Sequential Auctions with Decreasing Reserve Prices

Massimiliano Landi*
School of Economics
Singapore Management University

Domenico Menicucci†
Department of Economics and Management
University of Florence, Italy

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Abstract

We study sequential sealed bid auctions with decreasing reserve prices when there are two identical objects for sale and more than two unit-demand bidders. In the literature, an equilibrium with a strictly increasing bidding function in the stage one auction is found only when reserve prices are (weakly) increasing. Under decreasing reserve prices, bidders may have an incentive not to bid in the first auction and an equilibrium with a strictly increasing bidding function at stage one does not exist. However, we find that a symmetric pure strategy equilibrium always exists, and its shape depends on the distance between the two reserve prices. Moreover, the equilibrium exhibits some pooling at the stage one auction, which disappears in the limit as the number of bidders tends to infinity. We also show that revenue equivalence between first and second price sequential auctions holds under decreasing reserve prices. Finally, our results allow to shed some light on an optimal order problem (increasing versus decreasing reserve prices, under exogenous reserve prices) for selling the two objects.
Keywords: Sequential Auctions; First Price Auction; Second Price Auction; Revenue Equivalence

1 Introduction

We study a model of sequential sealed bid auctions with decreasing reserve prices. Sequential sealed bid auctions have received scarce attention: Milgrom and Weber (1999) and Weber (1983) provide theoretical analyses of sequential auctions for multiple identical objects under the assumption that there are no reserve prices. Gong et al. (2014) (GTX henceforth) study sequential auctions for identical objects, allowing for different reserve prices at different stages. When only two objects are for sale, they show that a symmetric pure-strategy equilibrium with a strictly increasing bidding function at stage one exists if and only if the reserve prices are

*School of Economics, Singapore Management University, 90 Stamford Road, 178903 Singapore. Email: landim@smu.edu.sg

†**Corresponding address**: Università degli Studi di Firenze, Dipartimento di Scienze per l'Economia e l'Impresa, Via delle Pandette 9, I-50127 Firenze, Italia. Phone: +39-055-2759666, email: domenico.menicucci@unifi.it

(weakly) ascending and they provide an equivalence result between sequential first and second price auctions. When reserve prices are descending, they identify an equilibrium only when the first auction's reserve price is sufficiently larger than the second auction's reserve price (see the equilibrium (a) below in the introduction) and suggest that mixed-strategy equilibria may exist in other cases, but they do not identify them.

Our paper essentially complements their analysis. We study sequential (first price and second price) auctions with two identical objects and more than two unit-demand bidders. We show that: a symmetric pure-strategy equilibrium exists for any pair of decreasing reserve prices; in the limit, as the number of bidders grows to infinity, the equilibrium bidding function at stage one converges to a strictly increasing function; sequential first price auctions are revenue equivalent to sequential second price auctions even under decreasing reserve prices. Last, we use our results to tackle a problem about the optimal order of sale for two objects.

In order to understand better our results, it is useful to recall a few properties of the equilibrium described by GTX for the case of ascending reserve prices. To fix the ideas, let us focus on sequential second price auctions, for which the equilibrium bidding at stage two is straightforward. Let r_1, r_2 denote the reserve prices for the first and second auction, respectively. Then, given $r_1 \leq r_2$, a bidder with value x participates in the first auction if and only if $x \geq r_1$, and bids as in a single-unit auction if $x \leq r_2$ because it is not profitable for him to participate in the second auction. Conversely, a bidder with value $x > r_2$ bids less aggressively than in a single-unit auction because he will have another opportunity to win, at stage two, if he loses at stage one.

With descending reserve prices, the incentive to shade bids at stage one is magnified because, all else being equal, $r_2 < r_1$ makes the stage two auction more profitable for a bidder that can participate both stages. In particular, a bidder may choose not to compete at stage one even though his value is greater than r_1, because winning the object at price r_1 may be less profitable than competing in the second auction.[1] This force must be taken into account when constructing an equilibrium, and we obtain that the shape of the equilibrium depends on the distance between r_1 and r_2. Specifically, in the pure strategy symmetric equilibrium of the second price sequential auction we have that:

(a) When r_1 is sufficiently larger than r_2, no bidder participates the first stage auction.

(b) When r_1 takes on intermediate values, there is a threshold $\gamma > r_1$ such that bidders with value smaller than γ do not bid in the first auction, and bidders with value at least γ bid r_1.

[1] A similar phenomenon occurs in McAfee and Vincent (1997), in which the seller of a single object auctions the object multiple times, until it is sold, and at each stage he chooses a reserve price given his beliefs on the bidders values, as determined by the information that the object has not been sold in the past. At each given stage, a bidder with value larger than the current reserve price may choose not to bid, and wait for a successive auction with a lower reserve price.

(c) If r_1 is close to r_2, then the equilibrium is characterized by two thresholds, γ (whose value is different from the previous case) and $\lambda > \gamma$, such that bidders with value less than γ do not bid in the first auction, bidders with value between γ and λ bid r_1, and bidders with value bigger than λ adopt a strictly increasing bidding function. This equilibrium arises because when r_1 becomes small, bidders with high value prefer to bid slightly more than r_1 if all the others active bidders are bidding r_1.

Our result is consistent with GTX's finding that no symmetric equilibrium characterized by a strictly increasing bidding function exists. Interestingly, for each $r_1 > r_2$ the equilibrium in (c) emerges if the number of bidders is sufficiently large. In addition, in such a case γ and λ are both close to r_1. Therefore (almost) each bidder with value above r_1 participates in the stage one auction and bids according to a strictly increasing function. This is the consequence of the fact that, given the large number of opponents, the expected payoff from participating in the second auction is quite small for a bidder with value greater than r_1, and thus it is suboptimal not trying to win at stage one.

Our findings allow us to deal with a problem introduced in GTX about the optimal order in which two objects should be auctioned. The setting considered has two sellers, each of whom owns one of the two objects, and it is commonly known that one seller has value $r > 0$ for her own object while the other seller has value zero for her own object. The objects are offered through sequential auctions such that at each stage the reserve price is equal to the seller's value for the object auctioned at that stage, and an auctioneer chooses the object which is put for sale first in order to maximize the sum of the sellers' profits. GTX use the equilibrium in (a) to show that decreasing reserve prices are optimal if r is large, whereas we use the equilibrium in (c) to show that increasing reserve prices are optimal when r is small. Moreover we obtain more specific results if the values are uniformly distributed.

Finally, we show that sequential first price and second price auctions are revenue equivalent even under decreasing reserve prices. This result is useful since sequential first price auctions are somewhat more complicated to deal with. Unlike in the sequential second price auctions, where at stage two a weakly dominant bid exists for each bidder who lost at stage one, and it depends only on the bidder's value, for sequential first price auctions the stage two equilibrium bid for each bidder who lost at stage one depends on his beliefs about the values of the other active bidders at stage two, which in turn depend on the information he learnt at stage one. The pooling of types at stage one, due to the decreasing reserve prices, makes the computation of those beliefs and the associated bids not straightforward even with just two objects for sale.

The remainder of the paper is organized as follows: Section 2 describes the model in detail. Section 3 provides the analysis for sequential second price auctions and for the optimal order problem. Section 4 proves that sequential first price auctions and sequential second price auctions are revenue equivalent. All the proofs are in the Appendix.

2 The Model

Two identical objects are offered to $n \geq 3$ bidders through two sequential (sealed bid) auctions with reserve prices r_1 and r_2, respectively. Specifically, one object is offered using an auction with reserve price r_1, and the winning bid is publicly announced. In case that the highest bid is submitted by $m \geq 2$ bidders, the winning bidder is selected randomly, with each highest bidder having probability $\frac{1}{m}$ to win. If no bid is submitted at stage one, then this information is revealed to each bidder before stage two, and the object remains unsold. The remaining object is offered using an auction with reserve price r_2.

We assume that each bidder is risk neutral, has no time discount and has unit demand, so that no bidder wants to buy more than one object, and, all else being equal, bidders are indifferent between getting the object at either stage. Moreover unit demand implies that the winner of the first object does not join the second auction. Last we assume that each bidder i has value X_i for the object and each X_i is i.i.d. on the non-negative support $[\underline{x}, \bar{x}]$, with c.d.f. F, and density $f \equiv F' > 0$ that is continuous in the support. We write x_i to denote a realization of X_i, which is privately observed by bidder i.

We are interested in analyzing sequential sealed bid first (F) and second (S) price auctions when the reserve prices are descending, that is when $\underline{x} \leq r_2 < r_1 \leq \bar{x}$.[2] To ease comparisons with the literature, we will also report the case of ascending reserve prices, but notice that except for Subsection 3.1 and Subsection 4.2, we assume $r_1 > r_2$.

A bidding strategy in auction $A = F, S$ for bidder i consists of a pair of functions $(b_{A,i}^{(1)}, b_{A,i}^{(2)})$ which specify bidder i's bids in stage one, $b_{A,i}^{(1)}$, as a function of i's value x_i, and in stage two, $b_{A,i}^{(2)}$ (conditional on i not winning in stage one), as a function of x_i and of any other information that bidder i has obtained in stage one. Since bidders are symmetric ex ante, we restrict the analysis to strategies that do not depend on bidders' identities. Therefore, a strategy will be indicated as a pair $(b_A^{(1)}, b_A^{(2)})$.

We are interested in equilibria that are sequentially rational, in the sense that the strategies in the second period form an equilibrium, given the information the bidders obtained in stage one, for any possible outcome of the stage one auction.

Before we proceed with the analysis, we recall from, e.g., Krishna (2010) a feature of a single stage first price auction with $k \geq 2$ bidders and reserve price r_2, in which each bidder's beliefs about the highest value among the other $k-1$ bidders are given by a c.d.f. G (with density g). In this game, the equilibrium bidding function β satisfies

$$\beta(r_2) = r_2 \qquad \text{and} \qquad \beta'(x) = (x - \beta(x))\frac{g(x)}{G(x)} \quad \text{for } x > r_2 \tag{1}$$

Under the specific assumption that values are i.i.d. random variables each with c.d.f. F and

[2] Notice that the stage two reserve price has no effect if $r_2 < \underline{x}$, just like $r_2 = \underline{x}$. The stage one reserve price prevents each bidder from participating in the stage one auction if $r_1 > \bar{x}$, just like $r_1 = \bar{x}$.

support $[\underline{x}, \overline{x}]$, it follows that $G(x) = F^{k-1}(x)$ and $\frac{g(x)}{G(x)} = \frac{(k-1)f(x)}{F(x)}$. Therefore from (1) we obtain the following equilibrium bidding function, which we use repeatedly in the paper:

$$\beta_{k,r_2}(x) = x - \frac{\int_{r_2}^{x} F^{k-1}(s)ds}{F^{k-1}(x)} = \frac{\int_{\underline{x}}^{x} \max\{r_2, s\}dF^{k-1}(s)}{F^{k-1}(x)} \quad \text{for } x \geq r_2 \tag{2}$$

In this equilibrium, the expected payoff of a bidder with type $x \geq r_2$ is

$$v_k(x) = \int_{r_2}^{x} F^{k-1}(s)ds \tag{3}$$

As it is well known, $v_k(x)$ is also the expected payoff of a bidder with type x in the unique undominated equilibrium in a single stage second price auction with the same information structure described above.[3]

We are now ready to proceed with the analysis of the equilibria of the sequential auctions. We will start from second price auctions, as they are simpler.

3 Second price auctions

We start the analysis by working backwards. We thus look at the equilibrium bid in the second stage auction, for all the active bidders. This is actually straightforward, since in a one-shot second price auction each bidder with value at least r_2 has a (weakly) dominant strategy, which consists in bidding his own value, regardless of the information he obtained at stage one. Hence, we introduce $b_S^{(2)} = b_S^{(2)*}$, with

$$b_S^{(2)*}(x) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, r_2) \\ x & \text{if } x \in [r_2, \bar{x}] \end{cases} \tag{4}$$

Conversely, the equilibrium bidding function at stage one is not as straightforward, and it depends on the relationship between r_1 and r_2.

In order to do our analysis, some additional notation is needed. Given a candidate equilibrium $(b_S^{(1)}, b_S^{(2)*})$, for each x and y in $[\underline{x}, \bar{x}]$ we use $u_S(x, y)$ to denote the payoff of a bidder with value x if he bids $b_S^{(1)}(y)$ in stage one (i.e., if he bids as a bidder with value y is supposed to do according to $b_S^{(1)}$), given that the other bidders follow $(b_S^{(1)}, b_S^{(2)*})$. Moreover, we let: $p(y)$ denote the probability to win at stage one with the bid $b_S^{(1)}(y)$; $t(y)$ denote a bidder's expected payment at stage one, conditional on winning at stage one with the bid $b_S^{(1)}(y)$; and $G_{b_S^{(1)}(y)}$ denote the expected c.d.f., conditional on losing with the bid $b_S^{(1)}(y)$, for the highest value among the other

[3] We use $v_k(x)$ rather than $v_{k,r_2}(x)$ in favour of lighter notation as there is no ambiguity: $v_k(x)$ is always related to the second stage auction whose reserve price is r_2.

bidders who lost at stage one. Then, given $x \geq r_2$, we have that

$$u_S(x,y) = p(y)(x - t(y)) + (1 - p(y)) \int_{r_2}^{x} G_{b_S^{(1)}(y)}(s)ds \tag{5}$$

The first term indicates the expected payoff from winning the first auction times the probability of winning it, while the second term indicates the expected payoff from participating the second auction times the probability of participating it. In particular, note that the payoff of a bidder with value $x \in [r_2, \bar{x}]$ in stage two, conditional on losing in stage one, is $\int_{\underline{x}}^{x}(x - \max\{r_2, s\})dG_{b_S^{(1)}(y)}(s)$, or $\int_{r_2}^{x} G_{b_S^{(1)}(y)}(s)ds$ after applying integration by parts. Notice that when $b_S^{(1)}$ is strictly increasing, it is straightforward to derive $G_{b_S^{(1)}(y)}$. Things are more complicated when $b_S^{(1)}$ is constant over an interval, as we will see below.

3.1 Ascending reserve prices

The case with (weakly) ascending reserve prices is solved by GTX. The following is an adaptation of their result to the two-object case, allowing for $r_1 > \underline{x}$. (GTX assume that $[\underline{x}, \bar{x}] = [0, 1]$ and $r_1 = 0$.)

Proposition 1 (Proposition 2 in GTX). *Suppose that two objects are offered through sequential second price auctions with ascending reserve prices r_1, r_2 such that $\underline{x} \leq r_1 \leq r_2 \leq \bar{x}$. Then there exists an equilibrium in which*

$$b_S^{(1)}(x) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, r_1) \\ x & \text{if } x \in [r_1, r_2) \\ \beta_{n-1,r_2}(x) & \text{if } x \in [r_2, \bar{x}] \end{cases} \tag{6}$$

and $b_S^{(2)}(x) = b_S^{(2)*}(x)$ from (4).

The rationale behind the equilibrium bidding function (6) is as follows. A bidder whose value is between r_1 and r_2 only participates in the first auction; from his perspective he is joining a one shot second price auction. As a result, it is still weakly dominant for him to bid his own value. A bidder with value above r_2 has two attempts at getting the object. At the second (and last) one he will bid his own value. In the first stage, he bids the expected payment he would make if he were to lose the first auction and win the second. This corresponds to the equilibrium bid in a one shot first price auction with reserve price r_2 and $n - 1$ bidders.[4] Krishna (2010) illustrates this result for sequential auctions with no reserve prices, so that the only difference in (6) comes from bidders with values below r_2.

[4] We notice that this result would not hold if bidders were not risk neutral or if there was some time discounting. Risk averse bidders will shade less their bids in the first stage to reduce the risk of losing the object when the winning bid is still below their value. Impatient bidders will also bid more aggressively in the first stage because to them, *de facto*, an object is more valuable today than tomorrow.

3.2 Descending reserve prices

In the equilibrium of Proposition 1, with $r_1 \leq r_2$, each bidder participates in the auction at stage j if and only if his value is grater (or equal) than r_j, for $j = 1, 2$. Conversely, when $r_1 > r_2$ we clearly have that bidders with value smaller than r_2 never bid, since they cannot make a positive payoff in either stage. For this reason, we will consider only $x \geq r_2$. However, when $r_1 > r_2$ not all bidders with value greater than r_1 bid at stage one, since it could be more profitable to try to win the second auction, which has a lower reserve price. As we will see, it is especially true when r_1 is much larger than r_2, but this principle holds in general. For instance, a bidder with type $x = r_1$ does not bid in the first auction because he cannot make a positive payoff in that auction, but has a positive payoff from participating in the stage two auction given that $r_2 < r_1$.

This suggests that the equilibrium analysis is more complicated with descending reserve prices. In fact, no equilibrium with a strictly increasing bidding function at stage one exists (about this, we provide an explanation in next subsection). We show that, nevertheless, a pure strategy equilibrium exists, and its features depend on the (relative) magnitude of r_1. In particular, when r_1 is *large*, we have an equilibrium in which nobody participates the first auction. The value of r_1 is so big that no type prefers to pay r_1 for the object when he can compete against all the other bidders for the remaining one in the second stage auction.

When r_1 takes on *intermediate values*, we find an equilibrium in which bidders with high values bid r_1 in the first auction, and the others do not bid. For intermediate values of r_1 bidders with high values are induced to participate the stage one auction but none of them wants to bid more than r_1.

Finally, when r_1 takes on *small values*, we have an equilibrium with two cutoffs: bidders with small values do not join the first auction, those with intermediate values bid r_1, and the remaining bidders bid according to β_{n-1,r_2}. As r_1 becomes small, a type close to $\overline{x}$ prefers to bid marginally above r_1 (if all the active bidders are expected to bid r_1). This breaks down the equilibrium computed for intermediate values of r_1 and the equilibrium is restored by introducing the second cutoff.

In order to formally state our results at the end of this section, it will be convenient to analyze these cases separately.

3.2.1 Large r_1

When r_1 is sufficiently large, GTX find an equilibrium in which no bidder bids at stage one. Precisely, we define $\bar{r}_1 \equiv \bar{x} - v_n(\bar{x})$ and consider

$$\bar{b}_S^{(1)}(x) = \text{no bid} \qquad \text{for each } x \in [\underline{x}, \bar{x}]$$

If $r_1 \geq \bar{r}_1$, there exists an equilibrium in which each bidder follows $\bar{b}_S^{(1)}$, essentially because bidding at stage one is unprofitable with respect to competing at stage two, when the reserve price is considerably lower. More in detail, given that all other bidders do not bid in stage one, (i) each type $x \geq r_2$ obtains the payoff $v_n(x)$ by not bidding at stage one; (ii) for each type $x \geq r_2$, the payoff from bidding r_1 in stage one is $x - r_1$; (iii) the first alternative is more profitable than the second since the inequality $v_n(x) \geq x - r_1$ holds for each $x \in [r_2, \bar{x}]$, given $r_1 \geq \bar{r}_1$.[5]

3.2.2 Intermediate r_1

When $r_2 < r_1 < \bar{r}_1$, it may be natural to inquire the existence of an equilibrium $(b_S^{(1)}, b_S^{(2)*})$ with a cutoff γ such that all bidders with value below γ do not participate at stage one, and the others follow a strictly increasing bidding function. In other words we are interested in whether there exists a $\gamma \in (\underline{x}, \bar{x})$ such that (i) $b_S^{(1)}(x) =$ no bid for $x \in [\underline{x}, \gamma)$ (i.e., types with value in $[\underline{x}, \gamma)$ do not bid in stage one); (ii) $b_S^{(1)}(\gamma) \geq r_1$; (iii) $b_S^{(1)}$ is strictly increasing in $[\gamma, \bar{x}]$ (i.e., types with value in $[\gamma, \bar{x}]$ bid according to a strictly increasing function, and γ is the smallest type to participate the first stage auction).

However, such an equilibrium fails to exist. Arguing as for ascending reserve prices, if $b_S^{(1)}$ were strictly increasing for $x > \gamma$, then $b_S^{(1)}(x)$ should be type x's expected payment in the stage two auction, conditional on winning at stage two, that is $\beta_{n-1,r_2}(x)$. Therefore we should have $\beta_{n-1,r_2}(\gamma) \geq r_1$, from which we conclude that $\gamma > r_1$. Moreover, type γ must be indifferent between bidding $b_S^{(1)}(\gamma)$ and not bidding. This indifference is equivalent to $\gamma - r_1 = \frac{v_n(\gamma)}{F^{n-1}(\gamma)}$. In fact, given the proposed equilibrium strategy, type γ wins an object if and only if he has the highest or the second highest value. If he has the highest value, his expected payoff is $\gamma - r_1$ if he bids $b_S^{(1)}(\gamma)$ at stage one, is $\int_{r_2}^{\gamma} \left(\frac{F(s)}{F(\gamma)}\right)^{n-1} ds = \frac{v_n(\gamma)}{F^{n-1}(\gamma)}$ if he does not bid at stage one. If type γ has the second highest value, his expected payoff is the same regardless of whether he bids $b_S^{(1)}(\gamma)$ or does not bid at stage one, as in both cases the highest value bidder wins the stage one auction and type γ's beliefs about his opponents' values are the same. Rearranging the indifference condition one gets $r_1 = \beta_{n,r_2}(\gamma)$. As $\gamma > r_2$ we obtain $r_1 = \beta_{n,r_2}(\gamma) > \beta_{n-1,r_2}(\gamma)$ which contradicts $\beta_{n-1,r_2}(\gamma) \geq r_1$ mentioned above.[6]

To overcome this problem, GTX suggest to look for a mixed strategy equilibrium but they do not compute it. Conversely, we find an equilibrium such that the bidding function at stage one is flat in a suitable interval. Specifically, we look at $b_S^{(1)}$ such that $b_S^{(1)}(x) = r_1$ for all x in an interval $[\gamma, \lambda]$ for some $\lambda > \gamma$ to be properly defined. Indeed, for r_1 slightly smaller than $\bar{r}_1$ we identify an equilibrium in which each type in $[\gamma, \bar{x}]$ bids r_1 (here $\lambda = \bar{x}$), that is $b_S^{(1)}$ is horizontal in the whole interval $[\gamma, \bar{x}]$, and type γ is indifferent between not bidding and bidding

[5] Notice that $\bar{r}_1$ can be viewed as the expected payment for type $\bar{x}$ from participating in the second period auction, given that he will face $n-1$ opponents. In case that $r_1 \geq \bar{r}_1$, type $\bar{x}$ prefers to wait for the stage two auction. If $r_1 < \bar{r}_1$, then type $\bar{x}$ prefers to purchase the object at the stage one auction by paying r_1.

