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Network interoperability and platform competition*

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Abstract

Network interoperability between platforms often comes in various possible configurations, including industry-wide, coalition-based, and pairwise interoperability arrangements. We present an approach to incorporate generalized configurations of network interoperability into the analysis of price competition among any number of symmetric platforms. Specifically, the network benefit received by consumers on each platform increases with the effective network size of the platform, which is determined by an interoperability matrix reflecting the connections between platforms. Four key factors—the strength of interoperability, the shape of the network externality function, the interoperability configuration, and the number of platforms—jointly determine the equilibrium prices. Our findings show, among other things, that increased interoperability strength tends to reduce prices and benefit consumers when: (i) the network externality function exhibits strong increasing returns to scale, or (ii) the interoperability configuration includes multiple coalitions.

Keywords: platforms, interoperability, interconnectivity, compatibility, data sharing, learning curve, coalitions

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1 Introduction

Many industries are characterized by competing platform services that exhibit network effects and network interoperability across platforms. In such settings, users who are exclusively using a given platform exert network externalities to fellow users on the same platform and, at a weaker extent, to users on other platforms.[1] To fix idea, consider the following examples:

- *Interoperability mandates.* Regulators may impose interoperability mandates such that users on different platforms can directly interact and communicate with each other. Examples include telecommunications, internet services (ISPs), and other communication protocols.[2]
- *Interface and format standardizations.* Some platforms may adopt a set of standardized protocols of service provision, which allow network effects to be shared within each coalition of platforms adopting the same standard. Examples include the VHS-Betamax videotape format rivalry, and hardware interface standardization (e.g., USB ports, audio jacks, and charging ports) for competing electronic devices.
- *Data sharing arrangements.* Data-driven network effects refer to virtuous cycles whereby a firm learns from its customer usage data to improves its product, which attracts more customers and allows it to collect even more data, and so on. In this context, network interoperability occurs when data collected by one firm partially allows another firm to improve its product, e.g., due to pairwise data sharing agreements between firms.[3]

This article analyzes the implications of network interoperability on the price competition outcome between platforms. There is by now a large literature on platforms and interoperability, which we will briefly review later. The existing works focus on the industry-wide interoperability, which is a reasonable description of some regulator-imposed interoperability mandates, but arguably less so for other commonly observed interoperability arrangements that arise from platforms engaging in coalition-based standardization or bilateral data sharing agreements. We develop a flexible modelling framework that captures these by incorporating arbitrary configurations of interoperability arrangements in a oligopolistic platform competition.

[1]Other existing terminologies include interconnectedness (Crémer et al., 2000), compatibility (Katz and Shapiro, 1985), and horizontal interoperability (Bourreau and Krämer, 2022).

[2]A recent case in point is the EU Digital Market Act (DMA), which includes an interoperability obligation for messenger services such as WhatsApp for basic communication functionalities (Scott Morton et al., 2023; Hovenkamp, 2023). In the context of social media platforms such as Facebook and Twitter, Scott Morton et al. (2023) also provide detail suggestions on how interoperability requirement can be implemented for a set of standard functionalities (e.g., exchange of text, images, video, or calendar).

[3]For example, Everis et al. (2018) reveals a growing practice of B2B data-sharing between firms. There are also several start-up firms that facilitate and support B2B data sharing (e.g., Bobsled, Priviti, Vivli, and iGrant.io) and policy efforts in fostering such data sharing between businesses (e.g., GAIA-X initiative in Europe).

Our framework is built upon the seminal price competition model pioneered by Armstrong (2006) and recently enriched by Tan and Zhou (2021). There are $n \geq 2$ one-sided ex-ante symmetric platforms that compete for singlehoming users by setting participation prices in a fully covered market.[4] Users on each platfom i enjoys network benefits that is increasing (and possibly non-linear) in the *effective network size* they have access to, which is a weighted sum of participation mass on platform i and also on other platforms. We specify such weights by introducing an n-by-n interoperability matrix: each of its entry specifies the strength of the interoperability connection between a given pair of platform. Moreover, the formulation nests various natural interoperability configurations as special cases, including zero interoperability, industry-wide, coalition-based, or pairwise interoperability arrangements.

Our first result shows that the symmetric equilibrium price that emerges from platform competition is conceptually similar to those obtained by Armstrong (2006) and Tan and Zhou (2021), despite our generalization in network interoperabilities. Specifically, the price equals a mark-up, due to market power associated with product differentiation, minus a subsidy, reflecting the *marginal network externalities* with respect to changes in effective network size. The key twist is our setting is that the subsidy term is augmented by a *translation ratio* coefficient: it measures by how much a unit change in a platform's participation mass (as a result of own price change) translate into a change in its effective network size. Intuitively, a smaller translation ratio means that the participation mass of a platform is less relevant in determining the effective network size enjoyed by its users, and so each platform has weaker incentives to subsidize and attract users, resulting in a higher equilibrium price.

Our second set of results show that an increase in interoperability strength influence the equilibrium price via two distinct channels of effects: (i) *changes in marginal externalities*, which depends on the shape of externality function; and (ii) *changes in the translation ratio*, which depends on the connectivity structure of the interoperability configuration considered. All else equal, if marginal externalities or translation ratio decreases, it implies less subsidization by platforms so that equilibrium price increases.

To gain further insights, we first specialize to the standard case of an industry-wide configuration, whereby the translation ratio is always decreasing in the interoperability strength. This simplicity allows us to focus on the role played by the shape of externality function. When the externality function is linear or concave, the marginal externality is weakly decreasing: the two channels of effects are aligned so that equilibrium price increases with interoperability strength. Conversely, when the externality function is highly convex, the two channels have opposing effects so that the equilibrium price can decrease with interoperability strength. We discuss how these findings relate to applications in data-driven network effects and data learning curve.

[4] For clarity and notational simplicity, the main bulk of our analysis focuses on a one-sided setting (with same-side network effects). We explain in Appendix A on how our analysis and insights easily extend to multi-sided settings with cross-side network effects.

We then specialize to the contrasting case of coalition-based configurations, whereby the translation ratio is instead increasing in the interoperability strength, provided that the network effect strong. When this happens, the equilibrium price will decrease with interoperability strength, even if the network externality is linear (i.e., constant marginal externality). Intuitively, a stronger interoperability within each coalition enhances the extent of product complementarity between member platforms in the same coalition (given the network effect is shared among themselves, and not with the rival coalitions). The stronger this complementarity is, the greater the translation ratio is, thus resulting in stronger incentives for each platform to cut price and expand its effective network size.

Our final set of results consider a comprehensive comparative statics by parameterizing all interoperability configurations that are admissible in our framework. For clarity, we focus on the setting of four platforms and yield the following findings. First, holding the total interoperability strength fixed, a more "narrow" allocation of the total strength across potential interoperability connections leads to a lower equilibrium price. As a case in point, the coalition and industry-wide configurations are respectively the most narrow and the least narrow configurations, meaning that the equilibrium prices in all other configurations are bounded between these two cases. Second, the results on coalition interoperability stated in the previous paragraph is robust to the presence of weak interoperability connections across coalitions. Finally, other pairwise interoperability configurations lead to similar conceptual insights as the industry-wide and coalition configurations.

1.1 Related Literature

This article contributes to the literature on one-sided and two-sided platform competition, which has examined the importance of network effects in driving the competition outcome. Prominent early works include Caillaud and Jullien (2003), Rochet and Tirole (2003, 2006), and Armstrong (2006), which has provided basic foundations for studying pricing by monopoly and duopoly platforms. Jullien et al. (2021) provide a recent survey on this literature. Our price competition model is built upon oligopolistic platform competition model with singlehoming users as recently contributed by Tan and Zhou (2021).[5] They provide important insights on the equilibrium pricing patterns and the impact of platform entry, but they do not allow for network interoperability, which is our focus.

Our study also relates to the literature on network interoperability or compatibility, pioneered by Katz and Shapiro (1985); Farrell and Saloner (1985, 1992). They use a static Cournot model to examine the competitive effects of compatibility and firms' incentives for compatibility. Subsequent work extend this Cournot-based framework to consider

[5] Other recent competition models involving singlehoming users on both sides include Jullien and Pavan (2019) and Karle et al. (2020), among others, while models featuring oligopolistic platform competition include Correia-da Silva et al. (2019); Anderson and Peitz (2020); Tremblay et al. (2023); Teh et al. (2023). We opt for the framework by Tan and Zhou (2021) primarily due to its tractability and flexibility in terms of the network externality functions allowed.

dynamic scenarios (Amir et al., 2021), asymmetric duopoly (Crémer et al., 2000), and varied business models (Shekhar et al., 2022).

The most closely related to our study is the branch of the network interoperability literature that is based on price competition (Doganoglu and Wright, 2006; Bourreau and Krämer, 2022; Ekmekci et al., 2023; Peitz and Sato, 2023). Doganoglu and Wright (2006) develop a one-sided Hotelling model, showing that symmetric firms have excessive incentives for compatibility due to reduced price competition. Bourreau and Krämer (2022) use a multi-period one-sided model to examine the trade-off between enhanced network effects resulting from compatibility and diminished contestability due to decreased incentives for multihoming. Two notable recent contributions are Ekmekci et al. (2023) and Peitz and Sato (2023), who introduce oligopolistic models with asymmetric platforms to study how does the extent of asymmetry affect the implications of interoperability on user participation, price, and platform profit.[6]

A recurring reasoning in these existing studies is that an industry-wide interoperability in a relatively symmetrical setting makes it less attractive for each individual platform to subsidize and attract consumers, thus resulting in a higher price. Our results show that, even in a symmetrical setting, this line of reasoning may be reversed in richer environments involving non-linear network externalities and general interoperability configurations that are not necessarily industry-wide. We then identify the mechanisms and conditions for such reversal to occur. In terms of methodological contributions, we introduce a flexible matrix-based approach to model arbitrary interoperability configurations. The interoperability matrix in our model is akin to an adjacency matrix in the analysis of social and economic networks (see e.g., Jackson, 2008), which allows us to apply the concepts and tools from that literature to obtain tractable results.

At a more general level, our application of data sharing between platforms also relates to the growing literature on data-driven network effects and data-enabled learning. Farboodi et al. (2019), Prüfer and Schottmüller (2021), and Hagiu and Wright (2023) consider dynamic competition when data accumulation over time serves as competitive advantage (in terms of either cost efficiency or product improvement) and examine the tendency of market tipping.[7] Bhargava et al. (2024) points out how an incumbent special-

[6]There are a few key differences in terms of how we capture network interoperability compared to these two papers. Ekmekci et al. (2023) adopt the "net-fee" conduct approach in analyzing their model, that is, the price charged to each user is such that the net utility is independent of the number of users, which differs from the Nash-in-prices conduct adopted by, e.g., Armstrong (2006) and Tan and Zhou (2021). Consequently, in their model all benefits of interoperability are accrued to the platforms. Peitz and Sato (2023) focus on logarithmic externalities to operationalize the aggregative-game modelling approach. They formulate interoperability as an additive sum of the values of network benefits across platforms, as opposed to our formulation based on effective network sizes.

[7]In a different vein, another branch of the literature studies negative implications of firms collecting and having access to more consumer data. Among others, recent contributions have examined privacy concern from selling data to third parties (e.g., Choi et al., 2019; Bergemann et al., 2022), ad-targeting (e.g., Athey and Gans, 2010; Bergemann and Bonatti, 2011; De Cornière and De Nijs, 2016), and price discrimination (e.g., Choe et al., 2024; Rhodes and Zhou, 2024). De Cornière and Taylor (2023) provide a comprehensive framework to show how different uses of data is pro- or anti-competitive, and briefly

ist firm can strategically commit ex-ante to data sharing to soften competitive pressure from a potential generalist entrant. Our broader framework abstracts away from these considerations and instead provide generalizations in terms of considering oligopolistic settings and richer interoperability configurations in the workhorse platform competition model of Armstrong (2006).

The remainder of this article is organized as follows: Section 2 presents the main model, the equilibrium of which is characterized in Section 3. Section 4 examines the implications of network interoperability. Section 5 concludes. All proofs and omitted derivations are relegated to the Appendix.

2 Benchmark model

There is a set $\mathcal{N} \equiv \{1, 2, \cdots, n\}$ of $n \geq 2$ symmetric platforms with a continuum of heterogenous consumers (of measure 1).[8] Denote p_i as the membership prices charged by platform $i \in \mathcal{N}$, and $x_i \in [0, 1]$ as the mass of consumers joining platform i. Each consumer knows her idiosyncratic match values (or membership benefits) $\boldsymbol{\epsilon} = (\epsilon_1, \cdots, \epsilon_n)$ with n platforms, where $\boldsymbol{\epsilon}$ is drawn from a joint distribution $\mathbf{G}(\cdot)$. We assume single-homing and full market coverage: each consumer joins one and only one platform.[9]

□ **Participation utility and interoperability.** The participation utility of a consumer from joining a platform $i \in \mathcal{N}$ is:

$$\epsilon_i - p_i + \phi(z_i), \tag{1}$$

which consists of match values, membership prices, and the *network externality function* $\phi : [0, 1] \to \mathbb{R}$ that indicates how a consumer benefits from interacting with other consumers. We assume $\phi(\cdot)$ is continuously differentiable with $\phi(0) = 0$ (normalization) and its first derivative is $\phi'(\cdot) \geq 0$ (positive network externalities). Meanwhile, z_i is the *effective network size* that a consumer on platform i can access and interact with.

