less weight on quality than the procurer's own payoff function $u(\cdot)$ and is flatter in quality. Under a weakly convex inverse reversed hazard rate, the optimal rule assigns weakly greater marginal weight to quality when the all-pay share rises or when more suppliers participate.

Finally, our equilibrium characterization bypasses the intermediate step of solving for equilibrium scores. We show that this direct approach extends naturally to minimum-score constraints, multi-dimensional quality bids, and nonlinear scoring rules.

[13] The details are available from the authors.

Appendix A

Proof of Price Bid in Equation (3)

If $q'(c_i) < 0$, we can directly verify that the single-crossing condition holds: $\frac{\partial^2 \pi_i(\tilde{c}_i, c_i)}{\partial \tilde{c}_i \partial c_i} > 0$.[14] Applying the envelope theorem to the incentive-compatible problem in (2) thus leads to

$$\frac{d\pi_i(c_i, c_i)}{dc_i} = -\{H(c_i)(1-\rho)\zeta(q(c_i)) + \rho\zeta(q(c_i))\}. \tag{25}$$

As stated in Section 3.1, bounded price bids imply that, regardless of the value of ρ, $\pi_i(\bar{c}, \bar{c}) = 0$, understood as the limit as $c_i \uparrow \bar{c}$.

Using the envelope condition (25), we have

$$\pi_i(c_i, c_i) = \int_{c_i}^{\bar{c}} \{H(t)(1-\rho)\zeta(q(t)) + \rho\zeta(q(t))\}dt.$$

Integrating the envelope condition yields the following price rule

$$p(c_i) = \frac{\int_{c_i}^{\bar{c}} \{H(t)(1-\rho)\zeta(q(t)) + \rho\zeta(q(t))\}dt + \rho c_i \zeta(q(c_i))}{H(c_i)} + (1-\rho)c_i\zeta(q(c_i)) \geq 0$$

which supports the quality provision rule $q(\cdot)$.

Derivation of the equilibrium quality bid

The corresponding Lagrangian for the optimization problem is

$$\max_{(q,p,\lambda)} L(q, p, \lambda) = H(c_i)[p - (1-\rho)c_i\zeta(q)] - \rho c_i \zeta(q) + \lambda[(q-p) - s(c_i)].$$

The first-order conditions are as follows.

$$\begin{aligned}
\frac{\partial L(q,p,\lambda)}{\partial q} &= -H(c_i)(1-\rho)c_i\zeta'(q) - \rho c_i\zeta'(q) + \lambda = 0, \\
\frac{\partial L(q,p,\lambda)}{\partial p} &= H(c_i) - \lambda = 0, \\
\frac{\partial L(q,p,\lambda)}{\partial \lambda} &= (q-p) - s(c_i) = 0.
\end{aligned}$$

[14] Note that $\frac{\partial^2 \pi_i(\tilde{c}_i, c_i)}{\partial \tilde{c}_i \partial c_i} = (N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i)(1-\rho)\zeta(q(\tilde{c}_i)) - H(\tilde{c}_i)(1-\rho)\zeta'(q(\tilde{c}_i))q'(\tilde{c}_i) - \rho\zeta'(q(\tilde{c}_i))q'(\tilde{c}_i) > 0$.

We thus have

$$\zeta'(q) = \frac{H(c_i)}{c_i\{H(c_i)(1-\rho)+\rho\}} = \frac{1}{c_i\left[(1-\rho)+\frac{\rho}{H(c_i)}\right]},$$

which is equivalent to the quality schedule in (5).

Proof of Lemma 1

Part (i): Note that from (3), we have

$$\begin{aligned} & H(c_i)p(c_i) \\ = & \int_{c_i}^{\bar{c}} \{H(t)(1-\rho)\zeta(q(t)) + \rho\zeta(q(t))\}dt \\ & +\rho c_i\zeta(q(c_i)) + H(c_i)(1-\rho)c_i\zeta(q(c_i)). \end{aligned}$$

By the first-order condition used to derive (5), $\frac{c_i[\rho+(1-\rho)H(c_i)]}{H(c_i)}\zeta'(q(c_i)) = 1$. We thus have

$$\begin{aligned} & [H(c_i)p(c_i)]' = \rho c_i\zeta'(q(c_i))q'(c_i) + c_i\frac{d\{H(c_i)(1-\rho)\zeta(q(c_i))\}}{dc_i} \\ \Leftrightarrow \quad & H(c_i)p'(c_i) - (N-1)(1-F(c_i))^{N-2}f(c_i)p(c_i) \\ = \quad & \rho c_i\zeta'(q(c_i))q'(c_i) + c_i\frac{d\{H(c_i)(1-\rho)\zeta(q(c_i))\}}{dc_i} \\ \Leftrightarrow \quad & p'(c_i) - \frac{(N-1)f(c_i)}{1-F(c_i)}p(c_i) \\ = \quad & q'(c_i)\frac{c_i[\rho+(1-\rho)H(c_i)]}{H(c_i)}\zeta'(q(c_i)) - \frac{(N-1)f(c_i)}{1-F(c_i)}c_i(1-\rho)\zeta(q(c_i)) \\ \Leftrightarrow \quad & p'(c_i) = q'(c_i) - \frac{(N-1)f(c_i)}{1-F(c_i)}c_i(1-\rho)\zeta(q(c_i)) + \frac{(N-1)f(c_i)}{1-F(c_i)}p(c_i) \\ \Leftrightarrow \quad & s'(c_i) = q'(c_i) - p'(c_i) = \frac{(N-1)f(c_i)}{1-F(c_i)}\{c_i(1-\rho)\zeta(q(c_i)) - p(c_i)\}. \end{aligned}$$

Note that from (3), we have $c_i(1-\rho)\zeta(q(c_i)) - p(c_i) < 0$. We thus have $s'(c_i) < 0$.

Part (ii): We show that a type c_i has no incentive to generate a score $s(c_i')$ where $c_i' \neq c_i$.

