Sharing sequential benefits in a network*

Ruben Juarez
University of Hawaii
rubenj@hawaii.edu

Chiu Yu Ko
National University of Singapore
kochiuyu@nus.edu.sg

Jingyi Xue
Singapore Management University
jyxue@smu.edu.sg

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Abstract

Consider a sequential process where agents have individual benefits at every possible step. A planner is in charge of choosing steps and distributing the accumulated aggregate benefits among agents. We model such a process by a directed network where each edge is associated with a vector of individual benefits. This model applies to several new and old problems, e.g., developing a connected public facility (such as rail-roads) and distributing total benefits received by surrounding districts, selecting a long-term production plan and sharing final profits among partners of a firm, choosing a machine schedule to serve different tasks and distributing total outputs among task owners.

We provide the first axiomatic study on path selection and benefit sharing in sequential processes. Surprisingly, we find that four sets of axioms from different perspectives characterize similar classes of solutions — selecting efficient path(s) and assigning to each agent a share of total benefits which is independent of the distribution of individual benefits. The four sets of axioms include those related to the additivity of agents' assignments over subnetworks, to the monotonicity of their assignments with respect to network expansion, to the independence of their assignments with respect to certain network transformations, and to implementation in the case where the planner has no information on the network and individual benefits. We also characterize more general classes of solutions, including one where the sharing rule depends, in a "rationalizable" way, on the distribution of individual benefits in a network.

Keywords: *Sequential Benefits, Sharing, Network, Redistribution*

***JEL* classification:** C72, D44, D71, D82.

1 Introduction

Axiomatic division of costs and benefits have been extensively studied over the past 60 years, Initiated from cooperative games (Shapley[35]) and followed by applications to problems such as rationing and bankruptcy (O'Neill[34], Aumann and Maschler[2], Thomson[39], Moulin[28, 29]), airport cost-sharing (Littlechild and Owen[24]), hierarchical ventures (Hougaard et al. [12]), and more general cost-sharing problems (e.g., Sprumont[36, 37], Moulin [30], Moulin and Friedman[10], Moulin and Shenker[31], Moulin and Sprumont[32]). Such studies have characterized a wide variety of sharing rules using axioms motivated by positive and normative perspectives. However, these studies are largely limited to a scenario where a fixed resource must be distributed among agents. Little is known regarding scenarios that are more general in two dimensions. First, the amount of the resource is no longer fixed but instead can be chosen. Second, the amount of the resource might be generated in a sequence of steps, where the availability of the resource in future steps depend on the choice in previous steps. The generalized problems require the selection of resource-generating steps to be determined together with the sharing of the generated resource. This "two-tiered" approach not only expands the types of problems to study, but also introduces a new question on the interdependence of the step selection rule and sharing rule. This interaction of rules does not exist when the resource is fixed, but naturally arises in our general framework.

To better illustrate our problem, consider a social planner in charge of developing a connected public facility (such as highways, rail-roads or irrigation canals). The project might be developed in different steps, each of which might benefit the agents in a society differently. The planner is in charge of choosing the steps and redistributing the benefits of the project among the agents. After proceeding along each step, a new problem is created. This new problem is different from the original problem and might be different across steps proceeded (Section 1.1 discusses several applications of this problem).

We represent the structure of our problem as a network, where each node in the network faces a forward problem. More precisely, our problem consists of an acyclic-directed-network with a common source and multiple sinks. Additionally, there is a finite number of agents and each node in the network contains a vector which represents the marginal contribution of these agents if this node is selected. A path connecting the source and a sink has to be chosen and the value of the path must be distributed among the agents.

Herein, we provide the first systematic study of this problem using a novel axiomatic approach. This approach encompasses axioms that are appropriate to a wide range of scenarios, including a complete information case (where the planner knows the network and marginal contribution of the agents) to an incomplete information case (where the planner might not know the marginal contribution of the agents). We impose new axioms as well as adapt more traditional axioms from other studies to our problem. Such axioms include those related to a fixed network (such as continuity or sequential composition), to a variable network (such as technology monotonicity and parallel composition), and to implementation (such as $k-$majority). Surprisingly, after a comprehensive analysis, we find that four sets of axioms from different perspectives characterize similar classes of solutions — selecting efficient path(s) and assigning to each agent a share of total benefits which is independent of the distribution of individual benefits.

1.1 Applications and Solutions

In order to illustrate the applicability of our problem and solutions consider the following examples.

Profit Sharing in Companies. Consider the case of profit-sharing in companies (Juarez and Nitta[21]). There are different production plans which might generate goods of different quality and require agents to generate profits differently. The manager is in charge of selecting a production plan (i.e. path), and make a redistribution of (a fixed percentage of) the profit among the agents.[1] Choosing an efficient path would be a natural solution for a company whose interest is to maximize its profit. The company might also elect not to perform any transfers and assign the marginal profits only to the agents who contributed it. The rule that selects the efficient path and assigns the marginal contribution to the agents is denoted by **EFF-MC**.

Sharing of Benefits (Costs) of Connected Public Facilities. Consider the case of construction of highways, rail-roads or irrigation canals by the government. In the case of highway construction, the government chooses from among several potential routes through which to build a highway. Benefits of this highway to individuals will largely depend on their access to it (i.e. route selected) and will vary accordingly. A typical solution for such a problem is for the social planner of the government to choose the most efficient path and perform no redistributions across individuals (EFF-MC). Such a solution is often employed by government officials and, while efficient, it is not necessarily equitable to all individuals.[2] Alternatively, a more fair solution that maintains efficiency could perform transfer between individuals. This can be achieved via a lump-sum tax to the individuals that benefit the most by the construction of the highway. For instance, a traditional solution would select the efficient path and equalize the benefits across the individuals (**EFF-ES**). EFF-ES is fair and has several other interesting properties, as we see below.

Dynamic time-sharing allocation. Consider an acyclic directed network with a single sink where each edge is associated with a time allocation of an object to different agents. Moreover, every two paths in the network connecting the source with the same terminal node have an equal aggregate time allocation. Thus, nodes are associated with the time that has passed. Each agent posses the capability to transfer the allocation of time into a given transferable

[1] Imagine that the company has committed to redistribute 10 % of the profit to his employees, as is the case of Chobani and other companies. Alternatively, we can imagine that the company is a joint business venture, such as a group of lawyers, which will re-distribute 100 % back to its employees.

[2] The failure of finding an equitable redistribution of the benefits can be seen in the construction of railways around the world, including China and Hawaii. Protest of agents with vested interests on certain routes often occur once the selected route does not meet the needs of such agents. For instance, in 2015, tens of thousands of people in Linshui county (in eastern Sichuan province) protested for being ruled out of high speed railway, such network is shown in Figure 1.

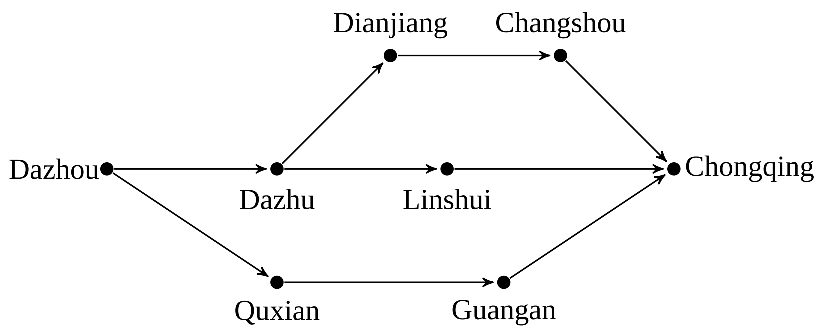

Figure 1: Network outlining the Chinese government plans to build a high-speed railway connecting Dazhou and Chongqing through Dazhu and Linshui (central route) in 2002. Tens of thousands of people from Linshui protested in 2015 when the government considers the two alternative routes.

good, such as money. The goal of the planner is to choose the sequence of time-shares and perform a redistribution of the resource. Examples of this problem include the allocation of time-shares in a super computer among scientists. The queueing problem, where a machine can only serve one agent at a time, is a particular case of this model (Chun [8]). A myopic path (MYO) is such that the edge that produces the largest aggregate profit (efficient edge) is selected at every node, disregarding the information about the profit produced at future steps. While this selection of the path has the advantage of being simple to implement, it does not necessarily select the efficient path. **MYO-MC** would select the myopic path and perform no transfers between the agents. **MYO-ES** selects the myopic path and equalizes the net benefits among agents.

Sequential Decision Making: Common pool resources management and related models. In general, our model can be applied to the problem of sequential decision making where the marginal benefits of every agent are observable at every step. The decisions chosen at every step influence the options available in the next step. For instance, consider the case where the government allocates temporary property rights of a natural resource to agents at every step of a process, such as quotas for the extraction of oil, lumber, or fish. The quota assigned to agents transform into direct benefits, and affect the stock of the resource available in the future. The selection of the path can take different forms, and would depend on the preferences of the planner, including a short term vision (MYO), long term vision (EFF) or something in between. Similarly, the transfers of benefits across agents depend on the path selected, and can be ES, MC or another of the multiple solutions discussed below.

1.2 Overview of the Results

We study two versions of the problem in relation to the information of the planner. For the first part of the paper, we consider the problem with complete information, where the planner knows the marginal contribution of the agents at every edge in the network and is interested in systematically selecting a path and share the profits meeting some axioms. We provide the first study of this problem, especially by providing three characterizations using new and old axioms that have been studied in other models.

Our first characterization uses three axioms that relate to a fix network. First, any problem that contains at least one path that has positive value should positively benefit at least one agent (**non-triviality**). Second, we require that small changes in the value of the vectors of marginal contributions have small impact in the final redistribution of the agents (**continuity**). Finally, we impose an independence of intermediate distributions along a selected path, where agent should be indifferent between receiving a lump-sum payment at the end of the process or receiving instalment at any point in time (**sequential composition**). Sequential composition has the advantage of being computationally easy and rule-out renegotiations of the agents at intermediate steps. The combination of these three axioms characterizes a family of path selection rules that selects a path(s) that has (have) the maximum value (efficient path(s)). The sharing of the value is distributed among the agents in fixed proportions that are independent of the network.

We also provide two alternative characterizations that relate to transformations of the network. Our second characterization requires that no agent shall get hurt from the technology improvement which brings a new edge and destination to the existing network (**technology monotonicity**). This axiom itself characterizes the class of solutions that selects in each network efficient path(s) and an allocation that depends only on the efficient value rather than individual values in the network.

Our third characterization relates to some independence principles with respect to certain network transformations. First, suppose that a step in process consists of two substeps. Then we require that the allocation be independent of representing the step by either one edge or two consecutive edges, as long as the value vectors of the two edges add up to the one of the single edge (**splitting invariance**). Second, suppose that there is a "component" of the network in which each path is Pareto dominated edge by edge compared with some path outside the component. Then removing this component from the network should not affect the allocation (**irrelevance of dominated paths**). Third, suppose that after solving a problem, an additional set of steps from the source is found to be available. Then the initial allocation can be cancelled and a new allocation calculated based on the augmented problem. Alternatively, the initial allocation can be saved as a value vector, an edge attached with this vector replace the initial problem in the augmented one, and a new allocation be calculated for the revised problem. We require that the two approaches lead to the same final allocation to avoid the dispute of agents on which way is better (**parallel composition**). The three axioms, together with continuity, characterize a general class of solutions. Each solution is associated with a redistribution function that assigns to each value vector a redistributed allocation, and a partial order over the set of all redistributed allocations that makes it a join-semilattice. It selects in a network efficient path(s) and the optimal allocation with respect to the partial order over the redistributed accumulated value vectors of all paths in the network. Further, we require the allocation in each "parallel network" depend only on the accumulated aggregate value vector of one path rather than all in a network (**irrelevance of parallel outside options**). This axiom, together with irrelevance of dominated paths and continuity, characterizes the subclass of solutions in which the sharing rules depend only on the efficient value of a network, like the class characterized by technology monotonicity.

For the second part of the paper, we study the problem with incomplete information, where the planner might not know the information of the problem, including the structure of the network or the values of the agents at every step. On the other hand, the agents have full information about the problem, including the network and values of the agents at every step. Contrary to the complete information case, we assume that the planer is interested in systematically selecting the efficient path, but lacking such information, can only delegate the agents to do so.[3] We provide the class of sharing rules that incentivize the agents to select the efficient path.

Our first axiom requires that for any two paths, the efficient path should give a larger share to at least k-agents (***k*-majority**), where k is larger than half the number of agents. This stability notion guarantees the existence of a Condorcet winner when making pairwise comparison of paths. Thus, the efficient path will be selected when agents are allowed to vote on the path using a Condorcet Voting rule. Second, we requires that the identity of the agents should not matter (**anonymity**). The combination of these two axioms characterizes a family of sharing rules such that the sharing of the value of a path has at least k agents getting the average value of the path. Furthermore, the equal sharing rule is characterized by adding either of the following axioms: Sequential composition, Lorentz monotonicity or transfer monotonicity.

1.3 Literature review

While the axiomatic study of sharing rules has been widely studied and applied in many settings, our general two-tiered framework that selects the path along with the sharing rule has not received much attention in the literature. Our framework can incorporate more stylistic two step problems such as the queuing problem (Chun [8]), the minimal cost spanning tree (Kar [23], Kar and Dutta[9], Bergantiños and Vidal-Puga[5], Hougaard and Moulin[13]) and other cost-sharing models (Juarez[18], Juarez[19], Juarez and Kumar[20]). In such problems, an ordering of the agents (queuing), a network meeting certain conditions (such as a spanning tree) or other decisions that affects the cost/benefits (such as the selection of a 'group' or a 'path') must be made and its cost divided among agents.[4]

The second part of the paper is related to the recent literature on implementation of the efficient graphs in networks. For instance, Juarez and Kumar[20] implements the efficient graph in connection networks, Hougaard and Tvede[14, 16] implement the minimal cost

[3]Different from the complete information case, the planner is information constrained such that only information about the marginal contribution on the path chosen is revealed but information about other paths not chosen remain unknown. Hence, it is only practical to consider sharing rule that disregard any information outside the chosen path.