In standard models with zero interoperability (e.g., Armstrong, 2006; Tan and Zhou, 2021), network externality depends only on participation on the same platform so that $z_i = x_i$. We model interoperability by allowing z_i to depend on participation mass on other platforms $j \neq i$:

$$z_i = x_i + \sum_{j \neq i} \lambda_{ij} x_j \tag{2}$$

where $\lambda_{ij} = \lambda_{ji} \in [0, 1]$ indicates strength of the two-way "interoperability link" between platforms i and j. The interoperability configuration between platforms can be summa-

examine the implications of data sharing, modelled as a one-off ex-ante transfer of data.

[8]In earlier versions of the paper, we allow $s \geq 1$ sides of consumers that exhibit cross-group externalities. We opt for $s = 1$ side for our main model here to simplify the exposition because the alternative model leads to similar insights but results in more complicated notations. See Appendix A for details.

[9]Formally, if there is an outside option which yields utility level v_0, then $v_0 \to -\infty$ implies that in equilibrium consumers always opt out of the outside option.

rized as an $n \times n$ *interoperability matrix* (or weighted adjacency matrix) $\mathbf{\Lambda} = (\lambda_{ij})$, where $\lambda_{ij} = 0$ means there is no interoperability link between i and j. We assume that matrix $\mathbf{\Lambda}$ is symmetric and that the diagonal entries $\lambda_{ii} = 0$, following the convention.[10]

□ **Platforms and symmetry.** Each platform $i \in \mathcal{N}$ makes independent pricing decisions to maximize its own profit, given by $\Pi_i = (p_i - c)x_i$, where we normalize the marginal cost $c = 0$ without loss of generality. We focus on the price as a membership fee that is not conditional on the participation of consumers on any side. We introduce the following two assumptions to ensure symmetry across platforms.

First, following Tan and Zhou (2021), we assume that the joint distribution function $\mathbf{G}(\cdot)$ is continuously differentiability and symmetric across n platforms, in the sense that the joint distribution of $(\epsilon_1, \cdots, \epsilon_n)$ is invariant under any permutation of the order of these n random variables. These assumptions are general enough to permit several specifications commonly used in the literature, including the case of independent and identically distributed (IID) shocks across platforms (Perloff and Salop, 1985) and spatial settings such as Hotelling model.

Second, we assume that matrix $\mathbf{\Lambda}$ is vertex-transitive (Godsil and Royle, 2001), meaning that the platforms are "equivalent" in terms of their positions on the interoperability configuration.[11] Vertex-transitivity implies that every platform has the same total (weighted) number of interoperability links, i.e.,

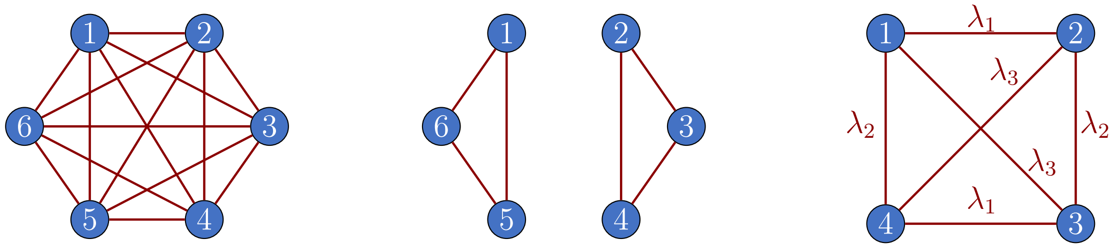
$$\sum_{j \in \mathcal{N}} \lambda_{ij} = \hat{\lambda} \in [0, n-1] \text{ for all } i \in \mathcal{N}. \tag{3}$$

We will refer to $\hat{\lambda}$ as the *total interoperability strength*. To fix idea, below are some notable vertex-transitive configurations $\mathbf{\Lambda}$, which we will use to refine some of our results:

- *Industry-wide interoperability.* For all $i \neq j$, $\lambda_{ij} = \lambda$, where $\lambda \in [0, 1]$ and so $\hat{\lambda} = (n-1)\lambda$. This configuration reduces to zero interoperability if $\lambda = 0$.
- *Duo-coalition interoperability.* Suppose all n platforms are partitioned into two mutually exclusive coalitions. For all $i \neq j$, $\lambda_{ij} = \lambda \in [0, 1]$ if i and j belong to the same coalition, and $\lambda_{ij} = 0$ otherwise. That is, there is an interoperability link of strength λ between all platforms within the same coalition, and zero interoperability across the two coalitions. Each coalition has $n/2$ members, and so $\hat{\lambda} = (n/2 - 1)\lambda$.
- *Parameterized interoperability.* Consider $n = 4$ platforms, and suppose we parameterize matrix $\mathbf{\Lambda}$ with exactly three paramaters $(\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_{12} = \lambda_{34} = \lambda_1$, $\lambda_{14} = \lambda_{23} = \lambda_2$, $\lambda_{13} = \lambda_{24} = \lambda_3$ (while applying symmetry $\lambda_{ij} = \lambda_{ji}$ and $\lambda_{ii} = 0$). Then, $\mathbf{\Lambda}$ is vertex-transitive for all $(\lambda_1, \lambda_2, \lambda_3) \in [0, 1]^3$.

[10] In vector notations, (2) becomes $\mathbf{z} = (\mathbf{I} + \mathbf{\Lambda})\mathbf{x}$, where $\mathbf{z} = (z_1, \cdots, z_n)^\intercal$ is the profile of effective network sizes, $\mathbf{x} = (x_1, \cdots, x_n)^\intercal$ is the profile of participation mass, and $\mathbf{I}$ is the identity matrix.

[11] In formal graph-theoretic terms, an adjancecy matrix $\mathbf{\Lambda}$ is vertex-transitive if any two vertices i and j (i.e., platforms i and j) are equivalent in the sense that there is an automorphism $\sigma : \mathcal{N} \to \mathcal{N}$ such that $\sigma(i) = j$. Here, automorphism is a permutation $\sigma : \mathcal{N} \to \mathcal{N}$ of the set of vertices $\mathcal{N}$ that results in an identical adjancecy matrix, i.e., $\lambda_{ij} = \lambda_{\sigma(i)\sigma(j)}$ for any $i, j \in \mathcal{N}$ (Godsil and Royle, 2001).



(a) Industry-wide (b) Coalitions (c) Parameterized interoperability

Figure 1: Examples of vertex-transitive configurations.

Figure 1 illustrate these examples of vertex-transitive configurations.

☐ **Timing**. (i) Platforms simultaneously choose their prices; (ii) Observing all prices, users simultaneously decide which platform to join. The solution concept is Subgame Perfect Nash Equilibrium (SPNE).

3 Equilibrium analysis

3.1 Participation equilibrium and preliminaries

Consider consumer decisions in the participation subgame. By definition, demand profile $\mathbf{X} = (x_1, \cdots, x_n)^\intercal$ in the equilibrium of the subgame is such that every consumer joins the platform that yields the highest utility specified in (1), while taking as given the participation decisions of other consumers. That is, $\mathbf{X} = (x_1, \cdots, x_n)^\intercal$ satisfies:

$$x_i = \Pr\left(\epsilon_i - p_i + \phi(z_i) \geq \max_{j \neq i}\{\epsilon_j - p_j + \phi(z_j)\}\right) \text{ for every } i \in \mathcal{N} \tag{4}$$

where recall z_i depends on $\mathbf{X}$ as per (2). The existence of a participation equilibrium satisfying (4) for any price profile is guaranteed by Brouwer's fixed point theorem.

For the subsequent analysis, it is convenient to define function

$$Q_i(u_1, u_2, ..., u_n) \equiv \Pr\left(\epsilon_i + u_i \geq \max_{j \neq i}\{\epsilon_j + u_j\}\right)$$

as the mass of consumers choosing platform 1 when the non-idiosyncratic component of consumer participation utility on each platform is given by $\mathbf{u} = (u_1, \cdots, u_n)$. Furthermore, denote $H(\cdot)$ and $h(\cdot)$ as the CDF and PDF of the distribution $\epsilon_i - \max_{j \neq i}\{\epsilon_j\}$ respectively.

Observe that (4) is equivalent to the system of equation $\mathbf{X} = \mathbf{Q}(\mathbf{u})$ using vector $\mathbf{Q}(\mathbf{u}) = (Q_1(\mathbf{u}), ..., Q_n(\mathbf{u}))$ and substituting $u_i = -p_i + \phi(z_i)$. Notice that full market coverage and a symmetric joint distribution $G(\cdot)$ imply that, at any symmetric outcome where $u_1 = u_2 = ...u_n$, we have

$$\frac{\partial Q_i}{\partial u_i} = h(0) \text{ and } \frac{\partial Q_i}{\partial u_j} = \frac{-1}{n-1}h(0) \text{ for every } j \neq i. \tag{5}$$

3.2 Pricing equilibrium

Let the symmetric equilibrium price profile and demand profile be such that each platform sets price p^* and has a participation mass of $x^* = 1/n$ and effective network size of $z^* = (1 + \hat{\lambda})/n$. In what follows, we first state the equilibrium price in Proposition 1 below, and then provide a sketch of our analysis to discuss the economic interpretations. Throughout, we denote $\mathbf{I}$ as $n \times n$ identity matrix, and superscript $\intercal$ as the transpose operator.

Toward pinning down p^*, suppose one of the platforms (say, platform $i = 1$ without loss of generality) deviates to $p_1 \neq p^*$ in an attempt to maximize its profit $\Pi_1 = p_1 x_1$. assuming that Π_1 is globally quasiconcave in own price p_1, we have:[12]

Proposition 1. *Define amplification matrix as*

$$\mathbf{A} \equiv (\mathbf{I} - \delta(\mathbf{I} + \mathbf{\Lambda}))^{-1}, \textit{ where scalar } \delta \equiv \frac{h(0)\phi'(z^*)n}{n-1}, \tag{6}$$

and $\phi'(z^*) \geq 0$ is the marginal externality evaluated at $z^* = \frac{1+\hat{\lambda}}{n}$, and assume $\delta < \frac{1}{1+\hat{\lambda}}$. There exists a SPNE with the outcome that all platforms have the same market share $x^* = 1/n$ and charge the same price $p^* = \frac{1/n}{\partial x_1/\partial p_1}|_{p_1=p^*} > 0$, where

$$\left(\frac{\partial x_1}{\partial p_1}, \frac{\partial x_2}{\partial p_1}, ..., \frac{\partial x_n}{\partial p_1}\right)^\intercal_{p_1=p^*} = \mathbf{A} \times \left(-h(0), \frac{h(0)}{n-1}, ..., \frac{h(0)}{n-1}\right)^\intercal. \tag{7}$$

The right-hand side of (7) consists only of model primitives: match distribution ($h(\cdot)$), number of platforms (n), marginal externality ($\phi'(z^*)$), and interoperability matrix ($\mathbf{\Lambda}$). Observe that if there is no network effect ($\phi' = 0$), then $\mathbf{A} = \mathbf{I}$ and so (7) implies $\partial x_1/\partial p_1 = -h(0)$ and $\partial x_j/\partial p_1 = \frac{1}{n-1}h(0)$, which reflects the standard demand substitutions following a price increase, with $h(0)$ measuring the extent of product substitutability in discrete choice models (Perloff and Salop, 1985; Zhou, 2017). Therefore, the amplification matrix $\mathbf{A}$ describes *how network externalities and interoperability amplify* standard demand substitutions.

[12]Without externalities, global quasiconcavity and the existence of an equilibrium with interior solutions hold under log-concavity of $1 - H(\cdot)$ (see, e.g., Caplin and Nalebuff, 1991). A recurring feature in the literature on platform competition is that such existence conditions typically hold when the network effect is not too strong relative to the extent of horizontal differentiation of the platforms (see, e.g., Jullien et al., 2021). In the case of logit demand (i.e., Gumbel distribution for match values) considered in Section 4, we numerically verified that global quasiconcavity holds in all examples when $\gamma < 1.84\beta$, where $\beta > 0$ is the scale parameter of logit demand.

To gain economic intuition on Proposition 1, we totally differentiate $x_1 = Q_1(\mathbf{u})$ with respect to p_1 and use (5) to get:

$$\begin{aligned}\frac{\partial x_1}{\partial p_1}|_{p_1=p^*} &= -h(0) + h(0)\phi'(z^*)\frac{1}{n-1}\sum_{j\neq 1}\left(\frac{\partial z_1}{\partial p_1} - \frac{\partial z_j}{\partial p_1}\right) \\ &= -h(0) + h(0)\phi'(z^*)\left(1+R\right)\frac{\partial z_1}{\partial p_1}, \qquad (8)\end{aligned}$$

where the second equality used $R \equiv \frac{-1}{n-1}\sum_{j\neq 1}\frac{\partial z_j/\partial p_1}{\partial z_1/\partial p_1} = \frac{1}{n-1}$. Here, R is the *average z-diversion ratio* measuring the decrease in platform 1's effective network size z_1 (in response to an increase in p_1) that are absorbed by the effective network of other platforms $j \neq 1$ on average.[13] Then, (8) leads to the following decomposition of the equilibrium price:

Corollary 1. *The equilibrium price in Proposition 1 can be stated as:*

$$p^* = \underbrace{\frac{1/n}{h(0)}}_{\textit{market power}} - \underbrace{(1+\frac{1}{n-1})\eta^*}_{\textit{generalized loop effect}} \times \underbrace{\phi'(z^*)}_{\textit{marginal externality}} \times \underbrace{\frac{1}{n}}_{\textit{market share}} > 0, \qquad (9)$$

where

$$\eta^* \equiv \frac{\partial z_i/\partial p_i}{\partial x_i/\partial p_i}|_{p_i=p^*} \geq 0 \qquad (10)$$

is the equilibrium translation ratio measuring how a change in platform i's participation mass translates into a change in its effective network size.

