Order on Types based on Monotone Comparative Statics*

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Abstract

This paper introduces an order on types by which the so-called monotone comparative statics is valid in *all* supermodular games with incomplete information. We fully characterize this order in terms of what we call the *common certainty of optimism*. We say that type t_i' is higher than type t_i in the order of the common certainty of optimism if t_i' is more optimistic about state than t_i; t_i' is more optimistic that all players are more optimistic about state than t_i; and so on ad infinitum. First, we show that whenever the common certainty of optimism holds, monotone comparative statics holds in all supermodular games. Second, we show the converse. We construct an "optimism-elicitation game" as a single supermodular game with the property that whenever the common certainty of optimism fails, monotone comparative statics fails as well.

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1 Introduction

In many economic problems, we are often interested in studying the effects of changes in certain variables ("parameters") on the behaviors of economic agents. This is well known as *comparative statics*. There are numerous examples showing that the comparative statics analysis is ubiquitous in economics, such as in the analyses of bidding strategies in auctions, portfolio choices in financial markets, optimal taxation policy, and so on. The literature on *supermodular games* shows that, given certain conditions[1] on the way the parameters enter the players' payoff functions, monotone changes in those parameters affect the players' equilibria in a monotonic way, a property called *monotone comparative statics*.

Although many papers in the literature study supermodular games with *complete information*, in practice, it is often the case that players do not observe some of the parameters of the game they play, or they have different information about them. In this paper, we study monotone comparative statics in supermodular games with such *incomplete information*. Athey (2001, 2002), McAdams (2003), Van Zandt (2010), and Van Zandt and Vives (2007) consider supermodular games with incomplete information. Often their main motivation is existence of equilibria (with nice properties such as in pure and monotone strategies), while our main focus is purely on monotone comparative statics. Also, they consider different assumptions with respect to the players' information structure, reflecting different levels of generality in this respect. In this sense, the setting in our paper is closest to that of Van Zandt and Vives (2007) where no restriction is made on each player's belief and higher-order beliefs (except for certain topological and order

[1] In particular, certain forms of complementarity (such as supermodularity, increasing difference, and single-crossing conditions) among economic variables and players' actions are shown to be important. See, for example, Milgrom and Roberts (1990), Milgrom and Shannon (1994), Topkis (1998), and Vives (1990).

structures). Specifically, we allow not only standard common-prior type spaces, but also non-common-prior cases where the players are allowed to enjoy arbitrary heterogeneous beliefs and higher-order beliefs. Indeed, in our setup, any type in the universal type space (and hence any belief hierarchy) is allowed.[2]

Including non-common-prior environments in our analysis is not just for technical generality. In some economic problems, it is well recognized that assuming a common prior may be too demanding. For example, the cerebrated no-trade theorem (Milgrom and Stokey (1982)) shows that, in certain trading contexts with common-value assets and a common prior,[3] all equilibria exhibit no trading, even though, in reality, many traders appear to be involved in speculative trading. As another example, the behavioral economics literature propose a number of ways real economic agents "wrongly" process information and evidence for them.[4]

With strategic interaction, such heterogeneity in (high-order) beliefs makes comparative statics much more subtle. Even if a trader becomes "more optimistic" about the fundamental, which makes him eager to "trade more" *ceteris paribus*, he may not want to do so if he believes that the other traders' beliefs change in the way that his "trading more" would hurt him. Our result could be useful in the analysis of such situations. Indeed, our result suggests that, even in such heterogenous-belief environments, there is a sense in which monotone comparative statics can still be conducted. For example, imagine that investors agree that certain news is "good news" for a startup even though they do not agree on "how good the news is" (because they may believe different underlying distributions). We show that this qualitative agreement may be sufficient to drive up the stock price of this startup.[5]

To explain our main idea more formally, imagine an incomplete information supermodular game with a parameter space Θ. Each player's interim belief is

[2]See Mertens and Zamir (1985) and Brandenberger and Dekel (1993).

[3]In fact, their no-trade result holds with *concordant* beliefs, and a common-prior environment is a special case.

[4]For example, see the cursed equilibrium of Eyster and Rabin (2005) and the analogy-based expectation equilibrium of Jehiel (2005).

[5]In Section 7, we also consider a trading environment where it is common knowledge that the asset has pure common value, but with heterogenous (high-order) beliefs about the exact value. Monotone comparative statics for the size / possibility of trade is conducted with respect to the size of the belief divergence between players.

identified by his *type*, which induces his *belief hierarchy*, that is, his first-order belief over Θ, his second-order belief (i.e., the joint belief over Θ and the other players' first-order beliefs), and so on, ad infinitum. We order two types of each player based on their belief hierarchies. Namely, we say that type $\hat{t}_i$ of player i is higher than t_i in the sense of *common certainty of optimism (henceforth, CCO)* if $\hat{t}_i$'s first-order belief on Θ stochastically dominates t_i's first-order belief; $\hat{t}_i$'s second-order belief (jointly about Θ and the first-order beliefs of the other players) stochastically dominates t_i's second-order belief; and so on ad infinitum. That is, $\hat{t}_i$ is more "optimistic" about the realization of $\theta \in \Theta$ than t_i; $\hat{t}_i$ is more "optimistic" about the "optimism" of the other players, and so on.

In Theorem 1, we show that the common certainty of optimism is sufficient for monotone comparative statics to hold in all supermodular games. More specifically, we show that if type $t_i^{'}$ is higher than type t_i in the CCO sense, then, $t_i^{'}$'s action in the least (greatest) equilibrium is higher than t_i's action in the least (greatest) equilibrium in all supermodular games.[6] The key observation for the proof of this theorem is that the least and greatest equilibria of an incomplete-information supermodular game (under certain regularity conditions) coincide with the game's least and greatest *interim correlated rationalizability* (henceforce, ICR) of Dekel, Fudenberg, and Morris (2007), which is fully identified by his belief hierarchy. Of course, different orders on these types may be induced if different games are considered. Theorem 1 establishes that our CCO order is the "coarsest" order of types such that, if a type of a player is higher than another in this order, then the former plays a higher (least and greatest) equilibrium action than the latter in *any* supermodular game.[7]

Theorem 2 shows its converse. Namely, we construct a supermodular game, which we refer to as an "optimism-elicitation game" such that, if type $t_i^{'}$ is *not* higher than t_i in the CCO sense, $t_i^{'}$'s action in the least (greatest) equilibrium of

[6]The set of equilibria in supermodular games is a complete lattice, and in particular, admits the least and greatest equilibria. Establishing monotone comparative statics for those extremal equilibria, monotone comparative statics for the set of equilibria (in an appropriate set-order sense) is established.

[7]Of course, this may not be the *only* interesting order. For example, one may have a specific supermodular game in mind, and desire to conduct comparative statics only in this game. Although we provide some preliminary argument on this issue in Section 6, we leave this for our future research.

this game is *not* higher than t_i's action in the least (greatest) equilibrium. In other words, the CCO order and monotone comparative statics in this game are equivalent. The construction is reminiscent of *scoring rules* in belief-elicitation procedures.[8] However, there are some important differences. First, although a scoring rule is usually constructed to elicit a single decision maker's belief about an uncertain variable, our game is to elicit multiple players' belief hierarchies. Second, because our goal is to construct a supermodular game, the action space of each player cannot be the space of all possible belief hierarchies, because it then does not constitute a lattice (with respect to first-order stochastic dominance). In this sense, our construction cannot be a straightforward extension of a standard (single-player) scoring rule to multiple players. Indeed, the action space of our game is based on the set of *capacities*,[9] in order to guarantee the (complete) lattice structure of the action space. A further complication arises with multiple players, because we must consider its "(infinitely-)higher-order" version. The key to our construction is a careful choice of a topological structure for the set of capacities.[10]

The condition we identify is purely based on the players' first and higher-order beliefs with respect to the state variables, and does not depend on which game we consider, as long as the game is supermodular. In this sense, our results may be useful in the context of mechanism design, where a game is not fixed but rather endogensouly constructed. The reader is referred to Mathevet (2010) for the study of designing supermodular mechanisms, motivated by the desirable features of supermodular games in terms of learning and bounded rationality in certain senses. We consider the application of our paper to mechanism design as one promissing direction of our future work.[11]

[8] Savage (1971) proposes the *proper scoring rule* to elicit an individual's belief.

[9] A capacity can be interpreted as a non-additive (hence not-necessarily probabilistic) belief. See Schmeideler (1986, 1989) for its application to decision theory.

[10] For example, a weak*-topology, which is standard to topologize the set of probability measures, is not applicable (at least in a straightforward manner) to topologize the set of capacities. Instead, our topology (or more specifically, a norm) is such that the action space is made isomorphic to a closed subset of a Hilbert cube, and hence a compact metric space. This nice structure is exploited to prove Theorem 2.

[11] For example, our result shows that one type of a player must play a higher action than another type of that player in any supermodular game. Viewing a game as a mechanism that implements certain allocation rule, such a condition may imply natural monotonicity structures on implementable social choice rules.

As briefly mentioned above, this paper is closest to Van Zandt and Vives (2007) who also investigate supermodular games with incomplete information. Van Zandt and Vives consider an implicit (Harsanyi) type space endowed with an exogeneously given partial order and then introduce each type's belief map that is consistent with this exogenous order. They establish the existence of the least and greatest equilibria that are monotone in types as well as the following monotone comparative statics result: the greatest and least equilibria are higher if there is a first-order stochastic dominant shift in the interim belief. Naturally, our CCO order and their exogenous order structure are quite related, discussed in detail in Section 6.

At a more abstract level, the motivation of our paper is closely related to Dekel, Fudenberg, and Morris (2006) and Chen, Di Tillio, Faingold, and Xiong (2010, 2016). Based on the observation that the interim correlated rationalizability (ICR) correspondence is not continuous with respect to the product topology on the universal type space,[12] Dekel, Fudenberg, and Morris (2006) introduce what they call the *strategic topology*, and show that this is the coarsest possible topology with respect to which the ICR correspondence is continuous (i.e. both upper and lower-hemi continuous) in all finite games. Chen, Di Tillio, Faingold, and Xiong (2010, 2016) further investigate the strategic topology based on ICR. They also propose the *uniform* strategic topology, which is described directly based on belief hierarchies, and study its relationship with the strategic topology. In this sense, Dekel, Fudenberg, and Morris (2006) and Chen, Di Tillio, Faingold, and Xiong (2010, 2016) establish "economically meaningful" topologies on the universal type space, thereby enhances a better understanding of this seemingly complicated mathematical object. Quite analogously, our attempt is to introduce a partial order over the universal type space (rather than topologies) that is "economically meaningful" in the sense that this order fully characterizes monotone comparative statics in all supermodular games.

The rest of the paper is organized as follows. In Section 2, we introduce the basic setup and definitions and identify the least and greatest equilibria via the iterated elimination of never best responses. Section 3 establishes the common certainty of optimism (CCO) as a sufficient condition for monotone comparative statics to hold in all supermodular games. In Section 4, by focusing on the single-

[12]More specifically, it satisfies upper-hemi continuity but not necessarily lower-hemi continuity.

person decision making problem, we show that the CCO order is also necessary for monotone comparative statics to hold in all supermodular games. We establish this by constructing a single supermodular game. In Section 5, we extend this result to the multi-player situation. Section 6 provides a detailed discussion about the relationship with Van Zandt and Vives (2007). In Section 7, we provide an application of our CCO order on types in the context of the no-trade result of Milgrom and Stokey (1982). Section 8 concludes the paper and the Appendix (Section 9) contains all the omitted proofs from the main body of the paper.

2 Preliminaries

We shall prepare the preliminary materials needed throughout the paper. Section 2.1 introduces first-order stochastic dominance. Section 2.2 introduces belief hierarchies and defines the concept of common certainty of optimism. We defines supermodular games in Section 2.3 and their Bayesian equilibria in Section 2.4. It is well-known that, in a supermodular game, the set of Bayesian equilibria has a lattice structure, and hence, admits the least and greatest equilibria. As is standard in the literature, monotone comparative statics are about those extremal equilibria.

2.1 First Order Stochastic Dominance

Let X be a separable, complete metric space.[13] Consider two Borel probability measures, b and b', on X. Let $\Delta(X)$ denote the set of all Borel-measurable probability distributions over X endowed with the weak*-topology. In Section 5, we introduce capacities (non-additive measures) over X endowed with a finer topology than the weak*-topology. In such a case, we avoid the use of the notation like $\Delta(X)$, which usually means the set of all probability distributions. We say that a partial order $\succeq$ on X is *closed* if, for any pair of sequences $\{x_n\}, \{y_n\} \in X$, whenever $x_n \succeq y_n$ for each n and $x_n \to x$, and $y_n \to y$ as $n \to \infty$, we have $x \succeq y$. We endow X with such a closed partial order $\succeq$.

We say that b' *first-order stochastically dominates* b if, for any increasing, mea-

[13] Examples include any finite set, $[0, 1]$, $\mathbb{R}^d$, and $L^p(\mathbb{R}^d)$.

surable, and bounded function $f : X \to \mathbb{R}$,

$$\int_x f(x)db' \geq \int_x f(x)db,$$

and b' *strictly* first-order stochastically dominates b if b' first-order stochastically dominates b and, in addition, the inequality is strict at least for some f that is increasing, measurable, and bounded.

Under the following two assumptions on X and $\succeq$, we have an alternative representation of first-order stochastic dominance, which is used in Section 4 and 5. We believe that they are mild requirements. For example, a Euclidean space with the usual component-wise partial order satisfies them.