[4]For instance, the queueing problem can be incorporated into our analysis by considering a tree where every node determines the agent who will be served next, and any two paths intersect after the same group of agents have been served in different orders. The value at every edge represents the benefit of the agent being served at that step and the cost incurred by the agents who have not been served.

spanning tree and Juarez and Nitta[21] implement the efficient time allocation in production economies. Hougaard, Moreno-Ternero, Tvede and Osterdal[12] study allocation rules of benefits in hierarchical ventures.

Our model is the first to jointly address the issue of selecting paths and dividing the benefits/cost axiomatically for sequential problems where the information of the marginal contributions of every agent is available at every step.

2 The model under complete information

Let $N = \{1, ..., n\}$ be a fixed finite set of agents, and $\mathcal{G}$ the set of finite directed multigraphs,[5] or simply, **network**, with a unique source (possibly multiple sinks) and no cycles.[6] For example, an element of $\mathcal{G}$ could represent the different steps in a production process. For each $G \in \mathcal{G}$, let $\mathcal{V}^G$ be the set of all functions that assign to each edge in G an element of $\mathbb{R}^n_+$, and we call elements of $\mathcal{V}^G$ **value functions** associated with G.[7] In the previous example, such a function represents the profit contributed by different agents at every step of such production process. A **problem** is a pair (G, v) where $G \in \mathcal{G}$ and $v \in \mathcal{V}^G$. For each $x \in \mathbb{R}^n_+$, we simply use (e, x) to denote a problem where the network contains a single edge e and the value function assigns x to the edge. Let $\mathcal{P}$ be the set of all problems.

For each $(G, v) \in \mathcal{P}$, each edge e and each path L in G,[8] let $v_N(e) := \sum_{i\in N} v_i(e)$ be the **value of *e***, $v_N(L) := \sum\limits_{e\in L} v_N(e)$ the **value of *L***, $v_N(G) := \max\limits_{L\in G} v_N(L)$ the **value of *G***, and L is called **efficient** if $v_N(L) = v_N(G)$.

A **solution** is a pair (φ, μ) of functions on $\mathcal{P}$ such that for each $(G, v) \in \mathcal{P}$, $\varphi(G, v)$ is a nonempty subset of paths in G with the same value, and $\mu(G, v)$ is an element of $\mathbb{R}^n_+$ with $\sum_{i\in N} \mu_i(G, v)$ being the value of $L \in \varphi(G, v)$.

Example 1 (Selection of Paths and Sharing-Rules)**.** *First, we discuss two general methods for path selection.*

1. *[Additively separable rules]. For a given utility function $u : \mathbb{R}^N_+ \to \mathbb{R}$, an additively separable rule selects path(s) that maximize the sum of the utilities of the value associated to the edges, $\varphi^u(G, v) \subset \arg\max_{L\in(G,v)} \sum_{e\in L} u(e)$. In particular, when the utility function equals to the value of the path, $u(e) = v_N(e)$, this selection rule selects the efficient path(s). We denote by EFF a rule that selects efficient path(s).*

2. *[Myopic separable rule] For a given $u : \mathbb{R}^N_+ \to \mathbb{R}$, a myopic rule selects the path(s) that lexicographically maximizes the utility of the value associated to the edges in the path. That is, for paths $L = [e^1, \ldots, e^k]$ and $L' = [\overline{e}^1, \ldots, \overline{e}^{k'}]$ we say that $L \succeq L'$ if and*

[5]A multigraph is a graph where there can be multiple edges with the same end nodes.

[6]Throughout the paper, we assume that the labels of nodes and edges in each network do not have identity.

[7]For simplicity, we restrict to non-negative values to keep the interpretation of benefits throughout the rest of the paper. This assumption is without loss of generality, as all results in the paper extend trivially to values in $\mathbb{R}$.

[8]In this paper, for each problem we only consider the paths from the source to a sink.

only if $u(e^1) > u(\overline{e}^1)$; or $u(e^1) = u(\overline{e}^1)$ and $u(e^2) > u(\overline{e}^2)$; $u(e^1) = u(\overline{e}^1)$, $u(e^2) = u(\overline{e}^2)$ and $u(e^3) > u(\overline{e}^3)$;... We denote by MYO the myopic solution under $u(e) = v_N(e)$.

Next, we discuss two traditional allocation rules for a given path selection rule $\varphi(G, v)$:

3. *[Equal-sharing] The equal-sharing rule μ^{ES} divides the value of the selected paths equally. That is, $\mu_i^{ES}(G, v) = \frac{v_N(L)}{n}\mathbf{1}$ for some $L \in \varphi(G, v)$.*

4. *[Average of Marginal Contributions] The average of the marginal contributions sharing-rule μ^{MC} assigns agents their marginal contribution over all selected paths.[9] That is, $\mu^{MC}(G, v) = \frac{\sum_{L\in\varphi(G,v)}\sum_{e\in L} v(e)}{|\varphi(G,v)|}$.*

A solution is determined by the combination of a path selection rule and a sharing rule.

2.1 Fixed Network Characterization

We consider the following axioms of a solution (φ, μ).

The first axiom requires that each problem that contains at least one path that has positive value should positively benefit at least one agent. This is a basic efficiency property that rules out that all agents get nothing when it is possible to distribute something.

Non-triviality: Given $(G, v) \in \mathcal{P}$, if $v_N(G) > 0$, then $\mu_i(G, v) > 0$ for some $i \in N$.

Non-triviality is satisfied by our four rules discussed above, $EFF - ES$, $EFF - MC$, $MYO - ES$, and $MYO - MC$. Furthermore, it is also satisfied by any additively separable path selection rule as long as the utility function $u(x) > 0$ for $x \gneq 0$. It is also satisfied by the myopic path selection rule as long as the function h selects a positive vector when available. For example, under the selection rule MYO which selects the maximal path.

Continuity states that in any network, small changes in the individual values have small impact on the final division. Such small changes often happen due to measurement errors, and we require that the division rule is robust with respect to such errors.

Continuity: Given $(G, v) \in \mathcal{P}$ and a sequence $\{v^k\}$ of elements of $\mathcal{V}^G$, if for each e in G, $\lim v^k(e) = v(e)$, then $\lim \mu(G, v^k) = \mu(G, v)$.

Continuity is a standard topological property that has been assumed in other models. In our setting it is a strong axiom that rules out a myopic path selection. It also rules out an efficient path selection with marginal contribution sharing as illustrated in the following example. There is, on the other hand, a large class of solutions meeting continuity.[10]

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[9] Alternatively, we can interpret this rule as selecting a path with equal probability, and assigning the marginal contribution every time a path is selected.

[10] Continuity has meaningful implications on both the path selection and the sharing rule. For instance, for path selection, consider solutions that pick paths continuously only depending in the value of the paths. That is, for every $k \in \mathbb{N}$, consider the continuous function $f^k : \mathbb{R}_+^k \to \mathbb{R}_+$ such that $f^k(x) \in \{x_1, x_2, \ldots, x_k\}$ for all

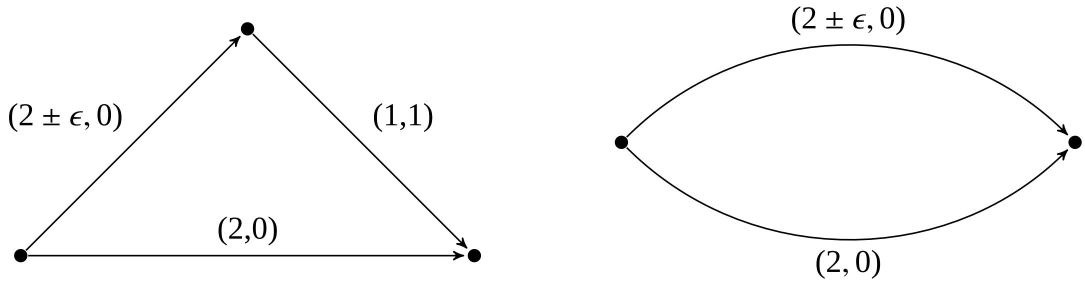

Figure 2: Networks Illustrating that MYO-ES and EFF-MC are not continuous.

Example 2. *In order to illustrate that MYO-ES does not meet continuity, we consider the network in Figure 2 (top). For any* $\epsilon > 0$*, for consider the case where the first edge in the top path has value* $(2+\epsilon, 0)$*. In this case, the solution MYO-ES selects the top path and allocates* $(2+\frac{\epsilon}{2}, 2+\frac{\epsilon}{2})$.

Alternative, consider the case where the first edge in the top path has a value $(2-\epsilon, 0)$*. In this case, the solution MYO-ES selects the bottom path and allocates* $(1, 1)$.

In order to illustrate that EFF-MC does not meet continuity, we consider the network in Figure 2 (bottom). When the top path has a value equal to $(2+\epsilon, 0)$*, the solution EFF-MC selects the top path and allocates* $(2+\epsilon, 0)$*. However, when the top path has a value* $(2+\epsilon, 0)$*, the solution EFF-MC selects the lower path and allocates* $(0, 2)$.

Given $(G, v) \in \mathcal{P}$ and a node d in G which is neither the source nor a sink, let $G|_d$ ($G|^d$) be the maximum sub-network with d being the sink (source), i.e., the sub-network which contains all the paths from the original source to node d (from node d to original sinks), and let $v|_d$ ($v|^d$) be the restriction of v to the edges in $G|_d$ ($G|^d$). We call such problems backward and forward sub-networks. For instance, in the network depicted in Figure 3 we illustrate $G|_d$ (red subnetwork) and $G|^d$ (blue subnetwork) for the node d.

The spirit of sequential composition is that the final division for a problem should be invariant with intermediate redistribution. Think about a process in which a selected plan involves at least two steps, and due to accounting practice interim payment has to be made in the middle of the conduction of the plan. CPN requires that the surplus sharing based on the whole process is equivalent to a step-by-step sharing where at each step the intermediate division is made for the restricted problem with the achieved node in the selected path as its source and the targeted node as its sink.

Sequential composition: Given $(G, v) \in \mathcal{P}$ and a path $L \in \varphi(G, v)$ with at least two edges. Let d be a node in L which is neither the source nor a sink. Then $\mu(G, v) = \mu(G|_d, v|_d) +$

$x \in \mathbb{R}^k_+$. For a network with value of paths $(v_1, \ldots, v_k)$, choose the path with value $f^k(v_1, \ldots, v_k)$. Examples of such continuous path selection rules include the path the the highest value, second highest value, median value or lowest value. Once a continuous path is chosen, divide its value among the agents in a continuous way using the information of the network, for instance, in proportion to the sum of the marginal contribution of the agents in the network.

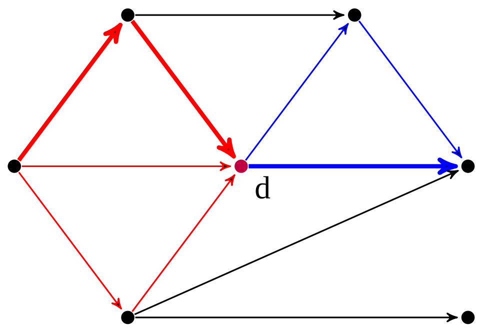

Figure 3: The Red and Blue subnetworks illustrate the backward sub-network $G|_d$ and forward sub-network $G|^d$ for the node d, respectively.

$\mu(G|^d, v|^d)$.

There is a large class of rules meeting sequential composition. Any of the rules introduced in Example 1, including MYO-ES, MYO-MC, EFF-ES and EFF-MC satisfy sequential composition.

We now characterize the solutions that satisfy the combination of the three axioms above.

Theorem 1. *A solution (φ, μ) satisfies non-triviality, continuity, and sequential composition if and only if φ only selects efficient path(s), and there is $\alpha \in \mathbb{R}^n_+$ with $\sum \alpha_i = 1$ such that for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = v_N(G)\alpha$.*

One implication of this result, perhaps surprising, is that the efficient path selection is guaranteed with three axioms that are seemingly unrelated to efficiency. Another implications of this result is that the way to share the value of a path is independent of the network configuration and their individual values. Sequential composition implies that the division is done edge by edge, thus disregarding any information about the network beyond the value of the chosen path.

The characterization above is tight. A class of solutions meeting non-triviality and continuity but not consistency will be discussed in Theorem 2 (see below). Some solutions meeting non-triviality and consistency but not continuity are EFF-MC, MYO-MC or MYO-EFF. One solution meeting continuity and consistency but not non-triviality is the solution that selects the path(s) with the smallest sum and divide the value equally among the agents.

It follows readily from Theorem 1 that if in addition the solution satisfies a basic symmetry requirement below, then μ assigns to each agent an equal share of the value of a problem.

Equal treatment of equals: For each $(G, v) \in \mathcal{P}$ and each pair $i, j \in \{1, ..., n\}$, if for each e in G, $v_i(e) = v_j(e)$, $\mu_i(G, v) = \mu_j(G, v)$.

Corollary 1. *A solution (φ, μ) satisfies non-triviality, continuity, sequential composition, and anonymity if and only if φ only selects efficient path(s), and $\forall (G, v) \in \mathcal{P}$, $\mu(G, v) = \frac{v_N(G)}{n}\mathbf{1}$.*

2.2 Changing Network Characterizations

2.2.1 Technology Monotonicity

Given $(G, v) \in \mathcal{P}$, we say that $(G', v') \in \mathcal{P}$ is a **one-step technology improvement** of (G, v) if (i) G' is constructed either by adding a parallel edge connecting two nodes in G or by adding a sink and an edge going out from a node in G to this sink, and (ii) for each e in G, $v'(e) = v(e)$.

Technology monotonicity: For each $(G, v), (G', v') \in \mathcal{P}$, if (G', v') is a one-step technology improvement of (G, v), then $\mu(G', v') \geq \mu(G, v)$.

Technology monotonicity says that no agent shall get hurt from the technology improvement which brings new paths and destinations to the existing network. Note that it is equivalent to assume in *technology monotonicity* that (G', v') is an one-step technology development of (G, v).