[6] When $r_1 = r_2$ the contradiction does not arise as we can set $\gamma = r_1$. See (6).

r_1 at stage one:

$$\hat{b}_S^{(1)}(x) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, \gamma) \\ r_1 & \text{if } x \in [\gamma, \bar{x}] \end{cases} \tag{7}$$

The formal statement of our result is reported in Proposition 2 (ii). In this equilibrium there is substantial pooling in stage one, as each two types in $[\gamma, \bar{x}]$ have the same probability to win in stage one. This plays a key role in complicating the updated beliefs at stage two of each stage one losing bidder.

For each x and y in $[\underline{x}, \bar{x}]$ we use $\hat{u}_S(x, y)$ to denote the payoff of bidder with value x if he bids $\hat{b}_S^{(1)}(y)$ in stage one (i.e., if he bids as a bidder with value y is supposed to do according to $\hat{b}_S^{(1)}$), given that each other bidder follows the strategy $(\hat{b}_S^{(1)}, b_S^{(2)*})$. Hence, $\hat{u}_S(x, \underline{x})$ is the payoff of a bidder with value x from not bidding at stage one; $\hat{u}_S(x, \gamma)$, or $\hat{u}_S(x, \bar{x})$, is his payoff from bidding r_1 at stage one. We now describe how $\hat{u}_S(x, \underline{x})$, $\hat{u}_S(x, \gamma)$ are derived, and how γ is determined. In order to shorten notation, we write $\Gamma = F(\gamma)$.

Remark that in the following the word *beliefs* indicates the updated beliefs at stage two of a losing bidder about the highest value among the other bidders who lost at stage one, and these beliefs will be represented by a c.d.f. G which depends on the stage one winning bid and the bidder's stage one bid.

Regarding $\hat{u}_S(x, \underline{x})$, if a bidder has not bid at stage one and learns that there has been no bid by any bidder, an event with probability Γ^{n-1} from his ex ante point of view, then $\hat{b}_S^{(1)}$ implies that his beliefs are given by the c.d.f. $\hat{G}(\cdot|\text{no,no})$ such that

$$\hat{G}(s|\text{no}, \text{no}) = \begin{cases} \frac{F^{n-1}(s)}{\Gamma^{n-1}} & \text{if } s \in [\underline{x}, \gamma] \\ 1 & \text{if } s \in (\gamma, \bar{x}] \end{cases} \tag{8}$$

On the other hand, if a bidder has not bid at stage one and learns that the winning bid has been r_1, an event with probability $1 - \Gamma^{n-1}$ from his point of view, his beliefs are given by the c.d.f. $\hat{G}(\cdot|\text{no}, r_1)$ such that (the details of the derivation of (9) and (11) below are in the proof of Proposition 2 in the Appendix.)

$$\hat{G}(s|\text{no}, r_1) = \begin{cases} \frac{(n-1)(1-\Gamma)}{1-\Gamma^{n-1}} F^{n-2}(s) & \text{if } s \in [\underline{x}, \gamma] \\ \frac{1-\Gamma}{1-\Gamma^{n-1}} \frac{F^{n-1}(s)-\Gamma^{n-1}}{F(s)-\Gamma} & \text{if } s \in (\gamma, \bar{x}] \end{cases} \tag{9}$$

At the time of choosing to make no bid, the bidder's expected c.d.f. for the highest value among the other losing bidders is $\hat{G}_{\text{no}}$, such that $\hat{G}_{\text{no}}(s) = \Gamma^{n-1}\hat{G}(s|\text{no,no}) + (1 - \Gamma^{n-1})\hat{G}(s|\text{no}, r_1)$. Therefore, in view of (5), the payoff from not bidding at stage one of type $x \geq r_2$ is

$$\hat{u}_S(x, \underline{x}) = \int_{r_2}^{x} \hat{G}_{\text{no}}(s) ds \tag{10}$$

Regarding $\hat{u}_S(x, \gamma)$, let $\hat{p}(\gamma)$ denote the probability to win at stage one for a bidder bidding

r_1, which we derive explicitly in (62) in the Appendix. In case the bidder loses, his beliefs are given by

$$\hat{G}(s|r_1,r_1) = \begin{cases} \frac{(n-1)(1-\Gamma)}{2(1-\hat{p}(\gamma))}F^{n-2}(s) & \text{if } s \in [\underline{x},\gamma] \\ \frac{1-\Gamma}{n}\frac{(n-1)F^n(s)-n\Gamma F^{n-1}(s)+\Gamma^n}{(1-\hat{p}(\gamma))(F(s)-\Gamma)^2} & \text{if } s \in (\gamma,\bar{x}] \end{cases} \tag{11}$$

Hence, the payoff of a type $x \geq r_2$ from bidding r_1 is

$$\hat{u}_S(x,\gamma) = \hat{p}(\gamma)(x-r_1) + (1-\hat{p}(\gamma))\int_{r_2}^{x} \hat{G}(s|r_1,r_1)ds \tag{12}$$

Now that $\hat{u}_S(x,\underline{x})$ and $\hat{u}_S(x,\gamma)$ are defined, we identify γ in $\hat{b}_S^{(1)}$ as the unique solution to the equation

$$\hat{u}_S(\gamma,\underline{x}) = \hat{u}_S(\gamma,\gamma) \tag{13}$$

In order to prove that $(\hat{b}_S^{(1)}, b_S^{(2)*})$ is an equilibrium, in the Appendix (see Section A) we show that: (i) there exists a unique solution to (13) in the interval $(r_2,\bar{x})$; (ii) $\hat{u}_S(x,\underline{x}) > \hat{u}_S(x,\gamma)$ for each $x \in [\underline{x},\gamma)$, and $\hat{u}_S(x,\underline{x}) < \hat{u}_S(x,\gamma)$ for each $x \in (\gamma,\bar{x}]$. Moreover, we also need to take into account that a bidder wins for sure at stage one if he bids more than r_1. This deviation is unprofitable if and only if the inequality $\hat{u}_S(x,\underline{x}) \geq x - r_1$ holds for each $x \in [\underline{x},\gamma)$, and $\hat{u}_S(x,\gamma) \geq x - r_1$ holds for each $x \in [\gamma,\bar{x}]$. It turns out that both inequalities are satisfied when r_1 is not too small, that is if r_1 belongs to the interval $(\tilde{r}_1,\bar{r}_1)$, for a suitable $\tilde{r}_1$ between r_2 and $\bar{r}_1$. Conversely, $\hat{u}_S(\bar{x},\gamma) < \bar{x} - r_1$ holds if r_1 is smaller than $\tilde{r}_1$: for a small r_1, a type $\bar{x}$ prefers to bid more than r_1 in order to secure a win at stage one. This suggests that if r_1 is small, then the equilibrium $b_S^{(1)}$ is strictly increasing for x close to $\bar{x}$. The next subsection is about this case.

3.2.3 Small r_1

Given r_1 between r_2 and $\tilde{r}_1$, we find an equilibrium with the following bidding function at stage one, in which the types γ and λ, with $r_1 < \gamma < \lambda < \bar{x}$, are identified by suitable indifference conditions described by (15) and (16) below:

$$\tilde{b}_S^{(1)}(x) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x},\gamma) \\ r_1 & \text{if } x \in [\gamma,\lambda] \\ \beta_{n-1,r_2}(x) & \text{if } x \in (\lambda,\bar{x}] \end{cases} \tag{14}$$

Hence, $\tilde{b}_S^{(1)}$ prescribes that: (i) bidders with value in $[\underline{x},\gamma)$ do not bid in stage one, but only bid in stage two (provided that $x \geq r_2$); (ii) bidders with value in $[\gamma,\lambda]$ bid r_1 in stage one, that is $\tilde{b}_S^{(1)}$ is constant in the interval $[\gamma,\lambda]$; (iii) bidders with value in $(\lambda,\bar{x}]$ bid their expected payment in the second auction conditional on losing at stage one and winning at stage two, as in (6).

In order to determine γ and λ, for each x and y in $[\underline{x},\bar{x}]$ we use $\tilde{u}_S(x,y)$ to denote the payoff of a bidder with value x if he bids $\tilde{b}_S^{(1)}(y)$ in stage one, given that all the other bidders follow the strategy $(\tilde{b}_S^{(1)}, b_S^{(2)*})$. For instance, $\tilde{u}_S(x, \underline{x})$ is the payoff of type x from not bidding in stage one; $\tilde{u}_S(x, \gamma)$, or $\tilde{u}_S(x, \lambda)$, is his payoff from bidding r_1 in stage one. We evaluate $\tilde{u}_S(x, y)$ in a way similar to that described in the previous subsection for $\hat{u}_S(x, y)$, using the beliefs of each losing bidder at stage two, given the information he obtains at stage one. Matters are slightly more complicated here because the equilibrium bids at stage one are not only *no bid* and r_1, but any bid in the range of $\tilde{b}_S^{(1)}$.

The values of γ and λ in $\tilde{b}_S^{(1)}$ are obtained as the unique solution of the following two equations:

$$\tilde{u}_S(\gamma, \underline{x}) = \tilde{u}_S(\gamma, \gamma) \tag{15}$$

$$\tilde{u}_S(\lambda, \gamma) = \lim_{y \downarrow \lambda} \tilde{u}_S(\lambda, y) \tag{16}$$

Equation (15) states that type γ is indifferent between not bidding, and bidding r_1. Equation (16) states that type λ is indifferent between bidding r_1, and bidding just above $\beta_{n-1,r_2}(\lambda)$ which is greater than r_1. As a result we have that $\tilde{b}_S^{(1)}$ is discontinuous at λ.[7]

We can therefore summarize our results as follows.

Proposition 2. *Suppose that the two objects are offered through sequential second price auctions with descending reserve prices, that is* $\underline{x} \leq r_2 < r_1 \leq \bar{x}$. *Let* $\bar{r}_1 \equiv \bar{x} - v_n(\bar{x})$. *There exists a unique* $\tilde{r}_1 \in (r_2, \bar{r}_1)$ *such that*

(i) if $r_1 \in (r_2, \tilde{r}_1)$, *then there exists an equilibrium in which each bidder follows the strategy* $\tilde{b}_S^{(1)}, b_S^{(2)*}$ *and* γ, λ *satisfy* (15)-(16);

(ii) if $r_1 \in [\tilde{r}_1, \bar{r}_1)$, *then there exists an equilibrium in which each bidder follows the strategy* $\hat{b}_S^{(1)}, b_S^{(2)*}$ *and* γ *satisfies* (13);

(iii) if $r_1 \in [\bar{r}_1, \bar{x}]$, *then there exists an equilibrium in which no bidder bids at stage one, and each bidder bids according to* $b_S^{(2)*}$ *at stage two.*

3.3 Large number of bidders

It is interesting to study the above equilibrium as the number of bidders increases. Remark first that $\lim_{n \to +\infty} \bar{r}_1 = \bar{x}$. The following proposition shows that the same limit result holds for $\tilde{r}_1$. In addition, it shows that the set of types that pool their bid becomes arbitrarily small.

Proposition 3. *As* $n \to +\infty$, *we have that* $\tilde{r}_1$ *tends to* $\bar{x}$, *and both* γ *and* λ *which solve* (15)-(16) *tend to* r_1.

The implications of Proposition 3 are straightforward: for a large n, the equilibrium described by Proposition 2(i) arises unless r_1 is very close to $\bar{x}$. Moreover, almost each type of bidder with

[7] Notice that $\hat{b}_S^{(1)}$ is a special case of $\tilde{b}_S^{(1)}$, obtained when $\lambda = \bar{x}$, and in such a case we find that (15) is equivalent to (13). The values of γ in cases (ii) and (iii) of Proposition 2 are otherwise different.

n	3	5	10	15	20	30	50	75
c_γ	1.57735	1.257289	1.111677	1.071574	1.052689	1.034499	1.020411	1.013514
c_λ	2.1547	1.365022	1.131076	1.079438	1.056924	1.036304	1.021040	1.013790
$\tilde{r}_1$	0.464102	0.732589	0.884114	0.926408	0.946142	0.964968	0.979393	0.986398
$\bar{r}_1$	$0.\bar{6}$	0.80	0.90	0.93	0.95	0.97	0.98	0.99

Table 1: Numerical solutions of the cutoffs values when bidders' values are uniformly distributed in the unit interval, $r_2 = 0$ and $r_1 > 0$.

value above r_1 bids in the stage one auction. The logic for this result is simple: given $r_1 > r_2$, some types of bidder do not bid in stage one because they prefer to compete under the more favourable terms of stage two, but if n is large then the intensity of the competition in both stages is mainly determined by the number of bidders, rather than by reserve prices, and then for a bidder it is unprofitable not to bid at stage one, unless his value is very close to r_1.

3.4 Example with uniformly distributed values

Let us provide a parametric example and compare the equilibrium bids under increasing and decreasing reserve prices. Suppose that values are uniformly distributed on $[0, 1]$ and $r_2 = 0$. For small r_1, using Step 2 in the proof of Proposition 2 (and (50)-(52) in particular) we find that (15)-(16) reduce to

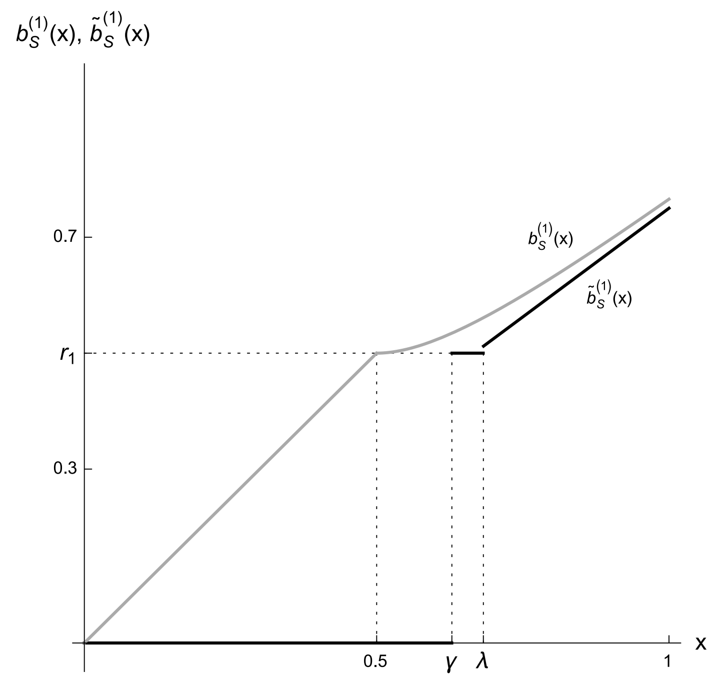
$$\frac{1}{n} \cdot \frac{\lambda^n - \gamma^n}{\lambda - \gamma}(\gamma - r_1) - \frac{1}{2}\gamma^{n-1}\lambda + \frac{n-2}{2n}\gamma^n = 0 \tag{17}$$

$$-\frac{n-2}{2}\gamma^{n-1} - \frac{(n-1)\lambda^n - n\lambda^{n-1}\gamma + \gamma^n}{(\lambda - \gamma)^2}(\lambda - r_1) + \lambda\frac{\lambda^{n-1} - \gamma^{n-1}}{\lambda - \gamma} = 0 \tag{18}$$

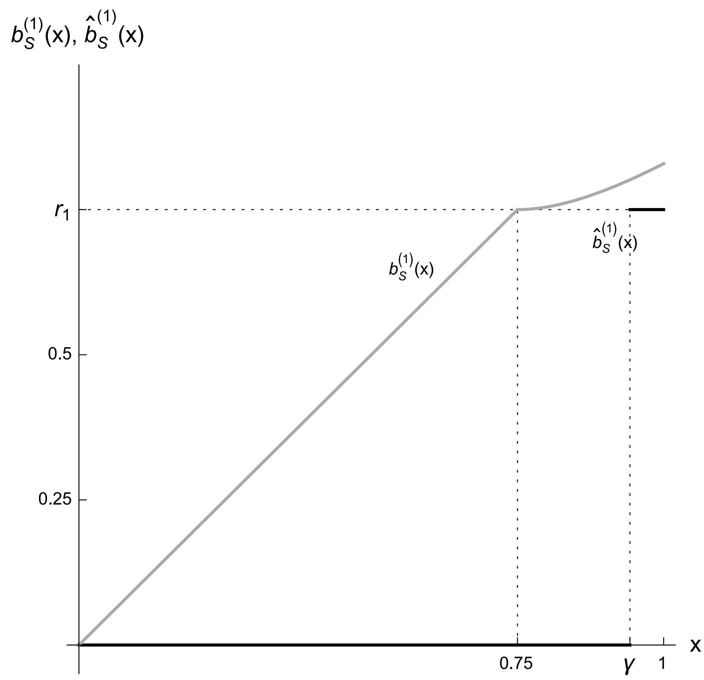
For $n = 3$ the system of equations (17)-(18) can be solved analytically and we obtain the solutions $\gamma = (1 + \frac{\sqrt{3}}{3})r$ and $\lambda = (1 + \frac{2\sqrt{3}}{3})r$. For larger n, analytical solutions are difficult or impossible to obtain. However, inspection of (17)-(18) reveals that, for $r_1 \in (r_2, \tilde{r}_1)$, the solution to (15)-(16) is homogeneous of degree one in r_1, that is $\gamma = c_\gamma r_1$ and $\lambda = c_\lambda r_1$ for suitable coefficients c_γ, and c_λ with $c_\lambda > c_\gamma > 1$. Therefore $\lambda < 1$ if and only if $r_1 < \frac{1}{c_\lambda}$. We can conclude that $\tilde{r}_1 = \frac{1}{c_\lambda}$. Finally, we have that $\bar{r}_1 = \frac{n-1}{n}$.

Table 1 reports the values of c_γ and c_λ obtained numerically for several values of n and the corresponding values of $\tilde{r}_1$ and $\bar{r}_1$. This allows us to visualize the convergence results in Proposition 3.

For intermediate values of r_1, Proposition 2(ii) applies and γ is obtained by solving (17) when $\lambda = 1$. Figure 1 reports the equilibrium bidding functions for the first stage under ascending ($r_1 = 0$ and $r_2 = r$) and descending ($r_1 = r$ and $r_2 = 0$) reserve prices with five bidders, and $r \in \{0.5, 0.75\}$. The equilibrium bidding function for the ascending reserve price auction is given by $b_S^{(1)}$ in (6) and is plotted in grey. When $r_1 = 0.5$, the equilibrium bidding function for the descending reserve price auction given by $\tilde{b}_S^{(1)}$ in (14) and is plotted in black in Figure 1(a)



(a) Equilibrium bidding functions when $r_1 = 0$ and $r_2 = 0.5$ (grey) or when $r_1 = 0.5$ and $r_2 = 0$ (black)



(b) Equilibrium bidding functions when $r_1 = 0$ and $r_2 = 0.75$ (grey) or when $r_1 = 0.75$ and $r_2 = 0$ (black)

Figure 1: Equilibrium bidding function in the first stage of a sequential second price auction when $r \in \{0.5, 0.75\}$. There are five bidders with values independently drawn from a uniform distribution in the unit interval

with $\lambda = \frac{1.257289}{2}$ and $\gamma = \frac{1.365022}{2}$. When $r_1 = 0.75$, the equilibrium bidding function for the descending reserve price auction is given by $\hat{b}_S^{(1)}$ in (7) with $\gamma = 0.942$ and is plotted in black in Figure 1(b).

3.5 The optimal order problem

In this section we use Proposition 2 to relook at the optimal order problem for exogenously given reserve prices analyzed by GTX. Specifically, we assume that values are randomly drawn from the unit interval, and there are two different sellers, each of whom owns one of the two objects that are auctioned. One seller has value zero for her own object, and the other seller has a commonly known value $r \in (0, 1)$ for her own object. GTX assume that the objects are offered through sequential second price auctions[8] such that at each stage the reserve price coincides with the seller's value for the object auctioned at that stage. An auctioneer then chooses the object to put on sale first in order to maximize the sum of each seller's expected profits, $\pi = \pi^0 + \pi^r$, where π^0 is just the expected revenue from the sale of the object that has no reserve price, while π^r is the difference between the expected revenue from the object with reserve price r, and the reserve price times the probability of making the sale.

Hence the auctioneer chooses between $r_1 = 0$, $r_2 = r$ (that is, the object with zero reserve price is auctioned first: reserve prices are increasing), and $r_1 = r$, $r_2 = 0$ (that is, the object with zero reserve price is auctioned second: reserve prices are decreasing). We use IRP, DRP to denote in a succinct way the case of increasing reserve prices and the case of decreasing reserve prices, respectively. Therefore $\pi_{\text{IRP}} = \pi^0_{\text{IRP}} + \pi^r_{\text{IRP}}$ will indicate the sellers' total profits given IRP. Likewise, $\pi_{\text{DRP}} = \pi^0_{\text{DRP}} + \pi^r_{\text{DRP}}$ will indicate the sellers' total profits given DRP.

With respect to this optimal order problem, GTX only study the case of $r > \bar{r}_1$ because they do not identify equilibria for the DRP when $r \leq \bar{r}_1$. Proposition 4(i) below slightly generalizes their results. In addition, we exploit Propositions 1 and 2 to obtain some results about the optimal order when $r < \bar{r}_1$.