Assumption 1. There exists a countable dense subset $X_0 \subseteq X$ for which for each $x \in X$ and $\varepsilon > 0$, there is $y \in X_0$ such that $y \geq x$ and $y \in B_\varepsilon(x)$.[14]

Because X is separable, it has a countable dense subset. But our assumption requires an additional condition, which is a sort of "local non-satiation".[15]

To introduce the second assumption, for each $x \in X$, let $up(x) \subseteq X$ be the smallest upper set that contains x, i.e., $up(x) = \{y \in X | y \geq x\}$. For each $Y \subseteq X$, let $up(Y) \subseteq X$ be the smallest upper set that contains Y, i.e., $up(Y) = \bigcup_{y \in Y} up(y)$. The second assumption says that the upper-set correspondence is continuous.

Assumption 2. For each $Y \subseteq X$ and $\varepsilon > 0$, there exists $\delta(Y, \varepsilon) > 0$ such that, for any $Z \subseteq X$ with $d(Y, Z) < \delta(Y, \varepsilon)$,[16] we have $d(up(Y), up(Z)) < \varepsilon$.

For each $Y \subseteq X$, let $clup(Y)$ denote the closure of $up(Y)$. The next result shows that, under the assumptions above, we can show that we do not need to check all increasing, measurable, and bounded functions to determine whether b first-order stochastically dominates b'. We only need to check a countable subclass

[14] $B_\varepsilon(x)$ denotes the open ball around x with radius ε.

[15] In their analysis of revealed preference theory, Chambers, Echenique, and Lambert (2017) use the *countable order property*, which is similar to our Assumption 1. See also Proposition 13 of their paper for two prominent cases where the countable order property is satisfied.

[16] By abuse of notation, we let

$$d(Y, Z) = \max\{\sup_{y \in Y} \inf_{z \in Z} d(y, z), \sup_{z \in Z} \inf_{y \in Y} d(y, z)\}$$

denote the Hausdorff metric between $Y, Z \subseteq X$.

of "clup" sets. The proof is in the appendix, but it is worth mentioning that the proposition (in particular, Lemma 1 as its intermediate step) requires that X be Polish (i.e., separable and completely metrizable) and $\succeq$ be a closed partial order.

Proposition 1. Let $b, b^{'} \in \Delta(X)$. b (first-order) stochastically dominates b' if and only if, for any $Y_0 \subseteq X_0$, $b(clup(Y_0)) \geq b'(clup(Y_0))$. In addition, b strictly stochastically dominates b' if and only if the inequality holds for any $Y_0 \subseteq X_0$ and it is strict at least for some $Y_0 \subseteq X_0$.

Proof. First, we state the following intermediate result, dues to Kamae, Krengel, and O'Brien (1977). Its proof is omitted.

Lemma 1. Let $b, b^{'} \in \Delta(X)$. $b^{'}$ first-order **stochastically dominates** b (denoted $b^{'} \succeq_{SD} b$) if and only if $b^{'}(Y) \geq b(Y)$ for any $Y \in U(X)$. In this case, we say that $b^{'}$ is more optimistic than b. In addition, $b^{'}$ **strictly** first-order stochastically dominates b if and only if $b^{'} \succeq_{SD} b$ and $b^{'}(Y) > b(Y)$ for some $Y \in U(X)$.

The result states that it is enough to consider all the closed upper sets (instead of all the increasing, measurable, and bounded functions) to establish a first-order stochastic dominance relation (and its strict variant) between two probability measures. We relegate the rest of the proof to the Appendix. □

2.2 Belief Hierarchies

Throughout this paper, let I denote the set of (finitely many) players, and let Θ denote the payoff-relevant state space. We assume that Θ is a separable, complete metric space, endowed with a closed partial order $\succeq_\Theta$, and that Assumption 1 and 2 are satisfied for $X = \Theta$ and $\succeq = \succeq_\Theta$, where the corresponding countable dense subset is denoted by Θ_0.[17]

It is often the case in practice that the players only partially and asymmetrically observe θ before they play a particular game. We represent their beliefs over θ and over each other's beliefs by *types*. Let $(T_i, \mathscr{T}_i, \pi_i)_{i \in I}$ be a *type space* where each T_i represents player i's set of types; each $\mathcal{T}_i$ represents a sigma-algebra over T_i and $\mathscr{T}$ and $\mathscr{T}_{-i}$ represent the product sigma-algebra over T and T_{-i}, respectively; and

[17]To be clear, these assumptions are not used for the results in Section 3, but used in Section 4 and 5.

a $\mathscr{T}_i$-measurable $\pi_i : T_i \to \Delta(\Theta \times T_{-i})$ is player i's interim belief map about the parameter and the other players' types.

Observe that, given any type space $(T_i, \mathscr{T}_i, \pi_i)_{i\in I}$, we can deduce the *belief hierarchy* of each type t_i of each player i as follows. Define $Z_j^1 = \Delta(\Theta)$ for $j \in I$, $Z_{-i}^1 = \prod_{j\neq i} Z_j^1$, and inductively for each $k \geq 1$, define $Z_j^{k+1} = \Delta(\Theta\times Z_{-j}^1 \times\cdots\times Z_{-j}^k)$ for $j \in I$, and $Z_{-i}^{k+1} = \prod_{j\neq i} Z_j^{k+1}$. Then, (i) his *first-order belief* is defined by $h_i^1(t_i) = \text{marg}_\Theta \pi_i(t_i) \in Z_i^1$, i.e., for each measurable $\tilde{\Theta} \subseteq \Theta$,

$$h_i^1(t_i)[\tilde{\Theta}] = \pi_i(t_i)[\tilde{\Theta} \times T_{-i}];$$

and inductively for each $k \geq 1$, (ii) his $(k+1)$*th-order belief* is defined by $h_i^{k+1}(t_i) = \text{marg}_{\Theta\times Z_{-i}^1\times\cdots\times Z_{-i}^k} \pi_i(t_i) \in Z_i^{k+1}$. The *belief hierarchy* of t_i is then defined by $(h_i^k(t_i))_{k=1}^\infty$.

We now introduce this paper's fundamental concept of *common certainty of optimism*. Let t_i and $t_i^{'}$ be two types of player i. Suppose that (i) $t_i^{'}$ is more optimistic about Θ than t_i; (ii) $t_i^{'}$ is more optimistic that all players are more optimistic about Θ than t_i; (iii) $t_i^{'}$ is more optimistic that all players are more optimistic that all players are more optimistic about Θ than t_i; and so on ad infinitum. In such a case, we say that $t_i^{'}$ is at least high as t_i in the order of common certainty of optimism. We formally define this as follows:

Definition 1. $t_i^{'}$ is at least as high as t_i in the order of **common certainty of optimism** (denoted by $t_i^{'} \succeq_{CCO} t_i$) if $h^k(t_i^{'}) \succeq_{SD} h^k(t_i)$ for each $k \in \mathbb{N}$.

In what follows, we often refer to this order as the CCO order.

2.3 Supermodular Games

The players in the set I play the following game. For each player $i \in I$, let A_i denote his action space, and let $u_i : A \times \Theta \to \mathbb{R}$ denote his payoff function, where $A = \prod_{j\in I} A_j$. Recall that Θ is the payoff-relevant state space introduced in the previous subsection.

Let X be a complete lattice and a partial order $\geq$. For each $Y \subseteq X$, let $\bigvee Y \in X$ denote the least upper bound ("join") of Y, and $\bigwedge Y \in X$ denote the greatest lower bound ("meet") of Y.[18] That X is a complete lattice means that the join and meet exist for any $Y \subseteq X$. In case Y is a binary set of the form $\{x, y\}$ with $x, y \in X$, following the standard notation, we denote its join by $x \vee y$ and its meet by $x \wedge y$.

We consider *supermodular games* as a domain of games, defined as follows. First, A_i is a complete lattice endowed with a partial order $\succeq_{A_i}$. Second, each $u_i(\cdot)$ is *supermodular* on A_i and has *increasing difference* in both (a_i, a_{-i}) and (a_i, θ). That is, for each $a_i, a_i' \in A_i$, $a_{-i}, a_{-i}^{'} \in A_{-i}$, and $\theta, \theta^{'} \in \Theta$, whenever $(a_{-i}, \theta) \geq (a_{-i}', \theta')$, it follows that

$$u_i((a;\theta) \vee (a';\theta')) + u_i((a;\theta) \wedge (a';\theta')) \geq u_i(a;\theta) + u_i(a';\theta'),$$

or equivalently,

$$u_i(a_i \vee a_i', a_{-i}; \theta) + u_i(a_i \wedge a_i', a_{-i}'; \theta') \geq u_i(a_i, a_{-i}; \theta) + u_i(a_i', a_{-i}'; \theta').$$

A tuple $G = (I, \Theta, (A_i, u_i, T_i, \pi_i)_{i \in I})$ comprises an (incomplete-information) supermodular game.

2.4 Equilibria

In an incomplete-information supermodular game G, we denote a *pure* strategy of each player i by a $\mathscr{T}_i$-measurable function $\sigma_i : T_i \to A_i$. We first define a pure strategy Bayesian equilibrium.

Definition 2. A strategy profile $\sigma^* = (\sigma_i^*)_{i \in I}$ is a (pure-strategy) **Bayesian equilibrium** if, for each $i \in I$, $t_i \in T_i$, and $a_i \in A_i$,

$$\int_{\Theta \times T_{-i}} u_i(\sigma_i^*(t_i), \sigma_{-i}^*(t_{-i}), \theta) d\pi_i(t_i)[\theta, t_{-i}] \geq \int_{\Theta \times T_{-i}} u_i(a_i, \sigma_{-i}^*(t_{-i}), \theta) d\pi_i(t_i)[\theta, t_{-i}].$$

Let Σ^* denote the set of "all" Bayesian equilibria of an incomplete information supermodular game $G = (g, (T_i), (\mathscr{T}_i), (\pi_i))_{i \in I}$. It may well be the case that Σ^* is empty. The interested reader should be referred to Van Zandt and Vives (2007) for

[18] $z \in X$ is an upper (a lower) bound of $Y \subseteq X$ if $z \succeq y$ ($z \preceq y$) for all $y \in Y$. $z \in X$ is the least upper bound of $Y \subseteq X$ if is an upper bound of Y, and moreover, we have $z' \succeq z$ for any upper bound z' of Y. Analogously, $z \in X$ is the greatest lower bound of $Y \subseteq X$ if is a lower bound of Y, and moreover, we have $z' \preceq z$ for any lower bound z' of Y.

a sufficient condition for Σ^* to be nonempty.[19] In what follows, we simply assume that Σ^* is nonempty.

We call $\underline{\sigma} \in \Sigma^*$ the *least equilibrium* if, for each $\sigma^* \in \Sigma^*$, $i \in I$, and $t_i \in T_i$, we have $\sigma_i^*(t_i) \succeq_{A_i} \underline{\sigma}_i(t_i)$, and similarly, call $\bar{\sigma} \in \Sigma^*$ the *greatest equilibrium* if, for each $\sigma^* \in \Sigma^*$, $i \in I$, and $t_i \in T_i$, we have $\overline{\sigma}_i(t_i) \succeq_{A_i} \sigma_i^*(t_i)$. As is usually the case for monotone comparative statics, this paper focuses on the least and greatest Bayesian equilibria in supermodular games. The following is our definition of monotone comparative statics with respect to the CCO order.

Definition 3. We say that *monotone comparative statics holds in a supermodular game G with respect to the CCO order* if, for each i and $t_i, t_i' \in T_i$ such that $t_i \succeq_{CCO} t_i'$, we have $\underline{\sigma}_i(t_i) \succeq_{A_i} \underline{\sigma}_i(t_i')$ and $\overline{\sigma}_i(t_i) \succeq_{A_i} \overline{\sigma}_i(t_i')$.

3 Sufficiency for Monotone Comparative Statics

In this section, up to technical regularity conditions guaranteeing the existence of the least and greatest equilibria, monotone comparative statics holds in *any* supermodular game G with respect to the common certainty of optimism (CCO) order. In what follows, we focus only on the least equilibrium of a supermodular game G, because the logic for the greatest equilibrium is similar.

The key observation is that, for each type t_i of each player i, his least equilibrium action is characterized by his least *interim correlated rationalizability* (ICR) of Dekel, Fudenberg, and Morris (2006).

The least ICR is identified by iterative elimination of never best responses "from below". First, for each $i \in I$, $t_i \in T_i$, let $A_i^0(t_i) = A_i$ and $\underline{a}_i^0(t_i) = \bigwedge A_i^0(t_i)$, and then, let

$$\underline{A}_i^1(t_i) = \arg \max_{a_i \in A_i^0(t_i)} \int_{\Theta \times T_{-i}} u_i(a_i, \underline{a}_{-i}^0(t_{-i}); \theta) d\pi_i(t_i)[\theta, t_{-i}],$$

and $\underline{a}_i^1(t_i) = \bigwedge \underline{A}_i^1(t_i)$. Later we assume that $\underline{A}_i^1(t_i)$ is a complete sublattice, implying that $\underline{a}_i^1(t_i) \in \underline{A}_i^1(t_i)$, and that $\underline{a}_i^1(\cdot)$ is a measurable mapping. Note that, whenever $\underline{a}_i^1(t_i) \succ_{A_i} a_i$, it follows from supermodularity that any such a_i is a never-best response.

[19]See also our Proposition 2 and Remark 1 right after the proposition.

By an induction argument, for each $k \geq 1$, for each $i \in I$, $t_i \in T_i$, let

$$\underline{A}_i^{k+1}(t_i) = \arg\max_{a_i \in A_i^k(t_i)} \int_{\Theta \times T_{-i}} u_i(a_i, \underline{a}_{-i}^k(t_{-i}); \theta) d\pi_i(t_i)[\theta, t_{-i}],$$

and $\underline{a}_i^{k+1}(t_i) = \bigwedge \underline{A}_i^{k+1}(t_i)$. Again, later we assume that $\underline{A}_i^{k+1}(t_i)$ is a complete sublattice, implying that $\underline{a}_i^{k+1}(t_i) \in \underline{A}_i^{k+1}(t_i)$, and that $\underline{a}_i^{k+1}(\cdot)$ is a measurable mapping. Note that, whenever $\underline{a}_i^{k+1}(t_i) \succ_{A_i} a_i$, it follows from supermodularity that any such a_i is a never-best response.