Theorem 2. *A solution (φ, μ) satisfies technology monotonicity if and only if φ only selects efficient path(s), and there is a non-decreasing function $f : \mathbb{R}_+ \to \mathbb{R}^n_+$ such that for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = f(v_N(G))$.*

Note that if *continuity* is also imposed, then the function f in Theorem 2 must be continuous. In addition, if we require *equal treatment of equals*, then μ must give equal division.

Corollary 2. *A solution (φ, μ) satisfies technology monotonicity and equal treatment of equals if and only if φ only selects efficient path(s), and for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = \frac{v_N(G)}{n}\mathbf{1}$.*

2.2.2 Independence with respect to network transformation

Suppose that the benefits of a step in a process is generated by several substeps. In the network representation of this process, such a step can either be represented by a single edge attached with the aggregate benefits, or by several consecutive edges attached with the substep benefits. We require that the two ways of formulating this problem have no impact on agents' assignments.

Splitting invariance: Let $G, G' \in \mathcal{G}$ be such that G' is constructed by splitting an edge e in G into two consecutive edges e_1 and e_2. If $v \in \mathcal{V}^G$ and $v' \in \mathcal{V}^{G'}$ are such that $v(e) = v'(e_1) + v'(e_2)$, and for each e' in G other than e, $v(e') = v'(e')$, then $\mu(G, v) = \mu(G', v')$.

Equivalently, this requirement can be formulated as a *merging invariance* axiom: in each problem, the allocation should not change after two consecutive edges, with no ingoing and outgoing edges at the node connecting them, are merged into one edge which is attached with the sum of the benefit vectors of both edges.

Splitting or merging invariance is familiar in the rationing problem (Banker [3], Moulin [26], De Frutos [11], Ju [**?**]), the cost sharing problem (Sprumont [38]), and the social

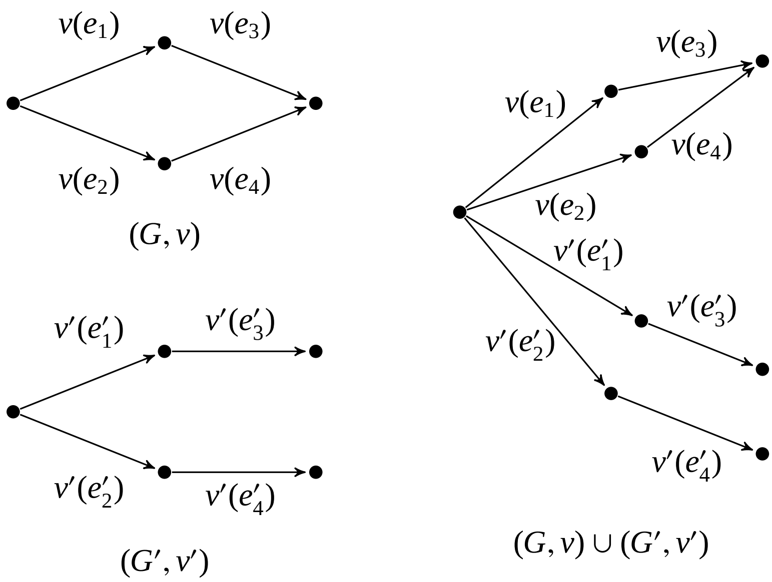

Figure 4: Union of two problems

choice problem (Moulin [26], Chun [7]).[11] In these problems, it is required that an allocation rule be immune to splitting an agent into several participation units in a problem or merging several agents into one participation unit. In comparison, we consider splitting and merging maneuvers not over agents, but over consecutive steps in a process. In our sequential setting, it could be desirable for a sharing rule to disregard how the individual benefits are accumulated through potential substeps in one step of a process.

Suppose that in a problem, each path in one "component" of the network generates for each agent a smaller benefit in each edge of the path compared with another path, outside this component, with the same number of edges. Out of an efficiency concern, one simplification of the problem is to remove the "dominated" component. The following axiom requires that the sharing rule gives the same allocation in the simplified problem as in the original problem.

Given $(G, v), (G', v') \in \mathcal{P}$, a path L in (G, v) is said to be **stepwise dominated** by a path L' in (G', v') if L and L' have the same number of edges, say $n \in \mathbb{N}$, and for each k-th edge e_k in L and e'_k in L', $k \in \{1, ..., n\}$, $v(e_k) \gneq v'(e'_k)$.

Given $(G, v), (G', v') \in \mathcal{P}$, let $(G, v) \cup (G', v')$ denote another problem given by merging the sources of G and G', and assigning the edges in the combined network the same benefit vectors as in the respective problems. An example is given in Figure 4.

Irrelevance of dominated paths: For each pair $(G, v), (G', v') \in \mathcal{P}$, if each path in (G, v) is stepwise dominated by a path in (G', v'), then $\mu((G, v) \cup (G', v')) = \mu(G', v')$.

Suppose that after an allocation being selected for a problem, a set of alternative paths,

[11] *Splitting or merging invariance* is investigated in a unified framework of allocation problems by Ju, Miyagawa and Sakai [17].

which have no intersection with the initial ones, is found to be available. Then the initial allocation can be cancelled and a new allocation be computed for the enlarged problem. Otherwise, the initial allocation can be taken as a benefit vector, an edge attached with this vector replace the initial part in the enlarged problem, and an allocation be computed for the revised problem. We require that both ways of dealing with the issue lead to the same allocations, so that agents will have no dispute on which way is better.

Parallel composition: For each pair $(G, v), (G', v') \in \mathcal{P}$, $\mu((G, v) \cup (G', v')) = \mu((e, \mu(G, v)) \cup (G', v'))$.

Parallel composition formulates the principle that a problem can be solved part by part. It is reminiscent of the "lower composition" axiom in the rationing model (Young [41]) and the "step by step negotiation" property in the axiomatic bargaining model (Kalai [22]).

These three axioms together with *continuity* characterize a class of "rationalizable" solutions defined as follows.

We call $r : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ a **redistribution function** if for each $x \in \mathbb{R}^n_+$, $\sum r_i(x) = \sum x_i$, and $r(r(x)) = r(x)$. Given a partial order $\succsim$ on a set $S \subseteq \mathbb{R}^n_+$ and a subset S' of S, we denote by $\max_{\succsim} S'$ the join of S when it exists. Given a redistribution function r and a partial order $\succsim$ on $r(\mathbb{R}^n_+)$, a solution (φ, μ) is said to be **(r, $\succsim$) - rationalizable** if (1) for each pair $x, y \in \mathbb{R}^n_+$ with $x \geq y$, $r(x) \succsim r(y)$, (2) $(r(\mathbb{R}^n_+), \succsim)$ is a join-semilattice, and (3) for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = x^* := \max_{\succsim}\{r(\sum_{e \text{ in } L} v(e)) : L \text{ in } G\}$, and $\sum x^*_i = v_N(L)$ where $L \in \varphi(G, v)$. Moreover, (φ, μ) is said to be **continuously (r, $\succsim$) - rationalizable** if (φ, μ) is **(r, $\succsim$) - rationalizable**, r is continuous and $g : r(\mathbb{R}^n_+)^2 \to r(\mathbb{R}^n_+)$, defined by setting for each $(x, y) \in r(\mathbb{R}^n_+)^2$, $g(x, y) = \max_{\succsim}\{x, y\}$, is continuous. Note that if (φ, μ) is continuously $(r, \succsim)$ - rationalizable, then $\succsim$ is continuous.

Theorem 3. *A solution (φ, μ) satisfies splitting invariance, irrelevance of dominated paths, independence, and continuity if and only if φ only selects efficient path(s), and there exist a redistribution function r and a partial order $\succsim$ on $r(\mathbb{R}^n_+)$ such that φ only select efficient path(s) and (φ, μ) is continuously $(r, \succsim)$ - rationalizable.*

Example 3 (Locally egalitarian solution with transfers)**.** *Imagine that the set of agents are divided into two groups according to their exogenous types, and a redistribution function divides equally within each group the sum of benefits of the group members. For instance, a partnership firm that runs two kinds of business may adopt the equal sharing rule respectively for its partners involved in each business (Burrows and Black [**?**], Baskenille-Morley and Beechey [**?**]). Note that when there are two agents in total, this redistribution function is simply the identity mapping that assigns each agent his individual benefits.*

A problem generates a set of allocations by applying this redistribution function to the aggregate benefit vector given by each path. A planner selects an allocation based on this set according to three criteria. First, if one allocation dominates another in the vector dominance sense, then the former should be selected over the latter. Second, if the sums of individual benefits of two allocations are the same while one is more egalitarian in the sense *that it is a convex combination of the other allocation and the equal-sharing allocation, then it should be selected over the other. Third, if one allocation is preferred to another and the second is preferred to a third allocation by either of the two previous criteria, then to be consistent, the first allocation should be chosen over the third one.*

Formally, let $S_1, S_2 \subseteq \{1, ..., n\}$ *be such that* $S_1, S_2 \neq \emptyset$, $S_1 \cap S_2 = \emptyset$, *and* $S_1 \cup S_2 = \{1, ..., n\}$. *Define* $r : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *by setting for each* $x \in \mathbb{R}^n_+$, *each* $k = \{1, 2\}$ *and each* $i \in S_k$, $r_i(x) = \frac{\sum_{i \in S_k} x_i}{|S_k|}$. *Define a binary relation* $\succsim$ *on* $r(\mathbb{R}^n_+)$ *by setting for each pair* $x, y \in r(\mathbb{R}^n_+)$, $x \succsim y$ *if there is* $z \in r(\mathbb{R}^n_+)$ *such that* $x \geq z$, $\sum z_i = \sum y_i$, *and there is* $\lambda \in [0, 1]$ *with* $z = \lambda y + (1 - \lambda)\frac{\sum y_i}{n}\mathbf{1}$. *Note that for each pair* $x, y \in r(\mathbb{R}^n_+)$, *if* $x \geq y$, *then* $x \succsim y$. *Moreover, if* $\sum x_i = \sum y_i$, *then* $x \succsim y$ *if and only if x is a convex combination of y and* $\frac{\sum x_i}{n}\mathbf{1}$. *In fact,* $\succsim$ *is the transitive closure of the binary relation that satisfies these two criteria.*[12]

Let (φ, μ) *be such that for each* $(G, v) \in \mathcal{P}$, φ *selects all efficient path(s) in* (G, v), *and* $\mu(G, v) = \max_{\succsim}\{r(\sum_{e \text{ in } L} v(e)) : L \text{ in } G\}$. *It can be shown that* (φ, μ) *is continuously* $(r, \succsim)$ *- rationalizable.*[13] *By Theorem 3, it satisfies splitting invariance, irrelevance of dominated paths, parallel composition, and continuity.*

Unlike the previous solutions that we have characterized, an allocation given by this solution may depend on all paths in a problem, even the inefficient ones. For example, suppose that there are two agents. Consider a problem consisting of two parallel paths that induce the aggregate benefit vectors $(1, 5)$ *and* $(3, 1)$. *Our solution will select the unique efficient path in this problem and choose the allocation* $(2, 4)$. *To see this, note that* $(2, 4) \succsim (1, 5)$ *by criterion two,* $(2, 4) \succsim (2, 2)$ *and* $(2, 2) \succsim (3, 1)$ *respectively by criteria one and two, and finally by transitivity,* $(2, 4) \succsim (3, 1)$. *Figure 3 shows the intersection of the upper contour sets of* $(1, 5)$ *and* $(3, 1)$ *is the upper contour set of* $(2, 4)$. *Thus,* $(2, 4) = \max_{\succsim}\{(1, 5), (3, 1)\}$. *Such an allocation serves as a comprise between the two allocations given by each path and favoring different agents.*

The previous example shows how all the paths (including inefficient ones) in a problem may together determine the final allocation. In real life, especially when facing a large number of options of feasible paths, for simplicity it could be desirable to determine the final allocation by just one option rather than many of them. The next axiom imposes this requirement for simple problems where the networks consist of parallel paths. It turns out that this axiom single outs the solutions that generate allocations based on only the value of the problem, like we obtain from the previous sections. Moreover, *splitting invariance* and *parallel composition* are endogenously implied.

Irrelevance of parallel outside options: For each $m \in \mathbb{N}$ and each set $\{(G^k, v^k) \in \mathcal{P} : G^k$ consists of a single path, $k = 1, ..., m\}$, there is $j \in \{1, ..., m\}$ such that $\mu(\bigcup_{k=1}^{m}(G^k, v^k)) = \mu(G^j, v^j)$.

[12]See the proof in Appendix.
[13]See the proof in Appendix.

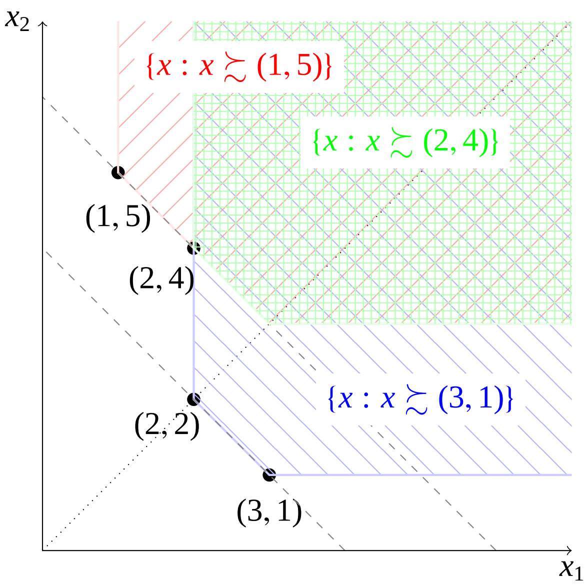

Figure 5: The upper contour sets of $(1, 5)$, $(3, 1)$, and $(2, 4)$

Theorem 4. *A solution (φ, μ) satisfies irrelevance of dominated paths, irrelevance of parallel outside options, and continuity if and only if φ only selects efficient path(s), and there is a continuous function $f : \mathbb{R}_+ \to \mathbb{R}^n_+$ such that for each $c \in \mathbb{R}_+$, $\sum f_i(c) = c$, and for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = f(v_N(G))$.*

The characterizations in both Theorem 3 and 4 are tight. Dropping *splitting invariance*, consider the solution in example 3 with a "redistribution function" r^p that depends on not only the benefit vector but also on the number of edges in the path that generate the benefit vector. That is, for each $(G, v) \in \mathcal{P}$ and each L in G, if the number of edges in L is 2, then r^p divides equally among agents in S_1 the aggregate benefits generated by agents of S_2 along the path, and among agents in S_2 those generated by agents of S_1; if the number of edges in L is other than 2, then r^p agrees with r in example 3. The solution with modified redistribution function satisfies *irrelevance of dominated paths*, *parallel composition*, and *continuity*. Dropping *irrelevance of dominated paths*, the solution that selects all the paths with the smallest benefits and divides the benefits equally among the agents satisfies *splitting invariance*, *parallel composition*, *irrelevance of parallel outside options*, and *continuity*. Dropping *parallel composition*, consider the solution that selects all the efficient paths and divides the value to each agent in proportion to the maximum aggregate benefits he can generate over all the paths in a problem. This solution satisfies *splitting invariance*, *irrelevance of dominated paths*, and *continuity*. Dropping *continuity*, consider a path selection rule that picks in the first round the paths among all the efficient ones that maximizes agent 1's individual aggregate benefits. Then, it picks among the selected ones in the first round those maximizing agent 2's individual aggregate benefits, and so on and so forth, to the n-th round. The solution that adopts this path selection rule and assigns to the agents their individual aggregate benefits along the selected path(s) satisfies *splitting invariance*, *irrelevance of dominated paths*, *parallel composition*, and *irrelevance of parallel outside options*. Dropping *irrelevance of parallel outside options*, the solution in example 3 satisfies *irrelevance of dominated paths* and *continuity*.