Suppose that a type c_i generates a score $s(c_i')$. Its optimal expected payoff is given by the following optimization problem:

$$\begin{aligned}(q(c_i', c_i), p(c_i', c_i)) &= \arg\max_{(q,p)}\{H(c_i')[p - (1-\rho)c_i\zeta(q)] - \rho c_i\zeta(q)\} \\ s.t. &: s = q - p = s(c_i').\end{aligned}$$

Equivalently, we have

$$q(c_i', c_i) = \arg\max_{q}\{H(c_i')[q - s(c_i') - (1-\rho)c_i\zeta(q)] - \rho c_i\zeta(q)\}.$$

Therefore, we have

$$q(c_i', c_i) = (\zeta')^{-1}\left(\frac{1}{c_i\{(1-\rho) + \frac{\rho}{H(c_i')}\}}\right). \tag{26}$$

The optimized expected payoff from this deviation to score $s(c_i')$ is thus

$$\pi_i(s(c_i'), c_i) = H(c_i')[q(c_i', c_i) - s(c_i') - (1-\rho)c_i\zeta(q(c_i', c_i))] - \rho c_i\zeta(q(c_i', c_i)).$$

We next establish the following single-crossing condition: $\frac{\partial^2\pi_i(s(c_i'), c_i)}{\partial c_i'\partial c_i} \geq 0$. By the envelope theorem, we have

$$\frac{\partial\pi_i(s(c_i'), c_i)}{\partial c_i} = -\{H(c_i')[(1-\rho)] + \rho\}\zeta(q(c_i', c_i)),$$

which is increasing in c_i'.

Similarly, by the envelope theorem, we have

$$\begin{aligned}\frac{\partial\pi_i(s(c_i'), c_i)}{\partial c_i'} &= \frac{\partial H(c_i')}{\partial c_i'}[q(c_i', c_i) - s(c_i') - (1-\rho)c_i\zeta(q(c_i', c_i))] \\ &\quad -H(c_i')s'(c_i').\end{aligned}$$

Recall that in part (i), we have $s'(c_i) = \frac{(N-1)f(c_i)}{1-F(c_i)}\{c_i(1-\rho)\zeta(q(c_i)) - p(c_i)\}$. For $c_i' = c_i$, we have

$$\begin{aligned}\frac{\partial\pi_i(s(c_i'), c_i)}{\partial c_i'}|_{c_i'=c_i} &= \frac{\partial H(c_i)}{\partial c_i}[q(c_i) - s(c_i) - (1-\rho)c_i\zeta(q(c_i))] \\ &\quad -H(c_i)s'(c_i) \\ &= -(N-1)[1-F(c_i)]^{N-2}f(c_i)[p(c_i) - (1-\rho)c_i\zeta(q(c_i))] \\ &\quad -H(c_i)\frac{(N-1)f(c_i)}{1-F(c_i)}\{c_i(1-\rho)\zeta(q(c_i)) - p(c_i)\} \\ &= 0.\end{aligned}$$

From the above results, we have $\frac{\partial \pi_i(s(c_i'),c_i)}{\partial c_i'}|_{c_i'=c_i} = 0$ and $\frac{\partial^2 \pi_i(s(c_i'),c_i)}{\partial c_i' \partial c_i} \geq 0$. Therefore, $\pi_i(s(c_i'), c_i)$ is globally maximized at $c_i' = c_i$. In other words, a type c_i has no incentive to generate a score $s(c_i')$ where $c_i' \neq c_i$. Scores below the equilibrium support lose with probability one and yield at most zero, whereas truthful equilibrium payoff is nonnegative. A score above the support wins with probability one, but the optimized payoff $\max_q\{q - c_i\zeta(q) - s\}$ is weakly decreasing in the required score s; hence it cannot improve on the highest equilibrium score. Thus no deviation outside the equilibrium score range is profitable.

Proof of Proposition 2

Fix $c_i < \bar{c}$ and let

$$x(\rho) = \frac{H(c_i)}{c_i\{(1-\rho)H(c_i) + \rho\}}.$$

Equation (6) implies

$$\left|\frac{d\pi_i(c_i, c_i)}{dc_i}\right| = \frac{H(c_i)}{c_i}\frac{\varphi(x(\rho))}{x(\rho)}.$$

For a convex function φ with $\varphi(0) = 0$, the ratio $\varphi(x)/x$ is nondecreasing for $x > 0$. Because $x(\rho)$ is nonincreasing in ρ, the absolute slope of the interim payoff is weakly decreasing in ρ. Together with $\pi_i(\bar{c}, \bar{c}) = 0$, this proves the result.

Proof of Lemma 2

For an equilibrium quality schedule $q(\cdot;\rho)$, expected total surplus is

$$\begin{aligned} TS(\rho) &= \int_{\underline{c}}^{\bar{c}} q(c;\rho) d[1-(1-F(c))^N] \\ &\quad -NE_c\left\{c[(1-\rho)H(c)+\rho]\zeta(q(c;\rho))\right\} \\ &= NE_c\left\{H(c)q(c;\rho) - c[(1-\rho)H(c)+\rho]\zeta(q(c;\rho))\right\}. \end{aligned}$$

A representative supplier's ex ante expected payoff is

$$\begin{aligned} \pi_S(\rho) &= \int_{\underline{c}}^{\bar{c}} \pi_i(c,c)\, dF(c) \\ &= \int_{\underline{c}}^{\bar{c}} \int_{c}^{\bar{c}} \{(1-\rho)H(t)+\rho\}\zeta(q(t;\rho))\, dt\, dF(c), \end{aligned}$$

and the procurer's expected payoff is $\pi_P(\rho) = TS(\rho) - N\pi_S(\rho)$.

Under $\zeta(q) = q^\gamma$, equation (5) implies

$$c[(1-\rho)H(c)+\rho]q(c;\rho)^\gamma = \frac{1}{\gamma}H(c)q(c;\rho).$$

Substituting this identity into the total-surplus expression gives

$$TS(\rho) = NE_c\left\{q(c;\rho)H(c)\left(1-\frac{1}{\gamma}\right)\right\},$$

which is the first formula in the lemma.

Changing the order of integration in the supplier-payoff expression gives

$$\begin{aligned} N\pi_S(\rho) &= N\int_{\underline{c}}^{\bar{c}}\{(1-\rho)H(c)+\rho\}q(c;\rho)^\gamma F(c)\,dc \\ &= NE_c\left\{\{(1-\rho)H(c)+\rho\}q(c;\rho)^\gamma\frac{F(c)}{f(c)}\right\} \\ &= NE_c\left\{q(c;\rho)H(c)\frac{1}{\gamma c}\frac{F(c)}{f(c)}\right\}, \end{aligned}$$

where the last equality uses the same identity. This is the second formula in the lemma. Finally, $\pi_P(\rho) = TS(\rho) - N\pi_S(\rho)$ gives the third formula.