Finally, for each $i \in I$, $t_i \in T_i$, define

$$\underline{a}_i^{\infty}(t_i) \quad = \quad \bigvee \{\underline{a}_i^1(t_i), \underline{a}_i^2(t_i), \ldots\}.$$

Since A_i is a complete lattice, we have $\underline{a}_i^{\infty}(t_i) \in A_i$. Thus, if $\underline{a}_i^{\infty}(t_i)$ is a best response to $\underline{a}_{-i}^{\infty}(\cdot)$ (given his belief $\pi_i(t_i)$ over $\Theta \times T_{-i}$), then $\underline{\sigma}$ defined by $\underline{\sigma}_i(t_i) = \underline{a}_i^{\infty}(t_i)$ for each i, t_i constitutes an equilibrium. By construction, $\underline{\sigma}$ must be the least equilibrium of the game, because in each step k of the induction, any action $a_i \prec_{A_i} \underline{a}_i^k(t_i)$ is shown to be a never-best response to the lowest selection of the others' actions from $\underline{A}_i^{k-1}(\cdot)$, and hence, a never-best response to any other strategy profile σ_{-i} of the other players such that $\sigma_{-i}(t_{-i}) \succeq_{A_{-i}} \underline{a}_{-i}^k(t_{-i})$. We note this result as a proposition.

Proposition 2. Assume that, for each i, t_i, and $k \geq 1$, (i) $\underline{A}_i^k(t_i)$ is a complete sublattice, (ii) $\underline{a}_i^k(\cdot) = \bigwedge \underline{A}_i^k(\cdot)$ is a measurable mapping, and (iii) $\underline{a}_i^{\infty}(t_i)$ is a best response to $\underline{a}_{-i}^{\infty}(\cdot)$. Then, $\underline{\sigma}$ defined by $\underline{\sigma}_i(t_i) = \underline{a}_i^{\infty}(t_i)$ for each i and t_i constitutes the least equilibrium.

Remark 1. Interested readers are referred to Van Zandt and Vives (2007) and Van Zandt (2010) for more primitive assumptions on the environment that guarantee the existence of the least (and analogously, greatest) equilibrium. Specifically, they assume that (i) A_i is a compact metric lattice;[20] (ii) u_i is bounded, continuous in a_i and measurable in θ; and (iii) $\pi_i(\cdot)$ is measurable (as a mapping from T_i to $\Delta(\Theta \times T_{-i})$).

[20] Compactness is used not for existence of best replies (thanks to the supermodularity), but for guaranteeing that $\underline{a}_i^k(\cdot)$ is a measurable mapping. For this point, see Footnote 3 of Van Zandt (2010) who mentions that compactness can be replaced by sigma-compactness for this measurable selection argument.

Now we prove that monotone comparative statics holds in any supermodular game G with respect to the CCO order.

Theorem 1. Let $G = (g, (T_i), (\mathscr{T}_i), (\pi_i))_{i \in I}$ be an incomplete information supermodular game that satisfies (as in Proposition 2): for each $i \in I$, $t_i \in T_i$, and $k \geq 1$, (i) $\underline{A}_i^k(t_i)$ is a complete sublattice; (ii) $\underline{a}_i^k(\cdot) = \bigwedge \underline{A}_i^k(\cdot)$ is a measurable mapping; and (iii) $\underline{a}_i^\infty(t_i)$ is a best response to $\underline{a}_{-i}^\infty$. Let t_i and $t_i^{'}$ be two types of player i such that $t_i^{'} \succeq_{CCO} t_i$. Then, for the least equilibrium of the game G, $\underline{\sigma}$, we have $\underline{\sigma}_i(t_i^{'}) \succeq_{A_i} \underline{\sigma}_i(t_i)$.

Proof. In the previous proposition, we show that the least equilibrium is fully characterized by the iterated elimination of never-best responses of interim correlated rationalizability "from below." Thus, it suffices to show that, for each $i \in I$, $k \geq 1$, and t_i, t_i' such that $t_i \succeq_{CCO} t_i'$, we have $\underline{a}_i^k(t_i) \succeq_{A_i} \underline{a}_i^k(t_i')$.

First, because $\underline{a}_i^1(t_i) \in \underline{A}_i^1(t_i)$ and $\underline{a}_i^1(t_i') \in \underline{A}_i^1(t_i')$, we have

$$\int_{\Theta \times T_{-i}} u_i(\underline{a}_i^1(t_i'), \underline{a}_{-i}^0(t_{-i}); \theta) d\pi(t_i')[\theta, t_{-i}] \geq \int_{\Theta \times T_{-i}} u_i(\underline{a}_i^1(t_i) \vee \underline{a}_i^1(t_i^{'}), \underline{a}_{-i}^0(t_{-i}); \theta) d\pi(t_i')[\theta, t_{-i}]$$

and

$$\int_{\Theta \times T_{-i}} u_i(\underline{a}_i^1(t_i), \underline{a}_{-i}^0(t_{-i}); \theta) d\pi(t_i)[\theta, t_{-i}] \geq \int_{\Theta \times T_{-i}} u_i(\underline{a}_i^1(t_i) \wedge \underline{a}_i^1(t_i^{'}), \underline{a}_{-i}^0(t_{-i}); \theta) d\pi(t_i)[\theta, t_{-i}]$$

Since $h_i^1(t_i') \succeq_{SD} h_i^1(t_i)$ and $\underline{a}_{-i}^0(t_{-i})$ does not depend on t_{-i}, the distribution over $\Theta \times A_{-i}$ induced by $\pi_i(t_i')$ first-order stochastically dominates that induced by $\pi_i(t_i)$. Therefore, by the supermodularity of the game, we have

$$\begin{aligned} & \int_{\Theta \times T_{-i}} u_i(\underline{a}_i^1(t_i) \vee \underline{a}_i^1(t_i^{'}), \underline{a}_{-i}^0(t_{-i}); \theta) d\pi(t_i^{'})[\theta, t_{-i}] - \int_{\Theta \times T_{-i}} u_i(\underline{a}_i^1(t_i'), \underline{a}_{-i}^0(t_{-i}); \theta) d\pi(t_i')[\theta, t_{-i}] \\ \geq \; & \int_{\Theta \times T_{-i}} u_i(\underline{a}_i^1(t_i), \underline{a}_{-i}^0(t_{-i}); \theta) d\pi(t_i)[\theta, t_{-i}] - \int_{\Theta \times T_{-i}} u_i(\underline{a}_i^1(t_i) \wedge \underline{a}_i^1(t_i^{'}), \underline{a}_{-i}^0(t_{-i}); \theta) d\pi(t_i)[\theta, t_{-i}]. \end{aligned}$$

Because the left-hand side of the above inequality is nonpositive and the right-hand side is nonnegative, we must have both equal to zero. In particular, this implies that $\underline{a}_i^1(t_i) \wedge \underline{a}_i^1(t_i^{'}) \in \underline{A}_i^1(t_i)$. However, because $\underline{a}_i^1(t_i) = \bigwedge \underline{A}_i^1(t_i) \in \underline{A}_i^1(t_i)$, we must have $\underline{a}_i^1(t_i^{'}) \succeq_{A_i} \underline{a}_i^1(t_i)$.

We move on to the next step. Let

$$\underline{A}_2^1(t_i) = \arg \max_{a_i \in A_i^1(t_i)} \int_{\Theta \times T_{-i}} u_i(a_i, \underline{a}_{-i}^1(t_{-i}); \theta) d\pi_i(t_i)[\theta, t_{-i}],$$

and $\underline{a}_i^2(t_i) = \bigwedge \underline{A}_i^2(t_i)$. Again, we assume that $\underline{A}_i^1(t_i)$ is a complete sublattice, implying that $\underline{a}_i^1(t_i)$, and that $\underline{a}_i^1(\cdot)$ is a measurable mapping.

Whenever $\underline{a}_i^2(t_i) \succ_{A_i} a_i$, it follows from supermodularity that any such a_i does not survive the iterative elimination of never-best responses. Recall that for any $j \neq i$ and $t_j, t_j^{'}$, if $h_j^1(t_j^{'}) \succeq_{SD} h_j^1(t_j)$, then $\underline{a}_j^1(t_j^{'}) \succeq_{A_j} \underline{a}_j^1(t_j)$. Since we assume that $t_i^{'} \succeq_{CCO} t_i$, we also have $h_i^2(t_i^{'}) \succeq_{SD} h_i^2(t_i)$. Define

$$\tilde{Y} = \left\{ (\theta, a_{-i}^1) \in \Theta \times A_{-i} \middle| \, \exists (\hat{\theta}, \hat{t}_{-i}) \text{ s.t. } (\theta, a_{-i}^1) \succeq (\hat{\theta}, \underline{a}_{-i}^1(\hat{t}_{-i})) \right\}.$$

Clearly, $\tilde{Y} \in U(\Theta \times A_{-i})$ where $U(\Theta \times A_{-i})$ denotes the set of all upper events of $\Theta \tilde{\times} A_{-i}$. By Lemma 1, we can conclude that the weight $h_i^2(t_i^{'})$ assigns to the event $\tilde{Y}$ is at least as high as the weight $h_i^2(t_i)$ does.

Due to the definition of $\underline{a}_i^2(t_i) \in \underline{A}_i^2(t_i)$ and $\underline{a}_i^2(t_i') \in \underline{A}_i^2(t_i')$, we have

$$\int_{\Theta \times T_{-i}} u_i(\underline{a}_i^2(t_i^{'}), \underline{a}_{-i}^1(t_{-i}); \theta) d\pi(t_i')[\theta, t_{-i}] \geq \int_{\Theta \times T_{-i}} u_i(\underline{a}_i^2(t_i) \vee \underline{a}_i^2(t_i^{'}), \underline{a}_{-i}^1(t_{-i}); \theta) d\pi(t_i')[\theta, t_{-i}],$$

and

$$\int_{\Theta \times T_{-i}} u_i(\underline{a}_i^2(t_i), \underline{a}_{-i}^1(t_{-i}); \theta) d\pi(t_i)[\theta, t_{-i}] \geq \int_{\Theta \times T_{-i}} u_i(\underline{a}_i^2(t_i) \wedge \underline{a}_i^2(t_i^{'}), \underline{a}_{-i}^1(t_{-i}); \theta) d\pi(t_i)[\theta, t_{-i}].$$

Since $h_i^2(t_i^{'}) \succeq_{SD} h_i^2(t_i)$, by the supermodularity of the game, we have

$$\begin{aligned} & \int_{\Theta \times T_{-i}} u_i(\underline{a}_i^2(t_i) \vee \underline{a}_i^2(t_i^{'}), \underline{a}_{-i}^1(t_{-i}); \theta) d\pi(t_i^{'})[\theta, t_{-i}] - \int_{\Theta \times T_{-i}} u_i(\underline{a}_i^2(t_i^{'}), \underline{a}_{-i}^1(t_{-i}); \theta) d\pi(t_i^{'})[\theta, t_{-i}] \\ \geq \; & \int_{\Theta \times T_{-i}} u_i(\underline{a}_i^2(t_i), \underline{a}_{-i}^1(t_{-i}); \theta) d\pi(t_i)[\theta, t_{-i}] - \int_{\Theta \times T_{-i}} u_i(\underline{a}_i^2(t_i) \wedge \underline{a}_i^2(t_i^{'}), \underline{a}_{-i}^1(t_{-i}); \theta) d\pi(t_i)[\theta, t_{-i}]. \end{aligned}$$

Because the left-hand side of the above inequality is nonpositive and the right-hand side is nonnegative, we must have both equal to zero. In particular, this implies that $\underline{a}_i^2(t_i) \wedge \underline{a}_i^2(t_i^{'}) \in \underline{A}_i^2(t_i)$. However, because $\underline{a}_i^2(t_i) = \bigwedge \underline{A}_i^2(t_i) \in \underline{A}_i^2(t_i)$, we must have $\underline{a}_i^2(t_i^{'}) \succeq_{A_i} \underline{a}_i^2(t_i)$.

By an induction argument, we can analogously show that $\underline{a}_i^k(t_i^{'}) \succeq_{A_i} \underline{a}_i^k(t_i)$ for each $k \in \mathbb{N}$, which implies that $\underline{a}_i^\infty(t_i^{'}) \succeq_{A_i} \underline{a}_i^\infty(t_i)$.

Since the least equilibrium $\underline{\sigma}$ is defined as $\underline{\sigma}_i(t_i) = \underline{a}_i^\infty(t_i)$ for every $i \in I$ and t_i, we complete the proof. □

4 Necessity for Monotone Comparative Statics: The Single Person Case

In this section, common certainty of optimism (CCO) is shown to be a necessary condition for monotone comparative statics to be valid in all supermodular games. We show this by constructing a specific supermodular game, which we call an *optimism-elicitation game*, which satisfies the following: for each player i and his types t_i and t'_i, if we do *not* have $t_i \succeq_{CCO} t'_i$, then for the least equilibrium of this optimism-elicitation game, denoted by $\underline{\sigma}$, we do *not* have $\underline{\sigma}_i(t_i) \succeq_{A_i} \underline{\sigma}_i(t'_i)$. Together with the previous theorem, we thus conclude that the CCO order is necessary and sufficient for monotone comparative statics in all supermodular games.

4.1 A Single-Person Game

We first consider the single-person environment to explain the key technical issue and the main intuition how we treat it. The restriction to the single-person case simplifies our analysis significantly because there is no need to consider interactive beliefs so that we lose nothing to focus on the first-order beliefs only. Thus, a naive candidate for our optimism-elicitation game is a so-called *scoring rule*, which is essentially a single-person decision problem where the decision maker reveals his belief over Θ (and his payoff function is such that the truthful revelation is uniquely optimal). That is, his action space is the set of all probability measures over Θ. Monotone comparative statics is obtained in a straightforward manner by endowing this action space with a partial order based on the first-order stochastic dominance.