Expression (9) highlights pricing incentives of the platforms. The first term is the standard measure of market power of the oligopolistic firms offering differentiated products (Perloff and Salop, 1985). What is important here is the second term in (9), which reflects subsidization incentives of platforms. It features a *loop effect* due to network externalities and inteoperabilities, which can be explained as follows. Specifically, whenever the initial price change expands i's participation mass x_i by a factor of Δ, it translates into an increase in *effective network size* z_i by $\Delta \times \eta^*$ by the definition of translation ratio. Then, the average z-diversion ratio being $R = \frac{1}{n-1}$ means that effective network size on every other $n-1$ platforms would decrease, *on average*, by $\frac{\Delta\eta^*}{n-1}$. Therefore, the total network benefits obtained by users on platform 1, relative to benefits that they can get from other platforms, is effectively increased by $\phi'(z^*) \times \left(1 + \frac{1}{n-1}\right)\Delta\eta^*$. The coefficient $\delta = \frac{h(0)\phi'(z^*)n}{n-1}$ of matrix $\mathbf{A}$ in (6), after normalizing away the translation ratio, exactly captures this loop effect.

[13] Observe that $R = \frac{1}{n-1}$ always holds because full market coverage and the definition of $\hat{\lambda}$ imply that the sum $\sum_{i\in\mathcal{N}} z_i = 1+\hat{\lambda}$, on and off the equilibrium path. Moreover, in our model the standard demand diversion ratio (Katz and Shapiro, 2003) equals $-\sum_{j\neq 1}\frac{\partial x_j/\partial p_1}{\partial x_1/\partial p_1} = 1$ which, after normalizing by $n-1$ (average across the $n-1$ competitors of platform 1), is the same as $R = \frac{1}{n-1}$.

It is useful to compare pricing expression (9) with the special case of zero interoperability by Tan and Zhou (2021). In their setting, the loop effect is evaluated based on the participation mass x_i (so $\eta^* = 1$) instead of the effective network size z_i. Their analysis approach leads to the loop effect term $1 + \frac{1}{n-1}$ by using the symmetric demand diversion ratio property in their setup ($\frac{\partial x_j/\partial p_1}{\partial x_1/\partial p_1} = \frac{-1}{n-1}$ for all $j \neq 1$). With our arbitrary interoperability configurations, the symmetric demand diversion ratio property does not hold, which motivates our alternative analysis approach. Hence, a non-trivial contribution from (9) is in identifying a *generalized loop effect term* $(1 + R)\eta^*$, redefined based on the *average z-diversion ratio* and the *translation ratio*.

☐ **Translation ratio and special cases.** Continue from (9), a notable observation is that the implications of different interoperability configurations can be summarized with translation ratio η^* defined in (10). It is an endogenous object that can be expressed as:

$$\eta^* = 1 + \sum_{j \neq i} \lambda_{1j} \frac{\partial x_j/\partial p_1}{\partial x_1/\partial p_1}. \tag{11}$$

That is, it is a weight sum of the standard demand diversion ratios between platform 1 and its "partners j", i.e., other platforms j to which platform 1 has interoperability links $\lambda_{1j} > 0$. In the special case of an industry-wide interoperability with strength $\lambda_{1j} = \lambda$, full market coverage (that is, $\sum_{j \neq i} \partial x_j/\partial p_1 = -\partial x_1/\partial p_1$) and (11) gives $\eta^* = 1 - \lambda$. Likewise, with zero interoperability, $\lambda_{1j} = 0$ and so (11) gives $\eta^* = 1$.

Beyond these special cases, the calculation of the translation ratio is non-trivial because we have to explicitly compute the demand diversion ratios associated with each partnered platform j. Nonetheless, Proposition 1 says that we can explicitly solve for $\partial x_1/\partial p_1$ and $\partial x_j/\partial p_1$ based on the equation (7). Table 1 summarizes closed-form solutions for the translation ratio in a few notable configurations, which we will return to in Section 4.

Configurations	Total strength $\hat{\lambda}$	Translation ratio η^*
Industry-wide	$(n-1)\lambda$	$1 - \lambda$
Duo-coalition with $\lambda = 1$	$n/2 - 1$	$[1 + (1 - n\delta)\left(1 - \frac{2}{n}\right)]^{-1}$
Multiple size-m coalitions with $\lambda = 1$	$m - 1$	$[1 + (1 - n\delta)\left(\frac{m-1}{n-m}\right)]^{-1}$
Multiple size-m coalitions	$(m-1)\lambda$	$[1 + \frac{1-\delta(1-\lambda+n\lambda)}{(\frac{1-\delta}{\lambda}+\delta)\frac{n-1}{m-1}+n\delta(\lambda-1)}]^{-1}$

Table 1: Translation ratios for each interoperability configuration in the symmetric equilibrium. These four cases are special cases of expression (20) which are proved in the Appendix.

☐ **A graph-theoretic interpretation.** We now briefly discuss how graph-theoretic properties of the graph of interoperability links (described by matrix $\mathbf{\Lambda}$) affect the equilibrium price. Readers who are not interested in such interpretations may proceed to the next section without loss.

Recall that the amplification matrix in Proposition 1 is $\mathbf{A} \equiv (\mathbf{I} - \delta(\mathbf{I} + \mathbf{\Lambda}))^{-1}$, and that $\delta < 1/(1+\hat{\lambda})$ reflects the feedback loop due to network externalities. This expression has a convenient graph-theoretic interpretation as the Leontief inverse of matrix $\mathbf{I} + \mathbf{\Lambda}$ with decaying factor δ.[14] In particular, $\mathbf{A}$'s top-left entry

$$a_{11} = 1 + \delta + \delta^2 \left(1 + \sum\nolimits_{j \neq 1} \lambda_{1j}\lambda_{j1}\right) + ... \tag{12}$$

is the *discounted sum of direct and indirect self-loops*, i.e., paths on graph $\mathbf{I} + \mathbf{\Lambda}$ that starts from node 1 (that represents platform 1) and ends at node 1, where each path of length l (i.e., how many nodes does the path passes by) is discounted by factor δ^l.

Intuitively, (12) captures the *sum of all direct and indirect influences from an initial increase in participation* x_1 (due to price change) to x_1 itself, via changes in effective network size. The direct influence reflects that an initial increase in platform-1 participation by Δ unit increases z_1 by the same unit, which then feeds back as a further increase in x_1 by $\delta\Delta$ units due to network effects, and so on; the indirect influence reflects that the initial increase in x_1 by Δ unit also raises participation on each linked platform x_j by $\delta\lambda_{1j}\Delta$ unit, which similarly feeds back as a further increase in x_1 by $\delta\lambda_{j1} \times \delta\lambda_{1j}\Delta$ unit, and so on.

Then, we show in the proofs that (7) simplifies to

$$\frac{\partial x_1}{\partial p_1}|_{p_1 = p^*} = -\left(na_{11} - \frac{1}{1-(1+\hat{\lambda})\delta}\right) \times \frac{h(0)}{n-1}, \tag{13}$$

where recall $\hat{\lambda}$ is the total interoperability strength defined in (3). Equation (13) shows that a greater sum of self-loops $a_{11} \geq 1$ (which is graph-based measure that depends only on model primitives $\mathbf{\Lambda}$ and δ) implies a stronger incentive for platform 1 to expand its participation mass via lowering its price.

4 Implications of interoperability

4.1 Overview: price and consumer surplus

□ **Prices and profits.** It suffices to focus on the price effects of interoperability because each platform's equilibrium profit is $\Pi^* = p^*/n$. Corollary 1 reveals that interoperability, as described by matrix $\mathbf{\Lambda}$, influences the equilibrium price p^* via two distinct forces on platforms' cross-subsidization incentives: *change in marginal externalities (ME)* and *change in loop effects*. To conceptually illustrate these two forces, we compare an

[14] In matrix form, the Leontief inverse can be written as an infinite sum: $\mathbf{A} = \mathbf{I} + \delta(\mathbf{I} + \mathbf{\Lambda}) + \delta^2(\mathbf{I} + \mathbf{\Lambda})^2 + ...$ See, e.g., Jackson (2008) for further details.

arbitrary interoperability matrix $\mathbf{\Lambda} > 0$ with a zero interoperability matrix $\mathbf{0}$:

$$\text{Change in ME:} \qquad \phi'(z^*) - \phi'(\frac{1}{n})$$
$$\text{Change in loop effect:} \qquad \frac{1}{n-1}(\eta^* - 1).$$

First, interoperability expands the effective network size accessed by each consumer from $1/n$ to $z^* = (1+\hat{\lambda})/n$, which affects the value at which the ME $\sigma_i(z)$ is evaluated at. This channel of effect depends only on the total interoperability strength $\hat{\lambda}$ and, more importantly, the *shape of the externality function* $\phi(\cdot)$. In particular, $\phi'(z^*) - \phi'(\frac{1}{n}) > (<) 0$ if $\phi(\cdot)$ is convex (concave) in the relevant range.

Second, interoperability modifies the loop effect term by changing the translation ratio from 1 to η^*. This channel of effect depends on the *structure of the interoperability configuration*. As shown in Table 1, in the industry-wide configuration, the translation ratio $\eta^* = 1 - \lambda < 1$, which implies a weakened loop effect. In contrast, in the coalition configuration, $\eta^* > 1$ if and only if $\delta > 1/n$, thus implying an amplified loop effect instead.

□ **Consumer surplus.** Under the symmetric equilibrium, the consumer surplus is

$$CS = \mathbb{E}[\max_{i \in \mathcal{N}} \epsilon_i] + \phi(z^*) - p^*,$$

where the first term is the expected maximum match value from n platforms and not affect by interoperability; the second term is the network externality enjoyed by consumers, evaluated at $z^* = (1+\hat{\lambda})/n$; and the third term is the price. Therefore, the consumer surplus implications follow from the price effects discussed above, with the additional benefit in terms of consolidating the network externality enjoyed by consumers:

$$\text{Change in network externality enjoyed: } \phi(z^*) - \phi(\frac{1}{n}) > 0.$$

This implies that price decrease is a sufficient condition for interoperability to raise consumer surplus.[15]

To derive further insights, we specialize the discussions above to three specific configurations: industry-wide (Section 4.2), coalition (Section 4.3), and completely parameterized coalitions with quadropoly platforms (Section 4.4).

[15] In our model, analyzing the total welfare is relatively less meaningful because $TW = E[\max_{i \in \mathcal{N}} \epsilon_i] + \phi(z^*)$ due to full market coverage, where TW is trivially increasing in the effective network size z^*. Following Doganoglu and Wright (2006), an alternative approach is to define TW as the weighted sum of the consumers' and the platforms' surpluses. The possibility that higher prices lower welfare can be captured by discounting platforms' profit relative to consumer surplus. This leads to similar insights as analyzing consumer surplus.

4.2 Industry-wide configuration and shapes of externalities

We first consider the implications of interoperability strength in an industry-wide interoperability with strength $\lambda_{ij} = \lambda \in [0,1]$. Continuing from (9) with $\eta^* = 1-\lambda$, we get

$$p^* = \frac{1/n}{h(0)} - \frac{1-\lambda}{n-1}\phi'(z^*), \tag{14}$$

and so

$$\frac{dp^*}{d\lambda} = \underbrace{\frac{1}{n-1}\phi'(z^*)}_{\text{weakened loop effect}} - \underbrace{\frac{1-\lambda}{n}\phi''(z^*)}_{\text{change in ME}}, \tag{15}$$

which reflects the two forces discussed in Section 4.1.