Lastly, as in the previous sections, *equal treatment of equals* single outs the equal sharing rule.

Corollary 3. *A solution* (φ, μ) *satisfies irrelevance of dominated paths, irrelevance of parallel outside options, continuity, and equal treatment of equals if and only if* φ *only selects efficient path(s), and for each* $(G, v) \in \mathcal{P}$, $\mu(G, v) = \frac{v_N(G)}{n}\boldsymbol{1}$.

3 The model under incomplete information

We turn our attention to the case where the planner has incomplete information about the problem while the agents have perfect information, including the network and the marginal contributions about themselves and other agents. In this scenario, the planner might want to *delegate* the agents to collectively decide the direction to continue the project on a day-to-day basis.[14] Imagine a scenario where the board of a company (owner) sets the compensation rule for the agents (sharing rule) before knowing the profits in the tree.[15] Thus, the planner observes the realized path and uses this information to assign the shares to the agents. On the other hand, the agents make a decision to choose a path that maximizes their payoff taking into account the information about the problem and the sharing rule.[16]

We focus in the case where the objective of the planner is to select the path that produces the largest profit.[17] When delegation is possible, the planner should find a compensation scheme that aligns his objective with the final payoffs of the agents who make the decision.

Our work in this section differ from classical implementation problems since we do not specify a game. We follow an axiomatization of the sharing rules, and thus our analysis works for a large class of games. We impose two simple and seemingly natural properties regarding the stability of our rule.

First, we want our rule to be anonymous, which is a natural axiom in companies and other public decision to build facilities.[18] Second, we want a notion of stability, where at

[14] A more traditional way to select a path is by eliciting the information from the agents. We focus in a more decentralized setting, where the planner does not have the ability to get such information from the agents, or even if he gets this information, the decision about the paths is made by the agents. However, Section 3.2 briefly discusses this issue and ways to solve it.

[15] E.g. The board of a company would set the overall payment structure of the agents before actually hiring the agents. This compensation scheme might depend on the actual value brought by the agents but needs to be set in advance for all potential scenarios.

[16] An alternative interpretation occurs in the case of a social planner in charge of building a connected public facilities. The planner would delegate the agents to collectively choose (e.g. by voting) the direction of the public facility after knowing the cost/benefit re-distribution function. While in this scenario the planner might not have incomplete information, our analysis in this section is robust to the information context of the planner.

[17] As this is a desirable path, from both the positive and normative sides (Theorems 1-2).

[18] Nonetheless, there is a large class of asymmetric rules that will fit our problem. The remarks about in-

least a majority of agents (we fixed a threshold $k > \frac{n}{2}$ of agents) should always prefer the efficient path to any other path. This stability notion guarantees the existence of a Condorcet winner when making pairwise comparison of paths. Thus, any Condorcet Voting rule would pick the efficient path.

3.1 The result under incomplete information

To formalize the problem, a **path** is a finite sequence of elements of $\mathbb{R}^n_+$. We denote a typical path by l, and the set of all paths by $\mathcal{L}$. For each path $l = \{x^k\}_{k=1}^K$ in $\mathcal{L}$, where $K \in \mathbb{N}$, let $l_N := \sum_k \sum_i x_i^k$ be the **value of *l***. A sharing rule is a function $\mu : \mathcal{L} \to \mathbb{R}^n_+$ such that for each $l \in \mathcal{L}$, $\sum \mu_i(l) = l_N$.

Our first axiom requires that for every pair of paths, at least k agents prefer the more efficient path. This axiom is necessary to guarantee the selection of the efficient path when agents are delegated to make a decision using a Condorcet Voting rule and other k-majoritarian rules, as we will see in Applications 1 and 2.

***k*-majority ($k > \frac{n}{2}$):** For each pair $l, l' \in \mathcal{L}$, if $l_N \geq l'_N$, then there is $N' \subset N$ such that $|N'| \geq k$ and for each $i \in N'$, $\mu_i(l) \geq \mu_i(l')$.

A solution meeting this axiom will be referred to as k-majoritarian. When $k = [n/2] + 1$ then a majority of people always prefer the efficient path. When $k = n$, all agents prefer the efficient path.[19]

Application 1 (Path Selection using a Condorcet Social Choice Function)**.** *Consider a situation where the path is selected using a Social Choice Function (SCF) that satisfies the Condorcet Property. That is, the SCF elects a Condorcet winner when available.*[20]

For each agent, a sharing rule determines a cardinal assignment for each path, which induces an ordinal ranking over paths. Any $k-$majoritarian sharing rule guarantees that the efficient path is a Condorcet winner at any problem. Therefore, any SCF that meets the Condorcet property picks the efficient path.

Conversely, if the rule is not $k-$majoritarian for any $k > \frac{n}{2}$, then the efficient path is not guaranteed to be selected by any voting function that meets the Condorcet Property.

Application 2 (Sequential voting for a public facility)**.** *Consider the dynamic game of complete information where agents vote to decide a path in a given problem. For instance, agents might vote on the route used by a railway or other connected public facilities. More precisely,*

formation in Section 3.2 uncovers a class of asymmetric delegation rules that would implement the efficient equilibrium.

[19] In the case of $k = n$ this property is reminiscent to Juarez and Kumar[20]. This property characterizes the sharing rules that only use the value of the path to divide the proceeds, as in Theorem 2.

[20] Formally, given the set of objects $\mathcal{M}$, let $\mathcal{R}$ be the set of ordinal preferences over $\mathcal{M}$. A social choice function $\Psi : \mathcal{R}^N \to \mathcal{M}$ meets the Condorcet property if for the preference profile $\succeq = (\succeq_1, \ldots, \succeq_n) \in \mathcal{R}^N$ there exists $l^* \in M$ such that for any $l \in M$, $|\{i \in N | l^* \succ l\}| > \frac{n}{2}$, then $\Psi(\succeq) = \{l^*\}$. A large class of SCFs that satisfy this property are discussed in Moulin[27].

in this game agents vote on the direction to continue at every node. A path is selected using a k-majoritarian voting rule at every node (i.e., if an outcome receives at least k-votes, then is chosen). The payoff of the agents is given by $\mu(l)$, where l is the realized path.

A k-majoritarian sharing rule will always have an efficient path as a strong subgame perfect Nash equilibrium, however, this equilibrium might not be unique due to indifferences. For problems where there does not exist two paths with the same value, k-majoritarian rules will have a unique strictly strong subgame perfect Nash equilibrium.[21] *Corollary 5 pins down the only anonymous k-majoritarian rule that generates a strictly strong subgame perfect Nash equilibrium for any problem.*

Finally, the selection of the efficient path(s) is robust to voting at only a subset of decision nodes. It is also robust to incomplete information about which are the decision nodes.[22]

Example 4 (Examples of k-majority rules)**.** *The following rules satisfy the k-majority axiom:*

1. *Consider a subset S of agents with at least k elements and consider a sharing rule ξ that awards the average value of the path to the agents in S, that is $\mu_i(l) = \frac{v_N(l)}{n}$ for $i \in S$.*

2. *Consider the rules ξ such that for a path l, where $v_{i_1}(l) \geq \cdots \geq v_{i_n}(l)$, it awards $\mu_i(l) = \frac{v_N(l)}{k}$ for $i = i_1, \ldots, i_k$ (break indifferences arbitrarily), and $\mu_j(l) = 0$ for $j > i_k$.*

The first class of rules illustrate the case where a fixed group of agents with at least k elements always get a fixed proportion of the share of the path. Thus clearly satisfying the k-majoritarian axiom.

The second example shows that the k-majority axiom can be adapted to fixing the top k agents. In general, there exists a large class of k-majoritarian sharing-rules. We will discuss more in the examples below.

For permutation π of N and each path $l = \{x^k\}_{k=1}^K$ in $\mathcal{L}$, where $K \in \mathbb{N}$, let $l^\pi \in \mathcal{L}$ be such that for each $k \in \{1, ..., K\}$ and each $i \in N$, $x_i^k = x_{\pi(i)}^k$.

We focus in compensation schemes that are independent of the names of the agents. For instance, on public facilities, agents often vote to potentially re-distribute their share.

Anonymity: For each permutation π of N and $l \in \mathcal{L}$, $\mu(l) = \mu(l^\pi)$.

Anonymity is a desirable property, as the model assumes that agents have symmetric information. Thus, agents are not discriminated solely on the base on the names. On the other hand, this does not prevent agents from being discriminated based on their contributions.

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[21] Recall that under strictly strong subgame perfect Nash equilibrium a deviating coalition is feasible if at least one agent strictly improves and the rest are not worse-off, whereas under the strong Nash equilibrium all agents in the deviation coalition should strictly improve.

[22] This problem often occur when voters want to re-evaluate a chosen route after it has been partially built, for instance in projects that take several years to construct, like the rail in Honolulu, from Ewa side to Waikiki via Downtown. While the decision to build the rail from Honolulu to Ewa was approved by voters, their construction stopped in the middle to re-evaluate the route chosen and be confirmed by the voters before further spending in the project occurs.

We now move to the main result of this section. The combination of the above two axioms characterizes a fairly large class of sharing rules.

Proposition 1. *If μ satisfies k-majority and anonymity, then for each $l \in \mathcal{L}$, there is $N' \subseteq N$ such that $|N'| \geq k$ and for each $i \in N'$, $\mu_i(l) = \frac{v_n(l)}{n}$.*

For small number of agents, the rules found in the above theorem can be described easily.

Example 5.
- *When $n = 3$ or $n = 4$, ES is the only solution meeting meeting k-majority and anonymity.*
- *When $n = 5$ or $n = 6$, for any solution meeting k-majority and anonymity there exists a function $g : [0, 1] \to [0, \frac{2}{n}]$ such that $g(0) = g(1) = \frac{1}{n}$ and for $x_1 \geq x_2 \geq \cdots \geq x_n$ we have that $\xi(x) = [g(x_1), \frac{1}{n}, \ldots, \frac{1}{n}, \frac{2}{n} - g(x_1)]$. In particular, ES is determined for $g(z) = \frac{1}{n}$ for all $z \in [0, 1]$.*

Finally, we provide several characterizations of the equal sharing solution by adding a third axiom to our Theorem above.

Theorem 5. *A rule μ is the equal sharing rule if and only if it satisfies k-majority, anonymity, and either of the following axioms:*

i. *Sequential composition: For each $l = \{x^k\}_{k=1}^K$ in $\mathcal{L}$, where $K \in \mathbb{N}$, $\mu(l) = \sum \mu(\{x^k\})$.*

ii. *Lorenz monotonicity: For each pair $l = \{x^k\}_{k=1}^K$ and $l' = \{x'^k\}_{k=1}^K$ in $\mathcal{L}$, where $K \in \mathbb{N}$, if for each $k \in \{1, ..., K\}$, $x^k \succ_{Lorenz} x'^k$, then $\mu(l') \not\succ_{Lorenz} \mu(l)$.*

iii. *Transfer monotonicity: For each pair $i, i' \in N$, and each pair $l = \{x^k\}_{k=1}^K$ and $l' = \{x'^k\}_{k=1}^K$ in $\mathcal{L}$, where $K \in \mathbb{N}$, if for each $k \in \{1, ..., K\}$, $x_i'^k - x_i^k = x_{i'}^k - x_{i'}'^k \geq 0$, and for each $j \in N \setminus \{i, i'\}$, $x_j^k = x_j'^k$, then $\mu_i(l') \geq \mu_i(l)$.*

iv. *The efficient Path is a Strictly Strong Nash equilibrium in the voting game $\Gamma(G, v)$ (Example 2).*

3.2 Remarks about Information

Given the symmetric information among the agents, our analysis in delegation focuses in groups making decisions. We can alternatively consider the issue of delegation of individual agents that make the decision about the continuation at individual nodes (or collections of nodes). It is easy to show that a rule that incentivize the delegate to make the efficient decision at every problem should allocate the agent a share of the total contribution using a monotone rule, while non-delegated agents can be given an arbitrary share of the value. The equal division rule is the only symmetric rule in this class.

Finally, we have considered the case where agents have symmetric information among them. We can alternatively consider the problem of asymmetric information among agents (e.g. for the case where agents know their own contribution). In this setting, a more traditional approach from the mechanism design literature would require agents to report their information to the planner, who will use this information to make an estimation of the value of the network and select a path —See Hougaard and Tvede [14, 16] for a related study of implementation of the efficient path in minimal cost spanning trees. A natural question in this setting is to find the mechanisms that incentivize agents to report their true information. When agents are critical, that is, when they have information about some nodes that no one else has, it is easy to prove that every estimation rule is manipulable. On the other hand, when agents are not critical, in particular when at least three agents have information about the value at a given edge, several sharing rules, such as rules that use the median of the reports to select the path, can achieve truth telling as an equilibrium.