Proof of Proposition 4

Substituting (5) into the third expression in Lemma 2 gives

$$\pi_P(\rho) = \frac{1}{\gamma}NE_c\left\{(\gamma c)^{-\frac{1}{\gamma-1}}\left[(1-\rho)+\frac{\rho}{H(c)}\right]^{-\frac{1}{\gamma-1}}H(c)\left[(\gamma-1)-\frac{1}{c}\frac{F(c)}{f(c)}\right]\right\},$$

and

$$\begin{aligned} \pi_P'(\rho) &= \frac{1}{\gamma}NE_c\Bigg\{(-\frac{1}{\gamma-1})\,(\gamma c)^{-\frac{1}{\gamma-1}}\left[(1-\rho)+\frac{\rho}{H(c)}\right]^{-\frac{1}{\gamma-1}-1}\left[\frac{1}{H(c)}-1\right] \\ &\quad \cdot H(c)\left[(\gamma-1)-\frac{1}{c}\frac{F(c)}{f(c)}\right]\Bigg\}. \end{aligned}$$

Therefore,

$$\pi_P(0) = \frac{1}{\gamma}NE_c\left\{(\gamma c)^{-\frac{1}{\gamma-1}}H(c)\left[(\gamma-1)-\frac{1}{c}\frac{F(c)}{f(c)}\right]\right\},$$

and

$$\pi_P'(0) = \frac{1}{\gamma}(\frac{1}{\gamma-1})NE_c\left\{(\gamma c)^{-\frac{1}{\gamma-1}}\left[H(c)-1\right]\left[(\gamma-1)-\frac{1}{c}\frac{F(c)}{f(c)}\right]\right\}.$$

Recall the definitions of $D(c)$, $g(c)$, $r(c)$, and c_0 preceding Proposition 4, and denote the upper endpoint in (7) by B_0. We first verify that this upper endpoint is strictly positive. Let

$$k = \frac{1}{\gamma-1} \qquad \text{and} \qquad H_0 = H(c_0).$$

Because $D(\underline{c}) = \gamma - 1 > 0$ and c_0 is an interior crossing point, $c_0 > \underline{c}$ and $H_0 \in (0,1)$. Hence

$$r(c_0) = \frac{1-H_0}{H_0} > 0.$$

Moreover, for $c < \bar{c}$,

$$r(c_0) - r(c) = \frac{1}{H_0} - \frac{1}{H(c)} = \frac{H(c)-H_0}{H_0 H(c)}.$$

Using $g(c) = (\gamma c)^{-k}H(c)D(c)$, we may therefore rewrite B_0 as

$$B_0 = \frac{N}{\gamma}\frac{E_c\big[(\gamma c)^{-k}D(c)\big(H(c)-H_0\big)\big]}{1-H_0}.$$

The function $H(c) = [1-F(c)]^{N-1}$ is strictly decreasing on the interior of the support. Thus $H(c) - H_0$ is positive for $c < c_0$ and negative for $c > c_0$. By the assumed single crossing, $D(c)$ has exactly the same sign. Consequently,

$$D(c)\big(H(c)-H_0\big) > 0$$

for every interior $c \neq c_0$. Since $f(c) > 0$ on the interior and $1 - H_0 > 0$, it follows that $B_0 > 0$.

We have

$$\pi_P(0) = \frac{N}{\gamma}E_c[g(c)], \qquad \pi_P'(0) = -\frac{N}{\gamma(\gamma-1)}E_c[g(c)r(c)].$$

Using the decomposition

$$E_c[g(c)r(c)] = r(c_0)E_c[g(c)] - E_c\left[g(c)\left(r(c_0)-r(c)\right)\right]$$

and the definition of B_0 gives

$$E_c[g(c)r(c)] = \frac{\gamma}{N}r(c_0)\left[\pi_P(0) - B_0\right].$$

Therefore,

$$\pi_P'(0) = \frac{r(c_0)}{\gamma-1}\left[B_0 - \pi_P(0)\right] > 0,$$

where the strict inequality follows from (7). It remains to justify right continuity at $\rho = 0$. For $\rho \in [0, 1/2]$, the factor $[(1-\rho) + \rho/H(c)]^{-1/(\gamma-1)-1}$ is bounded above by a constant, while $H(c)[1/H(c) - 1] = 1 - H(c) \leq 1$. Because c is bounded away from zero and $F(c)/f(c)$ has a continuous extension to the compact support, the remaining terms are bounded. Dominated convergence therefore implies that $\pi_P'(\rho)$ is right-continuous at zero. Hence there exists $\bar{\rho} \in (0, 1)$ such that $\pi_P'(\rho) > 0$ for every $\rho \in [0, \bar{\rho}]$.

Proof of Proposition 5

Recall

$$\pi_P(0) = \frac{1}{\gamma} N E_c \left\{ (\gamma c)^{-\frac{1}{\gamma-1}} H(c) \left[(\gamma - 1) - \frac{1}{c} \frac{F(c)}{f(c)} \right] \right\}.$$

Note

$$\pi_P(1) = \frac{1}{\gamma} N E_c \left\{ (\gamma c)^{-\frac{1}{\gamma-1}} H(c) \left[(\gamma - 1) - \frac{1}{c} \frac{F(c)}{f(c)} \right] [H(c)]^{\frac{1}{\gamma-1}} \right\}.$$

Therefore,

$$\begin{aligned} & \pi_P(1) - \pi_P(0) \\ = \; & -\frac{1}{\gamma} N E_c \left\{ (\gamma c)^{-\frac{1}{\gamma-1}} H(c) \left[(\gamma - 1) - \frac{1}{c} \frac{F(c)}{f(c)} \right] \left[1 - (H(c))^{\frac{1}{\gamma-1}} \right] \right\}. \end{aligned}$$

Recall the definitions of $D(c)$, $g(c)$, c_0, and $\widetilde{r}(c)$ preceding Proposition 5, and denote the upper endpoint in (8) by B_1. We first verify that this upper endpoint is strictly positive. Let

$$k = \frac{1}{\gamma - 1} \qquad \text{and} \qquad H_0 = H(c_0).$$

As in the proof of Proposition 4, $c_0 > \underline{c}$ and $H_0 \in (0, 1)$. Hence

$$\widetilde{r}(c_0) = 1 - H_0^k > 0.$$

Moreover,

$$\widetilde{r}(c_0) - \widetilde{r}(c) = H(c)^k - H_0^k.$$

Using $g(c) = (\gamma c)^{-k} H(c) D(c)$, we may therefore rewrite B_1 as

$$B_1 = \frac{N}{\gamma} \frac{E_c\left[(\gamma c)^{-k} H(c) D(c) \left(H(c)^k - H_0^k\right)\right]}{1 - H_0^k}.$$

Because $k > 0$ and $H(\cdot)$ is strictly decreasing on the interior of the support, $H(c)^k - H_0^k$ is positive for $c < c_0$ and negative for $c > c_0$. By the assumed single crossing, $D(c)$ has exactly the same sign. Thus,

$$D(c)\left(H(c)^k - H_0^k\right) > 0$$

for every interior $c \neq c_0$. Since $H(c) > 0$ and $f(c) > 0$ on the interior, while $1 - H_0^k > 0$, it follows that $B_1 > 0$.