To formally describe the shape of externality functions, we define the *curvature index* of the externality function $\phi(\cdot)$ as:

$$\rho(z) \equiv \frac{\phi''(z)}{\phi'(z)} \quad \text{for } z \in [0,1]. \tag{16}$$

Recall $\phi'(\cdot) > 0$, and so $\rho < 0$ indicates that $\phi(\cdot)$ is concave (diminishing returns in network effects), while $\rho > 0$ indicates the reverse. If $\rho = 0$, then we obtains the commonly adopted linear externality specification. We impose the following regularity assumption:

$$\rho(z) \text{ is weakly decreasing in } z \in [0,1]. \tag{17}$$

Assumption (17) holds for all log-linear and log-concave $\phi'(z)$ where $\rho(z)$ is constant or decreasing. A useful special case that satisfies (17), which we will invoke to illustrate some of our results below, is the class of *constant-curvature* externality functions:

$$\phi(z) = \begin{cases} \gamma\left(\frac{1-\exp\{\rho z\}}{-\rho}\right) & \text{if } \rho \neq 0 \\ \gamma z & \text{if } \rho = 0 \end{cases}, \tag{18}$$

where $\gamma > 0$ indicates the strength of externality and the curvature index $\rho(z) = \rho$ is a constant independent of z. Then, the following result follows from (15):

Proposition 2. *Suppose Assumption (17) holds.*

- *If $\rho(1/n) \leq \frac{n}{n-1}$, then $dp^*/d\lambda \geq 0$ for all $\lambda \in [0,1]$.*
- *If $\rho(1/n) > \frac{n}{n-1}$, then there exists a cutoff $L \in (0,1)$ such that $dp^*/d\lambda < 0$ if and only if $\lambda < L$. Moreover, L is increasing in n.*

Proposition 2 shows that when the curvature index ρ is low (e.g., linear externalities), a greater interoperability strength λ relaxes competition and raises the equilibrium price. However, when ρ is large (i.e., highly convex externalities), the implications of λ are reversed and result in lower prices.[16] The latter occurs when the extent of market

[16] If the model has negative network externalities such that $\phi'(z) \leq 0$ for all z, then the sign of the

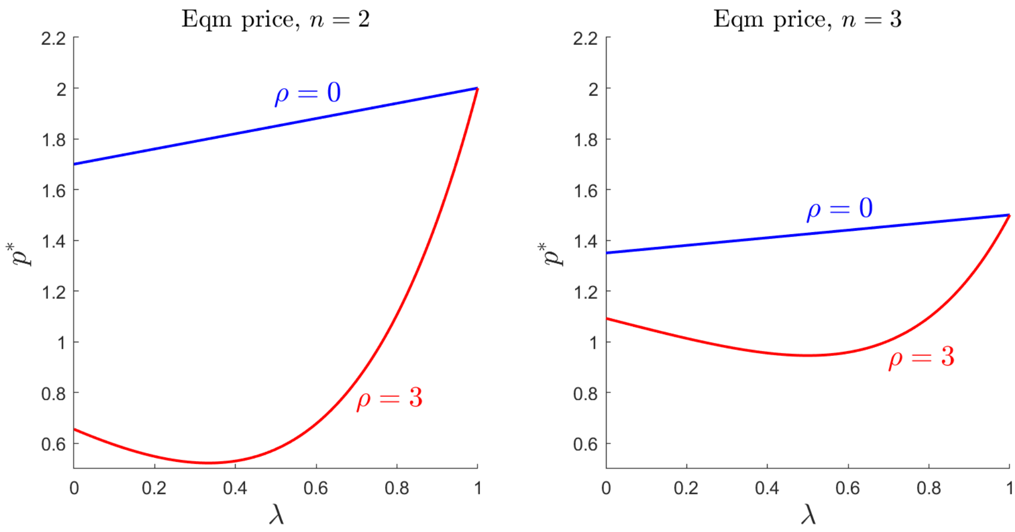


Figure 2: Interoperability and the equilibrium price, based on the constant-curvature externalities (18) with $\gamma = 0.3$, and the match values follow the Gumbel distribution with scale parameter $\beta = 1$ (logit demand form). We verified that global quasiconcavity holds for all $\rho < 10$ in both panels.

fragmentation n is large, or when the interoperability strength λ is small. Intuitively, both of these changes magnify the change in ME relative to the change in loop effect, as can be seen from (15). A notable consequence of Proposition 2 is a U-shape relationship between interoperability strength λ and the equilibrium price, which we illustrate in Figure 2 below, based on the constant-curvature externality function.

As for the equilibrium consumer surplus, we have $dCS/d\lambda = \frac{n-1}{n}\phi'(z^*) - dp^*/d\lambda$ as discussed in Section 4.1. Simplifying:

Proposition 3. *Suppose that Assumption (17) holds. There exists cutoffs $L_{CS} > L$ and L'_{CS} such that:*[17]

- *If $\rho(z) = 0$ for all z, then $dCS/d\lambda > 0$ if and only if $n \geq 3$.*
- *If $\rho(z) > 0$ for all z, then $dCS/d\lambda > 0$ if and only if $\lambda < L_{CS}$.*
- *If $\rho(z) < \min\{0, \rho'(z)\}$ for all z, then $dCS/d\lambda > 0$ if and only if $\lambda > L'_{CS}$.*

Moreover, if $dCS/d\lambda_{n=n'} > 0$ at some n', then $dCS/d\lambda_{n=n''} > 0$ for all $n'' > n'$.

The intuition of Proposition 3 is the easiest to understand by focusing on the class of constant-curvature externality function (18), where

$$\frac{dCS}{d\lambda} > 0 \Leftrightarrow n - 2 + (1 - \lambda)\rho > \frac{1}{n-1}. \tag{19}$$

From (19), $dCS/d\lambda > 0$ tends to hold when market fragmentation n is large (implies a greater gain from consolidating network externalities) or when index ρ is large (implies a

price change is reversed: if $\rho(1/n) \leq \frac{n}{n-1}$ then $\partial p^*/\partial\lambda \leq 0$; while if $\rho(1/n) > \frac{n}{n-1}$, then $\partial p^*/\partial\lambda > 0$ if and only if $\lambda < L$.

[17] It is possible that the cutoffs L_{CS} and $L'_{\text{CS}} \notin [0, 1]$, in which case $\partial CS/\partial\lambda > 0$ has the same sign for all $\lambda \in [0, 1]$ even when $\rho(z) \neq 0$.

price decrease by Proposition 2). Meanwhile, with linear externalities ($\rho = 0$), condition (19) does not depend on λ. With non-linear externalities of $\rho > 0$, condition (19) tends to hold when λ is small and is reversed when λ is large, implying an inverted U-shape relationship between CS and λ (an analogous discussion applies to $\rho < 0$). Figure 3 below illustrates Proposition 3.

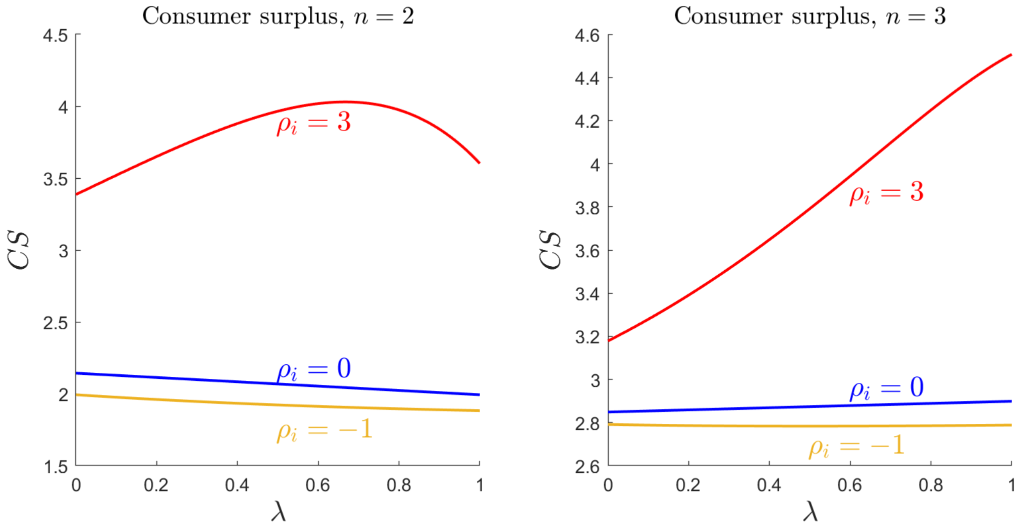


Figure 3: Interoperability and consumer surplus, with the same parameter specifications as Figure 2.

□ **Discussion.** The results in this subsection highlight the roles of the curvature of network externality function $\phi(\cdot)$. With concave and linear externalities (commonly considered in the literature), a stronger interoperability tends to relax platform competition and reduce consumer surplus. With highly convex externalities, the reverse is true.

To fix idea about curvature of network externality, let us consider the context of network effects via platforms' *data-enabled learning* (Hagiu and Wright, 2023). In this context, $\phi(\cdot)$ corresponds to the "learning curve" that describe how the input of consumer demand results in user data that improves the value of a platform's product. As an illustration, suppose the learning curve follows the typical S-shaped logistic form $\phi(z) = (1 + \exp(-(z + z_0)/\theta)^{-1}$, where θ is the scale parameter while z_0 can be interpreted as the "initial data stock" (accumulated over time) that is available for learning and product improvement or initial level of product value. By definition:

$$\rho(z) = \left(\frac{1 - \exp(\frac{z+z_0}{\theta})}{1 + \exp(\frac{z+z_0}{\theta})} \right) \frac{1}{\theta}$$

which is decreasing in z_0. Thus, if z_0 is small, e.g., industries at the "early segment" of the learning curve, then the learning curve tends to be locally convex, and vice-versa. Existing empirical and theoretical studies (Bajari et al., 2019; Lee and Wright, 2021; Peukert et al., 2023; Schaefer and Sapi, 2023) have pointed out that the learning curve tends to be concave in improving the accuracy of media content recommendations and search quality, but can be convex in high-stake forecasting tasks where forecast failures due to insufficient data are extremely punishing.[18]

4.3 Coalition configuration and amplified loop effect

Consider the following multi-coalition configuration. The set of all $\mathcal{N}$ platforms is partitioned into mutually exclusive coalitions where each coalition has m member platforms. For all $i \neq j$, $\lambda_{ij} = \lambda \in [0, 1]$ if i and j belong to the same coalition, and $\lambda_{ij} = 0$ otherwise. To focus on the non-trivial cases, we assume $m \in [1, n/2]$ so that there will be a total of $l \equiv n/m \in [2, n]$ coalitions, assuming that n/m is a well-defined integer. The total interoperability strength is $\hat{\lambda} = (m-1)\lambda$. Observe that $m = 1$ corresponds to zero interoperability.[19]

Given that the roles of non-linear externalities is well-understood from Section 4.2, in what follows we focus on linear network externality where $\phi'(z) = \gamma$ for all $z \in [0, 1]$ where scalar $\gamma > 0$.[20] Then, Proposition 1 yields the following equilibrium price:

$$\begin{aligned} p^* &= \frac{1/n}{h(0)} - \frac{\eta^*}{n-1}\gamma \\ &= \frac{1/n}{h(0)} \left(\frac{1-(1-\lambda)\delta}{1 + \left(\frac{n-m}{n-1}\right) \frac{\lambda\delta}{1-(1+\lambda(m-1))\delta}} \right), \end{aligned} \tag{20}$$

where $\delta \equiv \frac{h(0)\gamma n}{n-1}$, as defined in (6). Following Proposition 1, we assume $\delta < \frac{1}{1+(m-1)\lambda}$, which implies (20) is strictly positive.

In what follows, we first focus on comparative statics with respect to λ (the interoperability strength), and then briefly discuss how an analysis with respect to m (the size of each coalition) leads to similar insights. From equation (20), we get

$$\frac{dp^*}{d\lambda} = -\frac{\gamma}{n-1}\frac{d\eta^*}{d\lambda},$$

which is negative if and only if the translation ratio increases, i.e., $d\eta^*/d\lambda > 0$. In general, the sign of $d\eta^*/d\lambda$ can go either way, but it can easily be determined by examining a quadratic equation of λ. For example, with two coalitions (so $m = n/2$) and $n = 4$, we

[18]Examples include medical services that rely on AI-powered medical imaging for recommending conditions or treatments(e.g. Behold.ai), or airport security that rely on AI-based readings of X-rays of bags for threat detection (e.g. seeTrue.ai), as noted by Lee and Wright (2021).

[19]Technically, $m = n$ corresponds to an industry-wide interoperability that has been analyzed in Section 4.2, which is why we focus on $m \leq n/2$ here. In addition, there is a potential interpretation and discontinuity issue when one consider comparative statics with respect to m between $m = n/2$ and $m = n$, which we discuss at the end of this section.

[20]Using the approach shown in Section 4.2, the insights below readily extend to the case of non-linear externalities.

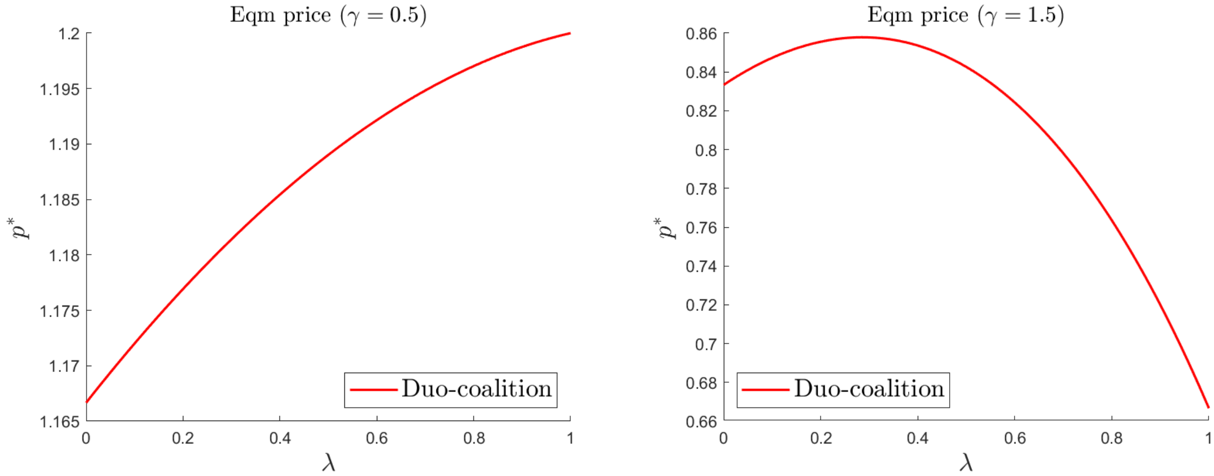


Figure 4: Interoperability and prices, based on the case of $n = 4$ and two coalition. We assume linear externality and logit scale parameter $\beta = 1$. These parameters imply $\delta = 1/8$ in the first panel, and $\delta = 3/8$ in the second panel.

have $\eta^* = \frac{3(1-\delta)-\lambda+3\delta\lambda^2}{3(1-\delta)+\delta\lambda}$ and so:

$$\frac{\partial \eta^*}{\partial \lambda} = -\frac{\delta^2\lambda^2 - 6(1-\delta)\delta\lambda + (1-\delta)^2}{3\left(1-\delta-\frac{\delta\lambda}{3}\right)^2}.$$

Generalizing this observation and ruling out the root involving $\lambda > 1$, we yield:

Proposition 4. *Suppose that $m \in [2, n/2]$ and that the network externality function is linear with slope $\gamma > 0$. Define cutoff*

$$L^{coal} \equiv \left(\frac{1}{\delta} - 1\right)\left(\frac{n-1-\sqrt{n(n-m)}}{n(m-2)+1}\right) > 0.$$

Then, $dp^*/d\lambda < 0$ if and only if $\lambda > L^{coal}$, where the cutoff L^{coal} is decreasing in $\gamma h(0)$, and increasing in m. Moreover, if $\delta > 1/n$, then $L^{coal} < 1$ and $p^*_{\lambda=1} < p^*_{\lambda=0}$.