4 Conclusion

We have introduced the problem of division of sequential benefits and provided a comprehensive study of it. In particular, we have addressed the problem from different angles, including the complete and incomplete information case, and used old and new axioms from other strands of literature to characterize a large class of solutions not uncovered elsewhere. Our paper highlights the robustness of the EFF-ES solution, in both the complete and incomplete information settings. This rule, however, is by no means the only rule when less stringent axioms are imposed.

The continuity axiom, while seemingly weak in other settings, was surprisingly strong in our setting. There is still a large class of solutions that are continuous, and we conjecture that the solutions described in Footnote 10 would lead to the description of most of them.

Sequential composition is also a very powerful axiom, with normative and positive consequences in the sharing of benefits. Several solutions meet this axiom, including all the solutions discussed in Example 1.

Our axiomatic analysis of the incomplete information case highlights the need to characterize all sharing rule meeting the k-majoritarian axiom without anonymity. It also shows that our analysis might be extended to other information settings where more traditional tools from mechanism design might be applied.

References

[1] Aumann, R.: Acceptable points in general cooperative n-person games. In: Tucker, A.W., Luce, R.D. (eds.) Contributions to the Theory of Games IV. Princeton University Press 1959.

[2] Aumann, R., Maschler, M.: Game theoretic analysis of a bankruptcy problem from the Talmud. Journal of Economic Theory 36, 195-213 (1985).

[3] Banker, R. 1981. Equity considerations in traditional full cost allocation practices: an axiomatic perspective. In Joint Cost Allocations, e.d. University of Oklahoma Press, pp.110-130.

[4] Bartling, B. and von Siemens, F.A., 2010. Equal sharing rules in partnerships. Journal of Institutional and Theoretical Economics, 166(2), pp.299-320.

[5] Bergantiños, G. and Vidal-Puga, J.J., 2007. A fair rule in minimum cost spanning tree problems. Journal of Economic Theory, 137(1), pp.326-352.

[6] Chambers, C. 2006. Asymmetric Rules for Claims Problems without Homogeneity. Games and Economic Behavior 54, pp.241-260.

[7] Chun, Y., 2000. Agreement, separability, and other axioms for quasi-linear social choice problems. Social Choice and Welfare, 17(3), pp.507-521.

[8] Chun, Y., 2016, Fair Queuing. Springer.

[9] Dutta, B. and Kar, A., 2004. Cost monotonicity, consistency and minimum cost spanning tree games. Games and Economic Behavior, 48(2), pp.223-248.

[10] Friedman, E. and Moulin, H., 1999. Three methods to share joint costs or surplus. Journal of Economic Theory, 87(2), 275-312.

[11] de Frutos, M.A., 1999. Coalitional manipulations in a bankruptcy problem. Review of Economic Design, 4(3), 255-272.

[12] Hougaard, J., Moreno-Ternero, J., Tvede, M., Osterdal, L., 2015. Sharing the proceeds from a hierarchical venture. Mimeo, University of Southern Denmark.

[13] Hougaard, J. L., H. Moulin and L. P. Osterdal., 2010. Decentralized pricing in minimum cost spanning trees. Economic Theory. 44(2), 293-306.

[14] Hougaard, J., Tvede, M., 2012. Truth-Telling and Nash Equilibria in Minimum Cost Spanning Tree Models. European Journal Operations Research.

[15] Hougaard, J., Tvede, M., Osterdal, L., 2013. Cost Sharing in Chains and Other Fixed Trees. Mimeo, University of Southern Denmark.

[16] Hougaard, J., Tvede, M., 2015. Minimum cost connection networks: Truth-telling and implementation. Journal of Economic Theory, 157, 76-99.

[17] Ju, B.G., Miyagawa, E. and Sakai, T., 2007. Non-manipulable division rules in claim problems and generalizations. Journal of Economic Theory, 132(1), pp.1-26.

[18] Juarez, R., 2008. The worst absolute surplus loss in the problem of commons: Random Priority vs. Average Cost. Econ Theory 34, 69-84.

[19] Juarez, R., 2013. Group strategyproof cost sharing: the role of indifferences. Games and Economic Behavior, 82, pp.218-239.

[20] Juarez, R. and Kumar, R., 2013. Implementing efficient graphs in connection networks. Economic Theory, 54(2), 359-403.

[21] Juarez, R. and Nitta, K., 2016. Profit-Sharing and Implementation of Efficient Outcomes. Mimeo University of Hawaii.

[22] Kalai, E., 1977. Proportional solutions to bargaining situations: interpersonal utility comparisons. Econometrica, pp.1623-1630.

[23] Kar, A., 2002. Axiomatization of the Shapley value on minimum cost spanning tree games. Games and Economic Behavior, 38(2), 265-277.

[24] Littlechild, S. C., and Owen, G., 1973. A simple expression for the Shapley value in a special case. Management Science, 20(3), 370-372.

[25] Moulin, H., 1985. The separability axiom and equal-sharing methods. Journal of Economic Theory, 36(1), pp.120-148.

[26] Moulin, H., 1987. Equal or proportional division of a surplus, and other methods. International Journal of Game Theory, 16(3), pp.161-186.

[27] Moulin, H., 1991. Axioms of cooperative decision making (No. 15). Cambridge University Press.

[28] Moulin, H., 2000. Priority rules and other asymmetric rationing methods. Econometrica 68 (3), pp.643-684.

[29] Moulin, H., 2002. Axiomatic cost and surplus sharing. In: In K. J. Arrow, A. K. Sen, K. Suzumura (ed.): Handbook of Social Choice and Welfare, Vol 1, pp.289-357, North Holland.

[30] Moulin, H., 1994. Serial cost-sharing of excludable public goods. The Review of Economic Studies, 61(2), pp.305-325.

[31] Moulin, H. and Shenker, S., 1992. Serial cost sharing. Econometrica: Journal of the Econometric Society, pp.1009-1037.

[32] Moulin, H. and Sprumont, Y., 2005. On demand responsiveness in additive cost sharing. Journal of Economic Theory, 125(1), pp.1-35.

[33] Moulin, H. and R. A. Velez, 2013. The price of imperfect competition for a spanning network. Games and Economic Behavior. 81, pp.11-26.

[34] O'Neill, B., 1982, A problem of rights arbitration from the Talmud. Mathematical Social Sciences 2, pp.345-371.

[35] Shapley, L.S., 1953. A value for n-person games. In Contributions to the Theory of Games II, ed. H. Kuhn and A.W. Tucker. Princeton: Princeton University Press.

[36] Sprumont, Y., 1982. The division problem with single-peaked preferences: A characterization of the uniform allocation rule. Econometrica 59, 2, pp.509-519.

[37] Sprumont, Y., 1998. Ordinal cost sharing. Journal of Economic Theory, 81(1), pp.126-162.

[38] Sprumont, Y., 2005. On the discrete version of the Aumann-Shapley cost-sharing method. Econometrica, pp.1693-1712.

[39] Thomson, W., 2003. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. Mathematical Social Sciences 45, pp.249-297.

[40] Thomson, W., 2007. On the existence of consistent rules to adjudicate conflicting claims: a constructive geometric approach. Review of Economic Design 11(3), pp.225-251.

[41] Young, H.P., 1988. Distributive justice in taxation. Journal of Economic Theory, 44(2), pp.321–335.

5 Appendix

Proof of Theorem 1. The necessity is readily seen. We only check the sufficiency. Let (φ, μ) satisfy *non-triviality*, *continuity*, and *sequential composition*. Let $(G, v) \in \mathcal{P}$. Suppose that there is a unique efficient path L in (G, v) and for each edge e in L, $v_N(e) > 0$. Let $d_1, e_1, ..., d_m, e_m, d_{m+1}$ be the consecutive nodes and edges in L.

We claim that $\varphi(G, v) = \{L\}$. To see it, let $v^1 \in \mathcal{V}^G$ be such that $v^1(e_1) = v(e_1)$, and for each $e \in G$ with $e \neq e_1$, $v^1(e) = \mathbf{0}$. By *non-triviality*, $\sum \mu_i(G, v^1) > 0$, and thus for each $L' \in \varphi(G, v^1)$, $e_1 \in L'$. For each $\lambda \in (0, 1]$, let $v^{2\lambda} \in \mathcal{V}^G$ be such that $v^{2\lambda}(e_2) = \lambda v(e_2) + (1 - \lambda)\mathbf{0}$,[23] and for each $e \in G$ with $e \neq e_2$, $v^{2\lambda}(e) = v^1(e)$. Thus, for each $e \in G$, $\lim_{\lambda \to 0} v^{2\lambda}(e) = v^1(e)$. By *continuity*, $\lim_{\lambda \to 0} \sum \mu_i(G, v^{2\lambda}) = \sum \mu_i(G, v^1(e))$. Thus, when λ is sufficiently small, for each $L' \in \varphi(G, v^{2\lambda})$, $e_1 \in L'$, and by *sequential composition* and *non-triviality*, $\sum \mu_i(G, v^{2\lambda}) = \sum \mu_i(G|_{d_2}, v^{2\lambda}|_{d_2}) + \sum \mu_i(G|^{d_2}, v^{2\lambda}|^{d_2}) > v_N(e_1)$, so $e_2 \in L'$. Let $\bar{\lambda} := \sup\{\lambda' \in (0, 1] :$ for each $\lambda \in (0, \lambda']$ and each $L' \in \varphi(G, v^{2\lambda})$, $e_1, e_2 \in L'\}$. Then, $\bar{\lambda} > 0$, and by *continuity*, $\sum \mu_i(G, v^{2\bar{\lambda}}) = v_N(e_1) + \bar{\lambda} v_N(e_2)$. If $\bar{\lambda} < 1$, then there is a sequence $\{\lambda_k\}$ of elements of $(\bar{\lambda}, 1]$ such that $\lim \lambda_k = \bar{\lambda}$, and for each k, there is $L' \in \varphi(G, v^{2\lambda_k})$ such that either $e_1 \notin L'$ or $e_2 \notin L'$. Thus, $\limsup \sum \mu_i(G, v^{2\lambda_k}) < v_N(e_1) + \bar{\lambda} v_N(e_2) = \sum \mu_i(G, v^{2\bar{\lambda}})$, which is a violation of *continuity*. Hence, $\bar{\lambda} = 1$, and $\sum \mu_i(G, v^2) = v_N(e_1) + v_N(e_2)$. Thus, for each $L' \in \varphi(G, v^2)$, $e_1, e_2 \in L'$.

[23] We denote by $\mathbf{t}$, $t \in \mathbb{R}$, the n-dimensional vector in which each coordinate equals t.

For each $\lambda \in [0,1]$, let $\hat{v}^\lambda \in \mathcal{V}^G$ be such that for each $e \in L$, $\hat{v}^\lambda(e) = v(e)$, and for each $e \notin L$, $\hat{v}^\lambda(e) = \lambda v(e) + (1-\lambda)\mathbf{0}$. By applying the above argument repeatedly, $L \in \varphi(G, \hat{v}^0)$. By *continuity*, when λ is sufficiently small, $L \in \varphi(G, \hat{v}^\lambda)$. Let $\hat{\lambda} := \{\lambda' \in [0,1] :$ for each $\lambda \in [0, \lambda'], L \in \varphi(G, \hat{v}^\lambda)\}$. By *continuity*, $L \in \varphi(G, \hat{v}^{\hat{\lambda}})$. If $\hat{\lambda} < 1$, then there is a sequence $\{\lambda_k\}$ of elements of $(\hat{\lambda}, 1]$ such that $\lim \lambda_k = \hat{\lambda}$ and for each k, $L \notin \varphi(G, \hat{v}^{\lambda_k})$. Since $\lim \max\limits_{L' \in G, L' \neq L} \hat{v}_N^{\lambda_k}(L') = \max\limits_{L' \in G, L' \neq L} \hat{v}_N^{\hat{\lambda}}(L') \leq \max\limits_{L' \in G, L' \neq L} \hat{v}_N^1(L') < v_N(L)$, then $\limsup \sum \mu_i(G, \hat{v}^{\lambda_k}) < v_N(L) = \sum \mu_i(G, \hat{v}^{\hat{\lambda}})$, which is a violation of *continuity*. Hence, $\hat{\lambda} = 1$, and $L \in \varphi(G, v)$. Since L is the unique efficient path in (G, v), then $\varphi(G, v) = \{L\}$.

To see that there is $\alpha \in \mathbb{R}^n_+$ with $\sum \alpha_i = 1$ such that $\mu(G, v) = v_N(G)\alpha$, let $G' \in \mathcal{G}$ be as in Figure 5. That is, G' is obtained by adding an outgoing edge e_{m+1} to the node d_{m+1} with a new sink d_{m+2}, and adding a parallel path L' with $d_1, e'_1, d'_2, e'_2, d_{m+2}$ being the consecutive nodes and edges in L'. For each $\lambda < \frac{v_N(G)}{n}$, let $v'^\lambda \in \mathcal{V}^{G'}$ be such that for

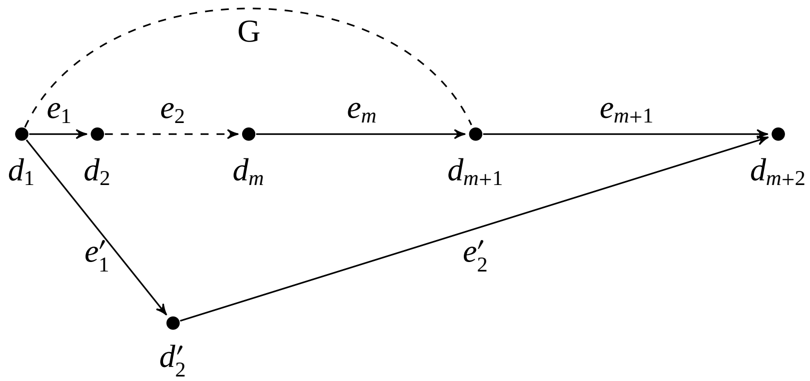

Figure 6: Incremented network G' based on G

each $e \in G$, $v'^\lambda(e) = v(e)$, $v'^\lambda(e_{m+1}) = v'^\lambda(e'_1) = \mathbf{0}$, and $v'^\lambda(e'_2) = (\frac{v_N(G)}{n} - \lambda)\mathbf{1}$. Note that whenever $\lambda > 0$, L incremented by e_{m+1} and d_{m+2} is the unique efficient path in (G', v'^λ). By *sequential composition*, $\mu(G', v'^\lambda) = \mu(G, v)$. By *continuity*, $\mu(G', v'^0) = \mu(G, v)$. Whenever $\lambda < 0$, L' is the unique efficient path in G'. By *continuity* and *sequential composition*, $\mu(G', v'^0) = \mu(e, \frac{v_N(G)}{n}\mathbf{1})$. Thus, $\mu(G, v) = \mu(e, \frac{v_N(G)}{n}\mathbf{1})$.