Moreover,

$$\pi_P(0) = \frac{N}{\gamma} E_c[g(c)], \qquad \pi_P(1) - \pi_P(0) = -\frac{N}{\gamma} E_c[g(c)\widetilde{r}(c)].$$

Using

$$E_c[g(c)\widetilde{r}(c)] = \widetilde{r}(c_0)E_c[g(c)] - E_c\left[g(c)\left(\widetilde{r}(c_0) - \widetilde{r}(c)\right)\right]$$

and the definition of B_1 gives

$$E_c[g(c)\widetilde{r}(c)] = \frac{\gamma}{N}\widetilde{r}(c_0)\left[\pi_P(0) - B_1\right].$$

Therefore,

$$\pi_P(1) - \pi_P(0) = \widetilde{r}(c_0)\left[B_1 - \pi_P(0)\right] > 0,$$

where the strict inequality follows from (8).

Proof of Lemma 4

Let $\tilde{\rho} = (1 - \alpha)\rho$ and, for $c < \bar{c}$, define

$$A(c, \tilde{\rho}) = (1 - \tilde{\rho}) + \frac{\tilde{\rho}}{H(c)}.$$

Under $\zeta(q) = q^\gamma$, equilibrium quality is

$$q(c; \tilde{\rho}) = [\gamma c A(c, \tilde{\rho})]^{-1/(\gamma - 1)}.$$

Using (11), total surplus can be written as

$$TS(\alpha) = N \int_{\underline{c}}^{\bar{c}} G(c, q(c; \tilde{\rho})) f(c)\, dc, \qquad G(c, q) = qH(c) - c[(1 - \rho)H(c) + \rho]q^\gamma.$$

For $c < \bar{c}$,

$$\frac{\partial q(c; \tilde{\rho})}{\partial \tilde{\rho}} = -\frac{q(c; \tilde{\rho})}{\gamma - 1} \frac{H(c)^{-1} - 1}{A(c, \tilde{\rho})} \leq 0.$$

Moreover, the equilibrium first-order condition gives

$$\frac{\partial G}{\partial q} = H(c) - \gamma c[(1-\rho)H(c) + \rho]q(c;\tilde{\rho})^{\gamma-1}$$

$$= -\frac{(\rho - \tilde{\rho})[1 - H(c)]}{A(c,\tilde{\rho})} \leq 0.$$

Consequently, $\partial G/\partial\tilde{\rho} \geq 0$. Since $d\tilde{\rho}/d\alpha = -\rho < 0$, $TS(\alpha)$ is weakly decreasing in α. For every $\alpha > 0$, the inequalities are strict on a set of types of positive measure, so $TS(\alpha) < TS(0)$.

Proof of Proposition 7

By Lemma 3, equilibrium bidding behavior under reimbursement proportion α coincides with that in the benchmark model with all-pay share $\tilde{\rho} = (1-\alpha)\rho$, which decreases in α. The procurer's expected payoff satisfies the accounting identity

$$\pi_P(\alpha) = TS(\alpha) - N\pi_S(\alpha),$$

where $\pi_S(\alpha)$ denotes a representative supplier's expected payoff under reimbursement proportion α. Lemma 4 establishes that $TS(\alpha)$ is weakly decreasing in α and is strictly lower for every $\alpha > 0$ than at $\alpha = 0$. Proposition 6 establishes that a higher α weakly raises every type's payoff and hence weakly raises $\pi_S(\alpha)$, since $\zeta(q) = q^\gamma$ implies that $\varphi = \zeta \circ (\zeta')^{-1}$ is convex. Hence $\pi_P(\alpha)$ decreases in α. In particular, reimbursing the losers' all-pay component (any $\alpha > 0$) generates lower total surplus and a lower procurer payoff than no reimbursement ($\alpha = 0$).

Proof of Lemma 5

Let $c_N = \min\{c_i\}$ and $F_N(c_N) = 1 - [1 - F(c_N)]^N$ be the cumulative distribution function of c_N. The total expected surplus of the procurer and suppliers is

$$\begin{aligned} TS &= \int_{\underline{c}}^{\bar{c}} [u(q(c_N)) - (1-\rho)c_N\zeta(q(c_N))]\, dF_N(c_N) - N\rho\int_{\underline{c}}^{\bar{c}} c\zeta(q(c))dF(c) \\ &= N\int_{\underline{c}}^{\bar{c}} \{[u(q(c)) - (1-\rho)c\zeta(q(c))]\, H(c) - \rho c\zeta(q(c))\}\, f(c)dc. \end{aligned} \tag{27}$$

Equation (6) gives the slope of a representative supplier's interim expected payoff as

$$\frac{d\pi_i(c,c)}{dc} = -\{H(c)(1-\rho) + \rho\}\zeta(q(c)).$$

Since $\pi_i(\bar{c},\bar{c}) = 0$, the representative supplier's interim expected payoff is

$$\pi_i(c,c) = \int_c^{\bar{c}} \{H(c')(1-\rho)+\rho\}\zeta(q(c'))dc'.$$

It follows that a representative supplier's expected payoff is

$$\begin{aligned}
E\pi_i &= \int_{\underline{c}}^{\bar{c}} \left[\int_c^{\bar{c}} \{H(c')(1-\rho)+\rho\}\zeta(q(c'))dc'\right] f(c)dc \\
&= \int_{\underline{c}}^{\bar{c}} \left[\int_{\underline{c}}^{c'} f(c)dc\right] \{H(c')(1-\rho)+\rho\}\zeta(q(c'))dc' \\
&= \int_{\underline{c}}^{\bar{c}} F(c')\{H(c')(1-\rho)+\rho\}\zeta(q(c'))dc' \\
&= \int_{\underline{c}}^{\bar{c}} \frac{F(c)}{f(c)}\{H(c)(1-\rho)+\rho\}\zeta(q(c))f(c)dc.
\end{aligned}$$