As illustrated in Figure 4 below, Proposition 4 shows that the equilibrium price can decrease with interoperability strength λ in the coalition configuration. This is true even with linear externalities, meaning that the mechanism involved here is distinct from Proposition 2, which we explain below.

To understand Proposition 4, recall that linear externality means that any decrease in price is purely driven by increases in the translation ratio (which measure the intensity of the loop effect):

$$\eta^* = 1 + (m-1)\lambda \frac{\partial x_{partner}/\partial p_1}{\partial x_1/\partial p_1} \tag{21}$$

whereby $\partial x_{partner}/\partial p_1$ indicates how a price increase by platform 1 changes the demand for the same-coalition platforms. To sharpen the discussion, let us compare on $\lambda = 1$ and

$\lambda = 0$. Then, a necessary and sufficient condition for $p^*_{\lambda=1} < p^*_{\lambda=0}$ is $\eta^*_{\lambda=1} > 1 = \eta^*_{\lambda=0}$, which is equivalent to $\frac{\partial x_{partner}}{\partial p_1}|_{\lambda=1} < 0$ (recall $\partial x_1/\partial p_1 < 0$). That is, $\lambda = 1$ causes the same-coalition platforms to become *gross complements* with respect to the market demand. This occurs when network effect (γ) is strong relative to the extent of platform differentiation:

$$p^*_{\lambda=1} < p^*_{\lambda=0} \Leftrightarrow \frac{\partial x_{partner}}{\partial p_1}|_{\lambda=1} < 0 \Leftrightarrow \frac{n^2}{n-1}\gamma > \frac{1}{h(0)}. \tag{22}$$

Recall that in standard settings with zero interoperability, competing platforms are necessarily *substitutes* ($\partial x_j/\partial p_1 > 0$) because a price drop by platform 1 would cause consumers on all other platforms to switch to platform 1. In contrast, with at least two coalitions, network interoperability can give rise to *product complementarity* within each coalition. To see this, suppose a platform i decreases its price p_1, which induces consumers to substitute away from other same-coalition platforms j and also away from rival-coalition platforms. Due to network interoperability, the latter substitution indirectly benefits other same-coalition platforms j by expanding j's effective network size, making platform j more attractive relative to the rival-coalition platforms, akin to product complementarities. When the network effect γ is large enough, a greater interoperability strength λ enhances this product complementarity (i.e., $\partial x_j/\partial p_1$ becomes lower or even negative).[21] From (21), this results in a greater translation ratio η^*, meaning that each platform has a stronger incentive to expand its effective network size by cutting price, and so $dp^*/d\lambda < 0$.

□ **Discussion.** A key insight from results above is that the industry-wide configuration, which is commonly considered in the literature, has restrictive properties that do not extend to other interoperability configurations. To see this, suppose the network externality is linear. Recall that in the industry-wide configuration, the translation ratio η^* is *always* below one and decreasing, and so any increase in strength λ always weakens loop effect and increases the equilibrium price. In contrast, in coalition configuration considered in this section, an increase in λ amplifies loop effect when the network effect parameter is large. When this occurs, an increase in λ would lower the equilibrium price, thus benefiting consumers.[22]

A natural question is, fixing the value of per-link strength λ, how does the coalition configuration compares with the industry-wide configuration? Given linear externality, we can express the price in (14) as $p^*_{wide} = \frac{1/n}{h(0)}(1-(1-\lambda)\delta)$ and compare it to the equilibrium price in (20), which we denote as p^*_{coal}. It is easily seen that $p^*_{coal} < p^*_{wide}$

[21] By this line of reasoning, gross complementarity $\partial x_j/\partial p_1 < 0$ occurs when, as a result of the initial decrease in p_1, the mass of consumers who switches from rival coalitions to platform j is more than those who switches from j to platform 1.

[22] Consumer surplus implications in this configuration has a similar flavor as Proposition 3. That is, given that network benefits enjoyed by consumers always increase with λ, it follows that $dp^*/d\lambda < 0$ is a sufficient condition for $dCS/d\lambda > 0$. Formally, it is easy to show the existence of a cutoff $L^{coal}_{\text{CS}} < L^{coal}$, such that $\lambda > L^{coal}_{\text{CS}}$ implies $dCS/d\lambda > 0$. We omit the details here for brevity.

for all $m \in [1, n/2]$, so that the coalition configuration always results in a more intense platform competition.

Alternatively, we can also consider an increase in the coalition size $m \in [1, n/2]$. This has a similar effect as an increase in the total interoperability strength $(m-1)\lambda$ associated with each platform. If we treat m as a continuous variable (recall that m is technically an integer), then the derivative dp^*/dm provides valid conclusions if dp^*/dm has the same sign over the relevant range. Assuming $\lambda = 1$, we find that for any $m \in [1, n/2]$, we have $dp^*/dm < 0$ if and only if $\frac{n^2}{n-1}\gamma > 1/h(0)$.[23] This is the same as condition (22), and the result follows a similar intuition: a greater coalition size m enhances within-coalition complementarity when the network effect parameter is large.

4.4 Completely parameterized configurations with $n = 4$

In this section, we consider richer configurations not covered by the previous two subsections. For expositional clarity, we focus on $n = 4$ platforms, which is the simplest setting that allows for a variety of interoperability configurations to arise while still satisfying our vertex-transitive assumption across platforms.[24] As noted in Section 2, in this case the class of interoperability matrix $\mathbf{\Lambda}$ that we consider is:

$$\mathbf{\Lambda} = \begin{pmatrix} 0 & \lambda_1 & \lambda_3 & \lambda_2 \\ \lambda_1 & 0 & \lambda_2 & \lambda_3 \\ \lambda_3 & \lambda_2 & 0 & \lambda_1 \\ \lambda_2 & \lambda_3 & \lambda_1 & 0 \end{pmatrix} \tag{23}$$

for any vector $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in [0, 1]^3$. Given that each element in $\vec{\lambda}$ is permutable, we can assume $\lambda_1 \geq \lambda_2 \geq \lambda_3$ without loss of generality. Denote the row sum as $\lambda_1 + \lambda_2 + \lambda_3 = \hat{\lambda}$. Graphically, matrix $\mathbf{\Lambda}$ correspond to Figure 5 below where each node corresponds to a platform and each link corresponds to an interoperability arrangement of the indicated strength λ_i.

For each given vector $\vec{\lambda}$ parameterizing $\mathbf{\Lambda}$, Proposition 1 leads to

$$p^*(\vec{\lambda}) = \frac{\gamma}{\delta} \left(\sum_{i=1,2,3} \frac{1}{1 - \delta + (\hat{\lambda} - 2\lambda_i)\delta} \right)^{-1} \tag{24}$$

[23] This result does not hold if we consider $m \in [n/2, n]$. To see this, suppose we interpret the industry-wide configuration as $m = n$. Then the previous paragraph implies $p^*_{m=n} > p^*_{m \leq n/2}$ always holds. The distinct results across the two intervals $[1, n/2]$ and $[n/2, n]$ is driven by a division-by-zero problem (or discontinuous domain) when evaluating dp^*/dm in the latter interval. This observation suggests that the model with $m = n$ (industry-wide) behaves very differently compared to those with $m \leq n/2$, hence explaning why we choose to focus on $m \leq n/2$ in this subsection.

[24] The only vertex-transitive $\mathbf{\Lambda}$ when $n = 2$ or $n = 3$ is an industry-wide configuration, which has been analyzed in Section 4.2.

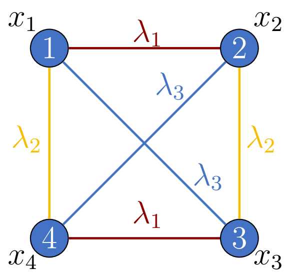


Figure 5: Interoperablity matrix $\mathbf{\Lambda}$ with $n = 4$ platforms

where the translation ratio writes

$$\eta^* = \frac{1}{\delta} - \frac{3}{\delta}\left(\sum_{i=1,2,3} \frac{1}{1-\delta+(\hat{\lambda}-2\lambda_i)\delta}\right)^{-1},$$

and $\delta = \frac{4}{3}h(0)\gamma$ in this setting. Following Proposition 1, we again assume $\delta < (1 + \lambda_1 + \lambda_2 + \lambda_3)^{-1}$ so that (24) is strictly positive.

☐ **Allocation of total interoperability strength across links.** Our first set of comparative statics compares the equilibrium prices across interoperability configurations parameterized by vector $\vec{\lambda}$. To ensure a fair comparison, we fix the total interoperability strength at a constant $\hat{\lambda} > 0$, and then compare different allocations of strength across each interoperability link.

Formally, we adopt the partial ordering of majorization (Marshall et al., 2011): for any given pairs of interoperability vectors that are sorted in descending order $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ and $\vec{\lambda}' = (\lambda'_1, \lambda'_2, \lambda'_3)$, we say that $\vec{\lambda}$ *majorizes* $\vec{\lambda}'$ (denoted as $\vec{\lambda} \succ \vec{\lambda}'$) if $\lambda_1 \geq \lambda'_1$, $\lambda_1 + \lambda_2 \geq \lambda'_1 + \lambda'_2$, and $\lambda_1 + \lambda_2 + \lambda_3 = \lambda'_1 + \lambda'_2 + \lambda'_3$.

Intuitively, $\vec{\lambda} \succ \vec{\lambda}'$ means that $\vec{\lambda}$ has a more unequal distribution of interoperability strength across the link, and so a "narrower"interoperability than $\vec{\lambda}'$. As a case in point, $\vec{\lambda}_{coal} \equiv (\hat{\lambda}, 0, 0) \succ (\hat{\lambda}/3, \hat{\lambda}/3, \hat{\lambda}/3) \equiv \vec{\lambda}_{wide}$ obviously holds. More generally, we have

$$\begin{aligned} &\vec{\lambda}_{coal} \succ \vec{\lambda} \succ \vec{\lambda}_{wide} \\ &\text{for any } \vec{\lambda} \in \Omega(\hat{\lambda}) \equiv \{(\lambda_1, \lambda_2, \lambda_3) \in [0,1]^3 : \lambda_1 + \lambda_2 + \lambda_3 = \hat{\lambda}, \lambda_1 \geq \lambda_2 \geq \lambda_3\} \end{aligned} \tag{25}$$

That is, the coalition configuration (Section 4.3) is the "narrowest" interoperability, while the industry-wide configuration (Section 4.2) is the "broadest " interoperability.

Fixing total interoperability strength $\hat{\lambda}$ as a constant implies that effective network size $z^* = (1 + \hat{\lambda})/4$ is constant across all configurations considered. Therefore, changes in consumer surplus will only reflect changes in the equilibrium price, and so it suffices to consider comparative statics with respect to prices.

Proposition 5. *Suppose the total interoperability strength is fixed at $\hat{\lambda}$. For any pairs of interoperability vectors such that $\vec{\lambda} \succ \vec{\lambda}'$, we have $p^*(\vec{\lambda}) < p^*(\vec{\lambda}')$. Consequently, (25)* *implies*

$$p^*(\vec{\lambda}_{coal}) \leq p^*(\vec{\lambda}) \leq p^*(\vec{\lambda}_{wide}).$$

for any $\vec{\lambda} \in \Omega(\hat{\lambda})$.

Proposition 5 says that, conditioned on the same interoperability strength $\hat{\lambda}$, a "narrower "configuration (in terms of majorization ordering) leads to a more intense price competition between platforms. To see the intuition, recall that we have fixed the effective network size in Proposition 5, and so the discussion Section 4.1 means that any decrease in p^* is purely driven by an increase in loop effect, as measured by translation ratio $\eta^* = \frac{\partial z_1/\partial p_1}{\partial x_1/\partial p_1}$. Hence, Proposition 5 reflects the idea that changes in platform 1's participation mass have substantial impacts on its effective network size when the configuration is more "narrow".[25] Formally, we prove that the equilibrium price $p^*(\vec{\lambda})$ is a Schur-concave function (Marshall et al., 2011) in interoperability vector $\vec{\lambda}$.

☐ **Other interoperability structures**. We now apply pricing equation (24) to explore two parameterized examples of economically interesting configurations.