Proposition 4. *Suppose $\underline{x} = 0$, $\overline{x} = 1$ and for only one unit we have a positive reserve price $r \in (0, 1)$. The following holds:*

(i) If r is close to 1, then $\pi_{\text{IRP}} < \pi_{\text{DRP}}$. Moreover, if $\pi_{\text{IRP}} \leq \pi_{\text{DRP}}$ holds when $r = \bar{r}_1$, then $\pi_{\text{IRP}} < \pi_{\text{DRP}}$ holds for each $r \in (\bar{r}_1, 1)$.

(ii) If r is close to 0, then $\pi_{\text{IRP}} > \pi_{\text{IRP}}$.

(iii) Suppose, in addition, that values are uniformly distributed. If $n = 3$, then $\pi_{\text{IRP}} > \pi_{\text{DRP}}$ for $r < 0.641$, and $\pi_{\text{IRP}} < \pi_{\text{DRP}}$ for $r > 0.641$. If $n \geq 4$, then $\pi_{\text{IRP}} > \pi_{\text{DRP}}$ for $r \leq \min\{\frac{5n-12}{5n-5}, \tilde{r}_1\}$, and $\pi_{\text{IRP}} < \pi_{\text{DRP}}$ for $r \geq \bar{r}_1$.

[8] Or sequential first price auctions: see the next section.

Proposition 4 allows us to conclude that $\pi_{\text{IRP}} < \pi_{\text{DRP}}$ when r is close to 1, whereas $\pi_{\text{IRP}} > \pi_{\text{DRP}}$ when r is close to 0. Under the assumption of uniform distribution of values, we can obtain more specific results. In particular, as n tends to infinity, we have that $\min\{\frac{5n-12}{5n-5}, \tilde{r}_1\}$ approaches 1. We then conclude that, for large n, IRP provide bigger profits at all r but those very close to 1.

The proof of Proposition 4 is made via standard mechanism design techniques (see, e.g. Krishna, 2010; Myerson, 1981) that allow to write the sum of the sellers' revenues as the expectation of the virtual values $\phi(x) = x - \frac{1-F(x)}{f(x)}$ of the winning bidders. For this reason it is convenient to introduce order statistics: given the n random variables $X_1, .., X_n$ which represent the bidders' values, with Y_h we denote the h-th highest order statistic, for $h = 1, ..., n$; y_h is a generic realized value for Y_h.

In particular, we obtain first that

$$\pi^r_{\text{IRP}} > \pi^r_{\text{DRP}} \qquad \text{for each } r \in (0,1) \tag{19}$$

that is, for each r, the profit from the object with reserve price r is higher under IRP. This result is immediate when $r \geq \tilde{r}_1$, since then $\pi^r_{\text{DRP}} = 0$ (the only positive bid is r when $r \in [\tilde{r}_1, \bar{r}_1)$ while nobody bids when $r \geq \bar{r}_1$) and $\pi^r_{\text{IRP}} = E[\max\{Y_3 - r, 0\}] > 0$. Some more care is needed to show that inequality (19) holds true also for $r < \tilde{r}_1$, and the analysis can be found in the Appendix. In general, inequality (19) holds because DRP discourage bidders' participation at stage one more than IRP do at stage two. Under DRP, a bidder knows that if he loses at stage one he will compete at stage two in a more favorable auction with $r_2 = 0 < r_1 = r$; under IRP, at stage two each bidder has his last opportunity to win an object. This generates higher bidding from a larger set of bidders under IRP.

On the other hand, we find that the comparison between π^0_{IRP} and π^0_{DRP} depends on the value of r: when r is close to 1, $\pi^0_{\text{DRP}} > \pi^0_{\text{IRP}}$ by a magnitude that outweighs (19), and therefore $\pi_{\text{DRP}} > \pi_{\text{IRP}}$ if r is close to 1 (Proposition 4(i)). When, instead, r is close to 0, $\pi^0_{\text{IRP}} > \pi^0_{\text{DRP}}$. This reinforces the effect from (19) and we conclude in Proposition 4(ii) that $\pi_{\text{IRP}} > \pi_{\text{DRP}}$ if r is close to 0.

In order to see why the comparison between π^0_{IRP} and π^0_{DRP} depends on r, notice that if r_1 is close to 1, $\pi^0_{\text{DRP}} = E(Y_2)$ because with DRP each bidder (does not bid at stage one and) bids his own value at stage two. With IRP, π^0_{IRP} would be equal to $E(Y_2)$ if each bidder was bidding his own value at stage one (this occurs if $r = 1$), but Proposition 1 reveals that each bidder with value greater than $r_2 = r$ bids less than his true value. Therefore when r is close to 1 we have that $\pi^0_{\text{IRP}} < \pi^0_{\text{DRP}}$. We then prove that overall $\pi_{\text{IRP}} < \pi_{\text{DRP}}$ by writing the expected revenues as expectation of the winners' virtual values, which yields

$$\pi_{\text{IRP}} - \pi_{\text{DRP}} = E[(\phi(Y_2) - r)\,\mathbf{1}_{\{Y_2 \geq r\}}]$$

where $\mathbf{1}_{\{Y_2 \geq r\}}$ is a function that assumes value 1 if $Y_2 \geq r$ and 0 otherwise. Following GTX's argument, we find that $E[(\phi(Y_2) - r)\,\mathbf{1}_{\{Y_2 \geq r\}}]$ is negative when r is close to 1. Hence $\pi_{\text{IRP}} < \pi_{\text{DRP}}$

because IRP generate a lower profit from the object with zero reserve price, and this effect dominates over the effect described by (19).

On the other hand, observe that $\pi^0_{\text{IRP}} = \pi^0_{\text{DRP}}$ if $r = 0$, and a small $r > 0$ increases π^0_{IRP} of a larger magnitude than it increases π^0_{DRP}. Specifically, we notice that Proposition 2(i) applies when r is close to 0 (and, in particular, λ is small too), and distinguish between $y_2 > \lambda$ (where we find that IRP prevail by a magnitude of order r^{n-1}) and $y_2 \leq \lambda$ (where we find that DRP *might* prevail, but by a magnitude of order r^n). In the first case the profit from the object with zero reserve price is $\beta_{n-1,r}(y_2)$ under IRP, and y_3 under DRP. One can verify that

$$\beta_{n-1,r}(y_2) > \beta_{n-1,0}(y_2) = E(Y_3|Y_2 = y_2) \tag{20}$$

Moreover, the expected profit difference is of order r^{n-1} since

$$\beta_{n-1,r}(y_2) - \beta_{n-1,0}(y_2) = \frac{\int_0^r F^{n-2}(s)ds}{F^{n-2}(y_2)}$$

(see (2)) and

$$E\left[\frac{\int_0^r F^{n-2}(s)ds}{F^{n-2}(Y_2)}\mathbf{1}_{\{Y_2>\lambda\}}\right] = \frac{n(n-1)(1-F(\lambda))^2}{2}\int_0^r F^{n-2}(s)ds$$

which is about equal to $\frac{nf^{n-2}(0)}{2}r^{n-1}$ for r close to 0.

Conversely, when $y_2 \leq \lambda$ in some cases the profit with IRP is smaller than with DRP because under DRP the stage one winner could be the bidder with the third (or fourth ...) highest value, and then the stage two revenue would be y_2, which is greater than $\beta_{n-1,r}(y_2)$. However, in expectation this profit difference is negligible with respect to $\frac{nf^{n-2}(0)}{2}r^{n-1}$ as (i) there exists a number $\xi > 1$ such that $\lambda < \xi r$ if r is close to 0, therefore $\Pr\{Y_2 \leq \lambda\} = F(\lambda)^n + nF(\lambda)^{n-1}(1 - F(\lambda))$ is of order r^{n-1}; (ii) for each profile of values such that $y_2 < \lambda$, the profit given DRP minus the profit given IRP is smaller than $\lambda - r$, which is of order r. From (i) and (ii) it follows that the expected profit difference in favor of DRP from the case $y_2 \leq \lambda$ is of order r^n and thus, given r close to 0, the sign of $\pi^0_{\text{IRP}} - \pi^0_{\text{DRP}}$ is determined by $\frac{nf^{n-2}(0)}{2}r^{n-1}$, which is positive.

4 First price auctions

In this section we prove that if the objects are offered through two sequential first price auctions, with reserve prices r_1, r_2 such that $r_1 > r_2$, then an equivalence result holds in the following sense: there exists an equilibrium for sequential first price auctions which generates the same outcome (in terms of allocation of the objects and of bidders' expected payments) as the equilibrium described by Proposition 2 for sequential second price auctions.[9] Therefore, in particular, the results in Subsections 3.3-3.5 apply also to sequential first price auctions.

[9] In fact, the equivalence result holds also if $r_1 \leq r_2$, as established by GTX.

As for the sequential second price auctions we focus on symmetric strategy profiles that are sequentially rational and we use $(b_F^{(1)}, b_F^{(2)})$ to denote each bidder's bidding functions at stage one and two. We remark that $b_F^{(1)}$ is a function only of the bidder's value x, whereas $b_F^{(2)}$ needs to specify the bid of a stage one losing bidder as a function of his value x, his stage one bid $\mathfrak{b}$, and the stage one winning bid $\mathfrak{b}_w$. We use the notation $b_F^{(2)}(x|\mathfrak{b}, \mathfrak{b}_w)$, and for example $b_F^{(2)}(x|\text{no}, r_1)$ is the bid of type x at stage two given that he has not bid at stage one ($\mathfrak{b} = no$), and given that the winning bid at stage one has been r_1 ($\mathfrak{b}_w = r_1$). With $\mathfrak{b}_w = no$ we represent the case in which no bid has been submitted at stage one.

Unlike in the second price auctions, there is no dominant strategy for the stage two auction. A bidder's equilibrium behavior at stage two must take into account his beliefs about the values of the other losing bidders at stage one, and these beliefs depend on $\mathfrak{b}$ and $\mathfrak{b}_w$. Therefore, the analysis of sequential first price auctions requires extra care, but this is mainly true for the case of decreasing reserve prices, because when $r_1 \leq r_2$ there exists an equilibrium in which the stage one bidding function is strictly increasing (for $x \geq r_1$), and this generates beliefs which are relatively simple to manage (see GTX, or Subsection 4.2).

4.1 Descending reserve prices

Given $r_1 > r_2$, we consider $\tilde{r}_1, \bar{r}_1$ defined in Proposition 2, and for each of the three cases considered in Proposition 2 we identify an equilibrium for sequential first price auctions which is equivalent to the equilibrium described by Proposition 2 for sequential second price auctions.

4.1.1 Intermediate r_1

We start with r_1 in the interval $[\tilde{r}_1, \bar{r}_1)$, for which Proposition 2(ii) identifies $(\hat{b}_S^{(1)}, b_S^{(2)*})$ where $\hat{b}_S^{(1)}$ is from (7) and $b_S^{(2)*}$ is from (4). For sequential first price auctions, we find an equilibrium given by the functions $(\hat{b}_F^{(1)}, \hat{b}_F^{(2)})$ below, in which γ is the unique solution to (13) as in Proposition 2(ii):

$$\hat{b}_F^{(1)}(x) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, \gamma) \\ r_1 & \text{if } x \in [\gamma, \bar{x}] \end{cases} \tag{21}$$

$$\hat{b}_F^{(2)}(x|\text{no}, \text{no}) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, r_2) \\ \beta_{n,r_2}(x) & \text{if } x \in [r_2, \gamma) \\ \beta_{n,r_2}(\gamma) & \text{if } x \in [\gamma, \bar{x}] \end{cases} \tag{22}$$

$$\hat{b}_F^{(2)}(x|\text{no}, r_1) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, r_2) \\ \beta_{n-1,r_2}(x) & \text{if } x \in [r_2, \gamma) \\ \hat{b}_F^{(2)}(\hat{y}(x)|r_1, r_1) \text{ such that } \hat{y}(x) \text{ is in} & \\ \arg\max_{y\in[\gamma,x]}(x - \hat{b}_F^{(2)}(y|r_1, r_1))\hat{G}(y|\text{no}, r_1) & \text{if } x \in [\gamma, \bar{x}] \end{cases} \tag{23}$$

$$\hat{b}_F^{(2)}(x|r_1,r_1) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, r_2) \\ \beta_{n-1,r_2}(x) & \text{if } x \in [r_2, \gamma) \\ \frac{\beta_{n-1,r_2}(\gamma)\hat{G}(\gamma|r_1,r_1)+\int_\gamma^x s\hat{g}(s|r_1,r_1)ds}{\hat{G}(x|r_1,r_1)} & \text{if } x \in [\gamma, \bar{x}] \end{cases} \tag{24}$$

$$\hat{b}_F^{(2)}(x|\mathfrak{b},\mathfrak{b}_w) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, r_2) \\ \beta_{n-1,r_2}(x) & \text{if } x \in [r_2, \bar{x}] \end{cases} \qquad \text{for each } \mathfrak{b}_w > r_1,\ \mathfrak{b}_w \geq \mathfrak{b} \tag{25}$$

The bidding functions in (22)-(25) refer to stage two and cover all the possible stage one outcomes.[10] In order to make sense of them, first notice that $\hat{b}_F^{(1)}$ coincides with $\hat{b}_S^{(1)}$, hence the beliefs at stage two of each losing bidder are the same as in sequential second price auctions, and they are given by (8), (9), and (11).

Consider $\hat{b}_F^{(2)}(\cdot|\text{no,no})$ in (22). Each bidder's beliefs are given by $\hat{G}(\cdot|\text{no,no})$ in (8), and $\frac{\hat{g}(s|\text{no,no})}{\hat{G}(s|\text{no,no})} = (n-1)f(s)/F(s)$ for $s \in (r_2, \gamma)$, therefore (1) reveals that the equilibrium bidding function for $x \in [r_2, \gamma)$ is $\beta_{n,r_2}(x)$, as specified by $\hat{b}_F^{(2)}(\cdot|\text{no,no})$. Moreover, $\hat{b}_F^{(2)}(\cdot|\text{no,no})$ needs to specify a bid also for each type $x \in [\gamma, \bar{x}]$, given $\mathfrak{b}_w = no$.[11] In this case such a bidder expects all the others to have a value in $[\underline{x}, \gamma)$. It is optimal for him to bid $\beta_{n,r_2}(\gamma)$ i.e. the minimum bid that guarantees a sure win at stage two, which is what (22) prescribes for $x \in [\underline{x}, \gamma)$.

The case in which the winning bid at stage one is r_1 is more involved, since a losing bidder's beliefs and bidding at stage two depend on the bidder's bid at stage one, which could be *no*, or r_1. In particular, a type who has not won at stage one expects an opponent of type x to bid $\hat{b}_F^{(2)}(x|\text{no}, r_1)$ if $x \in [r_2, \gamma)$, and to bid $\hat{b}_F^{(2)}(x|r_1, r_1)$ if $x \in [\gamma, \bar{x}]$. In order to see how these functions are determined, assume that the function

$$\hat{b}_{F,r_1}^{(2)}(x) = \begin{cases} \hat{b}_F^{(2)}(x|\text{no}, r_1) & \text{if } x \in [r_2, \gamma) \\ \hat{b}_F^{(2)}(x|r_1, r_1) & \text{if } x \in [\gamma, \bar{x}] \end{cases} \tag{26}$$

is strictly increasing and notice that the beliefs of a losing bidder are: $\hat{G}(\cdot|\text{no}, r_1)$ in (9) if the bidder has not bid at stage one (according to $\hat{b}_F^{(1)}$, these are the types in $[r_2, \gamma)$, neglecting the types in $[\underline{x}, r_2)$); and $\hat{G}(\cdot|r_1, r_1)$ in (11) if the bidder has bid r_1 at stage one (according to $\hat{b}_F^{(1)}$, these are the types in $[\gamma, \bar{x}]$).

Then solving

$$b(r_2) = r_2 \qquad \text{and} \qquad b'(x) = (x - b(x))\frac{\hat{g}(x|\text{no}, r_1)}{\hat{G}(x|\text{no}, r_1)} \quad \text{for } x \in (r_2, \gamma)$$

[10] Of course, in all these bidding functions no type $x \in [\underline{x}, r_2)$ bids at stage two.

[11] This occurs if the bidder did not follow $\hat{b}_F^{(1)}$ at stage one, perhaps because he made a mistake or because he chose to deviate from $\hat{b}_F^{(1)}$.

yields $\hat{b}_F^{(2)}(x|\text{no}, r_1) = \beta_{n-1,r_2}(x)$ for $x \in [r_2, \gamma)$, which is what (23) prescribes. Likewise, solving

$$b(\gamma) = \beta_{n-1,r_2}(\gamma) \qquad \text{and} \qquad b'(x) = (x - b(x))\frac{\hat{g}(x|r_1, r_1)}{\hat{G}(x|r_1, r_1)} \quad \text{for } (\gamma, \bar{x}]$$

yields $\hat{b}_F^{(2)}(x|r_1, r_1) = \frac{\beta_{n-1,r_2}(\gamma)\hat{G}(\gamma|r_1,r_1)+\int_\gamma^x s\hat{g}(s|r_1,r_1)ds}{\hat{G}(x|r_1,r_1)}$ for $x \in [\gamma, \bar{x}]$, which is what (24) prescribes. Hence the resulting function $\hat{b}_{F,r_1}^{(2)}$ in (26) is indeed strictly increasing.

We complete the description of $\hat{b}_F^{(2)}(x|\text{no}, r_1)$ and $\hat{b}_F^{(2)}(x|r_1, r_1)$ by looking at off-the-equilibrium play. Namely, $\hat{b}_F^{(2)}(x|\text{no}, r_1)$ for $x \in [\gamma, \bar{x}]$ is obtained by computing the payoff maximizing bid for a type $x \in [\gamma, \bar{x}]$ who has not bid at stage one, given the beliefs $\hat{G}(\cdot|\text{no}, r_1)$ and given that the opponents bid according to (26). In this case, we find that for such a type it is sub-optimal to bid less than $\hat{b}_F^{(2)}(\gamma|r_1, r_1)$. Likewise, $\hat{b}_F^{(2)}(x|r_1, r_1)$ is the payoff maximizing bid for a type $x \in [r_2, \gamma)$ who has bid r_1 in stage one. We find that $\hat{b}_F^{(2)}(x|r_1, r_1) = \hat{b}_F^{(2)}(x|\text{no}, r_1)$ (equal to $\beta_{n-1,r_2}(x)$) in the interval $[r_2, \gamma)$ because the equality $\frac{\hat{g}(s|\text{no},r_1)}{\hat{G}(s|\text{no},r_1)} = \frac{\hat{g}(s|r_1,r_1)}{\hat{G}(s|r_1,r_1)}$ (equal to $\frac{(n-2)f(s)}{F(s)}$) holds for $s \in [r_2, \gamma)$.

Finally, the equilibrium strategies include also (25), which covers the off-the-equilibrium case in which $\mathfrak{b}_w > r_1$. Then we suppose that the beliefs of each losing bidder are equal to the initial beliefs, and therefore the stage two auction is an ordinary first price auction with $n-1$ bidders and reserve price r_2, for which (25) is the equilibrium bidding function.

Given (22)-(24), we can move to stage one and evaluate the total expected payoff (over the two stages) for each type from not bidding at stage one, and from bidding r_1. This allows to prove that bidding according to (21) is a best reply for a bidder which expects all other bidders to follow (21). For instance, for a type $x \in [r_2, \gamma)$, the payoff from not bidding at stage one is

$$\hat{u}_F(x, \underline{x}) = \Gamma^{n-1}\left(x - \hat{b}_F^{(2)}(x|\text{no}, \text{no})\right)\hat{G}(x|\text{no}, \text{no}) + (1 - \Gamma^{n-1})\left(x - \hat{b}_F^{(2)}(x|\text{no}, r_1)\right)\hat{G}(x|\text{no}, r_1)$$

which is equal to $\hat{u}_S(x, \underline{x}) = v_n(x) + (n-1)(1-\Gamma)v_{n-1}(x)$ as it is obtained from (10). Moreover, the payoff from bidding r_1 is

$$\hat{u}_F(x, \gamma) = \hat{p}(\gamma)(x - r_1) + (1 - \hat{p}(\gamma))(x - \hat{b}_F^{(2)}(x|r_1, r_1))\hat{G}(x|r_1, r_1)$$

in which $\hat{p}(\gamma)$ is the probability to win at stage one after bidding r_1 given by (62). Using (12) we see that $\hat{u}_F(x, \gamma) = \hat{u}_S(x, \gamma) = \hat{p}(\gamma)(x - r_1) + \frac{n-1}{2}(1-\Gamma)v_{n-1}(x)$. We know from Proposition 2(ii) that $\hat{u}_S(x, \underline{x}) \geq \max\{\hat{u}_S(x, \gamma), x - r_1\}$ for each $x \in [r_2, \gamma)$, hence not bidding at stage one is a best reply for each type in $[r_2, \gamma)$.

Finally, remark that the equilibrium (21)-(25) generates the same allocation of the two objects as the equilibrium described by Proposition 2(ii) for sequential second price auctions, and since in both cases each bidder with type $\underline{x}$ has payoff equal to zero, the Revenue Equivalence Theorem implies that each type of bidder and the seller have the same payoff in both cases.

4.1.2 Small r_1, that is $r_1 \in (r_2, \tilde{r}_1)$, and large r_1, that is $r_1 \in [\bar{r}_1, \bar{x}]$

When $r_1 \in (r_2, \tilde{r}_1)$ we find an equilibrium which is equivalent to the equilibrium of Proposition 2(i) for sequential second price auctions. The analysis is similar to that for the case of $r_1 \in [\tilde{r}_1, \bar{r}_1)$, although somewhat more complicated as the equilibrium bidding function at stage one is strictly increasing for x close to $\bar{x}$. Therefore we have reported this part in the Appendix, in the proof of Proposition 5(i).