The procurer's expected payoff is therefore

$$\begin{aligned}
& PS(q(\cdot)) = TS - N\cdot E\pi_i \\
= \; & N\int_{\underline{c}}^{\bar{c}} \{[u(q(c)) - (1-\rho)c\zeta(q(c))]\, H(c) - \rho c\zeta(q(c))\}\, f(c)dc \\
& -N\int_{\underline{c}}^{\bar{c}} \frac{F(c)}{f(c)}\{H(c)(1-\rho)+\rho\}\zeta(q(c))f(c)dc \\
= \; & N\int_{\underline{c}}^{\bar{c}} \left\{u(q(c))H(c) - [(1-\rho)H(c)+\rho]\,(c+\frac{F(c)}{f(c)})\zeta(q(c))\right\} f(c)dc.
\end{aligned}$$

Lemma A1 *$\zeta'(q)/v^{*\prime}(q)$ is nondecreasing for $q \le q^*(\underline{c})$ and strictly increasing on the equilibrium-quality range for types $c < \bar{c}$.*

Proof of Lemma A1

On the equilibrium-quality range, $q^*(c)$ decreases strictly with c. It therefore suffices to show that $\zeta'(q^*(c))/v^{*\prime}(q^*(c))$ decreases with c. Note that

$$\begin{aligned}
\frac{\zeta'(q^*(c))}{v^{*\prime}(q^*(c))} &= \frac{\zeta'(q^*(c))}{u'(q^*(c))}\left(\frac{c(q^*(c);\rho)+\frac{F(c(q^*(c);\rho))}{f(c(q^*(c);\rho))}}{c(q^*(c);\rho)}\right) \\
&= \frac{H(c)}{[(1-\rho)H(c)+\rho]}\cdot\frac{1}{c+\frac{F(c)}{f(c)}}\left(\frac{c+\frac{F(c)}{f(c)}}{c}\right) \\
&= \frac{H(c)}{[(1-\rho)H(c)+\rho]}\cdot\frac{1}{c}.
\end{aligned}$$

Therefore, $\frac{\zeta'(q^*(c))}{v^{*\prime}(q^*(c))}$ decreases strictly in c: the all-pay wedge $H(c)/[(1-\rho)H(c)+\rho]$ is nonincreasing, while $1/c$ is strictly decreasing. Thus, the ratio is strictly increasing on the equilibrium-quality range for $c < \bar{c}$. When $\rho = 0$, the extension below $q^*(\bar{c})$ gives $\zeta'(q)/v^{*\prime}(q) = 1/\bar{c}$, so the ratio is nondecreasing on the full interval $[0, q^*(\underline{c})]$.

Proof of Lemma 6

The argument uses only monotonicity and integrated envelope identities. Under the maintained assumptions, $q^*(\cdot)$ is continuous and strictly decreasing on $[\underline{c}, \bar{c})$, but it need not be differentiable, because u' and F/f are only assumed to be continuous.

Preliminaries. For $c < \bar{c}$, let $A(c) = (1-\rho)H(c)+\rho$, $\lambda(c) = A(c)/H(c) = (1-\rho)+\rho/H(c)$, and $\kappa(c) = c\lambda(c)$. The function $\lambda(\cdot)$ is nondecreasing, and $\kappa(\cdot)$ is strictly increasing and continuously differentiable on $(\underline{c}, \bar{c})$. Condition (17) states that $\zeta'(q^*(c))/v^{*\prime}(q^*(c)) = 1/\kappa(c)$. Because $s^*(q,p)$ does not reward quality above $q^*(\underline{c})$, such quality is never optimal, and we restrict attention to $q \in [0, q^*(\underline{c})]$. For $\kappa > 0$, define

$$\Psi(\kappa) = \max_{0\le q\le q^*(\underline{c})}\{v^*(q) - \kappa\zeta(q)\}.$$

The derivative of the objective is $v^{*\prime}(q)[1-\kappa\zeta'(q)/v^{*\prime}(q)]$, where $v^{*\prime}(q) > 0$ for $q > 0$. By (18) and the continuity of $c(q;\rho)$, the ratio $\zeta'(q)/v^{*\prime}(q)$ is continuous on $(0, q^*(\underline{c})]$. By Lemma A1, it is nondecreasing and strictly increasing on the equilibrium-quality range; by (17), it equals $1/\underline{c}$ at $q^*(\underline{c})$. If $\rho > 0$, it tends to zero as $q \downarrow 0$, because $\zeta'(0) = 0$; if $\rho = 0$, it equals $1/\bar{c}$ on $(0, q^*(\bar{c})]$. Let $\mathcal{K} = \{c\lambda(c') : c, c' \in [\underline{c}, \bar{c})\}$, which equals $[\underline{c}, +\infty)$ if $\rho > 0$ and $[\underline{c}, \bar{c})$ if $\rho = 0$. For every $\kappa \in \mathcal{K}$, the equation $\zeta'(q)/v^{*\prime}(q) = 1/\kappa$ therefore has a unique solution $\hat{q}(\kappa)$ on the equilibrium-quality range; the objective is strictly increasing in q below $\hat{q}(\kappa)$ and strictly decreasing above it, so $\hat{q}(\kappa)$ is its unique maximizer. Hence $\hat{q}(\cdot)$ is nonincreasing on $\mathcal{K}$, and $\hat{q}(\kappa(c)) = q^*(c)$ by (17).