- *Interlinked coalitions*: $\vec{\lambda}_{intcoal} = (\lambda, \mu, \mu)$ such that $\lambda \geq \mu \geq 0$. This can be interpreted as generalizing $\vec{\lambda}_{coal} = (\lambda, 0, 0)$ to allow for weaker interoperability links across the coalitions. More technically, $\vec{\lambda}_{intcoal}$ is a convex combination between $(\lambda, 0, 0)$ and $(\lambda, \lambda, \lambda)$, i.e., an industry-wide interoperability.
- *Circular pairwise*: $\vec{\lambda}_{cir} = (\lambda, \lambda, \mu)$ such that $\lambda \geq \mu \geq 0$. If $\mu = 0$, then this configuration is a simple circular graph, which can be interpreted as platforms engaging in pairwise interoperability arrangement with each other without involving an industry-wide arrangement. By setting $\mu > 0$ we can allow for transitivity in these pairwise interoperability arrangements.[26]

In what follows, we *do not fix* the total interoperability strength, meaning that λ and μ are independent parameters.[27] We consider comparative statics with respect to λ and μ (which can be interpreted as having more interoperability links, in a weighted sense).

Corollary 2. *Consider the following configurations.*

- *Interlinked coalitions: equilibrium price* $p^*(\lambda, \mu, \mu)$ *is increasing in* μ*; it is decreasing in* λ *if and only if* $\lambda > L_{intcoal}$*, where the cutoff* $L_{intcoal} > 0$ *is increasing in* μ*.*

[25]In terms of the graph-theoretic notions developed in Section 3, a more "narrow"configuration corresponds to a higher discounted sum of direct and indirect self-loops for each node, as defined in (12).

[26]For example, consider data-enabled learning and suppose platform 1 has an explicit data-sharing arrangement with platform 2 but not with platform 3. In this case, data sharing arrangements between platforms 2 and platform 3 can potentially benefit platform 1 due to possible transmissions of data from platform 3 indirectly via platform 2, or other forms of data synergy.

[27]If we fix the total interoperability strength in each configuration below to a constant $\hat{\lambda}$, then the majorization ordering in (25) applies, which implies $p^*_{coal} < p^*_{intcoal} < p^*_{wide}$ and $p^*_{coal} < p^*_{cir} < p^*_{wide}$ by Proposition 5. However, even in this case, $\vec{\lambda}_{intcoal}$ and $\vec{\lambda}_{cir}$ are not generally comparable by the majorization ordering.

- *Circular pairwise: equilibrium price $p^*(\lambda, \lambda, \mu)$ is increasing in λ and μ.*

The first part of Corollary 2 says that across-coalition interoperability μ always relaxes overall competition ($dp^*/d\mu > 0$), while within-coalition interoperability λ intensifies overall competition ($dp^*/d\lambda < 0$) when $\lambda > L_{intcoal}$. Hence, the insight from Proposition 4 continues to hold as long as across-coalition interoperability $\mu > 0$ is not too strong, and a higher μ shrinks the parameter region where $dp^*/d\lambda < 0$ (in particular, $\mu \to 1$ implies $L_{intcoal} > 1$). The second part of Corollary 2 demonstrates that similar insights hold in the circular pairwise setting, except that a strengthened pairwise interoperability agreement will not reverse the positive relationship between λ and the pricing, as it does in the interlinked-coalition case.

5 Conclusion

In this paper, we propose a flexible conceptual approach to model a broad range of configurations of network interoperability in platform markets. A key modelling ingredient is the interoperability matrix, which determines the interoperability structure between platforms and the effective network size faced by consumers on each platform. In the canonical platform competition model, we find that the equilibrium pricing formula has an intuitive structure, comprising a market power term and a subsidy term. Conveniently, the price implications of different interoperability configurations can be captured in terms of translation ratio (i.e., the rate at which changes in a platform's participation mass affect its effective network size), which adjusts the subsidy term.

From a regulatory perspective, our analysis highlights conditions under which interoperability intensifies price competition and increases consumer surplus, even with a relatively symmetrical market structure. As noted in the introduction, an emerging insight from the literature is that an interoperability intervention intensifies competition in asymmetric settings, but has a potential downside of relaxing competition in symmetric settings. Our analysis offers a more nuanced view for the latter case, that is, the downside of competition relaxation does not necessarily hold in richer models with general network externality functions and interoperability configurations.

Our framework offers several promising avenues for further studies. First, the modelling approach of introducing interoperability matrix and effective network size can readily be applied to other frameworks of platform competition, e.g., the transaction fee models by Rochet and Tirole (2003, 2006), the net fee model by Ekmekci et al. (2023), and the aggregative game model by Peitz and Sato (2023). Our current analysis that is based on Armstrong (2006) can be seen as a proof of concept that demonstrates the feasibility of this research avenue.

Second, to ensure analytical tractability, we have focused on platforms that are ex-ante symmetric in terms of their underlying demand function (i.e., the match value distributions) as well as their "positions"in the interoperability configuration. Introducing platform asymmetry would broaden the policy applicability of our framework and allow us to examine other interesting configurations. For instance, in the presence of a dominant platform with multiple smaller platforms, one could compare two options of interoperability regulations to mitigate the dominance: a "star"configuration (where the dominant platform is central and has interoperability links with each smaller platforms) and compare it to a "alliance" configuration where the smaller platforms form a single interoperability coalition. We believe that the translation ratio would remain a useful object to assess platforms' pricing incentives in such settings.[28]

Finally, we have focused on price competition without considering the endogenous formation of coalitions. Endogenizing coalition formation would allow us to examine how industries transition from a no-interoperability benchmark to coalition-based interoperability and, ultimately, to industry-wide interoperability. This extension could shed light on the strategic factors that may prevent an industry from fully adopting industry-wide interoperability.

A Appendix: two-sided platforms

In our main model, we have assumed one-sided platforms to streamline the exposition. We now discuss how our approach and insights extend to case of two-sided platforms We label the two sides of users as B and S (buyers and sellers), but the model is not restricted to this particular interpretation. The analysis below easily extends to more than two sides of users.

Denote $\mathbf{p}_i = (p_i^B, p_i^S)$ as the buyer-side and seller-side membership prices charged by platform $i \in \mathcal{N}$, and $(x_i^B, x_i^S) \in [0,1]^2$ as the participation vector on platform i. Denote $(\epsilon_1^k, \cdots, \epsilon_n^k)$ as the idiosyncratic match values (or membership benefits) of a consumer on side $k \in \{B, S\}$ with n platforms. Both sides are singlehoming. We assume cross-side independence, in the sense that $(\epsilon_1^k, \cdots, \epsilon_n^k)$ are independent of $(\epsilon_1^k, \cdots, \epsilon_n^k)$ across the two sides.

The participation utility for each each buyer and seller joining a platform i are, respectively,

$$\epsilon_i^B - p_i^B + \phi^B\left(z_i^B, z_i^S\right) \quad \text{and} \quad \epsilon_i^S - p_i^S + \phi^S\left(z_i^B, z_i^S\right).$$

Here, $\phi^k : [0,1]^2 \to \mathbb{R}$ is the *network externality functions* that indicates the "inbound" network externalities (including same-side and cross-side externalities) enjoyed by a side-k consumer from having access to other consumers (on all sides). The effective network size $(z_i^B, z_i^S) \in [0,1]^2$ are determined as

$$z_i^B = x_i^B + \sum_{j \neq i} \lambda_{ij} x_j^B \quad \text{and} \quad z_i^S = x_i^S + \sum_{j \neq i} \lambda_{ij} x_j^S.$$

We assume ϕ^k is continuously differentiable and allow for the possibilities of negative network externalities. Other specifications remain the same as the main text (Section 2).

[28]We have examined this issue with extensive numerical simulations, and the results suggest that the former is superior in terms of mitigating the dominance and improving consumer surplus. Details are available upon request.

□ **Equilibrium analysis.** Similar to the main text, we define for, each side $k \in \{B, S\}$, function $Q_i^k(u_1^k, u_2^k, ..., u_n^k) \equiv \Pr\left(\epsilon_i^k + u_i^k \geq \max_{j \neq i}\{\epsilon_j^k + u_j^k\}\right)$. For each given profile of prices $(\mathbf{p}_1, ..., \mathbf{p}_n)$ by the platforms, the resulting demand system is pinned down by the following system of equations:

$$\begin{aligned} x_i^k &= Q_i^k(u_1^k, u_2^k, ..., u_n^k) \text{ for } i \in \mathcal{N} \text{ and } k \in \{B, S\} \\ \text{and } u_i^k &= -p_i^k + \phi^k\left(z_i^B, z_i^S\right). \end{aligned}$$

Let us denote $H^k(\cdot)$ and $h^k(\cdot)$ as the CDF and PDF of the distribution $\epsilon_i^k - \max_{j \neq i}\{\epsilon_j^k\}$ respectively. Then, at any symmetric outcome, we have

$$\frac{\partial Q_i^k}{\partial u_i^k} = h^k(0) \text{ and } \frac{\partial Q_i^k}{\partial u_j^k} = \frac{-1}{n-1} h^k(0) \text{ for every } j \neq i \text{ and } k \in \{B, S\}.$$

To proceed, we denote marginal externalities as on the equilibrium path as:

$$\begin{pmatrix} \gamma_{BB} & \gamma_{SB} \\ \gamma_{BS} & \gamma_{SS} \end{pmatrix} = \begin{pmatrix} \partial\phi^B/\partial z^B & \partial\phi^S/\partial z^B \\ \partial\phi^B/\partial z^S & \partial\phi^S/\partial z^S \end{pmatrix}_{(z_i^B, z_i^S) = (z^*, z^*)}$$

Let the symmetric equilibrium price profile and demand profile be such that each platform sets price $\mathbf{p}^* = (p^{B*}, p^{S*})$ and has a participation mass of $(x^{B*}, x^{S*}) = (1/n, 1/n)$ and effective network size of (z^*, z^*), $z^* = (1 + \hat{\lambda})/n$. Toward pinning down the symmetric equilibrium, suppose one of the platform (say, $i = 1$, without loss of generality) deviates to $\mathbf{p}_1 \neq \mathbf{p}^*$ in an attempt to maximize its profit $\Pi_1 = p_1^B x_1^B + p_1^S x_1^S$. Following Armstrong (2006) and Tan and Zhou (2021), a useful technique to analyze two-sided pricing strategies is to reframe platform 1's decision variables as directly choosing a target participation vector (x_1^B, x_1^S), and then pin down the corresponding price levels (p_1^B, p_1^S) implicitly from the demand system. In particular, if we totally differentiate $x_1^B = Q_1^B(\mathbf{u})$ and $x_1^S = Q_1^S(\mathbf{u})$ with respect to x_1^B, and evaluate the resulting expression at the symmetric outcome, we get:

$$1 = -h^B(0)\frac{\partial p_1^B}{\partial x_1^B} + \frac{h^B(0)}{n-1}\left(\gamma_{BB}\sum_{j \neq 1}\left(\frac{\partial z_1^B}{\partial x_1^B} - \frac{\partial z_j^B}{\partial x_1^B}\right) + \gamma_{BS}\sum_{j \neq 1}\left(\frac{\partial z_1^S}{\partial x_1^B} - \frac{\partial z_j^S}{\partial x_1^B}\right)\right)$$

and

$$0 = -h^S(0)\frac{\partial p_1^S}{\partial x_1^B} + \frac{h^S(0)}{n-1}\left(\gamma_{SS}\sum_{j \neq 1}\left(\frac{\partial z_1^S}{\partial x_1^B} - \frac{\partial z_j^S}{\partial x_1^B}\right) + \gamma_{SB}\sum_{j \neq 1}\left(\frac{\partial z_1^B}{\partial x_1^B} - \frac{\partial z_j^B}{\partial x_1^B}\right)\right).$$

Simplifying using full market coverage as in (8) in the main text and rearranging, we get

$$\frac{\partial p_1^B}{\partial x_1^B} = \frac{-1}{h^B(0)} + \frac{n}{n-1}\left(\gamma_{BB}\frac{\partial z_1^B}{\partial x_1^B} + \gamma_{BS}\frac{\partial z_1^S}{\partial x_1^B}\right) \tag{26}$$

$$\frac{\partial p_1^S}{\partial x_1^B} = \frac{n}{n-1}\left(\gamma_{SS}\frac{\partial z_1^S}{\partial x_1^B} + \gamma_{SB}\frac{\partial z_1^B}{\partial x_1^B}\right). \tag{27}$$

□ **Equilibrium price.** Utilizing (26) and (27), the first-order condition $\partial\Pi_1/\partial x_1^B = 0$ gives the equilibrium buyer-side price as

$$p^{B*} = \frac{1/n}{h^B(0)} - \frac{1}{n-1}\left((\gamma_{BB}+\gamma_{SB})\frac{\partial z_1^B}{\partial x_1^B} + (\gamma_{BS}+\gamma_{SS})\frac{\partial z_1^S}{\partial x_1^B}\right). \tag{28}$$

Likewise, $\partial\Pi_1/\partial x_1^S = 0$ gives the equilibrium seller-side price:

$$p^{S*} = \frac{1/n}{h^S(0)} - \frac{1}{n-1}\left((\gamma_{BB}+\gamma_{SB})\frac{\partial z_1^B}{\partial x_1^S} + (\gamma_{BS}+\gamma_{SS})\frac{\partial z_1^S}{\partial x_1^S}\right). \tag{29}$$

To see the relations with the one-sided pricing formula shown in Corollary 1 in the main text, it is useful to define *translation ratio matrix* as

$$\eta = \begin{pmatrix} \eta^*_{BB} & \eta^*_{SB} \\ \eta^*_{BS} & \eta^*_{SS} \end{pmatrix} = \begin{pmatrix} \partial z_1^B/\partial x_1^B & \partial z_1^S/\partial x_1^B \\ \partial z_1^B/\partial x_1^S & \partial z_1^S/\partial x_1^S \end{pmatrix}_{\mathbf{p}_1=\mathbf{p}^*}.$$

Matrix η is analogous to the scalar (one-sided) translation ratio (10) defined in the main text, except that here we do not state the ratios in terms of prices derivatives (recall we are considering decision variables (x_1^B, x_1^S) here). Then, (28) and (29) can be expressed in matrix form as:

$$\begin{pmatrix} p^{B*} \\ p^{S*} \end{pmatrix} = \begin{pmatrix} \frac{1/n}{h^B(0)} \\ \frac{1/n}{h^S(0)} \end{pmatrix} - \underbrace{(1+\frac{1}{n-1})\eta}_{\text{generalized loop effect}} \times \underbrace{\begin{pmatrix} \gamma_{BB}+\gamma_{SB} \\ \gamma_{SS}+\gamma_{BS} \end{pmatrix}}_{\text{outbound ME}} \times \underbrace{\frac{1}{n}}_{\text{market share}}. \tag{30}$$

Decomposition (30) is analogous to Corollary 1, with two key distinctions. First, the marginal externality term is now replaced by aggregate outbound marginal externalities (outbound ME) provided by users from each side $k \in \{B, S\}$ to all sides B and S. It is evaluated at vector (z^*, z^*) where platform 1's effective network size is $z^* = (1+\hat{\lambda})/n$ across both sides. Second, and more importantly, the augmentation factor in the generalized loop effect term is now a 2×2 diversion ratio matrix, reflecting the two-sidedness. To see why the matrix matters, we note that any changes in platform i's participation mass on one side (e.g. x_1^B) can affect platform 1's effective network size on the opposite side (e.g., z_1^S), even when x_1^S is being *held fixed* (in the profit-maximization problem with decision variables (x_1^B, x_1^S)). This is due to cross-side externalities and network interoperability, such that changes in x_1^B can potentially affect participation mass x_j^S on other platforms $j \neq 1$ that are connected with platform 1.