When $r_1 \in [\bar{r}_1, \bar{x}]$, we find the following equilibrium which is equivalent to the equilibrium of Proposition 2(iii) for sequential second price auctions:

$$\bar{b}_F^{(1)}(x) = \text{no bid for each } x \in [\underline{x}, \bar{x}] \tag{27}$$

$$\bar{b}_F^{(2)}(x|\text{no}, \text{no}) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, r_2) \\ \beta_{n,r_2}(x) & \text{if } x \in [r_2, \bar{x}] \end{cases} \tag{28}$$

$$\bar{b}_F^{(2)}(x|\mathfrak{b}, \mathfrak{b}_w) = \hat{b}_F^{(2)}(x|\mathfrak{b}, \mathfrak{b}_w) \qquad \text{for each } \mathfrak{b}_w > r_1,\ \mathfrak{b}_w \geq \mathfrak{b} \tag{29}$$

Given (27), when $\mathfrak{b}_w = no$ the beliefs of each bidder at stage two coincide with the initial beliefs, and therefore an ordinary first price auction with n bidders and reserve price r_2 is held, for which (28) is the equilibrium bidding function. In case that some bids have been submitted at stage one (an off-the-equilibrium event), we can argue as for $\hat{b}_F^{(2)}$ in (25).

Moving at stage one, we see that for each bidder it is a best reply not to bid if he expects the other bidders to follow (27)-(28): If a type x bids at stage one, his payoff is not larger than $x - r_1$, which is smaller than the payoff $v_n(x)$ he obtains from not bidding at stage one, since $r_1 \geq \bar{r}_1$.

Proposition 5. *Suppose that the two objects are offered through sequential first price auctions, with descending reserve prices, that is* $\underline{x} \leq r_2 < r_1 \leq \bar{x}$. *Let* $\tilde{r}_1, \bar{r}_1$ *be defined as in Proposition 2. Then*

(i) if $r_1 \in (r_2, \tilde{r}_1)$, *then there exists an equilibrium which generates the same outcome as the equilibrium described by Proposition 2*(i) *for sequential second price auctions;*

(ii) if $r_1 \in [\tilde{r}_1, \bar{r}_1)$, *then there exists an equilibrium in which each bidder follows the strategy* $\hat{b}_F^{(1)}, \hat{b}_F^{(2)}$ *in* (21)-(25)*;*

(iii) if $r_1 \in [\bar{r}_1, \bar{x}]$, *then there exists an equilibrium in which each bidder follows the strategy* $\bar{b}_F^{(1)}, \bar{b}_F^{(2)}$ *in* (27)-(29).

4.2 Ascending reserve prices

The ascending reserve prices case is solved by GTX. The following is an adaptation of their result to the two-object case, in which β_{k,r_1} is the equilibrium bidding function (for $x \geq r_1$) in a one-shot first price auction with k bidders, reserve price r_1, and i.i.d. values, each with the c.d.f. F.

Proposition 6 (Proposition 1 in GTX)**.** *Suppose that the two objects are offered through sequential first price auctions with ascending reserve prices r_1, r_2 such that $\underline{x} \leq r_1 \leq r_2 \leq \bar{x}$. Then there exists an equilibrium in which each bidder plays the following strategy:*

$$b_F^{(1)}(x) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, r_1) \\ \beta_{n,r_1}(x) & \text{if } x \in [r_1, r_2] \\ \frac{\beta_{n,r_1}(r_2)F^{n-1}(r_2)+\int_{r_2}^{x}\beta_{n-1,r_2}(s)dF^{n-1}(s)}{F^{n-1}(x)} & \text{if } x \in (r_2, \bar{x}] \end{cases} \tag{30}$$

$$b_F^{(2)}(x|no, no) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, r_2) \\ r_2 & \text{if } x \in [r_2, \bar{x}] \end{cases} \tag{31}$$

$$b_F^{(2)}(x|\mathfrak{b}, b_F^{(1)}(z)) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, r_2) \\ r_2 & \text{if } x \in [r_2, \bar{x}] \end{cases} \quad \text{for each } z \in [r_1, r_2],\ \mathfrak{b} \leq b_F^{(1)}(z) \tag{32}$$

$$b_F^{(2)}(x|\mathfrak{b}, b_F^{(1)}(z)) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, r_2) \\ \beta_{n-1,r_2}(x) & \text{if } x \in [r_2, z) \\ \beta_{n-1,r_2}(z) & \text{if } x \in [z, \bar{x}] \end{cases} \quad \text{for each } z \in (r_2, \bar{x}],\ \mathfrak{b} \leq b_F^{(1)}(z) \tag{33}$$

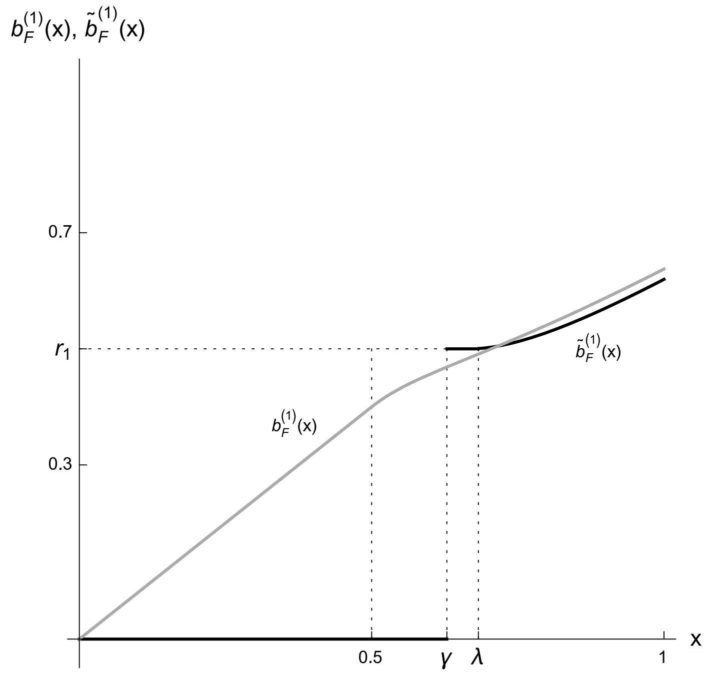
$$b_F^{(2)}(x|\mathfrak{b}, \mathfrak{b}_w) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, r_2) \\ \beta_{n-1,r_2}(x) & \text{if } x \in [r_2, \bar{x}] \end{cases} \quad \text{for each } \mathfrak{b}_w > b_F^{(1)}(\bar{x}),\ \mathfrak{b}_w \geq \mathfrak{b} \tag{34}$$

A crucial feature of $b_F^{(1)}$ is that it is strictly increasing for $x \geq r_1$, hence if at least a bid is submitted at stage one and $\mathfrak{b}_w = b_F^{(1)}(z)$ for some $z \geq r_1$, then each losing bidder's beliefs are given by the c.d.f. which takes value $\frac{F^{n-2}(s)}{F^{n-2}(z)}$ if $s \in [\underline{x}, z)$, and 1 if $s \in [z, \bar{x}]$. In the case that $\mathfrak{b}_w = b_F^{(1)}(z)$ for a $z \in [r_1, r_2]$, a losing bidder infers that each other bidder has value smaller than r_2. Therefore bidding r_2 suffices to win at stage two, and it is a best reply if the bidder has value $x \geq r_2$: see (32), and notice that the same principle holds for (31). Conversely, if $\mathfrak{b}_w = b_F^{(1)}(z)$ for a $z \in (r_2, \bar{x}]$ then each bidder with type $x \in [r_2, z)$ bids $\beta_{n-1,r_2}(x)$ as it follows from (1) with the above beliefs: see (33).

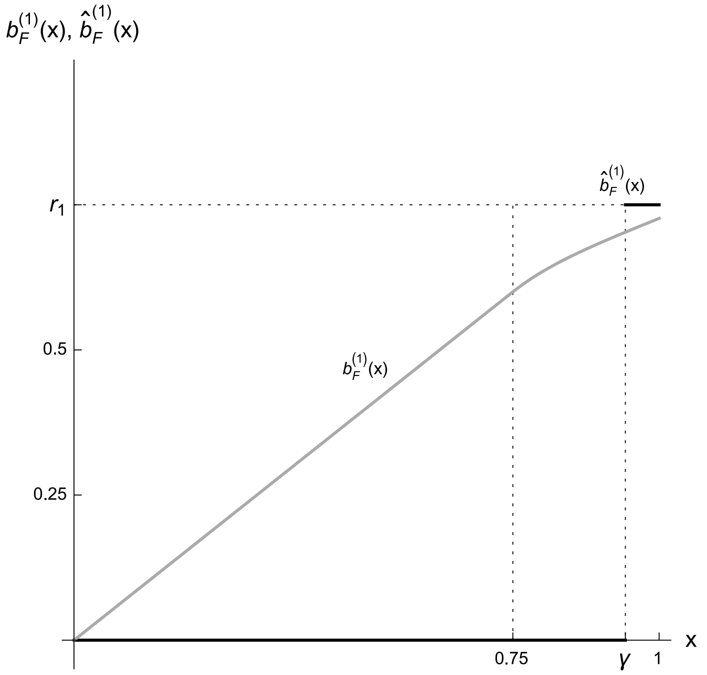
Moving to stage one, the rationale for $b_F^{(1)}$ is as follows: bidders whose value is between r_1 and r_2 only participate in the first auction. From their perspective, they are joining a one shot first price auction with reserve price r_1. As a result, they bid as in the equilibrium of that auction. Each bidder with value above r_2 has two attempts at getting an object. At the second and last one (along the equilibrium path) he will play the equilibrium bid of a one shot first price auction with reserve price r_2: see (32)-(33). This determines that the equilibrium bidding function for the first stage is obtained from the differential equation $b'(x) = (\beta_{n-1,r_2}(x) - b(x))\frac{(n-1)f(x)}{F(x)}$ for

$x > r_2$, with the boundary condition $b(r_2) = \beta_{n,r_1}(r_2)$. With respect to (1), the value of type x is replaced by his stage two bid, as illustrated by GTX, and consistently with the analysis of Krishna (2010) for a setting without reserve prices.

Figure 2 reports the plots of the equilibrium bidding functions in the first stage with sequential first price auctions when values are uniformly distributed, there are five bidders, one item has no reserve price and the other has a reserve price $r \in \{0.5, 0.75\}$. The equilibrium bidding function for the ascending reserve price auction is given by $b_F^{(1)}$ in (30) and is plotted in grey. When $r_1 = 0.5$, the equilibrium bidding function for the descending reserve price auction given by $\tilde{b}_F^{(1)}$ in (74) and is plotted in black in Figure 2(a) with $\lambda = \frac{1.257289}{2}$ and $\gamma = \frac{1.365022}{2}$. When $r_1 = 0.75$, the equilibrium bidding function for the descending reserve price auction is given by $\hat{b}_S^{(1)}$ in (21) with $\gamma = 0.942$ and is plotted in black in Figure 2(b). The cutoff values λ and γ are the same as in the sequential sealed bid second price auctions in light of Proposition 5.



(a) Equilibrium bidding functions when $r_1 = 0$ and $r_2 = 0.5$ (grey) or $r_1 = 0.5$ and $r_2 = 0$ (black)



(b) Equilibrium bidding functions when $r_1 = 0$ and $r_2 = 0.75$ (grey) or $r_1 = 0.75$ and $r_2 = 0$ (black)

Figure 2: Equilibrium bidding function in the first stage of a sequential sealed bid first price auction when reserve prices are 0 and 0.5 (left) or 0 and 0.75 (right). There are five bidders with values independently drawn from a uniform distribution in the unit interval

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A Proof of Proposition 2

A.1 Proof of Proposition 2(i)

We prove that there exists $\tilde{r}_1 \in (r_2, \bar{r}_1)$ such that if $r_1 \in (r_2, \tilde{r}_1)$, then there exists a unique solution to (15)-(16) and there exists an equilibrium in which each bidder bids according to the strategy $(\tilde{b}_S^{(1)}, b_S^{(2)*})$. We employ several steps to obtain this result.

Step 1: Derivation of $\tilde{u}_S(x, \underline{x})$, $\tilde{u}_S(x, \gamma)$ and $\tilde{u}_S(x, y)$

We start by illustrating how $\tilde{u}_S(x, \underline{x}), \tilde{u}_S(x, \gamma)$ and $\tilde{u}_S(x, y)$ are derived. To this end we need to determine the updated beliefs, for a bidder who lost at stage one (because either he did not participate the auction, or he bid r_1 or he bid $\beta_{n-1,r_2}(y)$), about the highest value among the other bidders who did not win at stage one, conditional on the information the bidder learns at stage one: his own bid at stage one (which we denote with $\mathfrak{b}$) and the winning bid at stage one (which we denote with $\mathfrak{b}_w$). In order to shorten the notation, we set $\Gamma \equiv F(\gamma)$ and $\Lambda \equiv F(\lambda)$.

Step 1.1: Updated beliefs for a bidder who has not bid at stage one, and $\tilde{u}_S(x, \underline{x})$. Consider a bidder with type x who has made no bid at stage one. Here we describe his beliefs upon learning $\mathfrak{b}_w$, and his expected payoff $\tilde{u}_S(x, \underline{x})$ from (5).

- In case there has been no bid by any bidder, an event with probability Γ^{n-1} from the bidder's ex ante point of view, his beliefs are given by the c.d.f. $\tilde{G}(\cdot|\text{no,no})$ such that

$$\tilde{G}(s|\text{no}, \text{no}) = \begin{cases} \frac{F^{n-1}(s)}{\Gamma^{n-1}} & \text{if } s \in [\underline{x}, \gamma) \\ 1 & \text{if } s \in [\gamma, \bar{x}] \end{cases} \tag{35}$$

- In case $\mathfrak{b}_w = r_1$, an event with probability $\Lambda^{n-1} - \Gamma^{n-1}$ from the bidder's ex ante point of view, his beliefs are given by the c.d.f. $\tilde{G}(\cdot|\text{no}, r_1)$ such that

$$\tilde{G}(s|\text{no}, r_1) = \begin{cases} \frac{(n-1)(\Lambda-\Gamma)}{\Lambda^{n-1}-\Gamma^{n-1}} F^{n-2}(s) & \text{if } s \in [\underline{x}, \gamma) \\ \frac{\Lambda-\Gamma}{\Lambda^{n-1}-\Gamma^{n-1}} \frac{F^{n-1}(s)-\Gamma^{n-1}}{F(s)-\Gamma} & \text{if } s \in [\gamma, \lambda] \\ 1 & \text{if } s \in (\lambda, \bar{x}] \end{cases} \tag{36}$$

  About the derivation of $\tilde{G}(s|\text{no}, r_1)$, consider the point of view of, say, bidder 1; the following probabilities refer to the $n-1$ bidders different from 1. For $s \in [\underline{x}, \gamma)$, $\tilde{G}(s|\text{no}, r_1)$ is obtained by evaluating the probability that exactly one of the other bidders has value in $[\gamma, \lambda]$ and each other bidder has value smaller than s. This probability is equal to $(n-1)(\Lambda - \Gamma)F^{n-2}(s)$.

For $s \in [\gamma, \lambda]$, $\tilde{G}(s|\text{no}, r_1)$ is obtained by evaluating the probability that at least one of the other bidders has value in $[\gamma, \lambda]$, none of them has value bigger than λ, and each of the non winning bidders with value $x \in [\gamma, \lambda]$ is such that $x \leq s$. Such a probability is given by[12]

$$(n-1)(\Lambda - \Gamma) \sum_{j=0}^{n-2} \frac{C_{n-2,j}}{j+1} \Gamma^{n-2-j} (F(s) - \Gamma)^j \tag{37}$$

Specifically, $\Lambda - \Gamma$ is the probability that a bidder (the winner) has value in $[\gamma, \lambda]$ and we have $n-1$ possible ways of picking a winner. If there are j other bidders (from the remaining $n-2$) whose value is greater than γ, we need each of them to have value less than s, and $\frac{1}{j+1}$ is the probability that our initially selected bidder wins. Remark that

$$\frac{C_{n-2,j}}{j+1} \Gamma^{n-2-j} (F(s) - \Gamma)^j = \frac{C_{n-1,j+1}}{(n-1)(F(s) - \Gamma)} \Gamma^{n-2-j} (F(s) - \Gamma)^{j+1} \tag{38}$$

for $j = 0, 1, ..., n-2$. The right hand side of (38) is equal to $\frac{C_{n-1,h}}{(n-1)(F(s)-\Gamma)} \Gamma^{n-1-h} (F(s) - \Gamma)^h$, for $h = 1, 2, ..., n-1$ (with $h = j+1$). Hence (37) is equal to

$$(n-1)(\Lambda - \Gamma) \sum_{h=1}^{n-1} \frac{C_{n-1,h}}{(n-1)(F(s) - \Gamma)} \Gamma^{n-1-h} (F(s) - \Gamma)^h = \frac{\Lambda - \Gamma}{F(s) - \Gamma} \left(F^{n-1}(s) - \Gamma^{n-1} \right)$$

- In case $\mathfrak{b}_w = \tilde{b}_S^{(1)}(z)$ for some $z \in (\lambda, \bar{x}]$, an event with probability $1 - \Lambda^{n-1}$ from the bidder's ex ante point of view, his beliefs are given by the c.d.f. with value $F^{n-2}(s)/F^{n-2}(z)$ if $s \in [\underline{x}, z)$, with value 1 if $s \in [z, \bar{x}]$. In fact, this c.d.f. applies as long as the winning bid has been $\tilde{b}_S^{(1)}(z)$ for some $z \in (\lambda, \bar{x}]$, for each stage one bid $\mathfrak{b} \leq \tilde{b}_S^{(1)}(z)$; hence we define $\tilde{G}(s|\mathfrak{b}, \tilde{b}_S^{(1)}(z))$ such that

$$\tilde{G}(s|\mathfrak{b}, \tilde{b}_S^{(1)}(z)) = \begin{cases} \frac{F^{n-2}(s)}{F^{n-2}(z)} & \text{if } s \in [\underline{x}, z) \\ 1 & \text{if } s \in [z, \bar{x}] \end{cases} \quad \text{for each } \mathfrak{b} \leq \tilde{b}_S^{(1)}(z) \tag{39}$$

When he decides to make no bid, the bidder's expected beliefs are represented by the c.d.f. $\tilde{G}_{\text{no}}$ such that

$$\begin{aligned} \tilde{G}_{\text{no}}(s) &= \Gamma^{n-1} \tilde{G}(s|\text{no}, \text{no}) + (\Lambda^{n-1} - \Gamma^{n-1}) \tilde{G}(s|\text{no}, r_1) + \int_{\lambda}^{\bar{x}} \tilde{G}(s|\text{no}, \tilde{b}_S^{(1)}(z)) dF^{n-1}(z) \\ &= \begin{cases} F^{n-1}(s) + (n-1)(1-\Gamma) F^{n-2}(s) & \text{if } s \in [\underline{x}, \gamma) \\ \Gamma^{n-1} + \frac{(\Lambda - \Gamma)(F^{n-1}(s) - \Gamma^{n-1})}{F(s) - \Gamma} + (n-1)(1-\Lambda) F^{n-2}(s) & \text{if } s \in [\gamma, \lambda] \\ (n-1) F^{n-2}(s) - (n-2) F^{n-1}(s) & \text{if } s \in (\lambda, \bar{x}] \end{cases} \end{aligned}$$

[12] For any pair of non negative integers $k \geq h$ we write $C_{k,h}$ to denote $\frac{k!}{h!(k-h)!}$.

using (35), (36) and (39). Hence the payoff of a type x from not bidding at stage one is

$$\tilde{u}_S(x, \underline{x}) = \int_{r_2}^{x} \tilde{G}_{\text{no}}(s) ds \tag{40}$$

Step 1.2: Updated beliefs for a bidder who has bid r_1 at stage one but has not won at stage one, and $\tilde{u}_S(x, \gamma)$. For future convenience, we introduce the following function M, defined for $a \in [0, 1]$ and $b \in [0, 1]$:

$$M(a, b) = \begin{cases} \frac{(n-1)a^n - na^{n-1}b + b^n}{(a-b)^2} & \text{if } a \neq b \\ \frac{n(n-1)}{2} a^{n-2} & \text{if } a = b \end{cases} \tag{41}$$

Multiplying $(a - b)^2$ by $(n - 1)a^{n-2} + (n - 2)a^{n-3}b + ... + 2ab^{n-3} + b^{n-2}$ reveals that

$$M(a, b) = (n - 1)a^{n-2} + (n - 2)a^{n-3}b + ... + 2ab^{n-3} + b^{n-2} \tag{42}$$

and therefore M is strictly increasing both with respect to a and with respect to b.

For a bidder bidding r_1, the probability to win at stage one is

$$\begin{aligned} \tilde{p}(\gamma) &= \sum_{j=0}^{n-1} \frac{C_{n-1,j}}{j+1} \Gamma^{n-1-j} (\Lambda - \Gamma)^j = \sum_{j=0}^{n-1} \frac{C_{n,j+1}}{n(\Lambda - \Gamma)} \Gamma^{n-1-j} (\Lambda - \Gamma)^{j+1} \\ &= \sum_{h=1}^{n} \frac{C_{n,h}}{n(\Lambda - \Gamma)} \Gamma^{n-h} (\Lambda - \Gamma)^h = \frac{\Lambda^n - \Gamma^n}{n(\Lambda - \Gamma)} \end{aligned} \tag{43}$$

Let $\tilde{p}_\ell$ denote the probability that another bidder wins at stage one with a bid of r_1. The probability that another bidder wins at stage one with a bid $\tilde{b}_S^{(1)}(z)$ for some $z \in (\lambda, \bar{x}]$ is $1-\Lambda^{n-1}$. Since $\tilde{p}(\gamma) + \tilde{p}_\ell + 1 - \Lambda^{n-1} = 1$, it follows that $\tilde{p}_\ell = \Lambda^{n-1} - \tilde{p}(\gamma)$, that is

$$\tilde{p}_\ell = \frac{\Lambda - \Gamma}{n} M(\Lambda, \Gamma) = (n - 1)(\Lambda - \Gamma) \sum_{j=0}^{n-2} \frac{C_{n-2,j}}{j+2} \Gamma^{n-2-j} (\Lambda - \Gamma)^j \tag{44}$$

Now consider a bidder who has bid r_1 at stage one but has not won. Then either $\mathfrak{b}_w = r_1$, or $\mathfrak{b}_w = \tilde{b}_S^{(1)}(z)$ for some $z \in (\lambda, \bar{x}]$.