Let $c, c' \in \mathbb{R}_+$, $\bar{G} \in \mathcal{G}$ be as in Figure 5, and $\bar{v} \in \mathcal{V}^{\bar{G}}$ be such that $\bar{v}(e_1) = \frac{c+c'}{n}\mathbf{1}$, $\bar{v}(e_2) = \mathbf{0}$, $\bar{v}(e'_1) = \frac{c}{n}\mathbf{1}$, and $\bar{v}(e'_2) = \frac{c'}{n}\mathbf{1}$. By *continuity* and *sequential composition*, $\mu(\bar{G}, \bar{v}) = \mu(e, \frac{c+c'}{n}\mathbf{1}) = \mu(e, \frac{c}{n}\mathbf{1}) + \mu(e, \frac{c'}{n}\mathbf{1})$. Thus, for each $i \in N$, $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ defined by setting for each $c \in \mathbb{R}_+$, $f_i(c) = \mu_i(e, \frac{c}{n}\mathbf{1})$ is additive, and by *continuity*, it is continuous. Hence, there is $\alpha_i \in \mathbb{R}$ such that for each $c \in \mathbb{R}_+$, $f_i(c) = \alpha_i c$. Since for each $c > 0$ and $i \in N$, $f_i(c) \geq 0$, and $\sum f_i(c) = c$, then for each $i \in N$, $\alpha_i \geq 0$, and $\sum \alpha_i = 1$. Hence, for each $i \in N$, $\mu_i(G, v) = f_i(\frac{v_N(G)}{n}) = \alpha_i v_N(G)$.

Lastly, suppose that there are multiple efficient paths in (G, v) or there are some edges of zero value in an efficient path. Let $\{v^k\}$ be a sequence of elements of $\mathcal{V}^G$ such that for each k, (G, v) has a unique efficient path, each edge of which has a positive value, and for

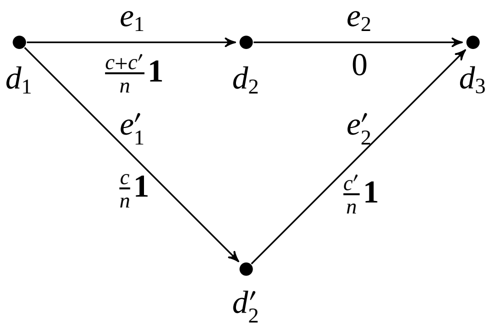

Figure 7: Problem $(\bar{G}, \bar{v})$

each edge e in G, $\lim v^k(e) = v(e)$. Thus, for each k, $\mu(G, v^k) = v^k_N(G)\alpha$. By *continuity*, $\mu(G, v) = v_N(G)\alpha$. This also shows that each path in $\varphi(G, v)$ is efficient.

□

Proof of Theorem 2. We shall only check the sufficiency. Suppose that (φ, μ) satisfies *technology improvement*. Define $f : \mathbb{R}_+ \to \mathbb{R}^n_+$ by setting for each $c \in \mathbb{R}_+$, $f(c) = \mu(e, \frac{c}{n}\mathbf{1})$. Let $(G, v) \in \mathcal{P}$. We claim that $\mu(G, v) = f(v_N(G))$.

Let $(G', v') \in \mathcal{P}$ be a one-step technology improvement of (G, v) where G' is constructed by adding an edge e' going out from the source to a sink in G, and $v'(e') = \frac{v_N(G)}{n}\mathbf{1}$. By *technology monotonicity*, $\mu(G', v') \geq \mu(G, v)$. Since $v'_N(G) = v_N(G)$, then $\mu(G', v') = \mu(G, v)$. Note that there is a finite sequence $\{(G^k, v^k)\}^K_{k=1}$, $K \in \mathbb{N}$, of elements of $\mathcal{P}$ such that $(G^1, v^1) = (e, \frac{v_N(G)}{n}\mathbf{1})$, $(G^K, v^K) = (G', v')$, and for each $k = 2, ..., K$, (G^k, v^k) is a one-step technological improvement of (G^{k-1}, v^{k-1}). By applying *technology monotonicity* repeatedly, $\mu(G', v') \geq \mu(e, \frac{v_N(G)}{n}\mathbf{1})$. Since $v'_N(G) = v_N(G)$, then $\mu(G', v') = \mu(e, \frac{v_N(G)}{n}\mathbf{1})$. Hence, $\mu(G, v) = \mu(e, \frac{v_N(G)}{n}\mathbf{1}) = f(v_N(G))$. This also shows that each path in $\varphi(G, v)$ is efficient.

□

Proof of Theorem 3. We shall only prove the "only if" direction. Define $r : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by setting for each $x \in \mathbb{R}^n_+$, $r(x) = y$ if $\mu(e, x) = y$. Clearly, $\sum x_i = \sum y_i$. By *continuity*, r is continuous. Suppose that $r(x) = y$ for some $x \in \mathbb{R}^n_+$. Then by *parallel composition* $\mu((e, x) \cup (e', x)) = \mu((e, y) \cup (e', x)) = \mu((e, y) \cup (e', y))$. By *irrelevance of dominated paths* and *continuity*, $\mu((e, x) \cup (e', x)) = \mu(e, x)$ and $\mu((e, y) \cup (e', y)) = \mu(e, y)$. Thus $\mu(e, y) = \mu(e, x) = y$. Hence, $r(y) = y$.

Define $\succsim$ on $r(\mathbb{R}^n_+)$ by setting for each pair $x, y \in r(\mathbb{R}^n_+)$, $y \succsim x$ if $\mu((e, x) \cup (e', y)) = y$. We claim that $\succsim$ is a partial order. Since for each $y \in r(\mathbb{R}^n_+)$, $\mu((e, y) \cup (e', y)) = y$, then $\succsim$ is reflexive. By definition, $\succsim$ is antisymmetric. To see $\succsim$ is transitive, let $x, y, z \in r(\mathbb{R}^n_+)$ be such that $y \succsim x$ and $z \succsim y$, and let $z' := \mu((e, x) \cup (e', z))$. By *parallel composition, irrelevance* *of dominated paths* and *continuity*,

$$\begin{aligned}&\mu((e_1,z)\cup(e_2,x)\cup(e_3,y)\cup(e_4,z))\\=&\mu((e_1,z)\cup(e_2,x)\cup(e_3,z))\\=&\mu((e_1,z)\cup(e_2,x))=z',\end{aligned}$$

and

$$\begin{aligned}&\mu((e_1,z)\cup(e_2,x)\cup(e_3,y)\cup(e_4,z))\\=&\mu((e_1,z)\cup(e_2,x)\cup(e_3,y))\\=&\mu((e_1,z)\cup(e_2,y))=z.\end{aligned}$$

Hence, $z'=z$. Thus, $z \succsim x$ as desired. By *parallel composition* and *irrelevance of dominated paths*, for each pair $x, y \in \mathbb{R}^n_+$ such that $x > y$, $\mu((e, r(x)) \cup (e', r(y))) = \mu((e, x) \cup (e', y)) = \mu(e, x) = r(x)$, so $r(x) \succsim r(y)$. Since $\succsim$ is continuous, for each pair $x, y \in \mathbb{R}^n_+$ such that $x \geq y$, $r(x) \succsim r(y)$.

To see that $(r(\mathbb{R}^n_+), \succsim)$ is a join-semilattice, let $x, y \in r(\mathbb{R}^n_+)$ and $z := \mu((e, x) \cup (e', y))$. We claim that $z = \max_{\succsim}\{x, y\}$. By *parallel composition*, *irrelevance of dominated paths*, and *continuity*,

$$\begin{aligned}&\mu((e_1,z))=\mu((e_1,z)\cup(e_2,z))\\=&\mu((e_1,x)\cup(e_2,y)\cup(e_3,x)\cup(e_4,y))\\=&\mu((e_1,x)\cup(e_2,y))=z,\end{aligned}$$

and

$$\begin{aligned}&\mu((e_1,x)\cup(e_2,z))\\=&\mu((e_1,x)\cup(e_2,y)\cup(e_3,x))\\=&\mu((e_1,x)\cup(e_2,y))=z.\end{aligned}$$

Hence, $z \in r(R^n_+)$ and $z \succsim x$. Similarly, $z \succsim y$. If there is $z' \in r(\mathbb{R}^n_+)$ such that $z' \succsim x$ and $z' \succsim y$, then by *parallel composition*,

$$\begin{aligned}&\mu((e_1,z')\cup(e_2,z))\\=&\mu((e_1,z')\cup(e_2,x)\cup(e_3,y))\\=&\mu((e_1,z')\cup(e_3,y))=z'.\end{aligned}$$

Hence, $z' \succsim z$. Thus, $z = \max_{\succsim}\{x, y\} \in r(\mathbb{R}^n_+)$ as desired.

Let $(G, v) \in \mathcal{P}$. We claim that φ only selects efficient path(s). If $v_N(G) = 0$, then we are done. Suppose that $v_N(G) > 0$. Let $\{L_1, ..., L_m\}$, $m \in \mathbb{N}$, be the set of paths in (G, v). For each $k \in \{1, ..., m\}$, we denote by (L_k, v^k) the problem consisted of a single path L_k such that for each edge e in L, $v^k(e) = v(e)$. By *irrelevance of dominated paths* and *continuity*, $\mu(G, v) = \mu(\bigcup_{k=1}^{m}(L_k, v^k) \cup (G, v)) = \mu(\bigcup_{k=1}^{m}(L_k, v^k))$.

For each $\lambda \in [0,1]$ and each $k \in \{1,...,m\}$, let $x^{\lambda k} := \sum_{e \text{ in } L_k} v(e)$ if $v_N(L_k) = v_N(G)$, and $x^{\lambda k} := \lambda \sum_{e \text{ in } L_k} v(e))) + (1-\lambda)\mathbf{0}$ if $v_N(L_k) < v_N(G)$. When $\lambda = 1$, we simply write x^k for x^{1k}. By *splitting invariance,* $\mu(G,v) = \mu(\bigcup_{k=1}^{m}(e_k, x^k))$. By *irrelevance of dominated paths,* the $\sum \mu_i(\bigcup_{k=1}^{m}(e_k, x^{0k})) = v_N(G)$. Let $\bar{\lambda} := \sup\{\lambda \in [0,1] : \sum \mu_i(\bigcup_{k=1}^{m}(e_k, x^{\lambda k})) = v_N(G)\}$. By *continuity,* $\sum \mu_i(\bigcup_{k=1}^{m}(e_k, x^{\bar{\lambda} k})) = v_N(G)$. Suppose that $\bar{\lambda} < 1$. Then there is $\epsilon > 0$ such that for each $\lambda > \bar{\lambda}$, $\sum \mu_i(\bigcup_{k=1}^{m}(e_k, x^{\lambda k})) \leq v_N(G) - \epsilon$, which is a violation of *continuity*. Hence, $\bar{\lambda} = 1$, and $\mu_i(G,v) = \mu_i(\bigcup_{k=1}^{m}(e_k, x^k)) = v_N(G)$.

Next, we claim $\mu(G,v) = \max_{\succsim}\{r(x^k) : k = 1,...,m\}$. Let $z := \mu(\cup_{k=1}^{m}(e_k, x^k))$. It suffices to show that $z = \max_{\succsim}\{r(x^k) : k = 1,...,m\}$. By *parallel composition, irrelevance of dominated paths,* and *continuity,* $\mu(e,z) = z$, and for each $j \in \{1,...,m\}$,

$$
\begin{aligned}
&\mu((e,z) \cup (e', r(x^j))) \\
=&\mu(\bigcup_{k=1}^{m}(e_k, x^k) \cup (e', x^j)) \\
=&\mu(\bigcup_{k=1}^{m}(e_k, x^k)) = z.
\end{aligned}
$$

Hence, $z \in r(\mathbb{R}^n_+)$ and for each $j \in \{1,...,m\}$, $z \succsim r(x^j)$. If there is $z' \in r(\mathbb{R}^n_+)$ such that for each $j \in \{1,...,m\}$, $z' \succsim r(x^j)$, then by *parallel composition,*

$$
\begin{aligned}
\mu((e,z) \cup (e',z')) = \mu(\bigcup_{k=1}^{m}(e_k, x^k) \cup (e',z')) \\
=\mu(\bigcup_{k=1}^{m}(e_k, r(x^k)) \cup (e',z')) = z',
\end{aligned}
$$

so $z' \succsim z$ as desired. Therefore, (φ,μ) is $(r, \succsim)$ - rationalizable. By *continuity,* (φ,μ) is continuously $(r, \succsim)$ - rationalizable. □

Proof for Example 3. To show the properties of $\succsim$, it is useful to observe that for each pair $x,y \in r(\mathbb{R}^n_+)$, $x \succsim y$ if and only if $x_1 + x_2 \geq y_1 + y_2$, $x_1 \geq \min\{y_1, \frac{\sum y_i}{n}\}$ and $x_2 \geq \min\{y_2, \frac{\sum y_i}{n}\}$. The "only if" direction is easy to check. To see the "if" direction, suppose that $y_1 \leq \frac{\sum y_i}{n}$. Then, $y_2 \geq \frac{\sum y_i}{n}$, $x_1 \geq y_1$, $x_2 \geq \frac{\sum y_i}{n}$, and $x_1 \leq \frac{\sum y_i}{n}$. If $x_2 \geq y_2$, then $x \geq y$, and thus $x \succsim y$. Suppose that $x_2 < y_2$. Since $x_1 + (n-1)\frac{\sum y_i}{n} \leq \sum y_i \leq x_1 + (n-1)x_2 = \sum x_i$, then there is $c \in [\frac{\sum y_i}{n}, x_2]$ such that $x_1 + (n-1)c = \sum y_i$. Let $z \in r(\mathbb{R}^n_+)$ be such that $z = (x_1, c, ..., c)$. Then,

$x \geq z$, $\sum z_i = \sum y_i$, and $y_1 \leq z_1 \leq \frac{\sum y_i}{n}$. Hence, $x \succsim y$. Similar arguments apply to the case of $y_1 > \frac{\sum y_i}{n}$.