However, as we observe in the next example, this decision problem is not a (single-person) supermodular game, because the action space, $\Delta(\Theta)$, is not a lattice, even if the parameter space Θ itself is. This means that we need a more careful choice of the action space. To illustrate this point, we go through the following example.

Example 1 (Kamae, Krengel, and O'Brien (1977)). [21]

[21] To be rigorous, it is a slightly different example from the one provided there, but essentially the same.

Let $\Theta = \{0,1\}^2$ be endowed with a component-wise partial order. Consider two probability measures $P, P' \in \Delta(\Theta)$ such that $P(0,0) = P(1,1) = 1/2$ and $P'(1,0) = P'(0,1) = 1/2$. Then, two probability measures $Q, Q' \in \Delta(\Theta)$ are upper bounds of $\{P, P'\}$: $Q(1,0) = Q(1,1) = 1/2$ and $Q'(0,1) = Q'(1,1) = 1/2$. Suppose that there exists a least upper bound Q''. Then, we have $Q''(1,1) = 1/2$ because we need $P(1,1) \leq Q''(1,1) \leq Q(1,1)$. Moreover, we have $Q''(0,1) = 0$ (or $Q''(0,1) + Q''(1,1) = 1/2$) because we need $Q''(0,1) + Q''(1,1) \leq Q(0,1) + Q(1,1)$. Similarly, we have $Q''(1,0) = 0$. However, then, such Q'' is equivalent to P, which does not first-order dominate P'. This contradicts that Q'' is an upper bound of $\{P, P'\}$.

Therefore, if we consider (the single-person version of) the optimism-elicitation game where an individual chooses his probability measure over Θ, the corresponding game, no matter how we define it, is "not" a supermodular game because his action space, the set of all probability measures over Θ (endowed with a stochastic dominance partial order), does not constitute a lattice.

The problem illustrated in the above example is that the set of all probability distributions (over Θ) is not closed in the meet and join operators. To elaborate on this point, we revisit the same example.

Example 2. We consider the same example as above, but now the agent chooses a function

$$\alpha : U(\Theta) \to [0,1],$$

where $U(\Theta) \subseteq 2^\Theta$ denotes the set of all subsets of Θ that are upper sets (recall that $Y \subseteq \Theta$ is an upper set if $[x \in Y$ and $y \geq x]$ implies $y \in Y$). In the current context, we have

$$\begin{aligned} U(\Theta) &= \Big\{\emptyset, \{(1,1)\}, \{(0,1),(1,1)\}, \{(1,0),(1,1)\}, \\ &\qquad \{(0,1),(1,0),(1,1)\}, \underbrace{\{(0,0),(0,1),(1,0),(1,1)\}}_{=\Theta}\Big\}. \end{aligned}$$

We may interpret each $\alpha(\Theta)$ as the agent's "belief" regarding the event Θ, and in fact, each belief corresponds to some mapping α.[22] However, other α may not

[22]For example, P in the previous example is equivalent to α_P such that (i) $\alpha(\emptyset) = 0$, (ii) $\alpha_P(Y) = 1/2$ for any nonempty $Y \in U(\Theta)$ with $(0,0) \notin Y$, and (iii) $\alpha_P(Y) = 1$ for any $Y \in U(\Theta)$ with $(0,0) \in Y$.

correspond to any probability measure. For example, let α be defined in such a way that, for each $Y \in U(\Theta)$,

$$\alpha(Y) = \max\{P(Y), P'(Y)\}.$$

That is,

$$\alpha(Y) = \begin{cases} 0 & \text{if } Y = \emptyset \\ 1/2 & \text{if } Y = \{(1,1)\}, \{(0,1),(1,1)\}, \{(1,0),(1,1)\}, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

If it corresponds to a probability measure Q^* over Θ, then $\alpha(\{1,1\}) = 1/2$ implies $Q^*(1,1) = 1/2$, which, together with $\alpha(\{(0,1),(1,1)\}) = \alpha(\{(1,0),(1,1)\}) = 1/2$, implies $Q^*(0,1) = Q^*(1,0) = 0$. However, $\alpha(\{(0,1),(1,0),(1,1)\}) = 1$ implies that $Q^*(0,1) + Q^*(1,0) = 1$, which is a contradiction. Therefore, this α does not correspond to any probability measure.

Consider a "modified" optimism-elicitation game with a single player who chooses any $\alpha : U(\Theta) \to [0,1]$. Let A^* denote the set of all such α. The player has a strictly larger strategy space than in the original optimism-elicitation game because some "non-additive" measures are allowed. Moreover, A^* is now a lattice (associated with the first-order stochastic dominance partial order), because, for any α, α', we have α'', α''' such that $\alpha''(Y) = \max\{\alpha(Y), \alpha'(Y)\}$ and $\alpha'''(Y) = \min\{\alpha(X), \alpha'(Y)\}$ for any $Y \in U(\Theta)$. In fact, it is even a complete lattice, because for any nonempty subset $A \subseteq A^*$, there are $\alpha', \alpha'' \in A^*$ such that $\alpha'(Y) = \sup_{\alpha \in A} \alpha(Y)$ and $\alpha''(Y) = \inf_{\alpha \in A} \alpha(Y)$ for any $Y \in U(\Theta)$.

The key for the construction of our optimism-elicitation game is two-fold. First, the action space of our game is based on non-additive beliefs such as α discussed above,[23] in order to make it a complete lattice. Second, as we see below in the formal construction, the action space of our game essentially comprises only countably many "test sets" to (partially) identify the agent's belief. We explain these features more in detail after formally introducing our optimism-elicitation game.

Formally, the optimism-elicitation game for the single agent case is defined as follows: (i) the agent chooses an action $\beta : F(\Theta_0) \times \mathbb{Q}_+ \to [0,1]$ (recall that Θ_0 denotes the countable dense subset of Θ where Assumption 1 is satisfied) where

[23] See Schmeideler (1986, 1989) for non-additive beliefs or *capacities* in decision theory.

- $F(\Theta_0)$ denotes the set of all *finite* subsets of Θ_0,
- $\mathbb{Q}_+$ denotes the set of nonnegative rational numbers, and
- β is nondecreasing (i.e., for any (γ, q) and (γ', q') with $clup(B_q(\gamma)) \subseteq clup(B_{q'}(\gamma'))$, we have $\beta(\gamma, q) \leq \beta(\gamma', q')$)

and (ii) given any realization $\theta \in \Theta$, the agent's payoff is given:

$$u(\beta, \theta) = \sum_{(\gamma,q)\in F(\Theta_0)\times\mathbb{Q}_+} \left[\beta(\gamma, q)I_{\{clup(B_q(\gamma))\}}(\theta) - \frac{\beta(\gamma, q)^2}{2}\right]\mu(\gamma, q),$$

where

- $B_q(\gamma) = \bigcup_{y\in\gamma} B_q(y)$;
- μ is a full-support distribution over a countable set $F(\Theta_0) \times \mathbb{Q}_+$[24]; and
- The indicator function is defined as:

$$I_{\{clup(B_q(\gamma))\}}(x) = \begin{cases} 1 & \text{if } x \in clup(B_q(\gamma)) \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\mathcal{B} = \left\{\beta : F(\Theta_0) \times \mathbb{Q}_+ \to [0, 1] \middle| \beta \text{ is nonderecasing}\right\}$$

denote the space of the agent's strategies. Note that $\mathcal{B}$ constitutes the set of *capacities* (i.e., non-additive measures) for closed upper sets generated by $F(\Theta_0) \times \mathbb{Q}_+$.[25] A capacity is often considered as a natural generalization of a probability measure.[26] In addition, the space of capacities has an advantageous feature that it is a complete lattice.

[24] We can set $h : F(\Theta_0) \times \mathbb{Q}_+ \to \mathbb{N}$ as an injection mapping because $F(\Theta_0) \times \mathbb{Q}_+$ is countable. Specifically, we define the full-support distribution μ by $\mu(\gamma, q) = (1/2)^{h(\gamma,q)} > 0$.

[25] A capacity is usually defined as a monotone set function as above, but with additional normalization conditions that it assigns probability zero (one) on the null (entire) set. Redefining $\mathcal{B}$ by adding these normalization conditions does not essentially change our arguments, and hence, we adopt the current definition to simplify the notation. Indeed, it should be clear that any type of any agent should find optimal to set $\beta(\gamma, q) = 0$ (1) if $clup(B_q(\gamma)) = \emptyset$ (Θ). In this sense, the definition above is without loss of generality. The same comment applies to the definitions of $\mathcal{B}^m$ and $\mathcal{B}^\infty$ in Section 5.1.

[26] See Schmeidler (1986, 1989) for more details on capacities.

As we mention above, another feature of our construction is that the action space of our game essentially comprises only countably many "test sets" to (partially) identify the agent's belief. Countability enables us to have a full-support distribution over the test sets, which makes the agent's incentive to tell the truth strict (and hence, the optimal decision is unique).[27]

Indeed, if a player has a belief $b \in \Delta(\Theta)$ over Θ,[28] then his unique optimal action is $\beta^*(b) : F(\Theta_0) \times \mathbb{Q}_+ \to [0,1]$ such that

$$\beta^*(\gamma, q|b) = b(clup(B_q(\gamma)))$$

for each $(\gamma, q) \in F(\Theta_0) \times \mathbb{Q}_+$.

Endowing $\mathcal{B}$ with a natural product order, we show that the game is a supermodular game. First, we claim that $\mathcal{B}$ is a complete lattice: for each $C \subseteq \mathcal{B}$, define two functions, $\bigvee(C)$ and $\bigwedge(C)$, so that

$$\begin{aligned} \bigvee(C)(\gamma, q) &= \sup_{\beta \in C} \beta(\gamma, q), \\ \bigwedge(C)(\gamma, q) &= \inf_{\beta \in C} \beta(\gamma, q), \end{aligned}$$

which makes both $\bigvee(C)$ and $\bigwedge(C)$ elements of $\mathcal{B}$, because they both take values in $[0,1]$ for any (γ, q), and they are both monotonic. Suppose, on the contrary, that $\bigvee(C)$ is not monotonic for some C. Then, there exist (γ, q), (γ', q') such that $clup(B_q(\gamma)) \subseteq clup(B_{q'}(\gamma'))$ and $\bigvee(C)(\gamma, q) > \bigvee(C)(\gamma', q')$. By definition, there exists $\beta \in C$ such that $\beta(\gamma, q)$ is close to $\bigvee(C)(\gamma, q)$, and in particular, $\beta(\gamma, q) > \bigvee(C)(\gamma', q') \geq \beta(\gamma', q')$. This contradicts the hypothesis that β is monotonic.

Second, the payoff function $u(\cdot)$ is supermodular on $\mathcal{B}$ and has increasing dif-

[27]Moreover, as we see in Section 5 to extend this construction to the case of multiple players, this countability plays another crucial role. There, we consider the "higher-order belief" version of the current construction as each player's action space (in order to elicit his belief hierarchy), and countability at each level of hierarchy (and certain continuity) is crucial to make the next level of hierarchy (and hence at any level of hierarchy) stay countable.

[28]Recall that $\Delta(\Theta)$ denotes the set of all probability measures over Θ.

ference in (β, x): for any $\beta, \beta' \in \mathcal{B}$, $x, x' \in \Theta$ with $x \geq x'$, we have

$$\begin{aligned}
& u(\beta \vee \beta', x) - u(\beta, x) + u(\beta \wedge \beta', x') - u(\beta', x') \\
= & \int_{(\gamma,q):\beta'(\gamma,q)>\beta(\gamma,q)} \left[(\beta'(\gamma,q) - \beta(\gamma,q)) I_{\{clup(B_q(\gamma))\}}(x) - \beta'(\gamma,q)^2 + \beta(\gamma,q)^2\right] d\mu \\
& - \int_{(\gamma,q):\beta'(\gamma,q)>\beta(\gamma,q)} \left[(\beta(\gamma,q) - \beta'(\gamma,q)) I_{\{clup(B_q(\gamma))\}}(x^{'}) - \beta(\gamma,q)^2 + \beta'(\gamma,q)^2\right] d\mu \\
= & \int_{(\gamma,q):\beta'(\gamma,q)>\beta(\gamma,q)} \left[(\beta'(\gamma,q) - \beta(\gamma,q)) I_{\{clup(B_q(\gamma))\}}(x)(1 - I_{\{clup(B_q(\gamma))\}}(x^{'}))\right] d\mu \\
\geq & \ 0.
\end{aligned}$$

We now examine monotone comparative statics for this supermodular game. The first result establishes the sufficiency of first-order stochastic dominance for monotone comparative statics in this supermodular game (as should be expected).

Proposition 3. Let $b, b' \in \Delta(\Theta)$. If $b^{'}$ first-order stochastically dominates b, then, for any $(\gamma, q) \in F(\Theta_0) \times \mathbb{Q}_+$, we have $\beta^*(\gamma, q|b^{'}) \geq \beta^*(\gamma, q|b)$.

Proof. While it is a corollary to Theorem 1, the proof is also straightforward once we notice that $\beta^*(\gamma, q|b) = b(clup(B_q(\gamma)))$ and $\beta^*(\gamma, q|b') = b'(clup(B_q(\gamma)))$. □

Next, we show the desired necessity of first-order dominance for monotone comparative statics in this supermodular game.

Proposition 4. Let $b, b' \in \Delta(\Theta)$. If $\beta^*(\gamma, q|b^{'}) \geq_{\mathcal{B}} \beta^*(\gamma, q|b)$ for each $(\gamma, q) \in F(\Theta_0) \times \mathbb{Q}_+$, then $b^{'} \succeq_{SD} b$.

Proof. We take the contrapositive of the statement. Then what we want to show is that if b' does "not" stochastically dominate b, then $\beta^*(\gamma, q|b')$ "cannot" be higher than $\beta^*(\gamma, q|b)$ in the sense of the partial order on $\mathcal{B}$. Thus, the rest of the proof is completed by the following lemma.