- In case $\mathfrak{b}_w = r_1$ and another bidder has won, an event with probability $\tilde{p}_\ell$ from the bidder's ex ante point of view, his beliefs are given by $\tilde{G}(\cdot|r_1, r_1)$ such that

$$\tilde{G}(s|r_1, r_1) = \begin{cases} \frac{(n-1)(\Lambda-\Gamma)}{2\tilde{p}_\ell} F^{n-2}(s) & \text{if } s \in [\underline{x}, \gamma) \\ \frac{\Lambda-\Gamma}{n\tilde{p}_\ell} M(F(s), \Gamma) & \text{if } s \in [\gamma, \lambda] \\ 1 & \text{if } s \in (\lambda, \bar{x}] \end{cases} \tag{45}$$

Considering the point of view of bidder 1, the derivation of $\tilde{G}(s|r_1, r_1)$ for $s \in [\underline{x}, \gamma)$ is similar to the derivation of $\tilde{G}(s|\text{no}, r_1)$ for $s \in [\underline{x}, \gamma)$, taking into account that bidder 1 has bid r_1 rather than abstaining from bidding.

For $s \in [\gamma, \lambda]$, $\tilde{G}(s|r_1, r_1)$ is obtained from the probability that none of the other bidders has value greater than λ, at least one of them has value in $[\gamma, \lambda]$ and wins, and each losing bidder with value $x \in [\gamma, \lambda]$ is such that $x \leq s$. This probability is equal to

$$(n-1)(\Lambda - \Gamma) \sum_{j=0}^{n-2} \frac{C_{n-2,j}}{j+2} \Gamma^{n-2-j} (F(s) - \Gamma)^j \tag{46}$$

From (44) we see that $\sum_{j=0}^{n-2} \frac{C_{n-2,j}}{j+2} \Gamma^{n-2-j} (\Lambda - \Gamma)^j = \frac{M(\Lambda, \Gamma)}{n(n-1)}$. Hence (46) is equal to

$$(n-1)(\Lambda - \Gamma) \sum_{j=0}^{n-2} \frac{C_{n-2,j}}{j+2} \Gamma^{n-2-j} (F(s) - \Gamma)^j = \frac{\Lambda - \Gamma}{n} M(F(s), \Gamma)$$

- In case $\mathfrak{b}_w = \tilde{b}_S^{(1)}(z)$ for some $z \in (\lambda, \bar{x}]$, an event with probability $1 - \Lambda^{n-1}$ from the bidder's ex ante point of view, then his beliefs are given by $\tilde{G}(\cdot|r_1, \tilde{b}_S^{(1)}(z))$ in (39).

When he decides to bid r_1 at stage one, the bidder expects to lose with probability $1 - \tilde{p}(\gamma) = \tilde{p}_\ell + 1 - \Lambda^{n-1}$, hence his expected beliefs are represented by the c.d.f. $\tilde{G}_{r_1}$ such that

$$\begin{aligned}
\tilde{G}_{r_1}(s) &= \frac{\tilde{p}_\ell \tilde{G}(s|r_1, r_1) + \int_\lambda^{\bar{x}} \tilde{G}(s|r_1, \tilde{b}_S^{(1)}(z)) dF^{n-1}(z)}{1 - \tilde{p}(\gamma)} \\
&= \frac{1}{1 - \tilde{p}(\gamma)} \begin{cases} \frac{(n-1)(2-\Gamma-\Lambda)}{2} F^{n-2}(s) & \text{if } s \in [\underline{x}, \gamma) \\ \frac{\Lambda - \Gamma}{n} M(F(s), \Gamma) + (n-1)(1-\Lambda) F^{n-2}(s) & \text{if } s \in [\gamma, \lambda] \\ (n-1) F^{n-2}(s) - (n-2) F^{n-1}(s) - \tilde{p}(\gamma) & \text{if } s \in (\lambda, \bar{x}] \end{cases}
\end{aligned}$$

using (45) and (39). Hence the payoff of a type x from bidding r_1 at stage one is

$$\tilde{u}_S(x, \gamma) = \tilde{p}(\gamma)(x - r_1) + (1 - \tilde{p}(\gamma)) \int_{r_2}^{x} \tilde{G}_{r_1}(s) ds \tag{47}$$

Step 1.3: Updated beliefs for a bidder who has bid $\tilde{b}_S^{(1)}(y)$, with $y \in (\lambda, \bar{x}]$, at stage one but has not won at stage one, and $\tilde{u}_S(x, y)$. If a bidder has bid $\tilde{b}_S^{(1)}(y)$ at stage one and has not won, then $\mathfrak{b}_w = \tilde{b}_S^{(1)}(z)$ for some $z \geq y$, and his beliefs are given by $\tilde{G}(\cdot|\tilde{b}_S^{(1)}(y), \tilde{b}_S^{(1)}(z))$ in (39). Hence, the payoff of a type x from bidding $\tilde{b}_S^{(1)}(y)$ at stage one, for $y \in (\lambda, \bar{x}]$, is

$$\tilde{u}_S(x, y) = \int_{\underline{x}}^{y} \left(x - \max\{r_1, \tilde{b}_S^{(1)}(z)\} \right) dF^{n-1}(z) + \int_y^{\bar{x}} \int_{r_2}^{x} \tilde{G}(s|\tilde{b}_S^{(1)}(y), \tilde{b}_S^{(1)}(z)) ds dF^{n-1}(z) \tag{48}$$

and notice that the second term is equal to

$$\begin{cases} (n-1)(1-F(x))v_{n-1}(x) + \int_y^x \left(\frac{v_{n-1}(z)}{F^{n-2}(z)} + x - z \right) dF^{n-1}(z) & \text{if } y < x \\ (n-1)(1-F(y))v_{n-1}(x) & \text{if } y \geq x \end{cases}$$

Step 2: Derivation of γ and λ: Existence of a unique solution for (15)-(16), and definition of $\tilde{r}_1$

Using (40), (47), and (48) we find

$$\begin{aligned} \tilde{u}_S(\gamma, \underline{x}) &= v_n(\gamma) + (n-1)(1-\Gamma)v_{n-1}(\gamma) \\ \tilde{u}_S(\gamma, \gamma) &= \tilde{p}(\gamma)(\gamma - r_1) + \frac{n-1}{2}(2 - \Gamma - \Lambda)v_{n-1}(\gamma) \\ \tilde{u}_S(\lambda, \gamma) &= \tilde{p}(\gamma)(\lambda - r_1) + \frac{(n-1)(\Lambda - \Gamma)}{2}v_{n-1}(\gamma) + \\ &\quad + \int_\gamma^\lambda \frac{\Lambda - \Gamma}{n} M(F(s), \Gamma)ds + (n-1)(1-\Lambda)v_{n-1}(\lambda) \\ \lim_{y \downarrow \lambda} \tilde{u}_S(x, y) &= \Lambda^{n-1}(x - r_1) + (n-1)(1-\Lambda)v_{n-1}(x) \equiv \tilde{u}_S(x, \lambda^+) \text{ for } x \leq \lambda \end{aligned} \tag{49}$$

Hence (15) and (16) reduce, after some rearranging, respectively to

$$A(\gamma, \lambda) = 0, \qquad B(\gamma, \lambda) = 0 \tag{50}$$

with

$$A(\gamma, \lambda) = \tilde{p}(\gamma)(\gamma - r_1) - \frac{(n-1)(\Lambda - \Gamma)}{2}v_{n-1}(\gamma) - v_n(\gamma) \tag{51}$$

$$B(\gamma, \lambda) = \frac{n(n-1)}{2}v_{n-1}(\gamma) - M(\Lambda, \Gamma)(\lambda - r_1) + \int_\gamma^\lambda M(F(s), \Gamma)ds \tag{52}$$

Step 2.1: Definition of λ^*. Define $\tau(\lambda) = F^{n-1}(\lambda)(\lambda - r_1) - v_n(\lambda)$, a strictly increasing function such that $\tau(r_1) < 0$ and $\tau(\bar{x}) = \bar{r}_1 - r_1 > 0$. Hence there exists λ in the interval $(r_1, \bar{x})$, which we denote λ^*, such that $\tau(\lambda) < 0$ for $\lambda \in (r_1, \lambda^*)$, $\tau(\lambda^*) = 0$, $\tau(\lambda) > 0$ for $\lambda \in (\lambda^*, \bar{x}]$.

Step 2.2: If $\lambda \in (r_1, \lambda^*)$, then there exists no $\gamma \in (r_1, \lambda]$ such that $A(\gamma, \lambda) = 0$; if $\lambda \in [\lambda^*, \bar{x}]$, then there exists a unique $\gamma \in (r_1, \lambda]$ such that $A(\gamma, \lambda) = 0$, denoted $\gamma_A(\lambda)$. Given a function h of two variables, here and in the remainder of the Appendix we write h_i to denote the partial derivative of h with respect to its i-th variable, $i = 1, 2$.

First we prove that the function A is strictly increasing with respect to γ:

$$A_1(\gamma, \lambda) = \frac{\partial \tilde{p}(\gamma)}{\partial \gamma}(\gamma - r_1) + \frac{n-1}{2}f(\gamma)v_{n-1}(\gamma) + \tilde{p}(\gamma) - \frac{(n-1)(\Lambda - \Gamma)}{2}\Gamma^{n-2} - \Gamma^{n-1}$$

From the definition of $\tilde{p}(\gamma)$ in (43), we see that

$$\tilde{p}(\gamma) - \frac{(n-1)(\Lambda-\Gamma)}{2}\Gamma^{n-2} - \Gamma^{n-1} = \sum_{j=2}^{n-1} \frac{C_{n-1,j}}{j+1}\Gamma^{n-j-1}(\Lambda-\Gamma)^j > 0$$

and, moreover, $\frac{\partial \tilde{p}(\gamma)}{\partial \gamma}(\gamma - r_1) > 0$ and $\frac{n-1}{2} f(\gamma) v_{n-1}(\gamma) > 0$.
Now we examine the sign of $A(r_1, \lambda)$ and of $A(\lambda, \lambda)$. We have that

$$A(r_1,\lambda) = -\frac{(n-1)(\Lambda - F(r_1))}{2} v_{n-1}(r_1) - v_n(r_1) < 0$$

and $A(\lambda,\lambda) = \tau(\lambda)$. Therefore, if $\lambda \in (r_1, \lambda^*)$ then $A(\lambda,\lambda) < 0$ and there is no solution to $A(\gamma,\lambda) = 0$ in the interval $(r_1, \lambda]$; if $\lambda \in [\lambda^*, \bar{x}]$, then there exists ($A$ is continuous in λ) a unique solution to $A(\gamma,\lambda) = 0$ in the interval $(r_1, \lambda]$, which we denote $\gamma_A(\lambda)$.

Step 2.3: There exists $\tilde{r}_1 \in (r_2, \bar{r}_1)$ such that the equation $B(\gamma_A(\lambda), \lambda) = 0$ has a unique solution in $(\lambda^*, \bar{x})$ if $r_1 \in (r_2, \tilde{r}_1)$; the equation $B(\gamma_A(\lambda), \lambda) = 0$ has no solution in $(\lambda^*, \bar{x})$ if $r_1 \geq \tilde{r}_1$. First we prove that $B(\gamma_A(\lambda), \lambda)$ is strictly decreasing in λ. Notice that Γ below is actually equal to $F(\gamma_A(\lambda))$. We have that

$$\frac{dB(\gamma_A(\lambda),\lambda)}{d\lambda} = f(\gamma_A(\lambda))\left(\int_{\gamma_A}^{\lambda}(M_2(F(s),\Gamma)ds - M_2(\Lambda,\Gamma)(\lambda - r_1)\right)\gamma_A'(\lambda) - M_1(\Lambda,\Gamma)(\lambda - r_1) f(\lambda)$$

and we prove that $\frac{dB(\gamma_A(\lambda),\lambda)}{d\lambda} < 0$. From the previous step we have that

$$\gamma_A'(\lambda) = -\frac{A_2(\gamma,\lambda)}{A_1(\gamma,\lambda)} = -\frac{\frac{\partial \tilde{p}}{\partial \lambda}(\gamma_A(\lambda) - r_1) - \frac{n-1}{2} v_{n-1}(\gamma_A(\lambda)) f(\lambda)}{\frac{\partial \tilde{p}}{\partial \gamma}(\gamma_A(\lambda) - r_1) + \frac{n-1}{2} v_{n-1}(\gamma_A(\lambda)) f(\gamma_A(\lambda)) + \tilde{p} - \frac{n-1}{2}(\Lambda-\Gamma)\Gamma^{n-2} - \Gamma^{n-1}} \tag{53}$$

From the proof of Step 2.2 we know that the denominator in the right hand side of (53) is positive. Therefore $\frac{dB}{d\lambda}$ has the same sign as

$$\begin{aligned} &-f(\gamma_A(\lambda))\left(\frac{\partial \tilde{p}}{\partial \lambda}(\gamma_A - r_1) - \frac{n-1}{2} v_{n-1}(\gamma_A(\lambda)) f(\lambda)\right)\left(\int_{\gamma_A(\lambda)}^{\lambda}(M_2(F(s),\Gamma)ds - M_2(\Lambda,\Gamma)(\lambda - r_1)\right) \\ &- M_1(\Lambda,\Gamma)(\lambda - r_1) f(\lambda)\left(\frac{\partial \tilde{p}}{\partial \gamma}(\gamma_A(\lambda) - r_1) + \frac{n-1}{2} v_{n-1}(\gamma_A) f(\gamma_A(\lambda)) + K\right) \end{aligned} \tag{54}$$

with $K = \tilde{p} - \frac{n-1}{2}(\Lambda - \Gamma)\Gamma^{n-2} - \Gamma^{n-1} > 0$. Moreover,

$$\frac{\partial \tilde{p}}{\partial \gamma} = \frac{M(\Gamma,\Lambda)}{n} f(\gamma_A(\lambda)), \qquad \frac{\partial \tilde{p}}{\partial \lambda} = \frac{M(\Lambda,\Gamma)}{n} f(\lambda)$$

hence (54) is smaller than

$$-f(\gamma_A(\lambda))f(\lambda)\left(\frac{M(\Lambda,\Gamma)}{n}(\gamma_A - r_1) - \frac{n-1}{2}v_{n-1}(\gamma_A(\lambda))\right)\left(\int_{\gamma_A}^{\lambda}(M_2(F(s),\Gamma)ds - M_2(\Lambda,\Gamma)(\lambda - r_1)\right)$$
$$- M_1(\Lambda,\Gamma)(\lambda - r_1)f(\lambda)f(\gamma_A(\lambda))\left(\frac{M(\Gamma,\Lambda)}{n}(\gamma_A(\lambda) - r_1) + \frac{n-1}{2}v_{n-1}(\gamma_A)\right)$$

which is equal to

$$-f(\gamma_A(\lambda))f(\lambda)\left[\frac{n-1}{2}v_{n-1}(\gamma_A(\lambda))\left((\lambda - r_1)M_1(\Lambda,\Gamma) + (\lambda - r_1)M_2(\Lambda,\Gamma) - \int_{\gamma_A}^{\lambda}(M_2(F(s),\Gamma)ds\right)\right.$$
$$\left. + \frac{\gamma_A(\lambda) - r_1}{n}\left(M(\Lambda,\Gamma)\int_{\gamma_A}^{\lambda} M_2(F(s),\Gamma)ds + (\lambda - r_1)\left[M_1(\Lambda,\Gamma)M(\Gamma,\Lambda) - M(\Lambda,\Gamma)M_2(\Lambda,\Gamma)\right]\right)\right] \tag{55}$$

We now prove that (55) is negative by showing that the terms inside the square brackets are positive.

- The inequality $\gamma_A(\lambda) > r_1$ implies $(\lambda - r_1)M_1(\Lambda,\Gamma) + (\lambda - r_1)M_2(\Lambda,\Gamma) - \int_{\gamma_A(\lambda)}^{\lambda}(M_2(F(s),\Gamma)ds > (\lambda - r_1)M_1(\Lambda,\Gamma) + \int_{\gamma_A(\lambda)}^{\lambda}(M_2(\Lambda,\Gamma) - M_2(F(s),\Gamma))\,ds$, and the right hand side is positive since $M_1 > 0$ and $M_2(a,b) = \sum_{j=0}^{n-3}(n-2-j)(j+1)a^{n-3-j}b^j$ is strictly increasing a.
- The term $M(\Lambda,\Gamma)\int_{\gamma_A(\lambda)}^{\lambda} M_2(F(s),\Gamma)ds$ is positive since $M_2 > 0$.
- The term $M_1(\Lambda,\Gamma)M(\Gamma,\Lambda) - M(\Lambda,\Gamma)M_2(\Lambda,\Gamma)$ is positive. In fact, from (41) we have $M_1(a,b) = \frac{(n-1)(n-2)a^n - 2b^n + n(n-1)a^{n-2}b^2 - 2(n-2)na^{n-1}b}{(a-b)^3}$, and $M_2(a,b) = \frac{(n-2)(a^n - b^n) + nab(b^{n-2} - a^{n-2})}{(a-b)^3}$. Therefore,

  $$M_1(\Lambda,\Gamma)M(\Gamma,\Lambda) - M(\Lambda,\Gamma)M_2(\Lambda,\Gamma) =$$
  $$\frac{n\Lambda^{2n-2}\Gamma}{(\Lambda - \Gamma)^4}\left(1 - (n-1)^2k^{n-2} + 2n(n-2)k^{n-1} - (n-1)^2k^n + k^{2n-2}\right)$$

  with $k = \frac{\Gamma}{\Lambda} \in (0,1)$. We now define

  $$\mu(k) = k^{2n-2} - (n-1)^2 k^n + 2n(n-2)k^{n-1} - (n-1)^2 k^{n-2} + 1 \tag{56}$$

  and show it is positive for each $k \in (0,1)$. Remark that $\mu(1) = 0$. We now prove that $\mu(k) > 0$ for each $k \in (0,1)$. We find that $\mu'(k) = k^{n-3}\nu(k)$, with $\nu(k) = 2(n-1)k^n - n(n-1)^2 k^2 + 2n(n-1)(n-2)k - (n-1)^2(n-2)$ and $\nu(1) = 0$. In addition, we have that $\nu'(k) = 2n(n-1)\left(k^{n-1} - (n-1)k + n - 2\right)$, with $\nu'(1) = 0$, and $\nu''(k) = -2n(n-1)^2(1 - k^{n-2}) < 0$ for each $k \in (0,1)$. Hence, ν' is strictly decreasing and since $\nu'(1) = 0$ we can conclude that $\nu'(k) > 0$ for each $k \in (0,1)$. This, in turn, implies that ν is strictly increasing, and since $\nu(1) = 0$, we obtain that $\nu(k) < 0$ for each $k \in (0,1)$.

Therefore $\mu'(k) < 0$ for each $k \in (0,1)$, and since $\mu(1) = 0$ we can conclude that $\mu(k) > 0$ for each $k \in (0,1)$.

Now we prove that $B(\gamma_A(\lambda^*), \lambda^*) > 0$ and then examine the sign of $B(\gamma_A(\bar{x}), \bar{x})$. Given that $B(\gamma_A(\lambda), \lambda)$ is a continuous function of λ, if $B(\gamma_A(\bar{x}), \bar{x}) < 0$ then there exists a unique $\tilde{\lambda} \in (\lambda^*, \bar{x})$ such that $B(\gamma_A(\tilde{\lambda}), \tilde{\lambda}) = 0$. Since $A(\gamma_A(\tilde{\lambda}), \tilde{\lambda}) = 0$, it follows that $\gamma_A(\tilde{\lambda}), \tilde{\lambda}$ is a solution to (50), i.e. to (15)-(16). We prove that there exists $\tilde{r}_1 \in (r_2, \bar{r}_1)$ such that $B(\gamma_A(\bar{x}), \bar{x}) < 0$ if and only if $r_1 \in (r_2, \tilde{r}_1)$.
Regarding $B(\gamma_A(\lambda^*), \lambda^*)$, since $A(\gamma_A(\lambda^*), \lambda^*) = 0 = \tau(\lambda^*) = A(\lambda^*, \lambda^*)$, we have that $\gamma_A(\lambda^*) = \lambda^*$; hence (42) and (52) imply that

$$\begin{aligned} B(\gamma_A(\lambda^*), \lambda^*) =& \frac{n(n-1)}{2} \left(v_{n-1}(\lambda^*) - F^{n-2}(\lambda^*)(\lambda^* - r_1) \right) \\ =& \frac{n(n-1)}{2F(\lambda^*)} \int_{r_2}^{\lambda^*} F^{n-2}(s) \left(F(\lambda^*) - F(s) \right) ds > 0 \end{aligned}$$

where the last equality follows from the definition of λ^*.
Regarding $B(\gamma_A(\bar{x}), \bar{x})$, we have that

$$B(\gamma_A(\bar{x}), \bar{x}) = \frac{n(n-1)}{2} v_{n-1}(\gamma_A(\bar{x})) - M(1, F(\gamma_A(\bar{x})))(\bar{x} - r_1) + \int_{\gamma_A(\bar{x})}^{\bar{x}} M(F(s), F(\gamma_A(\bar{x})))ds \tag{57}$$

We now take into account that $\gamma_A(\bar{x})$ is an increasing function of r_1 ($\frac{d\gamma_A(\bar{x})}{dr_1} > 0$ since $\frac{\partial A}{\partial \gamma} > 0$ and $\frac{\partial A}{\partial r_1} < 0$), and we view $B(\gamma_A(\bar{x}), \bar{x})$ as a function $\ell(r_1)$ of r_1 that is defined for $r_1 \in (r_2, \bar{r}_1)$. As $r_1 \uparrow \bar{r}_1$, we have that $\gamma_A(\bar{x}) \to \bar{x}$ and $F(\gamma_A(\bar{x})) \to 1$, hence

$$\lim_{r \uparrow \bar{r}_1} \ell(r_1) = \frac{n(n-1)}{2} v_{n-1}(\bar{x}) - \frac{n(n-1)}{2}(\bar{x} - \bar{r}_1) = \frac{n(n-1)}{2}(v_{n-1}(\bar{x}) - v_n(\bar{x})) > 0$$

As $r_1 \downarrow r_2$, we have that $\gamma_A(\bar{x}) \to r_2$, hence

$$\lim_{r_1 \downarrow r_2} \ell(r_1) = \int_{r_2}^{\bar{x}} \left(M(F(s), F(r_2)) - M(1, F(r_2)) \right) ds < 0$$

since $F(s) < 1$ for $s \in (r_2, \bar{x})$. The continuity of ℓ implies that there exists $\tilde{r}_1 \in (r_2, \bar{r}_1)$ such that $\ell(\tilde{r}_1) = 0$, and $\ell(r_1) < 0$ for $r_1 \in (r_2, \tilde{r}_1)$. The proof that a unique $\tilde{r}_1$ exists such that $\ell(\tilde{r}_1) = 0$ is long and is reported in E.