We claim that $\succsim$ is a partial order. By definition, $\succsim$ is reflexive. To see $\succsim$ is antisymmetric, let $x, y \in r(\mathbb{R}^n_+)$ be such that $x \succsim y$ and $y \succsim x$. Suppose that $y_1 \leq \frac{\sum y_i}{n}$. By the above observation, $\sum x_i = \sum y_i$. Moreover, since $x \succsim y$, then $x_1 \geq y_1$ and $x_1 \leq \frac{\sum y_i}{n}$. Since $y \succsim x$, then $y_1 \geq x_1$. Hence, $x_1 = y_1$, and thus $x = y$. Similar arguments hold when $y_1 > \frac{\sum y_i}{n}$. For transitivity, let $x, y, z \in r(\mathbb{R}^n_+)$ be such that $x \succsim y$ and $y \succsim z$. By the above observation, $\sum x_i \geq \sum y_i \geq \sum z_i$, $x_1 \geq \min\{y_1, \frac{\sum y_i}{n}\} \geq \min\{\min\{z_1, \frac{\sum z_i}{n}\}, \frac{\sum y_i}{n}\} = \min\{z_1, \frac{\sum z_i}{n}\}$, and $x_2 \geq \min\{y_2, \frac{\sum y_i}{n}\} \geq \min\{\min\{z_2, \frac{\sum z_i}{n}\}, \frac{\sum y_i}{n}\} = \min\{z_2, \frac{\sum z_i}{n}\}$. Hence, $x \succsim z$.

To see that $(r(\mathbb{R}^n_+), \succsim)$ is a join-semilattice, let $x, y \in r(\mathbb{R}^n_+)$. Suppose without loss of generality that $\sum x_i \geq \sum y_i$. We denote the median operator by *med*. Let $z^* \in \mathbb{R}^n_+$ be such that $z^*_1 = med\{\min\{y_1, \frac{\sum y_i}{n}\}, x_1, \sum x_i - (n-1)\min\{y_2, \frac{\sum y_i}{n}\}\}$ and $z^*_2 = ... = z^*_n = \frac{\sum x_i - z^*_1}{n-1}$.

We claim that $z^* = \max_{\succsim}\{x, y\}$. Let $z \in r(\mathbb{R}^n_+)$ be such that $z \succsim x$ and $z \succsim y$. Observe that $\sum z_i \geq \sum z^*_i = \sum x_i \geq \sum y_i$, $\min\{y_2, \frac{\sum y_i}{n}\} \leq \frac{\sum y_i}{n} \leq \frac{\sum x_i}{n}$, and

$$\sum x_i - (n-1)\min\{y_2, \frac{\sum y_i}{n}\} \geq \frac{\sum x_i}{n} \geq \min\{y_1, \frac{\sum y_i}{n}\}. \tag{1}$$

Thus, $z^* \in r(\mathbb{R}^n_+)$. Moreover, by (1), $z^*_1 \geq \min\{y_1, \frac{\sum y_i}{n}\}$ and $z^*_2 = \frac{\sum x_i - z^*_1}{n-1} \geq \frac{(n-1)\min\{y_2, \frac{\sum y_i}{n}\}}{n-1} = \min\{y_2, \frac{\sum y_i}{n}\}$. Hence, $z^* \succsim y$. If $z^*_1 = x_1$, then $z^* = x$, and thus $z \succsim z^* \succsim x$. Suppose that $z^*_1 = \min\{y_1, \frac{\sum y_i}{n}\}$. Then by (1), $\frac{\sum x_i}{n} \geq \min\{y_1, \frac{\sum y_i}{n}\} \geq x_1$. Thus, $z^*_1 \geq x_1 = \min\{x_1, \frac{\sum x_i}{n}\}$ and $z^*_2 = \frac{\sum x_i - z^*_1}{n-1} \geq \frac{\sum x_i - \frac{\sum x_i}{n}}{n-1} = \frac{\sum x_i}{n} \geq \min\{x_2, \frac{\sum x_i}{n}\}$. Moreover, $\min\{z^*_1, \frac{\sum z_i}{n}\} = \min\{y_1, \frac{\sum y_i}{n}\}$, and $\min\{z^*_2, \frac{\sum z_i}{n}\} = \min\{\frac{\sum x_i - z^*_1}{n-1}, \frac{\sum x_i}{n}\} \leq \min\{\frac{\sum x_i - x_1}{n-1}, \frac{\sum x_i}{n}\} = \min\{x_2, \frac{\sum x_i}{n}\}$. Hence, $z \succsim z^* \succsim x$. Suppose that $z^*_1 = \sum x_i - (n-1)\min\{y_2, \frac{\sum y_i}{n}\}$. By (1), $x_1 \geq \sum x_i - (n-1)\min\{y_2, \frac{\sum y_i}{n}\} \geq \frac{\sum x_i}{n}$. Thus, $z^*_1 \geq \min\{x_1, \frac{\sum x_i}{n}\}$ and $z^*_2 = \frac{\sum x_i - z^*_1}{n-1} \geq \frac{\sum x_i - x_1}{n-1} = x_2 \geq \min\{x_2, \frac{\sum x_i}{n}\}$. Moreover, $\min\{z^*_1, \frac{\sum z_i}{n}\} = \frac{\sum x_i}{n} = \min\{x_1, \frac{\sum x_i}{n}\}$, and $\min\{z^*_2, \frac{\sum z_i}{n}\} = \min\{\frac{\sum x_i - z^*_1}{n-1}, \frac{\sum x_i}{n}\} = \min\{\min\{y_2, \frac{\sum y_i}{n}\}, \frac{\sum x_i}{n}\} = \min\{y_2, \frac{\sum y_i}{n}\}$. Hence, $z \succsim z^* \succsim x$.

Let (φ, μ) be such that for each $(G, v) \in \mathcal{P}$, φ selects all efficient path(s) in (G, v), and $\mu(G, v) = \max_{\succsim}\{r(\sum_{e \text{ in } L} v(e)) : L \text{ in } G\}$. Note that for each pair $x \geq y$, $r(x) \geq r(y)$, and thus $r(x) \succsim r(y)$. Hence, (φ, μ) is $(r, \succsim)$ - rationalizable. To see that the solution is continuously $(r, \succsim)$ - rationalizable, let $g : r(\mathbb{R}^n_+)^2 \to r(\mathbb{R}^n_+)$, $x, y \in r(\mathbb{R}^n_+)$ and $\{x^n\}, \{y^n\}$ be two sequences of elements of $r(\mathbb{R}^n_+)$ such that $\lim x^n = x$ and $\lim y^n = y$. Let $z := \max\{x, y\}$ and for each $n \in \mathbb{N}$, $z^n = \max\{x^n, y^n\}$. Suppose without loss of generality that $\sum x_i \geq \sum y_i$. If $\sum x_i > \sum y_i$, then for sufficiently large $n \in \mathbb{N}$, $\sum x^n_i < \sum y^n_i$, and thus $\lim z^n = z$. If $\sum x_i = \sum y_i$, then $z_1 = med\{\min\{y_1, \frac{\sum y_i}{n}\}, x_1, \sum x_i - (n-1)\min\{y_2, \frac{\sum y_i}{n}\}\} = med\{\min\{x_1, \frac{\sum x_i}{n}\}, y_1, \sum y_i - (n-1)\min\{x_2, \frac{\sum x_i}{n}\}\}$. Hence, $\lim z^n = z$. □

Proof of Theorem 4. We shall only check the sufficiency. Let (φ, μ) satisfy *irrelevance of dominated paths*, *irrelevance of parallel outside options*, and *continuity*. For each $(G, v) \in \mathcal{P}$ and each L in G, we denote by (L, v^L) the problem consisting of the single path L and v^L

such that for each edge e in L, $v^L(e) = v(e)$. By *irrelevance of dominated paths* and *continuity*, for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = \mu(\bigcup_{L \text{ in } G} (L, v^L))$. Moreover, by a similar argument as in the the proof of Theorem 3, φ selects only efficient path(s). Let $(G, v), (G', v') \in \mathcal{P}$. Let L^* be an efficient path in $(G', v') \cup (G, v)$ and for each $\epsilon > 0$, $v^\epsilon \in \mathcal{V}^{L^*}$ be such that for each e in L^*, $v^\epsilon(e) = v(e) + \epsilon\mathbf{1}$ if L^* is in G, and $v^\epsilon(e) = v'(e) + \epsilon\mathbf{1}$ if L^* is in G'. Suppose that L^* is in G'. By *continuity*, $\mu((G, v) \cup (G', v')) = \lim_{\epsilon \downarrow 0} \mu((\bigcup_{L \text{ in } G} (L, v^L)) \cup (\bigcup_{L \text{ in } G', L \neq L^*} (L, v'^L)) \cup (L^*, v^\epsilon))$. Since φ selects only efficient path(s), then by *irrelevance of parallel outside options*, $\mu((G, v) \cup (G', v')) = \lim_{\epsilon \downarrow 0} \mu(L^*, v^\epsilon) = \mu(L^*, v^{L^*})$. Similarly, $\mu((e, \mu(G, v)) \cup (G', v')) = \mu(L^*, v^{L^*})$, and thus $\mu((G, v) \cup (G', v')) = \mu((e, \mu(G, v)) \cup (G', v'))$. Suppose that L^* is in G. Then $\mu((G, v) \cup (G', v')) = \lim_{\epsilon \downarrow 0} \mu((\bigcup_{L \text{ in } G, L \neq L^*} (L, v^L)) \cup (\bigcup_{L \text{ in } G'} (L, v'^L)) \cup (L^*, v^\epsilon)) = \mu(L^*, v^{L^*})$. Moreover, $\mu((e, \mu(G, v)) \cup (G', v')) = \lim_{\epsilon \downarrow 0} \mu((e, \mu(G, v) + \epsilon\mathbf{1}) \cup (G', v')) = \lim_{\epsilon \downarrow 0} \mu((e, \mu(G, v) + \epsilon\mathbf{1}) \cup (\bigcup_{L \text{ in } G'} (L, v'^L))) = \lim_{\epsilon \downarrow 0} \mu(e, \mu(G, v) + \epsilon\mathbf{1}) = \mu(e, \mu(G, v))$. For each $\epsilon > 0$, $\mu((e, \mu(G, v) + \epsilon\mathbf{1}) \cup (L^*, v^{L^*})) = \mu(e, \mu(G, v) + \epsilon\mathbf{1})$ and $\mu((e, \mu(G, v)) \cup (L^*, v^\epsilon)) = \mu(L^*, v^\epsilon)$. By *continuity*, $\mu((e, \mu(G, v)) \cup (L^*, v^{L^*})) = \mu(e, \mu(G, v)) = \mu(L^*, v^{L^*})$. Hence, $\mu((G, v) \cup (G', v')) = \mu((e, \mu(G, v)) \cup (G', v'))$. Thus, (φ, μ) satisfies *parallel composition*.

Moreover, we claim that (φ, μ) satisfies *splitting invariance*. It suffices to show that for each $(G, v) \in \mathcal{P}$, $\mu(\bigcup_{L \text{ in } G} (L, v^L)) = \mu(\bigcup_{L \text{ in } G} (e^L, \sum_{e \text{ in } L} v(e)))$. Let $(G, v) \in \mathcal{P}$ and L be a path in G. By *parallel composition*, it suffices to show that $\mu(L, v^L) = \mu(e^L, \sum_{e \text{ in } L} v(e))$. For each $\epsilon > 0$, let $v^\epsilon \in \mathcal{V}^L$ be such that for each e in L, $v^\epsilon(e) = v^L(e) + \epsilon\mathbf{1}$. Then, for each $\epsilon > 0$, $\mu((L, v^\epsilon) \cup (e^L, \sum_{e \text{ in } L} v(e))) = \mu(L, v^\epsilon)$ and $\mu((L, v^L) \cup (e^L, \sum_{e \text{ in } L} v(e) + \epsilon\mathbf{1})) = \mu(e^L, \sum_{e \text{ in } L} v(e) + \epsilon\mathbf{1})$. By *continuity*, $\mu((L, v^L) \cup (e^L, \sum_{e \text{ in } L} v(e))) = \mu(L, v^L) = \mu(e^L, \sum_{e \text{ in } L} v(e))$, as desired.

Since (φ, μ) satisfies both *splitting invariance* and *parallel composition*, then by Theorem 3, there exist a redistribution function r and a partial order $\succsim$ on $r(\mathbb{R}^n_+)$ such that for each pair $x, y \in \mathbb{R}^n_+$ with $x \geq y$, $r(x) \succsim r(y)$, and (φ, μ) is continuously $(r, \succsim)$ - rationalizable. By *irrelevance of parallel outside options*, for each pair $x, y \in r(\mathbb{R}^n_+)$, either $\mu((e, x) \cup (e', y)) = \mu(e, x) = x$ or $\mu((e, x) \cup (e', y)) = \mu(e', y) = y$. Since r is continuous, then $r(\mathbb{R}^n_+)$ is connected. Then by Eilenberg (1941),[24] there is a one-to-one and continuous mapping $g : r(\mathbb{R}^n_+) \to \mathbb{R}$.