To further understand (30) and the corresponding translation ratio matrix, we now specialize into two specific interoperability configurations considered in the main text.

☐ **Industry-wide configuration**. Following Section 4.2, we know in this case, $z_1^k = x_1^k + \lambda(1-x_1^k)$ on each side $k \in \{B, S\}$. Therefore,

$$\eta = \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}.$$

and so (28) and (29) becomes

$$p_1^{B*} = \frac{1/n}{h^B(0)} - \frac{1-\lambda}{n-1}(\gamma_{BB}+\gamma_{SB})$$

$$p_1^{S*} = \frac{1/n}{h^S(0)} - \frac{1-\lambda}{n-1}(\gamma_{BS} + \gamma_{SS}),$$

which nests the formula by Tan and Zhou (2021) as special case when $\lambda = 0$. Then, the analysis approach of Section 4.2 continues to hold after redefining the curvature index as in terms of the outbound ME.

□ **Coalition configuration**. Beyond the industry-wide configuration, we can still explicitly solve for the diversion ratio matrix by manipulating the system of demand derivatives, as in Proposition 1. We outline the general analysis below by focusing on the buyer-side pricing here (the derivation of the seller-side pricing is similar), and then specialize it into the case of coalition interoperability (with size m and strength λ, as in Section 4.3).

Continue (30), it remains to identify the system of equation that pins down the translation ratios $\partial z_1^B/\partial x_1^B$ and $\partial z_1^S/\partial x_1^B$ associated with the buyer-side pricing. Total differentiating the demand for platforms $i \neq 1$, $x_i^B = Q_i^B(\mathbf{u})$ and $x_i^S = Q_i^S(\mathbf{u})$ with respect to x_1^B, and apply the same simplifying technique due to full market coverage, we obtain

$$\frac{\partial x_i^B}{\partial x_1^B} = \frac{h^B(0)}{n-1}\frac{\partial p_1^B}{\partial x_1^B} + \frac{nh^B(0)}{n-1}\left(\gamma_{BB}\frac{\partial z_i^B}{\partial x_1^B} + \gamma_{BS}\frac{\partial z_i^S}{\partial x_1^B}\right) \text{ for } i \neq 1$$

$$\frac{\partial x_i^S}{\partial x_1^B} = \frac{h^S(0)}{n-1}\frac{\partial p_1^S}{\partial x_1^B} + \frac{nh^S(0)}{n-1}\left(\gamma_{SS}\frac{\partial z_i^S}{\partial x_1^B} + \gamma_{SB}\frac{\partial z_i^B}{\partial x_1^B}\right) \text{ for } i \neq 1$$

Subsituting away $\partial p_1^B/\partial x_1^B$ and $\partial p_1^S/\partial x_1^B$ using (26) and (27), we get

$$\frac{\partial x_i^B}{\partial x_1^B} = -\frac{1}{n-1} + \frac{nh^B(0)}{n-1}\left[\gamma_{BB}\left(\frac{\partial z_i^B}{\partial x_1^B} + \frac{\partial z_1^B}{\partial x_1^B}\right) + \gamma_{BS}\left(\frac{\partial z_i^S}{\partial x_1^B} + \frac{1}{n-1}\frac{\partial z_1^S}{\partial x_1^B}\right)\right] \text{ for } i \neq 1 \quad (31)$$

$$\frac{\partial x_i^S}{\partial x_1^B} = \frac{nh^S(0)}{n-1}\left[\gamma_{SS}\left(\frac{\partial z_i^S}{\partial x_1^B} + \frac{1}{n-1}\frac{\partial z_1^S}{\partial x_1^B}\right) + \gamma_{SB}\left(\frac{\partial z_i^B}{\partial x_1^B} + \frac{1}{n-1}\frac{\partial z_1^B}{\partial x_1^B}\right)\right] \text{ for } i \neq 1, \quad (32)$$

and recall $\partial x_1^B/\partial x_1^B = 1$ and $\partial x_1^S/\partial x_1^B = 0$ by definition.

Specializing into coalition configurations, we know $z_{partner}^k = (1 + \lambda(m-2))x_{partner}^k + \lambda x_1^k$ and $z_1^k = x_1^k + \lambda(m-1)x_{partner}^k$. Therefore:

$$\frac{\partial z_{partner}^B}{\partial x_1^B} + \frac{1}{n-1}\frac{\partial z_1^B}{\partial x_1^B} = \lambda + \frac{1}{n-1} + \left(1 + \lambda(m-2) + \frac{\lambda(m-1)}{n-1}\right)\frac{\partial x_{partner}^B}{\partial x_1^B}$$

$$\frac{\partial z_{partner}^S}{\partial x_1^B} + \frac{1}{n-1}\frac{\partial z_1^S}{\partial x_1^B} = \left(1 + \lambda(m-2) + \frac{\lambda(m-1)}{n-1}\right)\frac{\partial x_{partner}^S}{\partial x_1^B}.$$

Denote scalar $M = 1 + \lambda(m-2) + \frac{\lambda(m-1)}{n-1}$, (31) and (32) becomes

$$\frac{\partial x_{partner}^B}{\partial x_1^B} = -\frac{1}{n-1} + \frac{nh^B(0)}{n-1}\left[\gamma_{BB}\left(\lambda + \frac{1}{n-1}\right) + \left(\gamma_{BB}\frac{\partial x_{partner}^B}{\partial x_1^B} + \gamma_{BS}\frac{\partial x_{partner}^S}{\partial x_1^B}\right)M\right]$$

$$\frac{\partial x_{partner}^S}{\partial x_1^B} = \frac{nh^S(0)}{n-1}\left[\gamma_{SB}\left(\lambda + \frac{1}{n-1}\right) + \left(\gamma_{SS}\frac{\partial x_{partner}^S}{\partial x_1^B} + \gamma_{SB}\frac{\partial x_{partner}^B}{\partial x_1^B}\right)M\right],$$

Solving for the two-by-two simultaneous equation gives explicit solutions:[29]

$$\frac{\partial x^B_{partner}}{\partial x^B_1} = -\frac{1}{(n-1)M}\left(1-\lambda+\lambda n - \frac{(\delta_{SS}M-1)(\lambda-\lambda n-1+M)}{(\delta_{BB}\delta_{SS}-\delta_{BS}\delta_{SB})M^2-(\delta_{BB}+\delta_{SS})M+1}\right) \quad (33)$$

$$\frac{\partial x^S_{partner}}{\partial x^B_1} = -\frac{1}{n-1}\left(\frac{\delta_{SB}(\lambda-\lambda n-1+M)}{(\delta_{BB}\delta_{SS}-\delta_{BS}\delta_{SB})M^2-(\delta_{BB}+\delta_{SS})M+1}\right). \quad (34)$$

where we have denoted the amplification coefficients as

$$\begin{pmatrix} \delta_{BB} & \delta_{SB} \\ \delta_{BS} & \delta_{SS} \end{pmatrix} = \begin{pmatrix} \frac{nh^B(0)}{n-1}\gamma_{BB} & \frac{nh^S(0)}{n-1}\gamma_{SB} \\ \frac{nh^B(0)}{n-1}\gamma_{BS} & \frac{nh^S(0)}{n-1}\gamma_{SS} \end{pmatrix},$$

which just depend on the differentiation and the externality parameters.

We can then compute the required translation ratios as $\eta^*_{BB} = 1+\lambda(m-1)\frac{\partial x^B_{partner}}{\partial x^B_1}$ and $\eta^*_{SB} = \lambda(m-1)\frac{\partial x^S_{partner}}{\partial x^B_1}$. Then, (28) gives a closed-form solution for the equilibrium buyer-side price p^{B*}:

$$p^{B*} = \frac{1/n}{h^B(0)} - \frac{\gamma_{BB}+\gamma_{SB}}{n-1} - \lambda(\frac{m-1}{n-1})\left((\gamma_{BB}+\gamma_{SB})\frac{\partial x^B_{partner}}{\partial x^B_1} + (\gamma_{BS}+\gamma_{SS})\frac{\partial x^S_{partner}}{\partial x^B_1}\right),$$

after substituting (33) and (34). The seller-side equilibrium price can be explicitly solved analogously.

B Appendix: proofs

Proof. **(Proposition 1)**. As a preliminary step, we prove the following:

$$-\sum_{j\neq 1}\frac{\partial z_j/\partial p_1}{\partial z_1/\partial p_1} = -\sum_{j\neq i}\frac{\partial z_j/\partial p_1}{\partial z_i/\partial p_1} = 1, \quad (35)$$

which follows from

$$\sum_{i\in\mathcal{N}}\frac{\partial z_i}{\partial p_1} = \sum_{i\in\mathcal{N}}\left(x_i + \sum_{j\neq i}\lambda_{ij}x_j\right) = 1 + \sum_{i\in\mathcal{N}}\sum_{j\neq i}\lambda_{ij}x_j = 1+\bar{\lambda},$$

where we swapped the order of summation in the final step and used $\lambda_{ii} = 0$. Then, totally differentiating the system of equation $\mathbf{X} = \mathbf{Q}(\mathbf{u})$ with respect to p_1, we get:

$$\frac{\partial x_1}{\partial p_1}|_{p_1=p^*} = -h(0) + h(0)\phi'(z^*)\frac{1}{n-1}\sum_{j\neq 1}\left(\frac{\partial z_1}{\partial p_1} - \frac{\partial z_j}{\partial p_1}\right) = -h(0) + \delta\frac{\partial z_1}{\partial p_1} \quad (36)$$

$$\frac{\partial x_i}{\partial p_1}|_{p_1=p^*} = \frac{h(0)}{n-1} + h(0)\phi'(z^*)\frac{1}{n-1}\sum_{j\neq i}\left(\frac{\partial z_i}{\partial p_1} - \frac{\partial z_j}{\partial p_1}\right) = \frac{h(0)}{n-1} + \delta\frac{\partial z_i}{\partial p_1} \text{ for } i\neq 1 \quad (37)$$

[29] For general interoperability configurations, we recall that for each side k, we can express $\frac{\partial}{\partial x^B_1}\left(z^k_1,...,z^k_n\right)^\top = (\mathbf{I}+\mathbf{\Lambda})\cdot\frac{\partial}{\partial x^B_1}\left(x^k_1,...,x^k_n\right)^\top$. Let us continue from (31) and (32) and $\partial x^B_1/\partial x^B_1 = 1$ and $\partial x^S_1/\partial x^B_1 = 0$, and express them in a matrix form. Then, we get a system of $2n$ equations with $2n$ unknowns that is analogous to (7) in the main text.

where we used (35). Rearranging the system of equation in matrix form and using the definition $\mathbf{z} = (\mathbf{I} + \mathbf{\Lambda})\mathbf{X}$, we arrive at

$$(\mathbf{I}-\delta(\mathbf{I}+\mathbf{\Lambda}))\left(\frac{\partial x_1}{\partial p_1}, \frac{\partial x_2}{\partial p_1}, ..., \frac{\partial x_n}{\partial p_1}\right)^{\intercal}_{p_1=p^*} = \left(-h(0), \frac{h(0)}{n-1}, ..., \frac{h(0)}{n-1}\right)^{\intercal},$$

which gives (7) after inverting the symmetric matrix $\mathbf{B} \equiv \mathbf{I}-\delta(\mathbf{I}+\mathbf{\Lambda})$, so that $\mathbf{A} = \mathbf{B}^{-1}$.