Step 3: $\tilde{b}_S^{(1)}$ is strictly increasing in the interval $[\lambda, \bar{x}]$

It is immediate to see that $\tilde{b}_S^{(1)}$ is strictly increasing in $(\lambda, \bar{x}]$, and here we prove that $\lim_{x \downarrow \lambda} \tilde{b}_S^{(1)}(x) > r_1$, that is $(\lambda - r_1)\Lambda^{n-2} \geq v_{n-1}(\lambda)$. Since from $B(\gamma, \lambda) = 0$ in (50) we obtain $\lambda - r_1 =$

$\frac{1}{M(\Lambda,\Gamma)}\left(\frac{n(n-1)}{2}v_{n-1}(\gamma)+\int_\gamma^\lambda M(F(s),\Gamma)ds\right)$, it is sufficient to establish below that

$$\frac{n(n-1)\Lambda^{n-2}}{2}v_{n-1}(\gamma)+\Lambda^{n-2}\int_\gamma^\lambda M(F(s),\Gamma)ds \geq M(\Lambda,\Gamma)v_{n-1}(\gamma)+M(\Lambda,\Gamma)\int_\gamma^\lambda F^{n-2}(s)ds$$

Remark that: (i) the inequality $\frac{n(n-1)\Lambda^{n-2}}{2} \geq M(\Lambda,\Gamma)$ is satisfied as $\frac{n(n-1)\Lambda^{n-2}}{2} = M(\Lambda,\Lambda) > M(\Lambda,\Gamma)$; (ii) the inequality $\Lambda^{n-2}\int_\gamma^\lambda M(F(s),\Gamma)ds \geq M(\Lambda,\Gamma)\int_\gamma^\lambda F^{n-2}(s)ds$ reduces to

$$\int_\gamma^\lambda \sum_{j=0}^{n-2}(n-1-j)\Lambda^{n-2}F^{n-2-j}(s)\Gamma^j \geq \int_\gamma^\lambda \sum_{j=0}^{n-2}(n-1-j)F^{n-2}(s)\Lambda^{n-2-j}\Gamma^j$$

and it is satisfied since $\Lambda^{n-2}F^{n-2-j}(s) \geq F^{n-2}(s)\Lambda^{n-2-j}$ for $j=0,1,...,n-2$, for $s\in[\gamma,\lambda]$.

Step 4: Proof that no profitable deviation exists

Now that $\tilde{b}_S^{(1)}$ is well defined, we prove that if a bidder expects that each other bidder follows the strategy $(\tilde{b}_S^{(1)}, b_S^{(2)*})$, then no profitable deviation exists for him. Precisely, we prove that the following inequalities hold:[13]

$$\text{for each } x\in[r_2,\gamma),\quad \tilde{u}_S(x,\underline{x}) \geq \max\{\tilde{u}_S(x,\gamma),\tilde{u}_S(x,y)\}, \text{ for each } y\in(\lambda,\bar{x}] \tag{58}$$

$$\text{for each } x\in[\gamma,\lambda],\quad \tilde{u}_S(x,\gamma) \geq \max\{\tilde{u}_S(x,\underline{x}),\tilde{u}_S(x,y)\}, \text{ for each } y\in(\lambda,\bar{x}] \tag{59}$$

$$\text{for each } x\in(\lambda,\bar{x}] \text{ and } y\in(\lambda,\bar{x}],\quad \tilde{u}_S(x,x) \geq \max\{\tilde{u}_S(x,\underline{x}),\tilde{u}_S(x,\gamma),\tilde{u}_S(x,y)\} \tag{60}$$

Step 4.1: Proof of (59) Here we prove that each type in $[\gamma,\lambda]$ prefers to bid r_1 rather than not bidding, or bidding $\tilde{b}_S^{(1)}(y)$ for some $y\in(\lambda,\bar{x}]$.

Bidding r_1 is preferred to no bidding. We know from (15) that $\tilde{u}_S(\gamma,\gamma)=\tilde{u}_S(\gamma,\underline{x})$, and now we prove that $\tilde{u}_{S1}(x,\gamma) > \tilde{u}_{S1}(x,\underline{x})$ for each $x\in(\gamma,\lambda)$, which implies $\tilde{u}_S(x,\gamma) > \tilde{u}_S(x,\underline{x})$ for each $x\in(\gamma,\lambda]$. Using (40) and (47) we find that $\tilde{u}_{S1}(x,\underline{x})=\Gamma^{n-1}+(\Lambda-\Gamma)\frac{F^{n-1}(x)-\Gamma^{n-1}}{F(x)-\Gamma}+(n-1)(1-\Lambda)F^{n-2}(x)$, whereas $\tilde{u}_{S1}(x,\gamma)=\tilde{p}(\gamma)+\frac{\Lambda-\Gamma}{n}M(F(x),\Gamma)+(n-1)(1-\Lambda)F^{n-2}(x)$ for each $x\in(\gamma,\lambda)$. By using (43) for $\tilde{p}(\gamma)$ and (42) for $M(F(x),\Gamma)$, after some rearrangements we see that $\tilde{u}_{S1}(x,\underline{x}) < \tilde{u}_{S1}(x,\gamma)$ is equivalent to $M(\Gamma,\Lambda) > M(\Gamma,F(x))$, which holds since $\Lambda > F(x)$ for each $x\in(\gamma,\lambda)$.

Bidding r_1 is preferred to bidding like type $y>\lambda$. From (48) we see that the payoff of a type $x\in[\gamma,\lambda]$ from bidding $\tilde{b}_S^{(1)}(y)$ for some $y>\lambda$ is

$$\tilde{u}_S(x,y)=\int_{\underline{x}}^y\left(x-\max\{r_1,\tilde{b}_S^{(1)}(z)\}\right)dF^{n-1}(z)+(n-1)[1-F(y)]v_{n-1}(x) \tag{61}$$

[13] We can neglect bids between r_1 and $\lim_{x\downarrow\lambda}\tilde{b}_S^{(1)}(x)$, since each bid between r_1 and $\lim_{x\downarrow\lambda}\tilde{b}_S^{(1)}(x)$ has the same effect as bidding $\lim_{x\downarrow\lambda}\tilde{b}_S^{(1)}(x)$. We can also neglect bids strictly greater than $\tilde{b}_S^{(1)}(\overline{x})$ as they cannot increase the probability of winning while potentially increasing the price to be paid.

and $\tilde{u}_{S2}(x,y) = (n-1)f(y)\int_x^y (F^{n-2}(s) - F^{n-2}(y))ds < 0$. Therefore $\tilde{u}_S(x,y)$ is decreasing in y, and we consider $\tilde{u}_S(x,\lambda^+)$ defined in (49). We know from (16) that $\tilde{u}_S(\lambda,\gamma) = \tilde{u}_S(\lambda,\lambda^+)$ and now we prove that $\tilde{u}_{S1}(x,\lambda^+) > \tilde{u}_{S1}(x,\gamma)$ for $x \in (\gamma,\lambda)$, therefore $\tilde{u}_S(x,\gamma) > \tilde{u}_S(x,\lambda^+)$ for $x \in [\gamma,\lambda)$. Precisely, $\tilde{u}_{S1}(x,\lambda^+) = \Lambda^{n-1} + (n-1)(1-\Lambda)F^{n-2}(x)$ and $\tilde{u}_{S1}(x,\lambda^+) > \tilde{u}_{S1}(x,\gamma)$ is equivalent to $M(\Lambda,\Gamma) > M(F(x),\Gamma)$ which holds since $\Lambda > F(x)$ for each $x \in (\gamma,\lambda)$.

Step 4.2: Proof of (58) Here we prove that each type in $[r_2,\gamma)$ prefers not to bid rather than bidding r_1, or bidding $\tilde{b}_S^{(1)}(y)$ for some $y \in (\lambda,\bar{x}]$.

No bidding is preferred to bidding $\boldsymbol{r_1}$. We know from (15) that $\tilde{u}_S(\gamma,\underline{x}) = \tilde{u}_S(\gamma,\gamma)$, and now we prove that $\tilde{u}_{S1}(x,\underline{x}) < \tilde{u}_{S1}(x,\gamma)$ for each $x \in [r_2,\gamma)$, which implies $\tilde{u}_S(x,\underline{x}) > \tilde{u}_S(x,\gamma)$ for each $x \in [r_2,\gamma)$. From (40) and (47) we see that for each $x \in [r_2,\gamma)$ we have $\tilde{u}_{S1}(x,\gamma) = \tilde{p}(\gamma) + \frac{(n-1)(2-\Gamma-\Lambda)}{2}F^{n-2}(x)$, $\tilde{u}_{S1}(x,\underline{x}) = F^{n-1}(x) + (n-1)(1-\Gamma)F^{n-2}(x)$, and $\tilde{u}_{S1}(x,\gamma) > \tilde{u}_{S1}(x,\underline{x})$ reduces to $\tilde{p}(\gamma) > F^{n-1}(x) + \frac{n-1}{2}(\Lambda-\Gamma)F^{n-2}(x)$. We have proved in Step 2.2 that $\tilde{p}(\gamma) > \Gamma^{n-1} + \frac{(n-1)}{2}(\Lambda-\Gamma)\Gamma^{n-2}$, and $\Gamma > F(x)$ for each $x \in [r_2,\gamma)$; hence $\tilde{u}_{S1}(x,\gamma) > \tilde{u}_{S1}(x,\underline{x})$ for each $x \in [r_2,\gamma)$.

No bidding is preferred to bidding like type $\boldsymbol{y > \lambda}$. The payoff of each type $x \in [r_2,\gamma)$ from bidding $\tilde{b}_S^{(1)}(y)$ for some $y \in (\lambda,\bar{x}]$ is given by $\tilde{u}_S(x,y)$ as in (61) and is decreasing in y. Now we prove that $\tilde{u}_S(x,\lambda^+) < \tilde{u}_S(x,\underline{x})$ for each $x \in [r_2,\gamma)$. From (15) and the last remark in the proof of (59) we know that $\tilde{u}_S(\gamma,\underline{x}) = \tilde{u}_S(\gamma,\gamma) > \tilde{u}_S(\gamma,\lambda^+)$. We prove below that $\tilde{u}_{S1}(x,\lambda^+) = \Lambda^{n-1} + (n-1)(1-\Lambda)F^{n-2}(x)$ is greater than $\tilde{u}_{S1}(x,\underline{x}) = F^{n-1}(x) + (n-1)(1-\Gamma)F^{n-2}(x)$, which implies $\tilde{u}_S(x,\underline{x}) > \tilde{u}_S(x,\lambda^+)$ for each $x \in [r_2,\gamma)$. Precisely, $\tilde{u}_{S1}(x,\lambda^+) > \tilde{u}_{S1}(x,\underline{x})$ is equivalent to $\Lambda^{n-1} > F^{n-1}(x) + (n-1)(\Lambda-\Gamma)F^{n-2}(x)$ and holds for each $x \in [r_2,\gamma)$ if and only if $\Lambda^{n-1} - \Gamma^{n-1} - (n-1)(\Lambda-\Gamma)\Gamma^{n-2} \geq 0$. This inequality is satisfied since (i) the left hand side is 0 at $\Lambda = \Gamma$; (ii) the left hand side is increasing in Λ; (iii) $\Lambda > \Gamma$.

Step 4.3: Proof of (60) Here we prove that each type x in $(\lambda,\bar{x}]$ prefers to bid $\tilde{b}_S^{(1)}(x)$ rather than not bidding, or bidding r_1, or bidding $\tilde{b}_S^{(1)}(y)$ for some $y \in (\lambda,\bar{x}]$, $y \neq x$.

Bidding $\boldsymbol{\tilde{b}_S^{(1)}(x)}$ is preferred to bidding $\boldsymbol{\tilde{b}_S^{(1)}(y)}$ for some $\boldsymbol{y \neq x}$. For a type $x \in (\lambda,\bar{x}]$, the payoff from bidding as type $y \in (x,\bar{x}]$ is $\tilde{u}_S(x,y)$ as in (61), and is decreasing in y. Therefore, type x prefers to bid $\tilde{b}_S^{(1)}(x)$ rather than $\tilde{b}_S^{(1)}(y)$ for $y > x$. Moreover, from (48) we see that the payoff of type x from bidding as type $y \in (\lambda,x)$ is constant with respect to y, as $\tilde{u}_{S2}(x,y) = (n-1)f(y)\left(yF^{n-2}(y) - \tilde{b}_S^{(1)}(y)F^{n-2}(y) - v_{n-1}(y)\right) = 0$ for each $y \in (\lambda,x)$: Therefore bidding like type $y \in (\lambda,x)$ is no better than bidding $\tilde{b}_S^{(1)}(x)$.

Bidding $\boldsymbol{\tilde{b}_S^{(1)}(x)}$ is no worse than bidding $\boldsymbol{r_1}$. From (47) we see that $\tilde{u}_{S1}(x,\gamma) = (n-1)F^{n-2}(x) - (n-2)F^{n-1}(x)$, which is equal to $\frac{d\tilde{u}_S(x,x)}{dx}$. Therefore $\tilde{u}_S(x,\gamma)$ and $\tilde{u}_S(x,x)$ have parallel graphs in $(\lambda,\bar{x}]$, but (16) implies $\tilde{u}_S(\lambda,\gamma) = \tilde{u}_S(\lambda,\lambda^+) = \lim_{x\downarrow\lambda}\tilde{u}_S(x,x)$, hence $\tilde{u}_S(x,\gamma) = \tilde{u}_S(x,x)$ for each $x \in (\lambda,\bar{x}]$.

Bidding $\boldsymbol{\tilde{b}_S^{(1)}(x)}$ is preferred to no bidding. From (40) we see that $\tilde{u}_{S1}(x,\underline{x}) = (n-1)F^{n-2}(x) - (n-2)F^{n-1}(x)$, which is equal to $\frac{d\tilde{u}_S(x,x)}{dx}$. Therefore $\tilde{u}_S(x,\underline{x})$ and $\tilde{u}_S(x,x)$ have parallel graphs in $(\lambda, \bar{x}]$, but (16) and the proof of (59) imply $\lim_{x\downarrow\lambda} \tilde{u}_S(x,x) = \tilde{u}_S(\lambda, \lambda^+) = \tilde{u}_S(\lambda, \gamma) > \tilde{u}_S(\lambda, \underline{x})$, hence $\tilde{u}_S(x,x) > \tilde{u}_S(x, \underline{x})$ for $x \in (\lambda, \bar{x}]$.

A.2 Proof of Proposition 2(ii)

This proof is largely given by the proof of Proposition 2(i), after setting $\lambda = \bar{x}$, consistently with the remark in footnote 7. Precisely, $\hat{G}(s|\text{no}, r_1)$ in (9) and $\hat{G}(s|r_1, r_1)$ in (11) can be seen as special cases of (36) and (45) with $\lambda = \bar{x}$ and $\Lambda = 1$. Remark that the probability of winning when bidding r_1 is now given by (see (43) with $\Lambda = 1$)

$$\hat{p}(\gamma) = \sum_{j=0}^{n-1} \frac{C_{n-1,j}}{j+1} \Gamma^{n-1-j}(1-\Gamma)^j = \frac{1-\Gamma^n}{n(1-\Gamma)} \tag{62}$$

and in (11), $1 - \hat{p}(\gamma)$ replaces $\tilde{p}_\ell$ at the denominator because $1 - \hat{p}(\gamma)$ is a bidder's probability of losing after a bid of r_1 given $\hat{b}_S^{(1)}$, the analog of $\tilde{p}_\ell$.

The proofs that a unique solution to (13) exists and that $\hat{u}_S(x, \underline{x}) \geq \hat{u}_S(x, \gamma)$ for each $x \in [r_2, \gamma)$ and $\hat{u}_S(x, \underline{x}) \leq \hat{u}_S(x, \gamma)$ for each $x \in (\gamma, \bar{x}]$ are special cases of Steps 2.2, 4.1, and 4.2 in the proof of Proposition 2(i).

Finally, we need to explore the profitability of bidding slightly more than r_1 and to prove that $\max\{\hat{u}_S(x, \underline{x}), \hat{u}_S(x, \gamma)\} \geq x - r_1$ holds for each $x \in [r_2, \bar{x}]$. Since $\hat{u}_{S1}(x, \underline{x}) < 1$ for $x \in [r_2, \gamma)$ and $\hat{u}_{S1}(x, \gamma) < 1$ for $x \in [\gamma, \bar{x}]$, it suffices to prove that $\hat{u}_S(\bar{x}, \gamma) \geq \bar{x} - r_1$. Using (12) and rearranging the inequality we obtain

$$\frac{n(n-1)}{2} v_{n-1}(\gamma) + \int_{\gamma}^{\bar{x}} M(F(s), \Gamma) ds - M(1, \Gamma)(\bar{x} - r_1) \geq 0 \tag{63}$$

In examining this inequality, we need to take into account that γ is the unique solution to (13) given r_1. Since (13) is equivalent to (15) (i.e., to $A(\gamma, \lambda) = 0$ in (50) when $\lambda = \bar{x}$), it follows that the left hand side of (63) is a function of r_1 which coincides with $\ell(r_1)$ introduced in Step 2.3 in the proof of Proposition 2(i). From Step 2.3 we know that $\ell(r_1) \geq 0$ if and only if $r_1 \in [\tilde{r}_1, \bar{r}_1]$.

A.3 Proof of Proposition 2(iii)

The proof of this part (from GTX) has been already presented in subsection 3.2.1.

B Proof of Proposition 3

Step 1: $\lim_{n\to+\infty} \tilde{r}_1 = \bar{x}$.

Recalling Step 2.3 in the proof of Proposition 2(i), we here show that for each $r_1 < \bar{x}$, $B(\gamma_A(\bar{x}), \bar{x})$ is negative for a large n; this implies that $\tilde{r}_1 > r_1$ and that a solution to (50) exists. Precisely, given $r_1 < \bar{x}$ we find

$$B(\gamma_A(\bar{x}), \bar{x}) = \frac{n(n-1)}{2} \int_{r_2}^{\gamma_A(\bar{x})} F^{n-2}(s)ds - \frac{n-1-nF(\gamma_A(\bar{x})) + F^n(\gamma_A(\bar{x}))}{(1-F(\gamma_A(\bar{x})))^2}(\bar{x} - r_1) + \int_{\gamma_A(\bar{x})}^{\bar{x}} \frac{(n-1)F^n(s) - nF^{n-1}(s)F(\gamma_A(\bar{x})) + F^n(\gamma_A(\bar{x}))}{(F(s) - F(\gamma_A(\bar{x})))^2} ds$$

and the first and third term tend to zero; the second term tends to $-\infty$. Hence $B(\gamma_A(\bar{x}), \bar{x}) < 0$ if n is large.

Step 2: $\lim_{n\to+\infty} \gamma = \lim_{n\to+\infty} \lambda = r_1$.

Since for each n we have $r_1 < \gamma < \lambda$, it suffices to prove that $\lim_{n\to+\infty} \lambda = r_1$. Then, recalling Step 2.3 in the proof of Proposition 2(i), we now show that $B(\gamma_A(\lambda), \lambda)$ is negative at $\lambda = r_1 + \varepsilon$ for a large n; this implies that the λ solution is smaller than $r_1 + \varepsilon$ for a large n. For the sake of brevity, we write γ_A instead of $\gamma_A(r_1 + \varepsilon)$, and recall that $r_1 < \gamma_A < r_1 + \varepsilon$. The inequality $B(\gamma_A, r_1 + \varepsilon) < 0$ is equivalent to

$$\varepsilon > \frac{1}{M(F(r_1+\varepsilon), F(\gamma_A))} \left(\frac{n(n-1)}{2} v_{n-1}(\gamma_A) + \int_{\gamma_A}^{r_1+\varepsilon} M(F(s), F(\gamma_A)) ds \right) \tag{64}$$

and now we show that (64) is satisfied for a large n. First notice that

$$\frac{\frac{n(n-1)}{2} v_{n-1}(\gamma_A)}{M(F(r_1+\varepsilon), F(\gamma_A))} < \frac{\frac{n(n-1)}{2} v_{n-1}(\gamma_A)}{M(F(\gamma_A), F(\gamma_A))} = \int_{r_2}^{\gamma_A} \left(\frac{F(s)}{F(\gamma_A)} \right)^{n-2} ds$$

Hence the first term in the right hand side of (64) tends to zero. Regarding the second term in the right hand side of (64), we need to consider two cases:

- If $\gamma_A \le r_1 + \frac{\varepsilon}{2}$, then consider three numbers b, a, c in $[0,1]$, such that $b < a < c$ and notice that
$$\frac{M(a,b)}{M(c,b)} = \left(\frac{c-b}{a-b} \right)^2 \frac{(n-1)(\frac{a}{c})^n - n(\frac{a}{c})^{n-1}\frac{b}{c} + (\frac{b}{c})^n}{n-1-\frac{b}{c}n + (\frac{b}{c})^n}$$
tends to zero. Then we conclude that $\int_{\gamma_A}^{r_1+\varepsilon} \frac{M(F(s),F(\gamma_A))}{M(F(r_1+\varepsilon),F(\gamma_A))} ds$ tends to zero by taking $c = F(r_1+\varepsilon)$, $a = F(s)$, $b = F(\gamma_A)$. Hence, (64) is satisfied for a large n.