We claim that for each pair $x, y \in \mathbb{R}^n_+$ such that $\sum x_i = \sum y_i$, $r(x) = r(y)$. Suppose to the contrary that $x, y \in \mathbb{R}^n_+$, $\sum x_i = \sum y_i$, and $r(x) \neq r(y)$. Thus, $\sum x_i > 0$. Let $z \in \mathbb{R}^n_+$ be such that $\sum z_i < \sum x_i$, so $r(z) \neq r(x)$ and $r(z) \neq r(y)$. For each pair $(x', x'') \in \{(x, y), (y, z), (z, x)\}$, let $D_{(x', x'')} := \{\lambda x' + (1 - \lambda)x'' : \lambda \in [0, 1]\}$. Since r is continuous and $D_{(x', x'')}$ is convex, then $r(D_{(x', x'')})$ is path-connected. Since $r(D_{(x', x'')})$ is a Hausdorff space, then $r(D_{(x', x'')})$ is arc-connected. Thus, there is a function $h_{(x', x'')} : [0, 1] \to r(D_{(x', x'')})$ such that $h_{(x', x'')}(0) = r(x')$, $h_{(x', x'')}(1) = r(x'')$, and $h_{(x', x'')}$ is a homeomorphism between $[0, 1]$ and $h_{(x', x'')}([0, 1])$. Note that $h_{(x,y)}([0, 1]) \cap h_{(y,z)}([0, 1]) = \{r(y)\}$, $h_{(x,y)}([0, 1]) \cap h_{(z,x)}([0, 1]) = \{r(x)\}$, and $A := h_{(x,y)}([0, 1]) \cup h_{(y,z)}([0, 1]) \cup h_{(z,x)}([0, 1])$ is connected. Since g is continuous, then $g(A)$ is an interval. Since g is one-to-one, then there is $c^\circ \in (0, 1)$ such that $g(h_{(x,y)}(c^\circ))$ is an interior

[24] See his Theorem I and (6.1).

point of $g(A)$, and thus $g(A \setminus h_{(x,y)}(c^\circ))$ is not connected. Since $A \setminus h_{(x,y)}(c^\circ)$ is connected and g is continuous, then $g(A \setminus h_{(x,y)}(c^\circ))$ is connected, which is a contradiction.

Define $f : \mathbb{R}_+ \to \mathbb{R}^n_+$ by setting for each $c \in \mathbb{R}_+$, $f(c) = r(\frac{c}{n}\mathbf{1})$. For each $(G, v) \in \mathcal{P}$, since $\mu(G, v) = \max_{\succsim}\{r(\sum_{e \text{ in } L} v(e)) : L \text{ in } G\}$, then by the result proved in the previous paragraph, $\mu(G, v) = \max_{\succsim}\{f(v_N(L)) : L \text{ in } G\}$. Recall that for each pair $x, y \in \mathbb{R}^n_+$ with $x \geq y$, $r(x) \succsim r(y)$. Thus, $\mu(G, v) = f(v_N(G))$.

□

Lemma 1. *Let $c > 0$ and $X \subseteq \Delta = \{x \in \mathbb{R}^n_+ : \sum x_i = c\}$. Suppose that (i) $\frac{c}{n}\mathbf{1} \in X$, (ii) for each pair $x, y \in X$, there is $N' \subseteq N$ such that $|N'| \geq k$ and for each $i \in N'$, $x_i \geq y_i$, and (iii) for each permutation π of N and each $x \in X$, x^π defined by setting for each $i \in N$, $x^\pi_i = x_{\pi(i)}$ belongs to X. Then, for each $x \in X$, there is $N' \subseteq N$ such that $|N'| \geq k$ and for each $i \in N'$, $x_i = \frac{c}{n}$.*

Proof of Lemma 1. By assumption (ii), for each pair $x, y \in X$, $|\{i \in N : x_i > y_i\}| \leq n - k$, $|\{i \in N : x_i < y_i\}| \leq n - k$, and thus

$$|\{i \in N : x_i = y_i\}| \geq 2k - n. \tag{2}$$

For each $x \in X$, $N^x_> := \{i \in N : x_i > \frac{c}{n}\}$, $N^x_< := \{i \in N : x_i < \frac{c}{n}\}$, and $N^x_= := \{i \in N : x_i = \frac{c}{n}\}$. Let $x \in X$. By assumption (i), $|N^x_>| \leq n - k$, $|N^x_<| \leq n - k$, and $|N^x_=| \geq 2k - n$. Suppose to the contrary that $|N^x_=| < k$.

If $|N^x_=| \geq \frac{n}{2}$, then let π be a permutation on N such that for each $i \in N^x_> \cup N^x_<$, $\pi(i) \in N^x_=$. By assumption (iii), $x^\pi \in X$. Note that $N^x_> \cup N^x_< \subseteq N^{x^\pi}_=$ and $N^{x^\pi}_> \cup N^{x^\pi}_< \subseteq N^x_=$ (see Figure 5).

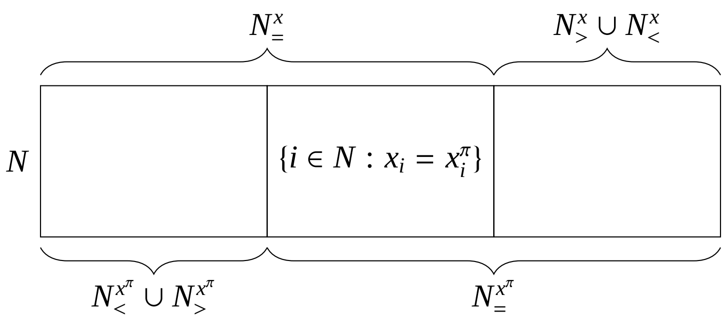

Figure 8: Permutation π of N

Thus, $|\{i \in N : x_i = x^\pi_i\}| = |N^x_=| - |N^x_>| - |N^x_<| = |N^x_=| - (n - |N^x_=|) < 2k - n$, which violates (2).

If $|N^x_=| < \frac{n}{2}$, assume that $|N^x_>| \leq |N^x_<| \leq |N^x_=|$. Let π be a permutation of N such that for each $i \in N^x_=$, $\pi(i) \in N^x_> \cup N^x_<$, and for each $i \in N^x_<$, $\pi(i) \in N^x_=$ (see Figure 5). Since $|N^x_>| \leq |N^x_=| < \frac{n}{2} < |N^x_> \cup N^x_<|$, then for each $i \in N^x_>$, $\pi(i) \in N^x_< \cup N^x_=$. Hence, for each $i \in N$, $x_i \neq x^\pi_i$, which violates (2). Similar contradiction can be obtained when $|N^x_<| \leq |N^x_>| \leq |N^x_=|$, and when $|N^x_=| < |N^x_>|$ or $|N^x_=| < |N^x_<|$, since both $|N^x_>|$ and $|N^x_<|$ are less than $\frac{n}{2}$.

□

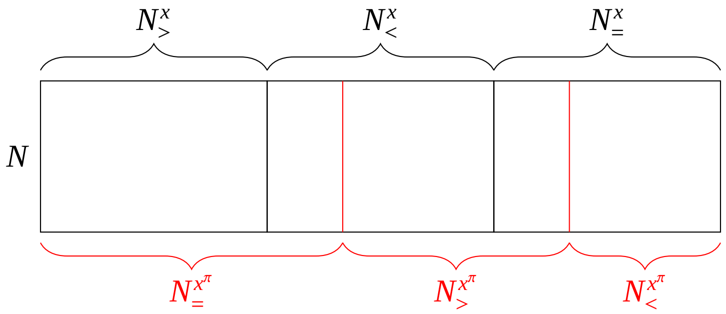

Figure 9: Permutation π of N

Proof of Proposition 1. Let $l \in \mathcal{L}$ and $c := l_N$. Let $X = \{\mu(l') : l' \in \mathcal{L}, l'_N = c\}$. By *anonymity*, assumptions (i) and (iii) of Lemma 1 are satisfied. By *k-majority*, assumption (ii) is also satisfied. By Lemma 1, there is $N' \subseteq N$ such that $|N'| \geq k$ and for each $i \in N'$, $\mu_i(l) = \frac{c}{n}$.

□

Proof of Theorem 5. We only check the sufficiency. Let μ satisfy *k-majority* and *anonymity*.

Suppose that μ satisfies *sequential composition*. For each $l \in \mathcal{L}$, let $N^l_= := \{i \in N : \mu_i(l) = \frac{l_N}{n}\}$ and $N^l_{\neq} := N \setminus N^l_=$. Suppose to the contrary that there is $l' \in \mathcal{L}$ such that $N^{l'}_{\neq} \neq \emptyset$. By Theorem 1, $|N^{l'}_=| \geq k \geq |N^{l'}_{\neq}|$. Let π be a permutation on N such that for each $i \in N^{l'}_{\neq}$, $\pi(i) \in N^{l'}_=$. Let $\bar{l} \in \mathcal{L}$ be a path that connects l' and l'^{π}. By *sequential composition*, $\mu(\bar{l}) = \mu(l') + \mu(l'^{\pi})$. Thus, $N^{\bar{l}}_= = N^{l'}_= \cap N^{l'^{\pi}}_=$, so $N^{\bar{l}}_{\neq} \neq \emptyset$ and $|N^{\bar{l}}_=| = |N^{l'}_=| - |N^{l'}_{\neq}| < |N^{l'}_=|$ (see Figure 5). Repeating the argument, there is $l'' \in \mathcal{L}$ such that $N^{l''}_{\neq} \neq \emptyset$ and $|N^{l''}_=| < |N^{l'}_=|$. Within finitely many steps, we can find $\hat{l} \in \mathcal{L}$ such that $|N^{\hat{l}}_=| < k$, which is a violation of Theorem 1.

Suppose that μ satisfies *Lorenz monotonicity*. Let $l = \{x^k\}_{k=1}^K$, where $K \in \mathbb{N}$. Let $l' = \{x'^k\}_{k=1}^K$ be such that for each $k \in \{1, ..., K\}$ and each $i \in \{1, ..., n-1\}$, $x'^k_i = 0$ and $x'^k_n = \sum_{i \in N} x'^k_i$. By construction, for each $k \in \{1, ..., K\}$, $x^k \succ_{Lorenz} x'^k$. By *anonymity*, for each pair $i, j \in \{1, ..., n-1\}$, $\mu_i(l') = \mu_j(l')$. By Theorem 1, for each $i \in \{1, ..., n-1\}$, $\mu_i(l') = \frac{l'_N}{n} = \frac{l_N}{n}$, and thus $\mu_n(l') = \frac{l_N}{n}$. By *Lorenz monotonicity*, $\frac{l_N}{n}\mathbf{1} \not\succ_{Lorenz} \mu(l)$. Hence, $\mu(l) = \frac{l_N}{n}\mathbf{1}$.

Suppose that μ satisfies *transfer monotonicity*. Let $l = \{x^k\}_{k=1}^K$, where $K \in \mathbb{N}$. For each $j \in N$, let $l^j = \{x'^k\}_{k=1}^K$ be such that for each $k \in \{1, ..., K\}$ and each $i \in N \setminus \{j\}$, $x'^k_i = 0$ and $x'^k_j = \sum_{i \in N} x'^k_i$. By *anonymity* and Theorem 1, $\mu_j(l^j) = \frac{l_N}{n}$. By *transfer monotonicity*, for each $j \in N$, $\mu_j(l) \leq \mu_j(l^j)$. Thus, $\mu(l) = \frac{l_N}{n}\mathbf{1}$.

Suppose that μ the efficient route is a *Strong Nash*. Suppose for a problem (G, v) that a path $l = \{x^k\}_{k=1}^K$ is chosen and it is not equal sharing among all agents. Denote $N^l_<$, $N^l_=$ and $N^l_>$ be the set of agents get less than, equal to and more than the average for value of the path l. First, we have $|N^l_<| > 0$ and $|N^l_>| > 0$. Second, by Theorem 1, we have $|N^l_<| + |N^l_=| \geq k$. Now consider a problem (G, v) consist of two parallel paths l and $\hat{l}$ where $\hat{l} = \{\hat{x}^k\}_{k=1}^K$ such that $\hat{x}^k_i = x^k_{\pi(i)}$ and π is a permutation of N such that $\pi(i) = i$ for all $i \in N^l_=$, and for some $i \in N^l_<$,

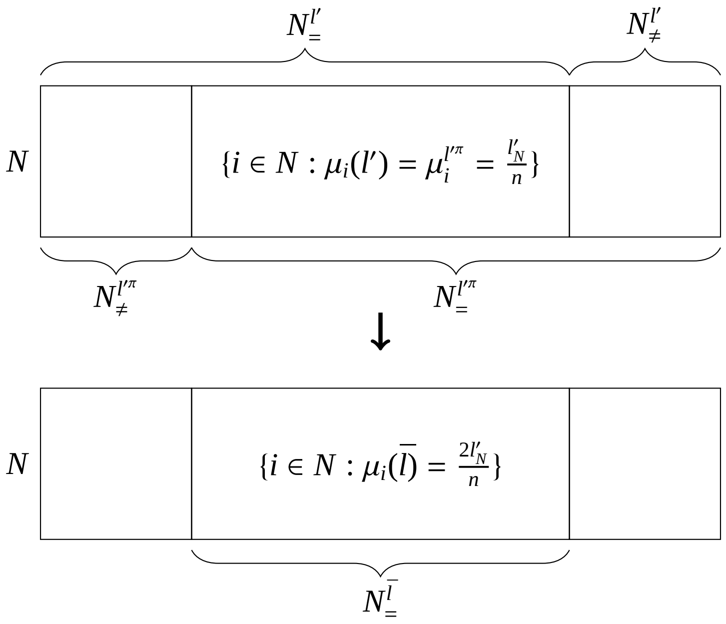

Figure 10: Permutation π of N

we have $\pi(i) = j$ where $j \in N^l_>$. Suppose l is chosen. By construction, $|N^{\hat{l}}_<| + |N^{\hat{l}}_=| \geq k$. This group of agent is weakly better off (with one strict) under $\hat{l}$, which contradicts group value monotonicity. We have the similar argument if $\hat{l}$ is chosen.

□