Lemma 2. Let $b, b' \in \Delta(\Theta)$. If b' does not first-order stochastically dominate b, then there exists $(\gamma, q) \in F(\Theta_0) \times \mathbb{Q}_+$ such that $\beta^*(\gamma, q|b) > \beta^*(\gamma, q|b')$.

Proof. We relegate the proof to the Appendix. □

With this lemma, we complete the proof of Proposition 4. □

4.2 Properties of $\mathcal{B}$

Recall the definition of $\mathcal{B} = \{\beta : F(\Theta_0) \times \mathbb{Q}_+ \to [0,1] |\ \beta \text{ is nonderecasing}\}$. Let $\overline{\mathcal{B}} = \{\beta : F(\Theta_0) \times \mathbb{Q}_+ \to [0,1]\}$ be the superset of $\mathcal{B}$ in which we only drop the property that β is nondecreasing from $\mathcal{B}$. In this subsection, we first introduce a metric for $\overline{\mathcal{B}}$, inducing a topology with respect to which $\overline{\mathcal{B}}$ is shown to be a compact metric space. Second, we claim that $\mathcal{B}$ is a closed subset of $\overline{\mathcal{B}}$ so that $\mathcal{B}$ is also a compact metric space. Note that every compact metric space is complete and separable. Thus, $\mathcal{B}$ has a countable dense subset $\mathcal{B}_0$. Finally, we will establish that $\mathcal{B}$ possesses its closed partial order, and satisfies Assumptions 1 and 2 (with replacement of X by $\mathcal{B}$ and X_0 with $\mathcal{B}_0$ in the statements). These properties are exploited in the next section when we consider the multi-player case.

First, we introduce a norm over $\overline{\mathcal{B}}$ to make it a normed space (and accordingly, its metric is induced by this norm).[29] For each $\beta \in \overline{\mathcal{B}}$, its norm is given by

$$\|\beta\| = \sum_{(\gamma,q)\in F(\Theta_0)\times\mathbb{Q}_+} |\beta(\gamma,q)|\mu(\gamma,q),$$

where μ is a full-support probability distribution over $F(\Theta_0) \times \mathbb{Q}_+$ such that we set $h : F(\Theta_0) \times \mathbb{Q}_+ \to \mathbb{N}$ as an injection map and $\mu(\gamma, q) = (1/2)^{h(\gamma,q)}$ for each $(\gamma, q) \in F(\Theta_0) \times \mathbb{Q}$. Because $\beta(\gamma, q) \in [0,1]$ for any (γ, q), we have $\|\beta\| \in [0,1]$ for any $\beta \in \overline{\mathcal{B}}$.

Lemma 3. $\mathcal{B}$ is a compact metric space.

Proof. We relegate the proof to the Appendix. □

Remark 2. The lemma implies that $\mathcal{B}$ is a separable and complete metric space.

Next, we show that $\mathcal{B}$ satisfies Assumption 1. First, for each $K \in \mathbb{N}$, we define $\mathcal{B}^{1,K} \subseteq \mathcal{B}$ as follows: $\beta \in \mathcal{B}^{1,K}$ if and only if there exists a K-element subset of Θ_0, say $X_K = \{x_1 \dots, x_K\} \in F(\Theta_0)$, such that for any $(\gamma, q) \in F(\Theta_0) \times \mathbb{Q}_+$, we have

$$\beta(\gamma,q) = \begin{cases} \min_{q'\in Q_K}\ \beta(X_K\cap\gamma, q') \text{ sub. to } B_{q'}(X_K\cap\gamma) \supseteq B_q(\gamma) & \text{if} \quad X_K\cap\gamma \neq \emptyset, \\ 1 & \text{if} \quad X_K\cap\gamma = \emptyset. \end{cases}$$

[29] A standard topology for the set of probability distributions is a weak*-topology (e.g., Brandenburger and Dekel (1993)), but note that $\mathcal{B}$ is not a set of probability distributions. In particular, some $\beta \in \mathcal{B}$ does not necessarily correspond to any probability measure over Θ. The norm above and its induced topology on $\mathcal{B}$ are well-defined despite this "non-probabilistic" nature of $\mathcal{B}$. The same comment applies when we discuss the objects like $\mathcal{B}^m$ and $\mathcal{B}^\infty$ later in Section 5.1.

where $Q_K = \{\frac{k}{K} | k = 0, 1, \ldots, K\}$.

Note that such β is fully identified by $(\beta(\tilde{X}, q))_{\tilde{X} \subseteq X_K, q \in Q_K}$. This implies that $\mathcal{B}^{1,K}$ contains countably many elements, and thus $\mathcal{B}_0 = \bigcup_{K \in \mathbb{N}} \mathcal{B}^{1,K}$ contains countably many elements. The next lemma shows that Assumption 1 is satisfied for $\mathcal{B}$, where in the statement, X is replaced by $\mathcal{B}$ and X_0 is replaced by $\mathcal{B}_0$.

Lemma 4. For any $\beta \in \mathcal{B}$ and $\varepsilon > 0$, there exists $\beta_0 \in \mathcal{B}_0$ such that $\|\beta_0 - \beta\| < \varepsilon$ and $\beta_0 \geq \beta$.

Proof. We relegate the proof to the Appendix. □

The next lemma shows that Assumption 2 is also satisfied for $\mathcal{B}$.

Lemma 5. For each $C \subseteq \mathcal{B}$ and $\varepsilon > 0$, there exists $\delta(C, \varepsilon) > 0$ such that, for any $D \subseteq \mathcal{B}$ with $d(C, D) < \delta(C, \varepsilon)$, we have $d(up(C), up(D)) < \varepsilon$.

Proof. The proof is relegated to the Appendix. □

Finally, we show that the partial order on $\mathcal{B}$ is a closed partial order.

Lemma 6. Let $\mathcal{B}$ be endowed with a natural product order $\geq_{\mathcal{B}}$. Then, $\geq_{\mathcal{B}}$ is a closed order.

Proof. Consider two sequences $\{\beta_n\}$ and $\{\beta'_n\}$ in $\mathcal{B}$, such that $\beta_n \to \beta$ and $\beta'_n \to \beta'$ as $n \to \infty$. Then, due to the continuity of β and β', for each $(\gamma, q) \in F(X_0) \times \mathbb{Q}_+$, we have $\beta_n(\gamma, q) \to \beta(\gamma, q)$ and $\beta'_n(\gamma, q) \to \beta'(\gamma, q)$ as $n \to \infty$. Now suppose that $\beta_n(\gamma, q) \geq_{\mathcal{B}} \beta'_n(\gamma, q)$ for any n. Then, we must have $\beta(\gamma, q) \geq_{\mathcal{B}} \beta'(\gamma, q)$ for each (γ, q). This means that the partial order on $\mathcal{B}$ is a closed partial order. □

5 Necessity for Monotone Comparative Statics: The Multi-Player Case

With multiple agents, we need an optimism-elicitation game where the equilibrium reflects each player's belief hierarchy (not only his first-order belief). Although one may think that the situation becomes prohibitively more complicated, we show in this section that the same technique as in the single-person case can be extended appropriately.

The goal of this section is to construct a (multi-player) supermodular game such that common certainty of optimism holds if and only if monotone comparative statics holds in this game (Theorem 2 in Section 5.2). The crucial step is where the construction of each player's action space where each player bets not only on the realization of $\theta \in \Theta$ but also on each other's betting behavior, reflecting his high-order beliefs (Section 5.1).

5.1 Preliminary

Let $X^1 = \Theta$, $X_0^1 = \Theta_0$, and $\mathcal{B}^1 = \mathcal{B}$. For $m \geq 2$, we inductively construct supermodular games where each player's m-th order belief is relevant. Specifically, for $m \geq 2$, assume that (i) X^{m-1} is a separable, complete metric space with a countable dense subset X_0^{m-1}, (ii) X^{m-1} satisfies Assumption 1 and 2 (with replacement of X by X^{m-1} and X_0 by X_0^{m-1} in the statements) with the corresponding closed partial order, (iii) $\mathcal{B}^{m-1}$ is a compact metric space with a countable dense subset $\mathcal{B}_0^{m-1}$, and (iv) $\mathcal{B}^{m-1}$ satisfies Assumption 1 and 2 (with replacement of X by $\mathcal{B}^{m-1}$ and X_0 by $\mathcal{B}_0^{m-1}$ in the statements) with the corresponding closed partial order.

We define $X^m = X^{m-1} \times (\mathcal{B}^{m-1})^{I-1}$, endowed with product topology and product partial order. Because both X^{m-1} and $\mathcal{B}^{m-1}$ are separable, complete metric spaces and satisfy Assumption 1 and 2, X^m also satisfies the same properties:

Lemma 7. X^m is a separable, complete metric space with a countable dense subset X_0^m such that Assumption 1 and 2 are satisfied with replacement of X by X^m and X_0 by X_0^m in the statements.

Next, we define

$$\mathcal{B}^m = \left\{ \beta : F(X_0^m) \times \mathbb{Q}_+ \to [0,1] \,\middle|\, \beta \text{ is nonderecasing } \right\}.$$

Then, applying the same logic in Section 4.2, we obtain the following (the proof omitted):

Lemma 8. $\mathcal{B}^m$ is a compact metric space with a countable dense subset $\mathcal{B}_0^m$ such that Assumption 1 and 2 are satisfied with replacement of X by $\mathcal{B}^m$ and X_0 by $\mathcal{B}_0^m$ in the statements.

Therefore, for any $m \geq 1$, (i)-(iv) are satisfied: (i) X^m is a separable, complete metric space with a countable dense subset X_0^m, (ii) X^m satisfies Assumption 1 and 2 (with replacement of X by X^m and X_0 by X_0^m in the statements) with the corresponding closed partial order, (iii) $\mathcal{B}^m$ is a compact metric space with a countable dense subset $\mathcal{B}_0^m$, and (iv) $\mathcal{B}^m$ satisfies Assumption 1 and 2 (with replacement of X by $\mathcal{B}^m$ and X_0 by $\mathcal{B}_0^m$ in the statements) with the corresponding closed partial order.

Finally, let $X^\infty = \prod_{m=1}^\infty X^m$. Then, we obtain the analogous properties for X^∞.

Lemma 9. X^∞ is a separable, complete metric space with a countable dense subset X_0^∞ such that Assumption 1 and 2 are satisfied with replacement of X by X^∞ and X_0 by X_0^∞ in the statements.

Similarly, let

$$\mathcal{B}^\infty = \left\{ \beta : F(X_0^\infty) \times \mathbb{Q}_+ \to [0,1] \;\middle|\; \beta \text{ is nonderecasing } \right\},$$

and we obtain the following analogous properties for $\mathcal{B}^\infty$.

Lemma 10. $\mathcal{B}^\infty$ is a compact metric space with a countable dense subset $\mathcal{B}_0^\infty$ such that Assumption 1 and 2 are satisfied with replacement of X by $\mathcal{B}^\infty$ and X_0 by $\mathcal{B}_0^\infty$ in the statements.

5.2 Optimism-Elicitation Game: The Multi-Player Case

Now we show that the necessity of the CCO order for monotone comparative statics.

Theorem 2. There is a supermodular game with the property that, for any player $i \in I$ and two types t_i, t_i', we have that $t_i' \succeq_{CCO} t_i$ if and only if $\underline{\sigma}_i(t_i') \succeq_{A_i} \underline{\sigma}_i(t_i)$, where $\underline{\sigma}$ is the least equilibrium of this supermodular game.

Proof. We construct an optimism-elicitation game such that: (i) each player i chooses an action from $\mathcal{B}_i = \mathcal{B}^\infty$ and (ii) given any realization $x \in X^\infty$ and action $\beta \in \mathcal{B}_i$, each player i's payoff is given:

$$u_i(\beta, x) = \sum_{(\gamma,q) \in F(X_0^\infty) \times \mathbb{Q}_+} \left[\sum_{m=1}^\infty \delta^{m-1} \left\{ \beta(\gamma, q) I^m_{\{clup(B_q(\gamma))\}}(x) - \frac{\beta(\gamma, q)^2}{2} \right\} \right] \mu(\gamma, q),$$

where

- $0 < \delta < 1$;
- $B_q(\gamma) = \bigcup_{y \in \gamma} B_q(y)$;
- μ is a full-support distribution over a countable set $F(X_0^\infty) \times \mathbb{Q}_+$[30]; and
- The indicator function is defined as:

$$I^m_{\{clup(B_q(\gamma))\}}(x) = \begin{cases} 1 & \text{if } x^m \in clup(B_q(\gamma)) \cap X^m \\ 0 & \text{otherwise,} \end{cases}$$

  where x^m denotes the truncation of x to X^m.

We can establish the following result by mimicking the argument for the case of single-person optimism-elicitation game. So, we only state the result.

Lemma 11. We obtain the following results:

1. $\mathcal{B}_i$ is a complete lattice;
2. $u(\cdot)$ is supermodular on $\mathcal{B}_i$; and
3. $u(\cdot)$ has increasing difference in (β, x).

Therefore, the game constructed above is indeed a supermodular game. The proposition below shows that player i reveals his probability assessment for each upper event (those generated by $F(X_0^\infty \times \mathbb{Q}_+)$) truthfully in this game, as his unique ICR action.

Proposition 5. For each player i with type t_i, we have $A_i^\infty(t_i) = \{\beta^*\}$, where for each $m \in \mathbb{N}$ and each $(\gamma, q) \in F(X_0^m) \times \mathbb{Q}_+$, we have

$$\beta^*(\gamma, q) = h^m(t_i)[clup(B_q(\gamma))],$$

where $h^m(t_i)$ is t_i's belief on X^m.