- If instead $\gamma_A > r_1 + \frac{\varepsilon}{2}$, we have that
$$\frac{\int_{\gamma_A}^{r_1+\varepsilon} M(F(s), F(\gamma_A)) ds}{M(F(r_1+\varepsilon), F(\gamma_A))} < \frac{M(F(r_1+\varepsilon), F(\gamma_A))(r_1+\varepsilon-\gamma_A)}{M(F(r_1+\varepsilon), F(\gamma_A))} = r_1 + \varepsilon - \gamma_A < \frac{\varepsilon}{2}$$
Since $\int_{r_2}^{\gamma_A} \left(\frac{F(s)}{F(\gamma_A)} \right)^{n-2} ds < \frac{1}{4}\varepsilon$ for a large n, it follows that the right hand side in (64) is smaller than $\frac{3}{4}\varepsilon$, and (64) is satisfied.

C Proof of Proposition 4

Let $\mathbf{x} = (x_1, ..., x_n)$, $\mathbf{X} = (X_1, ..., X_n)$, and given a sequence of reserve prices (increasing or decreasing) and an equilibrium for sequential second price auctions, for $i = 1, ..., n$ we use $q_i(\mathbf{x})$ to denote the probability that bidder i wins an object (either in stage one or in stage two) in that equilibrium, given the profile $\mathbf{x}$ of values. Moreover, in each equilibrium described by Propositions 1 and 2 the expected payoff of each bidder with value 0 is equal to zero, therefore arguing as in Myerson (1981) we find that the sellers' total payoff is given by

$$\pi = E\left[\phi(X_1)q_1(\mathbf{X}) + \cdots + \phi(X_n)q_n(\mathbf{X})\right] - r\Pr\{\text{the object with reserve price } r \text{ is sold}\} \quad (65)$$

We have introduced in the main text the order statistics. Here we introduce $\mathbf{y} = (y_1, \ldots, y_n)$, $\mathbf{Y} = (Y_1, \ldots, Y_n)$, and the joint density of $\mathbf{Y}$ is

$$f_{\mathbf{Y}}(y_1, \ldots, y_n) = \begin{cases} n! f(y_1)f(y_2)\cdots f(y_n) & \text{if } y_1 \geq y_2 \geq \cdots \geq y_n \\ 0 & \text{otherwise} \end{cases} \quad (66)$$

In case of increasing reserve prices, that is $r_1 = 0$, $r_2 = r$, let π_{IRP} denote the profit in (65) given $q_1, ..., q_n$, $\Pr\{\text{the object with reserve price } r \text{ is sold}\}$ resulting from the equilibrium described by Proposition 1. Neglecting ties, that are zero probability events, for each bidder i we have that $q_i(\mathbf{x}) = 1$ if $x_i = y_1$ or $x_i = y_2 \geq r$, otherwise $q_i(\mathbf{x}) = 0$. This reveals that π_{IRP} is equal to the expectation of the following function θ_{IRP}:

$$\theta_{\text{IRP}}(\mathbf{y}) = \begin{cases} \phi(y_1) & \text{if } y_2 < r \\ \phi(y_1) + \phi(y_2) - r & \text{if } r \leq y_2 \end{cases} \quad (67)$$

Likewise, if reserve prices are decreasing (that is $r_1 = r$, $r_2 = 0$), we denote the profit in (65) as $\bar{\pi}_{\text{DRP}}$, or $\hat{\pi}_{\text{DRP}}$, or $\tilde{\pi}_{\text{DRP}}$ depending on which case of Proposition 2 applies, and in each of these cases the profit can be written as the expectation of a suitable function of $\mathbf{y}$:

- If $\bar{r}_1 \leq r$, then $\bar{\pi}_{\text{DRP}} = E[\bar{\theta}_{\text{DRP}}(\mathbf{Y})]$ with

$$\bar{\theta}_{\text{DRP}}(\mathbf{y}) = \phi(y_1) \quad (68)$$

- If $\tilde{r}_1 \leq r < \bar{r}_1$, then $\hat{\pi}_{\text{DRP}} = E[\hat{\theta}_{\text{DRP}}(\mathbf{Y})]$ with

$$\hat{\theta}_{\text{DRP}}(\mathbf{y}) = \begin{cases} \phi(y_1) & \text{if } y_1 < \gamma \\ \phi(y_1) + \phi(y_2) - r & \text{if } y_3 < \gamma \leq y_1 \\ \phi(y_1) + \frac{2}{m}\phi(y_2) + \frac{1}{m}\phi(y_3) + \cdots + \frac{1}{m}\phi(y_m) - r & \text{if } y_{m+1} < \gamma \leq y_m,\ m \geq 3 \end{cases}$$

- If $r < \tilde{r}_1$, then $\tilde{\pi}_{\text{DRP}} = E[\tilde{\theta}_{\text{DRP}}(\mathbf{Y})]$ with

$$\tilde{\theta}_{\text{DRP}}(\mathbf{y}) = \begin{cases} \phi(y_1) & \text{if } y_1 < \gamma \\ \phi(y_1) + \phi(y_2) - r & \text{if } y_3 < \gamma < y_1 \leq \lambda, \text{ or } \lambda < y_1 \\ \phi(y_1) + \frac{2}{m}\phi(y_2) + \frac{1}{m}\phi(y_3) + \cdots + \frac{1}{m}\phi(y_m) - r & \text{if } y_{m+1} < \gamma \leq y_m \leq y_1 \leq \lambda,\ m \geq 3 \end{cases}$$

Notice that the last line in $\hat{\theta}_{\text{DRP}}$ and the last line in $\tilde{\theta}_{\text{DRP}}$ take into account that in stage one the winner is selected randomly when Proposition 2 (i)-(ii) applies and at least two bidders have values in $[\gamma, \bar{x}]$ (for $\hat{\theta}_{\text{DRP}}$) or in $[\gamma, \lambda]$ (for $\tilde{\theta}_{\text{DRP}}$).

C.1 Proof of Proposition 4(i)

From (67) and (68) we see that

$$\theta_{\text{IRP}}(\mathbf{y}) - \bar{\theta}_{\text{DRP}}(\mathbf{y}) = \begin{cases} 0 & \text{if } y_2 < r \\ \phi(y_2) - r & \text{if } r \leq y_2 \end{cases}$$

that is θ_{IRP} and $\bar{\theta}_{\text{DRP}}$ differ only when $y_2 \geq r$. From (66) it follows that the density of Y_2 is $f_2(y_2) = n(n-1)f(y_2)[1-F(y_2)]F^{n-2}(y_2)$, hence

$$\bar{\Delta}(r) \equiv \pi_{\text{IRP}} - \bar{\pi}_{\text{DRP}} = E[\theta_{\text{IRP}}(\mathbf{Y}) - \bar{\theta}_{\text{DRP}}(\mathbf{Y})] = \int_r^1 (\phi(y_2) - r) n(n-1) f(y_2)[1-F(y_2)]F^{n-2}(y_2)dy_2 \tag{69}$$

with $\bar{\Delta}(1) = 0$, and

$$\bar{\Delta}'(r) = (n-1)^2 F^n(r) - n(2n-3)F^{n-1}(r) + n(n-1)F^{n-2}(r) - 1$$
$$\bar{\Delta}''(r) = n(n-1)^2 F^{n-3}(r) f(r)[1-F(r)] \left[\frac{n-2}{n-1} - F(r)\right]$$

with $\bar{\Delta}'(1) = 0$. We let r' be such that $F(r') = \frac{n-2}{n-1}$, and distinguish the case of $r' \leq \bar{r}_1$ from the case of $\bar{r}_1 < r'$.

In the first case we have $\bar{\Delta}''(r) < 0$ for each $r \in (\bar{r}_1, 1)$, therefore $\bar{\Delta}'$ is strictly decreasing in $[\bar{r}_1, 1]$, and since $\bar{\Delta}'(1) = 0$ it follows that $\bar{\Delta}'(r) > 0$ for each $r \in (\bar{r}_1, 1)$. Then, from $\bar{\Delta}(1) = 0$ we conclude that $\bar{\Delta}(r) < 0$ for each $r \in [\bar{r}_1, 1)$.

In the second case we have $\bar{\Delta}''(r) > 0$ for $r \in (\bar{r}_1, r')$ and $\bar{\Delta}''(r) < 0$ for $r \in (r', 1)$, therefore $\bar{\Delta}'$ is strictly increasing in $(\bar{r}_1, r')$, is strictly decreasing in $(r', 1)$. Since $\bar{\Delta}'(1) = 0$, it follows that $\bar{\Delta}'$ is positive in an interval $(r'', \bar{x})$ such that $r'' < r'$ and r'' may be equal to $\bar{r}_1$; if $r'' = \bar{r}_1$ holds, then we conclude that $\bar{\Delta}(r) < 0$ for each $r \in [\bar{r}_1, 1)$, as in the first case. Conversely, if $\bar{r}_1 < r''$, then $\bar{\Delta}'(r) < 0$ in $(\bar{r}_1, r'')$, $\bar{\Delta}'(r) > 0$ in $(r'', \bar{x})$. Since $\bar{\Delta}(1) = 0$, it follows that $\bar{\Delta}(r)$ is negative in an interval $(r''', 1)$ such that $r''' < r''$ and r''' may be equal to $\bar{r}_1$. In any case, if $\bar{\Delta}(\bar{r}_1) \leq 0$ then $\bar{\Delta}(r) < 0$ for each $r \in (\bar{r}_1, 1)$.

C.2 Proof of (19)

For each $r \in (0,1)$, $\pi^r_{\text{IRP}} = E[\max\{Y_3 - r, 0\}]$. We show that for each $r \in (0,1)$, $\pi^r_{\text{DRP}} < \pi^r_{\text{IRP}}$ holds.

This is clearly true when $\pi^r_{\text{DRP}} = 0$, which happens for all $r \geq \tilde{r}_1$ (under DRP, in fact, no bidder bids at stage one for $r \geq \bar{r}_1$, and no bidder bids more than r for $r \in [\tilde{r}_1, \bar{r}_1)$) and for $r \in (0, \tilde{r}_1)$ if $y_2 \leq \lambda$ (here again the second largest bid does not exceed r).

Finally, when $r \in (0, \tilde{r}_1)$ and $y_2 > \lambda$, we have that the profits under DRP are given by $\beta_{n-1,0}(y_2) - r$. From (20) we can derive the following equality

$$\beta_{n-1,0}(y_2) - r = E(Y_3 - r | Y_2 = y_2) < E(\max\{Y_3 - r, 0\} | Y_2 = y_2)$$

while the inequality follows because when $y_2 > \lambda$ we have that Y_3 is smaller than r with positive probability. Therefore $\pi^r_{\text{DRP}} < \pi^r_{\text{IRP}}$ also if $r \in (0, \tilde{r}_1)$.

C.3 Proof of Proposition 4(ii)

Given (19), here it suffices to show that $\pi^0_{\text{IRP}} > \pi^0_{\text{DRP}}$. In order to do this, we denote with $\omega(y_2)$ the expected difference in profit (from the sale of the object with zero reserve price) between IRP and DRP given $Y_2 = y_2$: hence $\pi^0_{\text{IRP}} - \pi^0_{\text{DRP}} = \int_0^1 \omega(y_2) f_2(y_2) dy_2$. It is useful to distinguish between the following cases: $y_2 \in (0, r]$, $y_2 \in (r, \lambda]$, and $y_2 > \lambda$ and write

$$\int_0^1 \omega(y_2) f_2(y_2) dy_2 = \int_0^r \omega(y_2) f_2(y_2) dy_2 + \int_r^\lambda \omega(y_2) f_2(y_2) dy_2 + \int_\lambda^1 \omega(y_2) f_2(y_2) dy_2$$

Steps 2-4 below imply that $\int_0^1 \omega(y_2) f_2(y_2) dy_2 > 0$ for r close to zero, but first we need to prove a property of λ. For the rest of this proof we define $\underline{f} \equiv \min_{x \in [0,1]} f(x) > 0$, $\bar{f} \equiv \max_{x \in [0,1]} f(x) \geq \underline{f}$.

Step 1: There exists a $\xi > 1$ such that $\lambda < \xi r$ when r is close to zero

We can verify that B in (52) is strictly decreasing in both variables and from the proof of Proposition 2 (Step 2.3) we know that there exists a solution γ, λ to (50) such that $r < \gamma < \lambda$ exists. We now prove that $B(r, \xi r) < 0$ for a suitable $\xi > 1$, and this implies that $B(\gamma, z) < 0$ for each $\gamma \geq r$ and for each $z \geq \xi r$; therefore $\lambda < \xi r$. We have that

$$B(r,z) = \frac{n(n-1)}{2} \int_0^r F^{n-2}(s) ds - M(F(z), F(r))(z - r) + \int_r^z M(F(s), F(r)) ds$$

with $B(r,r) = \frac{n(n-1)}{2} \int_0^r F^{n-2}(s) ds > 0$ and $B_2(r,z) = -M_1(F(z), F(r)) f(z)(z-r) < 0$. Hence,

$$B(r,z) = \frac{n(n-1)}{2} \int_0^r F^{n-2}(s) ds - \int_r^z M_1(F(s), F(r)) f(s)(s - r) ds$$

and

$$\begin{aligned}M_1(F(s),F(r)) &= (n-1)(n-2)F^{n-3}(s)+(n-2)(n-3)F^{n-4}(s)F(r)+...+2F^{n-3}(r)\\ &= F^{n-3}(r)\left((n-1)(n-2)\left(\frac{F(s)}{F(r)}\right)^{n-3}+(n-2)(n-3)\left(\frac{F(s)}{F(r)}\right)^{n-4}+...+2\right)\\ &\geq F^{n-3}(r)\sum_{j=1}^{n-2}j(j+1)=\frac{n(n-1)(n-2)}{3}F^{n-3}(r)\end{aligned}$$

where the inequality follows because $s \geq r$. Since

$$F(r)=\int_0^r f(s)ds \leq \int_0^r \bar{f}ds = r\bar{f} \tag{70}$$

we have that

$$\begin{aligned}B(r,z) &< \frac{n(n-1)}{2}rF^{n-2}(r)-\frac{n(n-1)(n-2)}{3}F^{n-3}(r)\underline{f}\int_r^z (s-r)ds\\ &\qquad < \frac{n(n-1)}{2}F^{n-3}(r)\left(r^2\bar{f}-\frac{n-2}{3}\underline{f}(z-r)^2\right)\end{aligned}$$

and the right hand side is equal to zero if $z=\xi r$, with $\xi = 1+\sqrt{\frac{3\bar{f}}{(n-2)\underline{f}}}$.

Step 2: $\int_0^r \omega(y_2)f_2(y_2)dy_2 > 0$

Given $y_2 \in (0,r]$, we find that the profit under IRP is y_2, whereas the profit under DRP is y_3 if $y_1 \geq \gamma$, or y_2 if $y_1 < \gamma$; this implies that $\omega(y_2)=E[(y_2-Y_3)\mathbf{1}_{\{Y_1\geq\gamma\}}|Y_2=y_2]>0$ for $y_2 \leq r$, hence $\int_0^r \omega(y_2)f_2(y_2)dy_2 > 0$.

Step 3: $\int_r^\lambda \omega(y_2)f_2(y_2)dy_2 > -\eta_1 r^n$ for a suitable $\eta_1 > 0$

Given $y_2 \in (r,\lambda]$, we find that the profit under IRP is $\beta_{n-1,0}(y_2)$, whereas the profit under DRP is y_3 (if at stage one the highest or the second highest value bidder wins), or y_2 (if at stage one the highest or the second highest value bidder does not win). Therefore the profit difference is necessarily larger than $\beta_{n-1,0}(r)-\lambda = r-\lambda > -(\xi-1)r$. As a result, $\int_r^\lambda \omega(y_2)f_2(y_2)dy_2 > -(\xi-1)r\int_r^\lambda f_2(y_2)dy_2$. Now observe that $\int_r^\lambda f_2(y_2)dy_2 < \int_r^\lambda n(n-1)\bar{f}^{n-1}y_2^{n-2}dy_2 < n(\bar{f}\xi)^{n-1}r^{n-1}$, where the first inequality follows again from (70). The result is then obtained once we let $\eta_1=(\xi-1)n(\bar{f}\xi)^{n-1}$.

Step 4: $\int_\lambda^1 \omega(y_2)f_2(y_2)dy_2 \geq \eta_2 r^{n-1}$ for a suitable $\eta_2 > 0$

Given $y_2 > \lambda$, we have described in the main text (see page 16) that $\omega(y_2)=\frac{\int_0^r F^{n-2}(s)ds}{F^{n-2}(y_2)}$, and $\int_\lambda^1 \omega(y_2)f_2(y_2)dy_2 = \frac{n(n-1)(1-\Lambda)^2}{2}\int_0^r F^{n-2}(s)ds \geq \frac{n(n-1)(1-\Lambda)^2}{2}\int_0^r(\underline{f}s)^{n-2}ds=\eta_2 r^{n-1}$ where the inequality follows by using the logic in (70) and using $\underline{f}$ instead, and by letting $\eta_2 = \frac{n(1-\Lambda)^2 \underline{f}^{n-2}}{2}$.

C.4 Proof of Proposition 4(iii)

When the values are uniformly distributed, we know that $\bar{r}_1 = \frac{n-1}{n}$ and we can verify that $\phi(x) = 2x - 1$ is an increasing function.

C.4.1 Case of $\bar{r}_1 \leq r$

For each $n \geq 3$ we prove that $\bar{\Delta}(\bar{r}_1) < 0$, and then Proposition 4(i) implies that $\bar{\Delta}(r) < 0$ for each $r \in (\bar{r}_1, 1)$. From (69) we find

$$\bar{\Delta}(\bar{r}_1) = \frac{8n^2 - 7n + 1}{n\left(n^2 - 1\right)} \left[\left(1 - \frac{1}{n}\right)^n - \frac{3n^2 - 4n + 1}{8n^2 - 7n + 1}\right]$$

and $(1 - \frac{1}{n})^n - \frac{3n^2-4n+1}{8n^2-7n+1}$ is equal to $-\frac{4}{351} < 0$ for $n = 3$, is equal to $-\frac{267}{25\,856} < 0$ for $n = 4$. We now prove that $(1 - \frac{1}{n})^n - \frac{3n^2-4n+1}{8n^2-7n+1} < 0$ for each $n \geq 5$, which implies $\Delta(\bar{r}_1) < 0$. We have that

$$\left(1 - \frac{1}{n}\right)^n = \sum_{k=0}^{n} C_{n,k} \left(-\frac{1}{n}\right)^k = 1 - 1 + \sum_{k=2}^{4} C_{n,k} \left(-\frac{1}{n}\right)^k + \sum_{k=5}^{n} C_{n,k} \left(-\frac{1}{n}\right)^k$$

Now consider the case of n even, because when n is odd the last term in the sum above is negative and so makes our point easier. We have that

$$\sum_{k=5}^{n} C_{n,k} \left(-\frac{1}{n}\right)^k = \sum_{k=2}^{(n-2)/2} \left(-C_{n,2k+1} \frac{1}{n^{2k+1}} + C_{n,2k+2} \frac{1}{n^{2k+2}}\right)$$

and

$$-C_{n,2k+1} \frac{1}{n^{2k+1}} + C_{n,2k+2} \frac{1}{n^{2k+2}} = -\frac{(2k+1)C_{n+1,2k+2}}{n^{2k+2}} < 0$$

for $k = 2, ..., \frac{n-2}{2}$. Therefore

$$\left(1 - \frac{1}{n}\right)^n < \sum_{k=2}^{4} C_{n,k} \left(-\frac{1}{n}\right)^k = \frac{(n-1)\left(3n^2 + n + 2\right)}{8n^3}$$

Last,

$$\frac{(n-1)\left(3n^2 + n + 2\right)}{8n^3} - \frac{3n^2 - 4n + 1}{8n^2 - 7n + 1} = -(n-1)\frac{n\left(n-1\right)\left(5n-7\right) + 6n - 2}{8n^3(8n^2 - 7n + 1)} < 0$$

C.4.2 Case of $0 < r < \tilde{r}_1$

When $r < \tilde{r}_1$, recall that γ and λ are solutions of (50), and from Subsection 3.4 we know that $\gamma = c_\gamma r$, $\lambda = c_\lambda r$. Moreover we have that:

$$\theta_{\text{IRP}}(\mathbf{y}) - \tilde{\theta}_{\text{DRP}}(\mathbf{y}) = \begin{cases} 0 & \text{if } y_1 < \gamma \text{ and } y_2 < r, \text{ or } y_3 < \gamma \leq y_1 \leq \lambda, \text{ or } \lambda < y_1 \\ \phi(y_2) - r & \text{if } r \leq y_2 \leq y_1 < \gamma \\ r - \phi(y_2) & \text{if } y_2 < r < \gamma \leq y_1 \\ \frac{m-2}{m}\phi(y_2) - \frac{1}{m}\phi(y_3) - \cdots - \frac{1}{m}\phi(y_m) & \text{if } y_{m+1} < \gamma \leq y_m \leq y_1 \leq \lambda, \ m \geq 3 \end{cases} \tag{71}$$

Since ϕ is increasing, it follows that $\frac{m-2}{m}\phi(y_2) - \frac{1}{m}\phi(y_3) - \cdots - \frac{1}{m}\phi(y_m) \geq 0$. Therefore $\tilde{\Delta}(r) \equiv \pi_{\text{IRP}} - \tilde{\pi}_{\text{DRP}} = E[\theta_{\text{IRP}}(\mathbf{Y}) - \tilde{\theta}_{\text{DRP}}(\mathbf{Y})]$ is such that

$$\begin{aligned} \tilde{\Delta}(r) &\geq \int_r^\gamma \int_{y_2}^\gamma (\phi(y_2) - r) f_{1,2}(y_1, y_2) dy_1 dy_2 + \int_0^r \int_\gamma^1 (r - \phi(y_2)) f_{1,2}(y_1, y_2) dy_1 dy_2 \\ &= n(n-1)\left(\int_r^{c_\gamma r} (2y_2 - 1 - r)(c_\gamma r - y_2) y_2^{n-2} dy_2 + (1 - c_\gamma r)\int_0^r (r - 2y_2 + 1) y_2^{n-2} dy_2\right) \\ &= r^{n-1} W(c_\gamma, r) \end{aligned}$$

in which $f_{1,2}(y_1, y_2) = n(n-1)y_2^{n-2}$ is the joint density of Y_1, Y_2, obtained from (66), and

$$W(c_\gamma, r) \equiv n - (2n - 3 + c_\gamma^n) r + \frac{(n-1)^2 + (2c_\gamma(n-1) - n - 1)c_\gamma^n}{n+1} r^2$$

From the proof of Proposition 2(i) (Step 2.3), we know that the function B defined in (52) is strictly decreasing in both its variables. Therefore if z^* is the unique solution to $B(z, z) = 0$, then γ, λ solutions to (50) are such that $\gamma < z^* < \lambda$. The equation $B(z, z) = 0$ is equivalent to $\frac{n(n-1)}{2}\int_0^z s^{n-2} ds - \frac{n(n-1)}{2} z^{n-2}(z - r) = 0$, hence $z^* = \frac{n-1}{n-2} r$ and $c_\gamma < \frac{n-1}{n-2} < c_\lambda$, $\tilde{r}_1 = \frac{1}{c_\lambda} < \frac{n-2}{n-1}$.