Step 1 (integrated envelope identity). Let $\kappa_1 < \kappa_2$ in $\mathcal{K}$. By revealed preference, $\Psi(\kappa_2) \geq v^*(\hat{q}(\kappa_1)) - \kappa_2\zeta(\hat{q}(\kappa_1)) = \Psi(\kappa_1) - (\kappa_2 - \kappa_1)\zeta(\hat{q}(\kappa_1))$, and symmetrically $\Psi(\kappa_1) \geq \Psi(\kappa_2) + (\kappa_2 - \kappa_1)\zeta(\hat{q}(\kappa_2))$. Hence

$$-(\kappa_2 - \kappa_1)\zeta(\hat{q}(\kappa_1)) \leq \Psi(\kappa_2) - \Psi(\kappa_1) \leq -(\kappa_2 - \kappa_1)\zeta(\hat{q}(\kappa_2)).$$

Applying these bounds on the cells of a partition of $[\kappa_1, \kappa_2]$ and using that $\zeta(\hat{q}(\cdot))$ is monotone, and hence Riemann integrable, we obtain

$$\Psi(\kappa_2) - \Psi(\kappa_1) = -\int_{\kappa_1}^{\kappa_2} \zeta(\hat{q}(\kappa))\, d\kappa. \tag{28}$$

Step 2 (truthful payoff). Let $U^*(c) = \int_c^{\bar{c}} A(t)\zeta(q^*(t))\, dt$. By (19), $H(c)p^*(c) = U^*(c) + A(c)c\zeta(q^*(c))$. Since $v^*(q^*(c)) - s^*(c) = p^*(c)$ and $\hat{q}(\kappa(c)) = q^*(c)$, we have, for $c < \bar{c}$,

$$H(c)\big[\Psi(\kappa(c)) - s^*(c)\big] = H(c)p^*(c) - A(c)c\zeta(q^*(c)) = U^*(c).$$

Part (i). By the identity in Step 2, $s^*(c) = \Psi(\kappa(c)) - U^*(c)/H(c)$ for $c < \bar{c}$. Let $\underline{c} < c_1 < c_2 < \bar{c}$. By (28) and the change of variables $\kappa = \kappa(t)$,

$$\Psi(\kappa(c_2)) - \Psi(\kappa(c_1)) = -\int_{c_1}^{c_2} \zeta(q^*(t))\kappa'(t)\, dt, \qquad \kappa'(t) = \lambda(t) - \frac{\rho t H'(t)}{H(t)^2}.$$

Since $q^*(\cdot)$ is continuous, the function $U^*(\cdot)$ has the continuous derivative $-A(t)\zeta(q^*(t))$, and $H(\cdot)$ is continuously differentiable on $[c_1, c_2]$. Since $A(t)/H(t) = \lambda(t)$, it follows that

$$s^*(c_2) - s^*(c_1) = \int_{c_1}^{c_2} \frac{H'(t)}{H(t)^2}\big[U^*(t) + \rho t\zeta(q^*(t))\big]\, dt.$$

By (19), the integrand equals $\frac{(N-1)f(t)}{1-F(t)}\{t(1-\rho)\zeta(q^*(t)) - p^*(t)\}$. It is negative for every $t \in (\underline{c}, \bar{c})$, because $H'(t) = -(N-1)[1-F(t)]^{N-2}f(t) < 0$ and $U^*(t) > 0$. Hence $s^*(c_2) < s^*(c_1)$. Since $s^*(\cdot)$ is continuous on $[\underline{c}, \bar{c})$, it is strictly decreasing there.

Part (ii). By part (i), a supplier who submits the score $s^*(c')$ of a type $c' < \bar{c}$ wins with probability $H(c')$. Conditional on that score, he chooses $q \in [0, q^*(\underline{c})]$ and $p = v^*(q) - s^*(c')$, so the optimized expected payoff of type c is at most

$$\Pi(c', c) = \max_q \big\{H(c')[v^*(q) - s^*(c')] - A(c')c\zeta(q)\big\} = H(c')\big[\Psi(c\lambda(c')) - s^*(c')\big].$$

By the identity in Step 2, $\Pi(c, c) = U^*(c)$. Fix $c, c' \in [\underline{c}, \bar{c})$. Applying (28) with $\kappa = t\lambda(c')$ and using $H(c')\lambda(c') = A(c')$, we obtain

$$\Pi(c', c) - \Pi(c', c') = \int_c^{c'} A(c')\zeta\big(\hat{q}(t\lambda(c'))\big)\, dt.$$

Moreover, $U^*(c) - U^*(c') = \int_c^{c'} A(t)\zeta(\hat{q}(t\lambda(t)))\,dt$, because $\hat{q}(t\lambda(t)) = q^*(t)$. Since $\Pi(c', c') = U^*(c')$,

$$U^*(c) - \Pi(c', c) = \int_c^{c'} \Big[A(t)\zeta\big(\hat{q}(t\lambda(t))\big) - A(c')\zeta\big(\hat{q}(t\lambda(c'))\big)\Big]dt.$$

If $c < c'$, every $t \in [c, c']$ satisfies $A(t) \geq A(c') \geq 0$ and $\lambda(t) \leq \lambda(c')$, so $\hat{q}(t\lambda(t)) \geq \hat{q}(t\lambda(c'))$ because $\hat{q}(\cdot)$ is nonincreasing; the integrand and the integral are nonnegative. If $c > c'$, these inequalities reverse for $t \in [c', c]$, the integrand is nonpositive, and the integral, taken from c down to c', is again nonnegative. Hence $\Pi(c', c) \leq U^*(c)$, so no type gains by submitting another score in the equilibrium support.

Scores below the equilibrium support lose with probability one and yield at most zero, while truthful equilibrium payoff is nonnegative. A score above the support wins with probability one, but the optimized payoff $\max_q\{v^*(q) - s - c_i\zeta(q)\}$ is weakly decreasing in s and therefore cannot improve on the highest equilibrium score. Hence no deviation outside the equilibrium score range is profitable.

Lemma A2 *$s(c_i; \hat{c})$ decreases with $c_i \leq \hat{c}$.*

Proof of Lemma A2

For every $c_i \leq \hat{c}$, equation (21) gives

$$\begin{aligned} & p(c_i; \hat{c})H(c_i) \\ = & \int_{c_i}^{\hat{c}} \{H(t)(1-\rho)\zeta(q(t)) + \rho\zeta(q(t))\}dt + \rho c_i\zeta(q(c_i)) + H(c_i)(1-\rho)c_i\zeta(q(c_i)). \end{aligned}$$

Thus, we have

$$\frac{d\left\{p(c_i; \hat{c})H(c_i)\right\}}{dc_i} = \rho c_i\zeta'(q(c_i))q'(c_i) + c_i\frac{d\{H(c_i)(1-\rho)\zeta(q(c_i))\}}{dc_i}.$$