We note that $(1+\hat{\lambda})\delta < 1$ implies that matrix $\mathbf{B}$ is strictly diagonally dominant, which then immediately implies $\mathbf{B}$ is invertible. Indeed, for each row i, the entries of matrix $\mathbf{B}$ satisfies

$$|b_{ii}| - \sum\nolimits_{j\neq i} |b_{ij}| = 1 - \delta - \delta\hat{\lambda} > 0.$$

Finally, proving $p^* > 0$ is equivalent to showing $\frac{\partial x_1}{\partial p_1}|_{p_1=p^*} < 0$. We first note that matrix $\mathbf{B}$ being strictly diagonally dominant implies that its inverse matrix $\mathbf{A} = \mathbf{B}^{-1}$ satisfies the following weaker form of strict diagonal dominance: $|a_{ii}| > |a_{ij}|$ for all row i and column $j \neq i$. (see, e.g., Johnson et al. (2024)). Then, from (7), the first row of matrix $\mathbf{A}$ gives

$$\frac{\partial x_1}{\partial p_1}|_{p_1=p^*} = -\frac{h(0)}{n-1}\sum_{j=1}^{n}(a_{11} - a_{1j}) < 0, \tag{38}$$

where we used $a_{11} > 0$ as noted in (12), and so $p^* > 0$. Finally, observe that the $n \times 1$ column vector $(1, .., 1)^{\intercal}$ is clearly an eigenvector of matrix $\mathbf{B}$ with eigenvalue $1 - (1 + \hat{\lambda})\delta$, and so the same vector is also an eigenvector to matrix $\mathbf{A}$ with eigenvalue $(1 - (1 + \hat{\lambda})\delta)^{-1}$. This means $\sum_{j=1}^{n} a_{1j} = (1 - (1 + \hat{\lambda})\delta)^{-1}$, so that (38) implies (13). □

Proof. **(Proposition 2)**. Rearranging (15) gives

$$\frac{1/n}{\phi'(z^*)}\frac{dp^*}{d\lambda} = \frac{n}{n-1} - (1-\lambda)\,\rho\left(\lambda + \frac{1-\lambda}{n}\right),$$

where the RHS is monotonically increasing in λ. Hence, if $\frac{n}{n-1} \geq \rho\left(\frac{1}{n}\right)$, then $dp^*/d\lambda \geq 0$ for all $\lambda \geq 0$. If otherwise $\frac{n}{n-1} < \rho\left(\frac{1}{n}\right)$, then the intermediate value theorem implies the existence of the required unique cutoff $L \in (0, 1)$, pinned down by $\frac{n}{n-1} = (1-L)\,\rho\left(L + \frac{1-L}{n}\right)$, and a total derivative shows $dL/dn > 0$. □

Proof. **(Proposition 3).** Rearranging gives

$$\frac{1/n}{\phi'(z^*)}\frac{\partial CS}{\partial\lambda} = n - 2 - \frac{1}{n-1} + (1-\lambda)\rho\left(\lambda + \frac{1-\lambda}{n}\right). \tag{39}$$

If $\rho(z) = 0$ for all z, then (39) implies the result. If $\rho(z) > 0$ for all z, then the RHS of (39) is monotonically decreasing in λ by Assumption (17), and so the cutoff L_{CS} is pinned down by the solution of

$$n - 2 = \frac{1}{n-1} - (1-\lambda)\rho\left(\lambda + \frac{1-\lambda}{n}\right). \tag{40}$$

If $\rho(z) < \min\{0, \rho'(z)\}$ for all z, then the RHS of (39) is monotonically increasing in λ and the cutoff L'_{CS} is similarly pinned down by (40). Finally, differentiating (39) with respect to n shows $\frac{\partial CS}{\partial \lambda}$ is single-crossing in n, as required for the last statement of the proposition. $\square$

Proof. **(Pricing expression (20)).** Let $x_{partner}$ be the market share of each of remaining firms in the same coalition with platform 1. Continue from equations (36) and (37) in the proof of Proposition 1, we get

$$\begin{aligned}
\frac{\partial x_1}{\partial p_1} &= -h(0) + \delta \frac{\partial z_1}{\partial p_1} = -h(0) + \delta \left(\frac{\partial x_1}{\partial p_1} + \lambda(m-1) \frac{\partial x_{partner}}{\partial p_1} \right) \\
\frac{\partial x_{partner}}{\partial p_1} &= \frac{h(0)}{n-1} + \delta \frac{\partial z_{partner}}{\partial p_1} = \frac{h(0)}{n-1} + \delta \left((1 + \lambda(m-2)) \frac{\partial x_{partner}}{\partial p_1} + \lambda \frac{\partial x_1}{\partial p_1} \right)
\end{aligned}$$

Solving the simultaneous equation yields

$$\begin{aligned}
\frac{\partial x_1}{\partial p_1} &= -\frac{h(0)}{n-1} \left(\frac{n-1-\frac{\delta\lambda(m-1)}{1-\delta-\delta\lambda(m-2)}}{1-\delta-\frac{\delta\lambda(m-1)\delta\lambda}{1-\delta-\delta\lambda(m-2)}} \right) \\
&= -\frac{h(0)}{1-(1-\lambda)\delta} \left(1 + \left(\frac{n-m}{n-1} \right) \frac{\lambda\delta}{1-(1+\lambda(m-1))\delta} \right)
\end{aligned}$$

From the equilibrium condition $p^* = \frac{1/n}{\partial x_1/\partial p_1}|_{p_1=p^*}$, we pin down equation (20). $\square$

Proof. **(Proposition 4).** The first part follows from differentiating (20):

$$\frac{\partial p^*}{\partial \lambda} = \frac{\gamma(m-1)(n-1)}{[F_1(\lambda, m)]^2} F_2(\lambda, m)$$

where

$$\begin{aligned}
F_1(\lambda, m) &= (n-1)(\delta-1) + (1-2n+mn)\delta\lambda \\
F_2(\lambda, m) &= (1+mn-2n)\delta^2\lambda^2 + 2(n-1)(\delta-1)\delta\lambda + (\delta-1)^2.
\end{aligned}$$

Factorizing the quadratic equation F_2 in terms of λ, we get:

$$F_2(\lambda, m) = (1+mn-2n)\delta^2 \times (L_1 - \lambda)(L_2 - \lambda)$$

where

$$\begin{aligned}
L_1 &= \frac{n-1-\sqrt{(n-m)n}}{(mn-2n+1)} \left(\frac{1}{\delta} - 1 \right) \equiv L_{coal} \\
L_2 &= \frac{n-1+\sqrt{(n-m)n}}{(mn-2n+1)} \left(\frac{1}{\delta} - 1 \right)
\end{aligned}$$

and the recurring assumption $\delta < \frac{1}{1+(m-1)\lambda}$ implies $1/\delta > 1 + (m-1)\lambda \geq 1$. Therefore,

$$L_2 - \lambda > \frac{n-1+\sqrt{(n-m)n}}{(mn-2n+1)}(m-1)\lambda - \lambda$$

$$
\begin{aligned}
&> \frac{n-1+n-m}{(mn-2n+1)}(m-1)\lambda - \lambda \\
&= \frac{(n-m)m\lambda}{mn-2n+1} > 0.
\end{aligned}
$$

Consequently, $F_2(\lambda, m) > 0$ (equivalent to $\partial p^*/\partial\lambda > 0$) if and only if $\lambda < L_1 \equiv L_{coal}$. By examining L_{coal}, it is clear that it is decreasing in $\delta \equiv \frac{h(0)\gamma n}{n-1}$ and also increasing in m.

To prove the last statement, we note $\delta > 1/n$ implies

$$
p^*_{\lambda=1} = \frac{\frac{n-1}{nh(0)}}{n-1+\frac{(n-m)\gamma}{\frac{n-1}{nh(0)}-m\gamma}} < \frac{\frac{n-1}{nh(0)}-\gamma}{n-1} = p^*_{\lambda=0}.
$$

where recall $\delta \equiv \frac{h(0)\gamma n}{n-1}$. Then, $p^*_{\lambda=1} < p^*_{\lambda=0}$ implies that $dp^*/d\lambda < 0$ at least over some segment in $\lambda \in [0,1]$, which implies $L_{coal} < 1$. □

Proof. **(Proposition 5).** Define function $F_0(\lambda) \equiv 1 - \delta + (\hat{\lambda} - 2\lambda)\delta$, which is linear and decreasing in its scalar argument (recall $\hat{\lambda}$ is a constant in this proposition). Therefore,

$$
\tilde{F}(\vec{\lambda}) \equiv \frac{1}{F_0(\lambda_1)} + \frac{1}{F_0(\lambda_2)} + \frac{1}{F_0(\lambda_3)},
$$

where $F_0(\lambda_i) > 0$ for all $i = 1, 2, 3$ because $\delta \leq 1/(1+\hat{\lambda})$ by Proposition 1. Clearly, $\tilde{F}(\vec{\lambda})$ is convex and symmetric in $\vec{\lambda}$ and so Schur-convex in $\vec{\lambda}$ (see Chapter 3 of Marshall et al. (2011) for references on the properties of Schur-convexity and Schur-concavity). Given that Schur-convexity is an ordinal property preserved under increasing transformation, we conclude $-1/\tilde{F}(\vec{\lambda})$ is Schur-convex, and so $p^*(\vec{\lambda}) = 1/\tilde{F}(\vec{\lambda})$ is Schur-concave, which implies by definition that $p^*(\vec{\lambda}) \leq p^*(\vec{\lambda}')$ whenever $\vec{\lambda} \succ \vec{\lambda}'$. The remaining results follow from Schur-concavity of $p^*(\vec{\lambda})$. □

Proof. **(Corollary 2).** We continue from pricing equation (24). For this proof, it is convenient to substitute $\beta = \frac{1}{3h(0)}$, which gives:

$$
\begin{aligned}
p^*(\lambda, \mu, \mu) &= \frac{1}{\frac{1}{4\beta - \gamma(\lambda - 2\mu + 1)} + \frac{2}{4\beta + \gamma(\lambda - 1)}}, \\
\frac{\partial p^*(\lambda, \mu, \mu)}{\partial \mu} &= \frac{2\gamma(4\beta + \gamma(\lambda-1))^2}{[\gamma(\lambda - 4\mu + 3) - 12\beta]^2} \geq 0.
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
&\frac{\partial p^*(\lambda, \mu, \mu)}{\partial \lambda} \\
=&\frac{\gamma\left\{4\beta + \gamma\left[(4+2\sqrt{2})\mu - (3+2\sqrt{2})\lambda - 1\right]\right\}\left\{4\beta + \gamma\left[(4-2\sqrt{2})\mu - (3-2\sqrt{2})\lambda - 1\right]\right\}}{[12\beta - \gamma(\lambda - 4\mu + 3)]^2}.
\end{aligned}
$$

The assumption of $\delta < \frac{1}{1+\hat{\lambda}} = \frac{1}{1+\lambda+2\mu}$ is equivalent to $\beta \geq \frac{4}{9}(1+\lambda+2\mu)\gamma$, which implies

$$\begin{aligned}
&4\beta + \gamma\left[(4-2\sqrt{2})\mu - (3-2\sqrt{2})\lambda - 1\right] \\
\geq &\underbrace{(\frac{16}{9}-1)}_{>0}\gamma + \underbrace{\left[\frac{16}{9} - (3-2\sqrt{2})\right]}_{>0}\lambda\gamma + \underbrace{\left[\frac{32}{9} + (4-2\sqrt{2})\right]}_{>0}\mu\gamma \\
> &0
\end{aligned}$$

Hence, $\frac{\partial p^*(\lambda,\mu,\mu)}{\partial\lambda} \geq 0$ if and only if

$$\lambda \leq L_{intcoal} \equiv (3-2\sqrt{2})\frac{4\beta-\gamma}{\gamma} + (4-2\sqrt{2})\mu.$$

Observe that if $\mu \to 1$ (then $\lambda \to 1$ by assumption so $\beta > \frac{16}{9}\gamma$) or $\gamma \to 0$, $L_{intcoal} > 1$.

Meanwhile, $p^*(\lambda,\lambda,\mu)$ is mathematically equivalent to $p^*(\lambda,\mu,\mu)$ given that the arguments of $p^*(\cdot,\cdot,\cdot)$ are symmetrical in permutations:

$$\begin{aligned}
p^*(\lambda,\lambda,\mu) &= \frac{1}{\frac{1}{4\beta+\gamma(2\lambda-\mu-1)} + \frac{2}{4\beta+\gamma(\mu-1)}}, \\
\frac{\partial p^*(\lambda,\lambda,\mu)}{\partial\lambda} &= \frac{2\gamma(4\beta+\gamma(\mu-1))^2}{[12\beta+\gamma(4\lambda-\mu-3)]^2} \geq 0,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial p^*(\lambda,\lambda,\mu)}{\partial\mu} \\
=&\frac{\gamma\left\{4\beta+\gamma\left[(4+2\sqrt{2})\lambda-(3+2\sqrt{2})\mu-1\right]\right\}\left\{4\beta+\gamma\left[(4-2\sqrt{2})\lambda-(3-2\sqrt{2})\mu-1\right]\right\}}{[12\beta-\gamma(\mu-4\lambda+3)]^2}.
\end{aligned}$$

Different from previous discussion on $p^*(\lambda,\lambda,\mu)$, with the assumption $\lambda \geq \mu$, we yield

$$4\beta+\gamma\left[(4+2\sqrt{2})\lambda-(3+2\sqrt{2})\mu-1\right] > 4\beta+\gamma\left[(4-2\sqrt{2})\lambda-(3-2\sqrt{2})\mu-1\right] > 0.$$

Hence, $\frac{\partial p^*(\lambda,\lambda,\mu)}{\partial\mu} \geq 0$. □

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