Proof. We relegate the proof to the Appendix. □

[30] We can set $h : F(X_0^\infty) \times \mathbb{Q}_+ \to \mathbb{N}$ as an injection mapping because $F(X_0^\infty) \times \mathbb{Q}_+$ is countable. Specifically, we define the full-support distribution μ by $\mu(\gamma, q) = (1/2)^{h(\gamma,q)} > 0$.

This means that "any" interim correlated rationalizable strategy of each player i induces his true belief about any upper event $U(X^\infty)$. We now examine monotone comparative statics for this supermodular game. The first result shows the sufficiency for monotone comparative statics in this supermodular game (as should be expected).

Proposition 6. For each i and t_i, t_i' such that $t_i' \succeq_{CCO} t_i$, we have $\beta' \succeq_{\mathcal{B}_i} \beta$ where β and β' satisfy $A_i^\infty(t_i) = \{\beta\}$ and $A_i^\infty(t_i') = \{\beta'\}$, respectively.

Proof. While this is a corollary to Theorem 1, the proof is also straightforward once we notice that, by Proposition 5, for each $m \in \mathbb{N}$ and $(\gamma, q) \in F(X_0^\infty) \times \mathbb{Q}_+$, we obtain

$$\beta(\gamma, q) = h^m(t_i)[clup(B_q(\gamma))] \text{ and } \beta'(\gamma, q) = h^m(t_i')[clup(B_q(\gamma))].$$

□

Next, we show the desired necessity for monotone comparative statics in this supermodular game.

Proposition 7. For each i and t_i, t_i' such that $A_i^\infty(t_i) = \{\beta\}$ and $A_i^\infty(t_i') = \{\beta'\}$, if $\beta \succeq_{\mathcal{B}_i} \beta'$, then $t_i' \succeq_{CCO} t_i$.

Proof. We take the contrapositive of the statement: if there is some $m \in \mathbb{N}$ such that $h^m(t_i')$ does "not" stochastically dominate $h^m(t_i)$ (so that $h^m(t_i)[Y] > h^m(t_i')[Y]$ for some closed upper set $Y \subseteq X^m$), then β' "cannot" be higher than β in the sense of the partial order on $\mathcal{B}_i = \mathcal{B}^\infty$. This can be shown quite analogously as in Lemma 2, by replacing (i) X by X^m; (ii) X_0 by X_0^m; (iii) b with $h^m(t_i)$; and (iv) b' with $h^m(t_i')$, respectively. This completes the proof. □

Propositions 6 and 7 together complete the proof of Theorem 2. □

6 Relation to Van Zandt and Vives (2007)

In this section, we discuss the relationship with Van Zandt and Vives (henceforth, VZV, 2007). Both VZV and our paper attempt to represent the supermodular games with a (possibly non-common-prior) general type space, and discuss monotone comparative statics with respect to the orders on types. However, there are several differences in these two approaches:

- VZV consider an implicit (Harsanyi) type space endowed with a partial order and then introduce each type's belief map that is consistent with those implicitly given structures. On the other hand, our order on types is based on belief hierarchies constructed from the fundamentals space Θ, and in this sense, our order is based on the given order on Θ (rather than giving an order directly on a type space).

- Both papers order types based on the first-order stochastic dominance relation, but their formal relationship is not clear because they have different constructions of type spaces and their orders.

- VZV and our paper make different assumptions on the primitives: We consider a Polish space Θ so that each belief hierarchy is a Borel probability measure on a Polish space (thanks to the universal-type-space construction of Brandenburger and Dekel (1993)). VZV do not assume that their type space is a topological space.

- These two papers consider somewhat different classes of games. We consider a class of games where only θ is payoff-relevant information, and the players' (first and higher-order) beliefs are not directly payoff-relevant. In VZV, however, the agents' types can be directly payoff-relevant.

Despite those differences, both of the papers provide (their versions of) monotone comparative statics. In this sense, it seems natural to conjecture that one obtains some formal relationship between the two approaches. We formalize this relationship as the following propositions. First, we show that the order on types in VZV can be seen as our CCO order on types.

Proposition 8 (VZV $\Rightarrow$ CCO)**.** Fix an arbitrary supermodular game studied in VZV. Recall that, in VZV, each player's (Harsanyi) type space T_i is endowed with an exogenously given partial order. Fix any player i and any pair of types t_i and $\hat{t}_i$ such that t_i is a higher type than $\hat{t}_i$ in the sense of VZV (i.e., $p_i(t_i) \in \Delta(T_{-i})$ first-order stochastically dominates $p_i(\hat{t}_i) \in \Delta(T_{-i})$).

Define Θ as $\Theta = T$, endowed with the same partial order as that for T. Then, t_i is higher than $\hat{t}_i$ in the sense of CCO.

Proof. By definition, $b_i = (b_i^1, b_i^2, \ldots)$ is given as follows: for each $i \in \mathcal{I}$ and $\tilde{\theta} = (\tilde{t}_i, \tilde{t}_{-i}) \in \Theta$,

$$b_i^1(\tilde{\theta}) = \begin{cases} p_i(t_i)[\tilde{t}_{-i}] & \text{if} \quad \tilde{t}_i = t_i, \\ 0 & \text{otherwise,} \end{cases}$$

and inductively, for each $k \geq 1$ and $\tilde{b}_{-i}^k \in B_{-i}^1$,

$$b_i^{k+1}(\tilde{\theta}, \tilde{b}_{-i}^k) = \begin{cases} p_i(t_i)[\tilde{t}_{-i}] & \text{if} \quad \tilde{t}_i = t_i, \text{ and } \tilde{b}_{-i}^k = p_{-i}(\tilde{t}_{-i}) = (p_j(\tilde{t}_j))_{j \neq i}, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, $\hat{b}_i = (\hat{b}_i^1, \hat{b}_i^2, \ldots)$ is given as follows: for each $i \in \mathcal{I}$ and $\tilde{\theta} = (\tilde{t}_i, \tilde{t}_{-i}) \in \Theta$,

$$\hat{b}_i^1(\tilde{\theta}) = \begin{cases} p_i(\hat{t}_i)[\tilde{t}_{-i}] & \text{if} \quad \tilde{t}_i = \hat{t}_i, \\ 0 & \text{otherwise,} \end{cases}$$

and inductively, for each $k \geq 1$ and $\tilde{b}_{-i}^k \in B_{-i}^1$,

$$\hat{b}_i^{k+1}(\tilde{\theta}, \tilde{b}_{-i}^1) = \begin{cases} p_i(\hat{t}_i)[\tilde{t}_{-i}] & \text{if} \quad \tilde{t}_i = \hat{t}_i, \text{ and } \tilde{b}_{-i}^k = p_{-i}(\tilde{t}_{-i}) = (p_j(\tilde{t}_j))_{j \neq i}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we can prove that $t_i \succeq_{CCO} \hat{t}_i$ inductively. □

In this sense, the CCO order is (weakly) finer than the VZV order, and hence it admits (weakly) more comparative statics. One way to interpret this result is that (modulo some technical differences mentioned above) the class of games considered by VZV is a special case of our games where each player's belief hierarchy is degenerated in the above sense.

The next proposition shows that, conversely, the CCO order on types can be interpreted as the VZV order on types.

Proposition 9 (CCO $\Rightarrow$ VZV)**.** Fix an arbitrary supermodular game studied in our paper. Define $T_0 = \Theta$; $T_i = T_i^*$ for each $i \in \mathcal{I}$, where T_i^* represents the universal type space constructed based on Θ in the sense of Brandenberger and

Dekel (1993); and $T = \prod_{i=0}^{I} T_i$. For each i and his belief hierarchy $b_i \in T_i^*$, let $p_i(b_i) \in \Delta(T_{-i})$ be his belief map for T_{-i} induced by b_i.

Then, if $b_i \in T_i^*$ is a higher type than $\hat{b}_i \in T_i^*$ in the CCO sense, then $b_i \in T_i$ is higher than $\hat{b}_i \in T_i$ in the VZV sense.

Proof. By construction of the universal type space, we have $p_i(b_i) = b_i = (b_i^1, b_i^2, \ldots)$, and therefore, the CCO order and the VZV order coincide. □

Similarly to the previous proposition, one interpretation of this result is that the class of games we consider is a special case of that considered by VZV, where t_i is not directly payoff-relevant.

7 Application

Although our contribution is primarily theoretical, we suggest a situation where our results could be potentially useful. Consider a simple trading game between two parties with pure common values. According to the celebrated *no-trade theorem* of Milgrom and Stokey (1982), when the initial allocation of the goods is Pareto efficient and the parties share a common belief about how the prices of the goods are determined ex post, i.e., the rational expectations hypothesis is satisfied, they never trade ex post. However, in practice, traders may enjoy heterogeneous beliefs about how the prices of the goods are determined so that some traders may be systematically more optimistic than others. More specifically, the rational expectations hypothesis may be violated. Such belief divergence may admit some possibility of trading. Then, a natural question arises as to the relationship between belief divergence and volume of (or possibility of) trading. We introduce a partial order with respect to the size of belief divergence, and show that the corresponding CCO order admits monotone comparative statics.

There are two traders, a seller ($i = 1$) and a buyer ($i = 2$). Let $I = \{1, 2\}$. The seller has an asset whose common value is $v \in \mathbb{R}$. Due to this common value assumption, the initial allocation of the asset is trivially Pareto efficient. Each trader decides to "enter a market" or not. Let $A_i = \{0, 1\}$, where $a_i \in A_i$ is the indicator for i's entrance. If he enters, he has to pay a fixed cost $c > 0$. Unless both traders enter, there is no trade. Thus, a trader's payoff is $-c$ if he enters and there is no trade, and 0 if he does not. After both traders enter, we assume that they trade the asset at price $p \in \mathbb{R}$. Then, the seller's (ex post) payoff is $p - v - c$, and the buyer's (ex post) payoff is $v - p - c$. Hence, their ex post payoffs are given by the following matrix:

	$a_2 = 1$	$a_2 = 0$
$a_1 = 1$	$(p - v - c, v - p - c)$	$(-c, 0)$
$a_1 = 0$	$(0, -c)$	$(0, 0)$

At the timing of the (simultaneous) entry decision, imagine that v and p could be uncertain for the players. Let $\Theta = \mathbb{R}^2$ represent the payoff-state space so that, given $\theta = (\theta_1, \theta_2) \in \Theta$, θ_i denotes i's trading payoff. That is, $\theta_1 = p - v$ and $\theta_2 = v - p$. It is assumed to be common knowledge that $\theta_1 + \theta_2 = 0$ (i.e., the asset has a pure common value), but the players may not agree on the exact value of θ_1 (and hence that of θ_2). Player i's ex post payoff can be written as follows: for any $\theta = (\theta_1, \theta_2) \in \Theta$ and $a \in \{0, 1\}^2$,

$$u_i(a, \theta) = \theta_i a_1 a_2 - c a_i.$$

Observe that the constructed game $g = (I, \prod_{i \in I} A_i, \Theta, (u_i)_{i \in I})$ is a (complete-information) supermodular game.

Let $\mathcal{T} = (T_1, T_2, b_1, b_2)$ denote a Harsanyi type space, where a measurable space T_i denotes player i's type set, and a measurable map $b_i : T_i \to \Delta(\Theta \times T_{-i})$ denotes his belief map. We assume that, for any $i \in I$ and $t_i \in T_i$,

$$b_i(t_i) \left[\{\theta \in \Theta | \theta_1 + \theta_2 = 0\} \times T_{-i} \right] = 1,$$

that is, it is common knowledge that the asset has a pure common value.

First, consider the case where the players share a common prior. That is, there exists $\mu \in \Delta(\Theta \times T_1 \times T_2)$ such that each i's belief map b_i is a system of conditional probabilities induced by μ in the following sense: for all $i \in I$ and measurable events $\tilde{\Theta} \subseteq \Theta$, $\tilde{T}_1 \subseteq T_1$, and $\tilde{T}_2 \subseteq T_2$, we have

$$\mu(\tilde{\Theta} \times \tilde{T}_1 \times \tilde{T}_2) = \int_{\tilde{T}_i} b_i(t_i) \left[\tilde{\Theta} \times \tilde{T}_{-i} \right] d\mu_i(t_i),$$

where $\mu_i \in \Delta(T_i)$ is the marginal of μ on T_i.

Observation 1. The ex ante probability of trading is zero for any Bayesian (Nash) equilibrium $\sigma = (\sigma_i(t_i))_{i \in I,\ t_i \in T_i}$.

Proof. Suppose contrarily that the ex ante probability of trading is strictly positive for some equilibrium σ. In what follows, we only consider the case where σ is a pure strategy equilibrium, although the conclusion holds for mixed-strategy equilibria as well.

Let $\hat{T}_i \subseteq T_i$ denote the set of all types of player i who plays $a_i = 1$. Focus on player $i = 1$. For any $t_1 \in \hat{T}_1$, we have

$$E[\theta_1 \in \Theta_1 | t \in \{t_1\} \times \hat{T}_2] - c \geq 0,$$

which implies $E[\theta_1 | t \in \hat{T}_1 \times \hat{T}_2] > 0$. Similarly, we have $E[\theta_2 | t \in \hat{T}_1 \times \hat{T}_2] > 0$. However, it clearly contradicts the pure-common-value assumption. □

Next, imagine an alternative situation where the players enjoy heterogeneous beliefs. For each player $i \in I$ and types $t_i, t_i' \in T_i$, we write $t_i' \succeq_{CCO} t_i$ if $h^k(t_i') \succeq_{SD} h^k(t_i)$ for any $k \in \mathbb{N}$. That is, t_i' is more optimistic about θ_i than t_i; t_i' is more optimistic that all players are more optimistic about θ_i than t_i', and so on, ad infinitum.[31]

As a corollary of Theorem 1, we establish monotone comparative statics with respect to this partial order. We state the result without its proof.

Corollary 1. If $t_i' \succeq_{CCO} t_i$, then in the least and greatest equilibrium of the game, player i with type t_i' plays a higher action than with t_i.