It follows that

$$\begin{aligned} & H(c_i)\frac{dp(c_i; \hat{c})}{dc_i} - (N-1)(1-F(c_i))^{N-2}f(c_i)p(c_i; \hat{c}) \\ = & \rho c_i\zeta'(q(c_i))q'(c_i) + c_i\frac{d\{H(c_i)(1-\rho)\zeta(q(c_i))\}}{dc_i}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \frac{dp(c_i;\hat{c})}{dc_i} - \frac{(N-1)f(c_i)}{1-F(c_i)}p(c_i;\hat{c}) \\ = \; & q'(c_i)\frac{c_i[\rho+(1-\rho)H(c_i)]}{H(c_i)}\zeta'(q(c_i)) - \frac{(N-1)f(c_i)}{1-F(c_i)}c_i(1-\rho)\zeta(q(c_i)). \end{aligned}$$

By the first-order condition used to derive (5), $\frac{c_i[\rho+(1-\rho)H(c_i)]}{H(c_i)}\zeta'(q(c_i)) = 1$. Therefore, we have

$$\frac{dp(c_i;\hat{c})}{dc_i} = q'(c_i) - \frac{(N-1)f(c_i)}{1-F(c_i)}c_i(1-\rho)\zeta(q(c_i)) + \frac{(N-1)f(c_i)}{1-F(c_i)}p(c_i;\hat{c}),$$

which gives

$$\frac{ds(c_i;\hat{c})}{dc_i} = q'(c_i) - \frac{dp(c_i;\hat{c})}{dc_i} = \frac{(N-1)f(c_i)}{1-F(c_i)}\{c_i(1-\rho)\zeta(q(c_i)) - p(c_i;\hat{c})\}.$$

Note that from (21), we have $c_i(1-\rho)\zeta(q(c_i)) - p(c_i;\hat{c}) < 0$ for every $c_i \leq \hat{c}$. Hence $ds(c_i;\hat{c})/dc_i < 0$ for every $c_i \leq \hat{c}$.

Lemma A3 *Suppose the entry threshold is $\hat{c}$. The procurer's expected payoff $PS(\hat{c})$ is given by (22).*

Proof of Lemma A3

Suppose the entry threshold is $\hat{c}$. Let $c_N = \min\{c_i\}$ and $F_N(c_N) = 1 - [1-F(c_N)]^N$ be the cumulative distribution function of c_N. Then the total expected surplus of the procurer and suppliers is

$$\begin{aligned} TS &= \int_{\underline{c}}^{\hat{c}} [q(c_N) - (1-\rho)c_N\zeta(q(c_N))]\, dF_N(c_N) - N\rho\int_{\underline{c}}^{\hat{c}} c\zeta(q(c))dF(c) \\ &= N\int_{\underline{c}}^{\hat{c}} \{[q(c) - (1-\rho)c\zeta(q(c))]\, H(c) - \rho c\zeta(q(c))\}\, f(c)dc. \end{aligned}$$

Equation (25) gives the slope of a representative supplier's interim expected payoff as

$$\frac{d\pi_i(c,c)}{dc} = -\{H(c)(1-\rho)+\rho\}\zeta(q(c)).$$

Since $\pi_i(\hat{c},\hat{c}) = 0$, a participating supplier's interim expected payoff is

$$\pi_i(c,c) = \int_c^{\hat{c}} \{H(c')(1-\rho)+\rho\}\zeta(q(c'))dc', \forall c \leq \hat{c}.$$

A representative supplier's expected payoff is therefore

$$\begin{aligned} E\pi_i &= \int_{\underline{c}}^{\hat{c}} \left[\int_{c}^{\hat{c}} \{H(c')(1-\rho)+\rho\}\zeta(q(c'))dc' \right] f(c)dc \\ &= \int_{\underline{c}}^{\hat{c}} \left[\int_{\underline{c}}^{c'} f(c)dc \right] \{H(c')(1-\rho)+\rho\}\zeta(q(c'))dc' \\ &= \int_{\underline{c}}^{\hat{c}} F(c')\{H(c')(1-\rho)+\rho\}\zeta(q(c'))dc' \\ &= \int_{\underline{c}}^{\hat{c}} \frac{F(c)}{f(c)}\{H(c)(1-\rho)+\rho\}\zeta(q(c))f(c)dc. \end{aligned}$$

The procurer's expected payoff is therefore

$$\begin{aligned} & PS(\hat{c}) = TS - N \cdot E\pi_i \\ = \; & N\int_{\underline{c}}^{\hat{c}} \left\{ \left[q(c) - (1-\rho)c\zeta(q(c)) \right] H(c) - \rho c\zeta(q(c)) \right\} f(c)dc \\ & -N\int_{\underline{c}}^{\hat{c}} \frac{F(c)}{f(c)}\{H(c)(1-\rho)+\rho\}\zeta(q(c))f(c)dc \\ = \; & N\int_{\underline{c}}^{\hat{c}} \left\{ q(c)H(c) - [(1-\rho)H(c)+\rho] \left(c + \frac{F(c)}{f(c)} \right) \zeta(q(c)) \right\} f(c)dc. \end{aligned}$$

Appendix B

This appendix extends the two-step characterization to a differentiable score function $s(q,p)$ with $s_q(q,p) > 0$ and $s_p(q,p) < 0$. Price bids are bounded, so, as stated in Section 3.1, the limit equilibrium payoff is zero as $c \uparrow \bar{c}$, and for $\rho > 0$, the limit equilibrium quality is zero as $c \uparrow \bar{c}$. For the boundary-value argument below, take $\rho > 0$; the winner-pay benchmark has the different endpoint discussed in Section 3.1.

Implication of the first observation

Suppose $(q(c), p(c))$ generates scores decreasing in type. A type-c_i supplier who reports $\tilde{c}_i$ obtains

$$\pi_i(\tilde{c}_i, c_i) = H(\tilde{c}_i)[p(\tilde{c}_i) - (1-\rho)c_i\zeta(q(\tilde{c}_i))] - \rho c_i\zeta(q(\tilde{c}_i)).$$

The incentive-relevant allocation index is the all-pay-adjusted cost exposure $K(c)$ defined in (12), not quality alone. Incentive compatibility requires $K(c)$ to be nonincreasing and gives the envelope condition

$$U'(c) = -K(c), \qquad U(\bar{c}) = 0.$$

Thus,

$$U(c) = \int_c^{\bar{c}} [(1-\rho)H(t) + \rho]\zeta(q(t))\, dt,$$

and, for $c < \bar{c}$, the supporting price rule is

$$p(c) = \frac{\int_c^{\bar{c}} \{H(t)(1-\rho)\zeta(q(t)) + \rho\zeta(q(t))\}\, dt + \rho c\zeta(q(c))}{H(c)} + (1-\rho)c\zeta(q(c)). \tag{29}$$

The endpoint is understood by continuity whenever the limit exists.

Implication of the second observation

Let $s(c) = s(q(c), p(c))$. Conditional on submitting the equilibrium score of type c, the pair $(q(c), p(c))$ must solve

$$\max_{q,p} \; H(c)[p - (1-\rho)c\zeta(q)] - \rho c\zeta(q) \quad \text{subject to} \quad s(q,p) = s(c).$$

The first-order conditions imply

$$\zeta'(q(c)) = -\frac{s_q(q(c), p(c))}{s_p(q(c), p(c))} \frac{1}{c\{(1-\rho) + \rho/H(c)\}}, \tag{30}$$

where $p(c)$ is given by (29). For $\rho > 0$, equations (29) and (30), together with $q(\bar{c}) = 0$, determine a candidate schedule. To verify an equilibrium, one must additionally check that $K(c)$ is nonincreasing, the induced score is decreasing in type, no type profits from a score deviation, and the price bid is bounded. If the uniquely determined candidate fails one of these checks, no monotone symmetric pure-strategy equilibrium exists in the maintained class.

An application to the ratio score

Consider the all-pay case $\rho = 1$ and the ratio score $s(q, p) = q/p$, defined for $p > 0$. Equations (29) and (30) imply, for $c < \bar{c}$,

$$cq(c)\zeta'(q(c)) = \int_c^{\bar{c}} \zeta(q(t))\, dt + c\zeta(q(c)). \tag{31}$$

Differentiating gives

$$q'(c) = -\frac{\zeta'(q(c))}{c\zeta''(q(c))} < 0. \tag{32}$$

Equivalently, $c\zeta'(q(c))$ is constant. This does not rely on differentiability of $q(\cdot)$: conditional on submitting the score of type c', a type-c supplier solves $\max_q\{\kappa(c')q - c\zeta(q)\}$ with $\kappa(c') = H(c')/s(c')$, whose value strictly increases in $\kappa(c')$ when the maximizer is positive, as it is in equilibrium; hence equilibrium requires $\kappa(\cdot)$, and therefore $c\zeta'(q(c)) = \kappa(c)$, to be constant. The zero limit quality boundary condition as $c \uparrow \bar{c}$ therefore gives

$$\zeta'(q(c)) = \frac{\bar{c}\zeta'(0)}{c}.$$

If $\zeta'(0) = 0$, this condition yields only $q(c) = 0$, which cannot generate strictly decreasing ratio scores. Hence no equilibrium with strictly decreasing scores exists in the maintained class.

This nonexistence result relies on the boundedness of price bids. Without it, the zero limit quality boundary condition as $c \uparrow \bar{c}$ need not hold, and equilibria with strictly decreasing scores can exist. For example, let $N = 2$, let types be uniform on $[1, 2]$, and let $\zeta(q) = q^2$, so that $\zeta'(0) = 0$.[15] Consider

$$q(c) = \frac{1}{4c}, \qquad p(c) = \frac{1}{8c(2-c)}, \qquad s(c) = 2(2-c), \qquad c < 2.$$

The opponent's score is uniform on $(0, 2)$, so a score $s \in (0, 2]$ wins with probability $s/2$. A type-c supplier who submits it obtains $\max_q\{(s/2)(q/s) - cq^2\} = \max_q\{q/2 - cq^2\} =$

[15] We thank the Advisory Editor for suggesting this example.

$1/(16c)$, attained at $q = 1/(4c)$, whatever the score. Scores above the support win with probability one but offer a smaller coefficient on quality, $1/s < 1/2$, and are unprofitable. Hence this profile is a symmetric equilibrium with strictly decreasing scores for all types below $\bar{c} = 2$. Its price bid diverges as $c \uparrow 2$, its quality converges to $1/8$, and the highest-cost type retains the limiting payoff $1/32 > 0$. Bounded price bids exclude this equilibrium.

If $\zeta'(0) > 0$, the candidate quality and price bids are, for $c < \bar{c}$,

$$q(c) = (\zeta')^{-1}\left(\frac{\bar{c}\zeta'(0)}{c}\right), \qquad p(c) = \frac{\int_c^{\bar{c}} \zeta(q(t))\,dt + c\zeta(q(c))}{H(c)}. \tag{33}$$

Using (31), the induced score is

$$s(c) = \frac{H(c)}{\bar{c}\zeta'(0)},$$

which is strictly decreasing for $c < \bar{c}$. The formulas are stated on the interior because the endpoint type has zero probability; its strategy can be completed by a limit whenever one exists. By (31), $p(c) = \bar{c}\zeta'(0)q(c)/H(c)$, so the price bid is bounded if and only if $q(c)/H(c)$ is bounded as $c \uparrow \bar{c}$. If the right limit $\zeta''(0)$ is finite and positive, $q(c)$ vanishes linearly in $\bar{c} - c$, while $H(c) = [1 - F(c)]^{N-1}$ vanishes linearly for $N = 2$ and faster for $N \geq 3$. Thus the price bid is bounded for $N = 2$, whereas for $N \geq 3$ it diverges, so the candidate lies outside the maintained class and no equilibrium exists in that class.

To verify score deviations, suppose type c_i submits $s(c_i')$. Its optimal quality solves

$$\max_q \ \frac{H(c_i')}{s(c_i')}q - c_i\zeta(q) = \max_q \ \bar{c}\zeta'(0)q - c_i\zeta(q),$$

which is independent of c_i' and is uniquely maximized at $q(c_i)$. Hence every score in the equilibrium support gives type c_i the same optimized payoff. Scores above the support win with probability one but offer a smaller coefficient on quality and are not profitable. Thus, (33) constitutes a weak symmetric equilibrium. Whenever its price bid is bounded, it belongs to the maintained class and is the unique symmetric pure-strategy equilibrium with strictly decreasing scores in that class, because the conditions used to derive it, namely the conditional first-order condition, the constancy of $c\zeta'(q(c))$, and the boundary condition $q(\bar{c}) = 0$, are necessary for every equilibrium in the class and determine it uniquely; indifference among scores in the support makes the equilibrium weak but does not generate other equilibria.

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