Recall that it is common knowledge that $\theta_1 + \theta_2 = 0$. Nevertheless, trade can sometimes occur, because the players do not agree on the exact level of $\theta_1 (= -\theta_2)$. The CCO order introduced in this paper captures the connection between the size of the belief divergence and the trading probability.[32]

8 Concluding Remark

In this paper, we introduce an order on types over a universal type space. We consider it as a natural order in the sense that monotone comparative statics is

[31] Recall that any type t_i believes that $\theta_1 + \theta_2 = 0$ throughout the entire belief hierarchies.

[32] We can also reproduce essentially the same result if we make the cost of entry zero but instead assume that the players are strictly risk averse.

valid in a class of supermodular games with incomplete information. We fully characterize this order in terms of common certainty of optimism, that is, type $t_i^{'}$ is higher than type t_i if $t_i^{'}$ is more optimistic about state than t_i; more optimistic that all players are more optimistic about state than t_i; and so on ad infinitum. First, we show that whenever the common certainty of optimism holds, monotone comparative statics holds in all supermodular games. Second, as its converse, we construct an "optimism-elicitation game" as a single supermodular game with the property that whenever the common certainty of optimism fails, monotone comparative statics fails.

Although our CCO order characterizes monotone comparative statics in *any* supermodular game, in some cases, one may be more interested in a *fixed* supermodular game. In such a case, the CCO order continues to be a sufficient condition for monotone comparative statics of that game, but may not be necessary. That is, monotone comparative statics may hold even between types which are not ordered in the CCO sense. To see this, we consider a simple example where there are no strategic interactions. It is clearly a supermodular game. Imagine a pair of types of player i, t_i and t_i', such that the first-order belief of t_i (over Θ) first-order stochastically dominates t_i', while the second-order belief of t_i does "not" first-order stochastically dominates t_i'. Then, t_i and t_i' are not ordered in the CCO sense, but clearly t_i plays a higher equilibrium action than t_i'. More generally, in any supermodular game that is solvable by $R(< \infty)$ rounds of iterative elimination of strictly dominated strategies, only up to R-th order beliefs matter for monotone comparative statics. Hence, the CCO order is "too restrictive."[33]

Establishing a possibly finer order on types that is both necessary and sufficient for monotone comparative statics in a given supermodular game is interesting but challenging. Although we leave it as a future research question, here we briefly

[33]On the other hand, suppose that a researcher who analyzes such a $R(< \infty)$-round dominance solvable game fears a possibility of misspecification or over-simplification of the game, so that the "actual" environment is not R-round dominance solvable, although she is fine to assume that the actual game is supermodular. In such cases, the CCO order would be a "safe" way to introduce an order on types because monotone comparative statics holds uniformly in all (supermodular) games. Similarly, the CCO order would be relevant if one considers a mechanism design situation, where any game can be designed as long as it is supermodular. See Mathevet (2010) for supermodular mechanism design.

explain our conjecture, on which we are currently working.[34] The basic idea is to introduce "indifference" relations on types of each player as follows. Consider the first round of elimination of never best responses. If we have $\underline{a}_i^1(t_i) \succeq_{A_i} \underline{a}_i^1(t_i')$, then we let $t_i \succeq_i^1 t_i'$.[35] This order is richer than the first-order stochastic dominance order: if t_i first-order stochastically dominates t_i' in terms of their first-order beliefs, then we have $t_i \succeq_i^1 t_i'$, but the converse may not be true.

As the next step, if we have $\underline{a}_i^2(t_i) \succeq_{A_i} \underline{a}_i^2(t_i')$, then we let $t_i \succeq_i^2 t_i'$. We conjecture that this order is richer than the first-order stochastic dominance order: if t_i first-order stochastically dominates t_i' in terms of their second-order beliefs, then we have $t_i \succeq_i^2 t_i'$. If this logic goes through up to any level of iterative elimination, then in the limit, we conjecture that this alternative order is (i) richer than the CCO order; (ii) is implied by monotone comparative statics in this game; and (iii) implies monotone comparative statics in this game.

9 Appendix

In this appendix, we provide all the omitted proof from the main body of the paper.

9.1 Proof of Proposition 1

Proposition 1: Let $b, b' \in \Delta(X)$. b (first-order) stochastically dominates b' if and only if, for any $Y_0 \subseteq X_0$, $b(clup(Y_0)) \geq b'(clup(Y_0))$. In addition, b strictly stochastically dominates b' if and only if the inequality holds for any $Y_0 \subseteq X_0$ and it is strict at least for some $Y_0 \subseteq X_0$.

Proof. ($\Leftarrow$) First, suppose that b does not stochastically dominate b'. Then, there exists a closed upper set Y such that $b(Y) < b'(Y)$. We show that, in such a case, there exists $Y_0 \subseteq X_0$ such that $clup(Y_0) = Y$. Then this implies that $b(clup(Y_0)) < b'(clup(Y_0))$. To show this, we establish the following result:

[34] We thank Takashi Ui because this conjecture stems from a discussion with him.

[35] Technically, we may not be able to interpret $\succeq_i^1$ as a partial order because both $t_i \succeq_i^1 t_i'$ and $t_i \preceq_i^1 t_i'$ are possible even if $t_i \neq t_i'$. In that case, we can interpret those types as equivalent (in the sense of $\succeq_i^1$) and can consider a quotient space based on this equivalence class. Then, $\succeq_i^1$ is a partial order on this quotient space.

Lemma 12. For any $Y \subseteq X$, $up(Y) \cap X_0$ is dense in $up(Y)$, i.e., the closure of $up(Y) \cap X_0$ is $up(Y)$. In particular, if Y is itself an upper set, then $Y \cap X_0$ is dense in Y.

Proof. Fix $Y \subseteq X$. The lemma is trivially true if $up(Y)$ is empty. So let us assume not. Let $y \in up(Y)$. Then, by Assumption 1, for any $\varepsilon > 0$, there is $x \in X_0$ such that $x \geq y$ (and hence $x \in up(Y)$) and $x \in B_\varepsilon(y)$. This shows that $up(Y)$ is dense. □

($\Rightarrow$) Next, suppose that b stochastically dominates b'. Fix $Y_0 \subseteq X_0$. If $clup(Y_0)$ is a closed upper set, then we have $b(clup(Y_0)) \geq b'(clup(Y_0))$ by the previous lemma. Since $clup(Y_0)$ is closed by definition, it remains to show in the next lemma that $clup(Y_0)$ is an upper set, which completes the proof. In fact, we can further show that, for any $Y \subseteq X$ (not only for any $Y_0 \subseteq X_0$), $clup(Y)$ is a closed upper set, which turns out to be useful later.

Lemma 13. For any $Y \subseteq X$, $clup(Y)$ is a closed upper set.

Proof. Suppose, on the contrary, that $clup(Y)$ is not an upper set. Then, there exist $x \in clup(Y)$ and $y \geq x$ such that $y \notin clup(Y)$. Since $clup(Y)$ is closed, one can find $\varepsilon > 0$ such that $d(y, clup(Y)) \geq \varepsilon$.

By the previous lemma, $up(Y) \cap X_0$ is dense in $up(Y)$, and hence, $up(Y) \cap X_0$ is dense in $clup(Y)$. Thus, for any $\delta > 0$, there is $z \in up(Y) \cap X_0$ such that $d(x, z) < \delta$. By Assumption 2, we can set $\delta = \delta(x, \varepsilon)$ so that we have $d(up(x), up(z)) < \varepsilon$. This contradicts our hypothesis that $d(y, clup(Y)) \geq \varepsilon$ because we can deduce the following implication:

$$
\begin{aligned}
\varepsilon &\leq d(y, clup(Y)) \\
&= \inf_{y_0 \in clup(Y)} d(y, y_0) \\
&\leq \inf_{y_0 \in up(z)} d(y, y_0) \ \ (\because\ up(z) \subseteq clup(Y)) \\
&= d(y, up(z)) \ \ \text{(due to the definition of the Hausdorff metric)} \\
&\leq \sup_{y' \in up(x)} d(y', up(z)) \ \ (\because\ y \in up(x)) \\
&\leq d(up(x), up(z)) \ \ \text{(due to the definition of the Hausdorff metric)} \\
&< \varepsilon. \ \ \text{(Contradiction!)}
\end{aligned}
$$

This completes the proof of Lemma 13. □

With Lemmas 12 and 13, we thus complete the proof of Proposition 1.

□

9.2 Proof of Lemma 2

Lemma 2. Let $b, b' \in \Delta(\Theta)$. If b' does not first-order stochastically dominate b, then there exists $(\gamma, q) \in F(\Theta_0) \times \mathbb{Q}_+$ such that $\beta^*(\gamma, q|b) > \beta^*(\gamma, q|b')$.

Remark 3. Suppose that there exists some closed upper set $Y \subseteq X$ such that $b(Y) > b'(Y)$. By Proposition 1, there exists some $Y_0 \subseteq X_0$ such that $Y = clup(Y_0)$. If this Y_0 is finite, i.e., $Y_0 \in F(X_0)$, then we trivially have $\beta^*(Y_0, 0|b) > \beta^*(Y_0, 0|b')$. Thus, the subtlety of the proof of Lemma 2 lies in the possibility that Y_0 is (countably) infinite.

Proof. Suppose that there exists some closed upper set $Y \subseteq X$ such that $b(Y) > b'(Y)$. Then, we fix $\varepsilon \in (0, (b(Y) - b'(Y))/2)$. First, by the "inner regularity" property, there exist two compact sets $Z_1, Z_2 \subseteq X$ such that $b(Z_1) \geq 1 - \varepsilon$ and $b'(Z_2) \geq 1 - \varepsilon$. Let $Z = Z_1 \cup Z_2$. This Z is again compact, and we have that $b(Z) \geq 1 - \varepsilon$ and $b'(Z) \geq 1 - \varepsilon$.

Let $\{\eta_j\}_{j=1}^{\infty}$ be a decreasing sequence such that $\eta_j > 0$ for each $j \in \mathbb{N}$ and $\eta_j \to 0$ as $j \to \infty$. For each j, define

$$\delta_j = \frac{\delta(Y \cap Z, \eta_j)}{2},$$

where $\delta(Y \cap Z, \eta_j)$ is given as $\delta(Y, \varepsilon)$ in Assumption 2. By construction, we have that $\delta_j \to 0$ as $j \to \infty$.

Because Y is closed, $Y \cap Z$ is compact. Fix $j \in \mathbb{N}$. Let $\{B_{\delta_j}(x)\}_{x \in Y \cap Z}$ be an open cover of $Y \cap Z$. Since $Y \cap Z$ is compact, we can take a finite subcover $\{B_{\delta_j}(x_n)\}_{n=1}^{N_j}$ such that $x_n \in Y \cap Z$ for each $n = 1, \ldots, N_j$. Since X_0 is dense in X, for each $n = 1, \ldots N_j$, we can take $y_n \in X_0$ so that $y_n \in B_{\delta_j}(x_n)$.

Define $\gamma_j = \{y_1, \ldots, y_{N_j}\} \in F(X_0)$. Then, for each $n = 1, \ldots, N_j$, we have $d(y_n, B_{\delta_j}(x_n)) < 2\delta_j$. This implies that $B_{2\delta_j}(y_n) \supseteq B_{\delta_j}(x_n)$. Therefore,

$$B_{2\delta_j}(\gamma_j) = \bigcup_{n=1}^{N_j} B_{2\delta_j}(y_n) \supseteq \bigcup_{n=1}^{N_j} B_{\delta_j}(x_n) \supseteq Y \cap Z.$$

Define also

$$D_j = \bigcup_{k=j}^{\infty} B_{2\delta_k}(\gamma_k).$$

By construction, we observe that $D_j \supseteq B_{2\delta_j}(\gamma_j)$ for each $j \in \mathbb{N}$, and $D_1 \supseteq D_2 \supseteq \cdots \supseteq Y \cap Z$. Moreover, we have that $d(Y \cap Z, D_j) < 2\delta_j$. Since Assumption 2 guarantees that the upper set correspondence is continuous with respect to the Hausdorff metric, we obtain

$$d(clup(Y \cap Z), clup(D_j)) = d(up(Y \cap Z), up(D_j)) < \eta_j.$$

Fix $x \notin clup(Y \cap Z)$ arbitrarily. Then, we have $d(x, clup(Y \cap Z)) > 0$ because $clup(Y \cap Z)$ is closed. Let $j(x) \in \mathbb{N}$ be defined in such a way that $d(x, clup(Y \cap Z)) \geq \eta_{j(x)}$. Then we have that $x \notin clup(D_j)$ for any $j \geq j(x)$, implying that $x \notin \bigcap_{j=1}^{\infty} clup(D_j)$. Therefore, we have $\bigcap_{j=1}^{\infty} clup(D_j) \subseteq clup(Y \cap Z)$. However, because $clup(D_j) \supseteq clup(Y \cap Z)$ for any $j \in \mathbb{N}$, we obtain $\bigcap_{j=1}^{\infty} clup(D_j) = clup(Y \cap Z)$. Thus, we have $\lim_{j\to\infty} b(clup(D_j)) = b(clup(Y \cap Z))$.

Now, recall that, for each $j \in \mathbb{N}$,

$$clup(Y \cap Z) \subseteq clup(B_{2\delta_j}(\gamma_j)) \subseteq clup(D_j),$$

and thus,

$$\begin{aligned} b(clup(B_{2\delta_j}(\gamma_j))) &\in [b(clup(Y \cap Z)), b(clup(D_j))], \\ b'(clup(B_{2\delta_j}(\gamma_j))) &\in [b'(clup(Y \cap Z)), b'(clup(D_j))]. \end{aligned}$$

Regarding b', first observe that

$$\lim_{j\to\infty} b'(clup(B_{2\delta_j}(\gamma_j))) = b'(clup(Y \cap Z)).$$

Thus, by our hypothesis, there must exist $J \in \mathbb{N}$ such that $b'(clup(B_{2\delta_J}(\gamma_J))) \leq b'(clup(Y \cap Z)) + \varepsilon$. Define $\gamma = \gamma_J \in F(X_0)$ and $q \in \mathbb{Q}_+$ such that $q \in (0, 2\delta_J]$. Then, we deduce the following implication:

$$\begin{aligned} \beta^*(\gamma, q|b') &= b'(clup(B_q(\gamma))) \text{ (by the optimality of } \beta^* \text{ given } b') \\ &\leq b'(clup(Y \cap Z)) + \varepsilon \text{ (by our hypothesized inequality)} \\ &\leq b'(clup(Y)) + \varepsilon \ (\because\ Y \cap Z \subseteq Y) \\ &= b'(Y) + \varepsilon \ (\because\ Y \text{ is a closed upper set}). \end{aligned}$$

Regarding b, we have

$$\begin{aligned}\beta^*(\gamma, q|b) &= b(clup(B_q(\gamma))) \text{ (by the optimality of } \beta^* \text{ given } b)\\ &\geq b(clup(Y \cap Z)) \ (\because \ clup(Y \cap Z) \subseteq clup(B_q(\gamma)))\\ &\geq b(Y) - \varepsilon,\end{aligned}$$

where the last inequality is obtained because

$$\begin{aligned}b(Y) &= b(Y \cap Z) + b(Y \setminus Z) \ (\because \ b \text{ is a probability measure})\\ &\leq b(clup(Y \cap Z)) + \varepsilon\\ &\quad (\because \ Y \cap Z \subseteq clup(Y \cap Z) \text{ and } b(Z) \geq 1 - \varepsilon \Rightarrow b(Y/Z) \leq \varepsilon).\end{aligned}$$

Because $0 < \varepsilon < (b(Y) - b'(Y))/2$, we conclude that $\beta^*(\gamma, q|b) > \beta^*(\gamma, q|b')$. □

9.3 Proof of Lemma 3

Lemma 3: $\mathcal{B}$ is a compact metric space.

Proof. Since $\overline{\mathcal{B}}$ is made isomorphic to Hilbert cube, we confirm that $\overline{\mathcal{B}}$ is a compact metric space. Thus, it suffices to show that $\mathcal{B}$ is a closed subset of $\overline{\mathcal{B}}$. Therefore, our task here reduces to showing that $\overline{\mathcal{B}} \backslash \mathcal{B}$ is open. Fix $\beta \in \overline{\mathcal{B}} \backslash \mathcal{B}$ arbitrarily. Then, we know that there exist $(\gamma', q'), (\gamma'', q'') \in F(X_0) \times \mathbb{Q}_+$ such that $clup(B_{q'}(\gamma')) \subseteq clup(B_{q''}(\gamma''))$ and $\beta(\gamma', q') > \beta(\gamma'', q'')$. What we want to show is that there exists an open ball containing β that does not intersect with $\mathcal{B}$.

Define

$$\varepsilon = (\beta(\gamma', q') - \beta(\gamma'', q'')) \min\{\mu(\gamma', q'), \mu(\gamma'', q'')\}.$$

By our hypothesis, we have $\varepsilon > 0$. It then suffices to show that an open ball $B_\varepsilon(\beta) = \{\beta' \in \overline{\mathcal{B}} | \|\beta - \beta'\| < \varepsilon\}$ does not intersect with $\mathcal{B}$. Suppose, on the contrary, that there is $\beta' \in B_\varepsilon(\beta) \cap \mathcal{B}$. Then,

$$\begin{aligned}
\|\beta - \beta'\| &= \sum_{(\gamma,q)\in F(X_0)\times\mathbb{Q}_+} |\beta(\gamma,q) - \beta'(\gamma,q)|\mu(\gamma,q) \\
&\geq |\beta(\gamma',q') - \beta'(\gamma',q')|\mu(\gamma',q') + |\beta(\gamma'',q'') - \beta'(\gamma'',q'')|\mu(\gamma'',q'') \\
&= |\beta(\gamma',q') - \beta'(\gamma',q')|\mu(\gamma',q') + |\beta'(\gamma'',q'') - \beta(\gamma'',q'')|\mu(\gamma'',q'') \\
&\geq \Big\{|\beta(\gamma',q') - \beta'(\gamma',q')| + |\beta'(\gamma'',q'') - \beta(\gamma'',q'')|\Big\} \min\{\mu(\gamma',q'),\mu(\gamma'',q'')\} \\
&\geq |\beta(\gamma',q') - \beta'(\gamma',q') + \beta'(\gamma'',q'') - \beta(\gamma'',q'')| \min\{\mu(\gamma',q'),\mu(\gamma'',q'')\} \\
&\geq (\beta(\gamma',q') - \beta(\gamma'',q''))\min\{\mu(\gamma',q'),\mu(\gamma'',q'')\} \\
&\quad (\because\ \beta' \in \mathcal{B} \text{ and } clup(B_{q'}(\gamma')) \subseteq clup(B_{q''}(\gamma'')) \Rightarrow \beta'(\gamma',q') \leq \beta'(\gamma'',q'')) \\
&= \varepsilon,
\end{aligned}$$

which contradicts that $\|\beta - \beta'\| < \varepsilon$. □

9.4 Proof of Lemma 4

Lemma 4. For any $\beta \in \mathcal{B}$ and $\varepsilon > 0$, there exists $\beta_0 \in \mathcal{B}_0$ such that $\|\beta_0 - \beta\| < \varepsilon$ and $\beta_0 \geq \beta$.

Proof. Fix $\beta \in \mathcal{B}$ and $\varepsilon > 0$. For each $N \in \mathbb{N}$, let $\Gamma_N = \bigcup_{h(\gamma,q)\leq N} \gamma$ (recall that $h : F(X_0) \times \mathbb{Q}_+ \to \mathbb{N}$ is an injection). Because each γ is a finite subset of X_0, so is Γ_N. Hence, we denote Γ_N by $\{x_1, \ldots, x_{|\Gamma_N|}\}$.

We first construct $\beta_0^N \in \mathcal{B}_0$ as an approximation of $\beta \in \mathcal{B}$ such that β_0^N approaches β as $N \to \infty$.[36] For each $\tilde{X} \subseteq \Gamma_N$ and $q \in Q_N$, we set $n \in \mathbb{N}$ with the following three properties: (i) $\beta(\tilde{X}, q) \in ((n-1)/N, n/N]$; (ii) $\beta_0^N(\tilde{X}, q) = n/N$; and (iii) for each $(\gamma, q) \in F(X_0) \times \mathbb{Q}_+$,

$$\begin{aligned}
\beta_0^N(\gamma,q) &= \inf_{q'\in Q_N} \ \beta_0^N(\Gamma_N \cap \gamma, q') \\
&\quad \text{subject to} \quad B_{q'}(\Gamma_N \cap \gamma) \supseteq B_q(\gamma).
\end{aligned}$$

Second, we have that $\beta_0^N \in \mathcal{B}_0^1$ because $\beta_0^N(\gamma, q) \in Q_N$ and β_0^N is monotonic. Third, we claim that $\beta_0^N \geq \beta$. For any $(\gamma, q) \in F(X_0) \times \mathbb{Q}_+$, we have

$$\begin{aligned}
\beta_0^N(\gamma,q) &\geq \inf_{q'\in Q_N} \ \beta(\Gamma_N \cap \gamma, q') \\
&\quad \text{subject to} \quad B_{q'}(\Gamma_N \cap \gamma) \supseteq B_q(\gamma),
\end{aligned}$$

[36]To be precise, $\beta_0^N \in \mathcal{B}^{1,M}$, where $M = N \times |\Gamma_N|$.

while, by monotonicity of β, we have

$$\beta(\gamma, q) \leq \beta(\Gamma_N \cap \gamma, q'),$$

for any $q' \in Q_N$ satisfying $B_{q'}(\Gamma_N \cap \gamma) \supseteq B_q(\gamma)$. The above two inequalities together imply $\beta_0^N(\gamma, q) \geq \beta(\gamma, q)$.

Finally, we show that there exists $N \in \mathbb{N}$ such that $\|\beta - \beta_0^N\| < \varepsilon$. For each $(\gamma, q) \in F(X_0) \times \mathbb{Q}_+$, whenever $h(\gamma, q) \leq |\Gamma_N|$, we have $\gamma \subseteq \Gamma_N$, and hence, $0 \leq (\beta_0^N(\gamma, q) - \beta(\gamma, q)) \leq 1/N$. Thus,

$$\begin{aligned} \|\beta - \beta_0^N\| &\leq \frac{1}{N} + \sum_{n=N+1}^{\infty} \mu(h^{-1}(n)) \\ &= \frac{1}{N} + \frac{1}{2^N} \ \ (\because\ \mu(\gamma, q) = (1/2)^{h(\gamma,q)}). \end{aligned}$$

By taking N large enough so that $N > \max\{2/\varepsilon, 1 + \log_2(1/\varepsilon)\}$, we obtain $\|\beta - \beta_0^N\| < \varepsilon$. This completes the proof. □

9.5 Proof of Lemma 5

Lemma 5: For each $C \subseteq \mathcal{B}$ and $\varepsilon > 0$, there exists $\delta(C, \varepsilon) > 0$ such that, for any $D \subseteq \mathcal{B}$ with $d(C, D) < \delta(C, \varepsilon)$, we have $d(up(C), up(D)) < \varepsilon$.

Proof. Fix $C \subseteq \mathcal{B}$, $\varepsilon > 0$, and $D \subseteq \mathcal{B}$ with $d(C, D) < \varepsilon$. We show that $d(up(C), up(D)) < \varepsilon$ (i.e., we show that $\delta(C, \varepsilon) = \varepsilon$ works for any C).

Take any $\beta \in C$ and $\beta' \geq \beta$. Because $d(C, D) < \varepsilon$, there exists $\beta'' \in D$ such that $d(\beta, \beta'') < \varepsilon$.

Let $\beta^* = \beta' \vee \beta'' \in up(D)$. Then we have

$$\begin{aligned} d(\beta^*, \beta') &= \|\beta^* - \beta'\| \\ &= \sum_{(\gamma,q)} (\beta^*(\gamma,q) - \beta'(\gamma,q))\mu(\gamma,q) \\ &= \sum_{(\gamma,q)|\beta'(\gamma,q)<\beta''(\gamma,q)} (\beta''(\gamma,q) - \beta'(\gamma,q))\mu(\gamma,q) \\ &\leq \sum_{(\gamma,q)|\beta'(\gamma,q)<\beta''(\gamma,q)} (\beta''(\gamma,q) - \beta(\gamma,q))\mu(\gamma,q) \\ &\leq \sum_{(\gamma,q)} |\beta''(\gamma,q) - \beta(\gamma,q)|\mu(\gamma,q) \\ &= d(\beta, \beta'') \\ &< \varepsilon. \end{aligned}$$

By a symmetric argument, taking any $\beta \in D$ and $\beta' \geq \beta$, there exists $\beta'' \in C$ such that $d(\beta, \beta'') < \varepsilon$, and we have $d(\beta' \vee \beta'', \beta') < \varepsilon$.

Therefore, we conclude that $d(up(C), up(D)) < \varepsilon$. □

9.6 Proof of Proposition 5

Proposition 5. For each player i with type t_i, we have $A_i^\infty(t_i) = \{\beta^*\}$, where for each $m \in \mathbb{N}$ and each $(\gamma, q) \in F(X_0^m) \times \mathbb{Q}_+$, we have

$$\beta^*(\gamma, q) = h^m(t_i)[clup(B_q(\gamma))],$$

where $h^m(t_i)$ is t_i's belief on X^m.

Proof. Fix $m = 1$. Then, player i effectively plays a single-person game in which he reveals his first-order belief only. Suppose by way of contradiction that there is $\beta \in \mathcal{B}_i$ such that $\beta(\hat{\gamma}, \hat{q}) \neq h^1(t_i)[clup(B_{\hat{q}}(\hat{\gamma}))]$ for some $(\hat{\gamma}, \hat{q}) \in F(X_0^1) \times \mathbb{Q}_+$. Then, β is strictly dominated by another $\beta' \in \mathcal{B}_i$, where $\beta'(\gamma, q) = \beta(\gamma, q)$ for any $(\gamma, q) \neq (\hat{\gamma}, \hat{q})$ and $\beta'(\hat{\gamma}, \hat{q}) = h^1(t_i)[clup(B_{\hat{q}}(\hat{\gamma}))]$. Note that such β' is feasible because we impose no coherency condition among across different orders of beliefs. Thus, β^* must satisfy the truth-telling condition:

$$\beta^*(\gamma, q) = h^1(t_i)[clup(B_q(\gamma))].$$

The rest of the proof is by induction. Fix $m \geq 2$, and assume that, up to $(m-1)$th order, each type of each agent behaves truthfully. Assume by way of contradiction that there is an action $\beta \in \mathcal{B}_i$ such that $\beta(\hat{\gamma}, \hat{q}) \neq h^1(t_i)[clup(B_{\hat{q}}(\hat{\gamma}))]$ for some $(\hat{\gamma}, \hat{q}) \in F(X_0^m) \times \mathbb{Q}_+$. Then, β is strictly dominated by another $\beta' \in \mathcal{B}_i$, where $\beta'(\gamma, q) = \beta(\gamma, q)$ for any $(\gamma, q) \neq (\hat{\gamma}, \hat{q})$ and $\beta'(\hat{\gamma}, \hat{q}) = h^1(t_i)[clup(B_{\hat{q}}(\hat{\gamma}))]$. Again, such β' is feasible because we impose no coherency condition among across different orders of beliefs. Therefore, for any m and $(\gamma, q) \in F(X_0^m) \times \mathbb{Q}_+$ we have

$$\beta^*(\gamma, q) = h^m(t_i)[clup(B_q(\gamma))].$$

□

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