Step 1: W is strictly decreasing with respect to its first variable. Therefore, $W(c_\gamma, r) > W(\frac{n-1}{n-2}, r)$. We have that $W_1(c_\gamma, r) = c_\gamma^{n-1} r\,(2c_\gamma r(n-1) - n - nr)$, and $2c_\gamma r(n-1) - n - nr < 2\frac{n-1}{n-2}\frac{n-2}{n-1}(n-1) - n - n\frac{n-2}{n-1} = -\frac{n-2}{n-1} < 0$ since $c_\gamma < \frac{n-1}{n-2}$ and $r < \frac{n-2}{n-1}$.

Step 2: For each $n \geq 4$ we have $W(\frac{n-1}{n-2}, r) > 0$ for each $r \leq \frac{5n-12}{5n-5}$. We find that $W(\frac{n-1}{n-2}, \frac{5n-12}{5n-5}) > 0$ for each $n \geq 4$ since

$$W\left(\tfrac{n-1}{n-2}, \tfrac{5n-12}{5n-5}\right) = \frac{(5n-12)(17n^2 - 61n + 58)}{25(n+1)(n-2)(n-1)^2}\left(\frac{2(127n - 162)(n-1)(n-2)}{(5n-12)(17n^2 - 61n + 58)} - \left(\frac{n-1}{n-2}\right)^n\right)$$

and, therefore, $W(\frac{n-1}{n-2},\frac{5n-12}{5n-5})=\frac{344}{1125}>0$ for $n=4$, while for $n\geq 5$ we have that

$$\frac{2\,(127n-162)\,(n-1)\,(n-2)}{(5n-12)\,(17n^2-61n+58)}-\left(\frac{n-1}{n-2}\right)^n>\frac{2\,(127n-162)\,(n-1)\,(n-2)}{(5n-12)\,(17n^2-61n+58)}-\frac{275}{100}\left(\frac{n-1}{n-2}\right)^2=$$
$$\frac{(n-1)\left(\left((27n-8)^2+5534\right)(n-5)^2+23\,742(n-5)+3168\right)}{36\,(5n-12)\,(17n^2-61n+58)\,(n-2)^2}>0$$

Moreover, $W(\frac{n-1}{n-2},1)=-\frac{2(n-3)}{(n+1)(n-2)}\left(\left(\frac{n-1}{n-2}\right)^n-2-\frac{2}{n-3}\right)$ is negative for each $n\geq 4$. Since $W(\frac{n-1}{n-2},r)$ is a convex quadratic function of r, it follows that $W(\frac{n-1}{n-2},r)>0$ for each $r\in(0,\frac{5n-12}{5n-5}]$.

Step 3: $\tilde{\boldsymbol{\Delta}}(r)>0$ for each $r\leq\min\{\frac{5n-12}{5n-5},\tilde{r}_1\}$. If $\tilde{r}_1\leq\frac{5n-12}{5n-5}$, then Steps 1 and 2 establish that $W(c_\gamma,r)>W(\frac{n-1}{n-2},\tilde{r}_1)>0$ for each $r\in(0,\tilde{r}_1)$. If $\tilde{r}_1>\frac{5n-12}{5n-5}$, then Steps 1 and 2 establish that $W(c_\gamma,r)>W(\frac{n-1}{n-2},\frac{5n-12}{5n-5})>0$ for each $r\in(0,\frac{5n-12}{5n-5})$.

C.4.3 Case of $n=3$

Given $n=3$, we computed (see Table 1) $\tilde{r}_1=2\sqrt{3}-3$ and $\bar{r}_1=\frac{2}{3}$. We have proved above that $\pi_{\text{IRP}}-\tilde{\pi}_{\text{DRP}}<0$ for each $r\in[\bar{r}_1,1]$ (for each $n\geq 3$). Here we consider $r\in(0,\frac{2}{3})$.

When $r\in(0,2\sqrt{3}-3)$, we use (71) to find that $\tilde{\Delta}(r)=\pi_{\text{IRP}}-\tilde{\pi}_{\text{DRP}}$ is equal to

$$\int_r^\gamma\int_{y_2}^\gamma(2y_2-1-r)6y_2dy_1dy_2+\int_0^r\int_\gamma^1(r-2y_2+1)6y_2dy_1dy_2+\int_\gamma^\lambda\int_\gamma^{y_1}\int_\gamma^{y_2}(\frac{2}{3}y_2-\frac{2}{3}y_3)6dy_3dy_2dy_1$$
$$=\frac{7}{6}\gamma^4-(r+\frac{2}{3}\lambda+1)\gamma^3+\lambda^2\gamma^2-\frac{2}{3}\lambda^3\gamma+r^4-3r^3+3r^2+\frac{1}{6}\lambda^4$$

and we know from subsection 3.4 that $\gamma=(1+\frac{1}{3}\sqrt{3})r$, $\lambda=(1+\frac{2}{3}\sqrt{3})r$, hence $\tilde{\Delta}(r)=r^2(\frac{36\sqrt{3}+115}{54}r^2-\frac{45+10\sqrt{3}}{9}r+3)$ is positive for each $r\in(0,2\sqrt{3}-3)$.

When $r\in[2\sqrt{3}-3,\frac{2}{3})$, in order to evaluate $\hat{\Delta}(r)\equiv\pi_{\text{IRP}}-\hat{\pi}_{\text{DRP}}=E[\theta_{\text{IRP}}(\mathbf{Y})-\hat{\theta}_{\text{DRP}}(\mathbf{Y})]$ we use

$$\theta_{\text{IRP}}(\mathbf{y})-\hat{\theta}_{\text{DRP}}(\mathbf{y})=\begin{cases}0 & \text{if } y_1<\gamma \text{ and } y_2<r\text{, or } y_3<\gamma\leq y_1 \text{ and } r\leq y_2\\ \phi(y_2)-r & \text{if } r\leq y_2\leq y_1<\gamma\\ r-\phi(y_2) & \text{if } \gamma\leq y_1 \text{ and } y_2<r\\ \frac{1}{3}\phi(y_2)-\frac{1}{3}\phi(y_3) & \text{if } \gamma\leq y_3\end{cases}$$

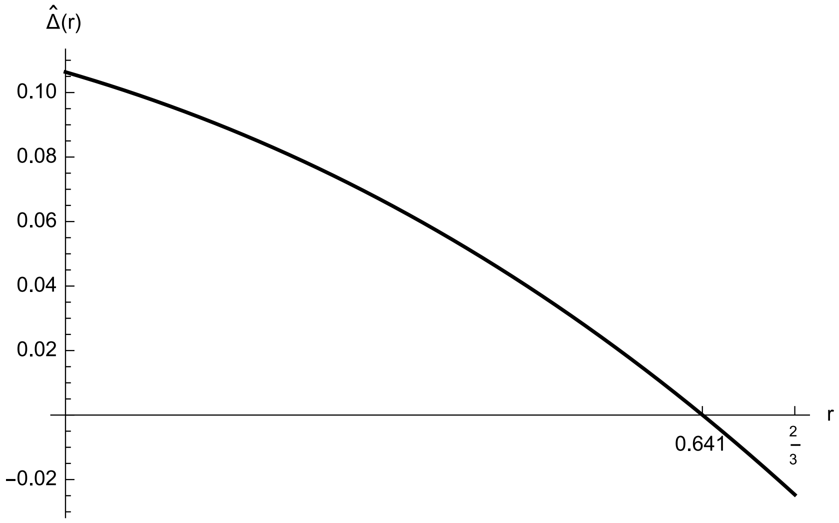

Figure 3: Plot of $\hat{\Delta}(r)$ when $n = 3$ and $r \in (2\sqrt{3} - 3, 2/3)$.

Therefore

$$\begin{aligned}\hat{\Delta}(r) &= \int_r^{\gamma}\int_r^{y_1}(2y_2 - 1 - r)6y_2dy_2dy_1 + \int_{\gamma}^{1}\int_0^{r}(r - 2y_2 + 1)6y_2dy_2dy_1 \\ &+ \int_{\gamma}^{1}\int_{\gamma}^{y_1}\int_{\gamma}^{y_2}(\frac{2}{3}y_2 - \frac{2}{3}y_3)6dy_3dy_2dy_1 \\ &= \frac{7}{6}\gamma^4 - (r + \frac{5}{3})\gamma^3 + \gamma^2 - \frac{2}{3}\gamma + r^4 - 3r^3 + 3r^2 + \frac{1}{6} \end{aligned} \tag{72}$$

The value of γ is determined as the unique solution in the interval $(r, 1)$ to the equation $3\gamma^3 - \gamma^2 + 2\gamma - 2(1 + \gamma + \gamma^2)r = 0$ (from (51) with $\lambda = 1$). Hence we find

$$\begin{aligned}\gamma = \frac{1}{9}(2r + 1) &+ \frac{1}{9}\sqrt[3]{8r^3 + 66r^2 + 222r - 26 + 9\sqrt{3}\sqrt{12r^4 + 112r^3 + 276r^2 - 126r + 23}} \\ &+ \frac{4r^2 + 22r - 17}{9\sqrt[3]{8r^3 + 66r^2 + 222r - 26 + 9\sqrt{3}\sqrt{12r^4 + 112r^3 + 276r^2 - 126r + 23}}}\end{aligned} \tag{73}$$

Inserting (73) into (72) reveals that the graph of $\hat{\Delta}$ in the interval $[2\sqrt{3} - 3, \frac{2}{3})$ is as in Figure 3, and $\hat{\Delta}(r) = 0$ if and only if $r = 0.641$.

D Proof for Proposition 5

D.1 Proof of Proposition 5(i)

Consider $r_1 \in (r_2, \tilde{r}_1)$, and let γ, λ be the unique solution to (15)-(16). Here we prove that the following bidding functions constitute an equilibrium:[14]

$$\tilde{b}_F^{(1)}(x) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, \gamma) \\ r_1 & \text{if } x \in [\gamma, \lambda] \\ \int_{\underline{x}}^{x} \frac{\max\{r_1, \tilde{b}_S^{(1)}(s)\} dF^{n-1}(s)}{F^{n-1}(x)} & \text{if } x \in (\lambda, \bar{x}] \end{cases} \tag{74}$$

$$\tilde{b}_F^{(2)}(x|\text{no}, \text{no}) = \begin{cases} \beta_{n,r_2}(x) & \text{if } x \in [r_2, \gamma) \\ \beta_{n,r_2}(\gamma) & \text{if } x \in [\gamma, \bar{x}] \end{cases} \tag{75}$$

$$\tilde{b}_F^{(2)}(x|\text{no}, r_1) = \begin{cases} \beta_{n-1,r_2}(x) & \text{if } x \in [r_2, \gamma) \\ \tilde{b}_F^{(2)}(\tilde{y}(x)|r_1, r_1) \text{ such that } \tilde{y}(x) \text{ is in} & \\ \arg\max_{y \in [\gamma, \lambda]} (x - \tilde{b}_F^{(2)}(y|r_1, r_1)) \tilde{G}(y|\text{no}, r_1) & \text{if } x \in [\gamma, \bar{x}] \end{cases} \tag{76}$$

$$\tilde{b}_F^{(2)}(x|r_1, r_1) = \begin{cases} \beta_{n-1,r_2}(x) & \text{if } x \in [r_2, \gamma) \\ \frac{\beta_{n-1,r_2}(\gamma)\tilde{G}(\gamma|r_1,r_1) + \int_\gamma^x s\tilde{g}(s|r_1,r_1)ds}{\tilde{G}(x|r_1,r_1)} & \text{if } x \in [\gamma, \lambda] \\ \frac{\beta_{n-1,r_2}(\gamma)\tilde{G}(\gamma|r_1,r_1) + \int_\gamma^\lambda s\tilde{g}(s|r_1,r_1)ds}{\tilde{G}(\lambda|r_1,r_1)} & \text{if } x \in (\lambda, \bar{x}] \end{cases} \tag{77}$$

$$\tilde{b}_F^{(2)}(x|\mathfrak{b}, \tilde{b}_F^{(1)}(z)) = \begin{cases} \beta_{n-1,r_2}(x) & \text{if } x \in [r_2, z) \\ \beta_{n-1,r_2}(z) & \text{if } x \in [z, \bar{x}] \end{cases} \quad \text{for each } z \in (\lambda, \bar{x}],\ \mathfrak{b} \leq \tilde{b}_F^{(1)}(z) \tag{78}$$

$$\tilde{b}_F^{(2)}(x|\mathfrak{b}, \mathfrak{b}_w) = \begin{cases} \beta_{n-1,r_2}(x) & \text{if } x \in [r_2, \bar{x}] \end{cases} \quad \text{for each } \mathfrak{b}_w > \tilde{b}_F^{(1)}(\bar{x}),\ \mathfrak{b}_w \geq \mathfrak{b} \tag{79}$$

Remark that, in light of $\tilde{b}_F^{(1)}(x)$, $\tilde{b}_F^{(2)}(x|\text{no,no})$ for $x \in [\gamma, \overline{x}]$, $\tilde{b}_F^{(2)}(x|\text{no}, r_1)$ for $x \in [\gamma, \overline{x}]$, $\tilde{b}_F^{(2)}(x|r_1, r_1)$ for $x \notin [\gamma, \lambda]$, $\tilde{b}_F^{(2)}(x|\mathfrak{b}, \tilde{b}_F^{(1)}(z))$ for $x \in [z, \overline{x}]$, and $\tilde{b}_F^{(2)}(x|\mathfrak{b}, \mathfrak{b}_w)$ for $x \in [\underline{x}, \overline{x}]$ relate to off-the-equilibrium play. Remark also that $\tilde{b}_F^{(2)}(x|r_1, r_1)$ is constant for $x \in (\lambda, \overline{x}]$.

Step 1: Proof for stage two

In this first step we prove that for each possible outcome at stage one, the bidding specified by (75)-(79) constitutes an equilibrium at stage two. We start by noticing that $\tilde{b}_F^{(1)}$ generates the same stage two beliefs for losing bidders as $\tilde{b}_S^{(1)}$. Precisely, by comparing (74) with (14), we see that this property is true if $\mathfrak{b}_w = no$, or if $\mathfrak{b}_w = r_1$; in these cases the updated beliefs are given by (35), (36), and (45). But the property is true also if $\mathfrak{b}_w = \tilde{b}_F^{(1)}(z)$ for some $z \in (\lambda, \bar{x}]$,

[14] For the sake of brevity, in each bidding function relative to stage two we consider only $x \geq r_2$, since each type with value smaller than r_2 does not bid at stage two, regardless of the outcome of stage one.

as $\tilde{b}_S^{(1)}$ is strictly increasing in the interval $(\lambda, \bar{x}]$: in this case the updated beliefs are given by the c.d.f.

$$\tilde{G}(s|\mathfrak{b}, \tilde{b}_F^{(1)}(z)) = \begin{cases} \frac{F^{n-2}(s)}{F^{n-2}(z)} & \text{if } s \in [\underline{x}, z) \\ 1 & \text{if } s \in [z, \bar{x}] \end{cases} \quad \text{for each } \mathfrak{b} \leq \tilde{b}_F^{(1)}(z) \tag{80}$$

which is essentially equivalent to (39) for each $z \in (\lambda, \bar{x}]$, $\mathfrak{b} \leq \tilde{b}_F^{(1)}(z)$.

Regarding $\tilde{b}_F^{(2)}(\cdot|\text{no,no})$ in (75), we can argue as for (22), and regarding $\tilde{b}_F^{(2)}(\cdot|\mathfrak{b}, \mathfrak{b}_w)$ in (79) we can argue as for (25).

In order to consider the case in which $\mathfrak{b}_w = r_1$ (the bidding functions (76) and (77)) we first prove a stochastic dominance relation between $\tilde{G}(\cdot|r_1, r_1)$ and $\tilde{G}(\cdot|\text{no}, r_1)$.

Step 1.1: $\tilde{G}(\cdot|r_1, r_1)$ dominates $\tilde{G}(\cdot|\text{no}, r_1)$ in terms of the reverse hazard rate.

$$\frac{\tilde{g}(s|\text{no}, r_1)}{\tilde{G}(s|\text{no}, r_1)} = \frac{\tilde{g}(s|r_1, r_1)}{\tilde{G}(s|r_1, r_1)} \quad \text{for } s \in [\underline{x}, \gamma), \qquad \frac{\tilde{g}(s|\text{no}, r_1)}{\tilde{G}(s|\text{no}, r_1)} < \frac{\tilde{g}(s|r_1, r_1)}{\tilde{G}(s|r_1, r_1)} \quad \text{for } s \in (\gamma, \lambda]$$

It is immediate to verify that $\frac{\tilde{g}(s|\text{no},r_1)}{\tilde{G}(s|\text{no},r_1)} = \frac{\tilde{g}(s|r_1,r_1)}{\tilde{G}(s|r_1,r_1)}$ for each $s \in [\underline{x}, \gamma)$. Now consider $s \in (\gamma, \lambda]$, and in order to prove that $\frac{\tilde{g}(s|\text{no},r_1)}{\tilde{G}(s|\text{no},r_1)} < \frac{\tilde{g}(s|r_1,r_1)}{\tilde{G}(s|r_1,r_1)}$, notice that

$$\begin{aligned} \frac{\tilde{g}(s|\text{no}, r_1)}{\tilde{G}(s|\text{no}, r_1)} &= \frac{f(s)}{F(s) - \Gamma} \frac{(n-1)F^{n-2}(s)(F(s) - \Gamma) - (F^{n-1}(s) - \Gamma^{n-1})}{F^{n-1}(s) - \Gamma^{n-1}}, \\ \frac{\tilde{g}(s|r_1, r_1)}{\tilde{G}(s|r_1, r_1)} &= \frac{f(s)}{F(s) - \Gamma} \frac{n(n-1)F^{n-2}(s)(F(s) - \Gamma)^2 - 2[(n-1)F^n(s) - nF^{n-1}(s)\Gamma + \Gamma^n]}{(n-1)F^n(s) - nF^{n-1}(s)\Gamma + \Gamma^n} \end{aligned}$$

After defining $k \equiv \frac{\Gamma}{F(s)} \in (0,1)$, we can write $\frac{\tilde{g}(s|r_1,r_1)}{\tilde{G}(s|r_1,r_1)} - \frac{\tilde{g}(s|\text{no},r_1)}{\tilde{G}(s|\text{no},r_1)}$ as

$$\frac{f(s)}{F(s) - \Gamma} \left(\frac{n(n-1)(1-k)^2 - 2(n-1-nk+k^n)}{n-1-nk+k^n} - \frac{(n-1)(1-k) - 1 + k^{n-1}}{1-k^{n-1}} \right)$$

and rearranging the last expression, we see that it has the same sign as

$$k^{2n-2} - (n-1)^2 k^n + 2n(n-2) k^{n-1} - (n-1)^2 k^{n-2} + 1$$

Remark that this is the equal to $\mu(k)$ in (56) that we know is positive for each $k \in (0,1)$.

Step 1.2: The bidding function $\tilde{b}_F^{(2)}(\cdot|\text{no}, r_1)$. Consider a bidder of type $x \geq r_2$ who has submitted no bid at stage one, and has learned that $\mathfrak{b}_w = r_1$. Then his beliefs on the highest value among the other losing bidders are given by $\tilde{G}(s|\text{no}, r_1)$ in (36), and we prove that it is optimal for him to bid $\tilde{b}_F^{(2)}(x|\text{no}, r_1)$ as specified in (76) if he expects each other losing bidder with value in $[r_2, \gamma)$ to bid according to $\tilde{b}_F^{(2)}(\cdot|\text{no}, r_1)$, and each other losing bidder with value in

$[\gamma, \lambda]$ to bid according to $\tilde{b}_F^{(2)}(\cdot|r_1, r_1)$ in (77).[15]

In detail, we formulate his bidding problem as the problem of selecting optimally $y \in [r_2, \lambda]$, with the interpretation that choosing $y \in [r_2, \gamma)$ is equivalent to bidding $\tilde{b}_F^{(2)}(y|\text{no}, r_1)$, and choosing $y \in [\gamma, \lambda]$ is equivalent to bidding $\tilde{b}_F^{(2)}(y|r_1, r_1)$. Therefore, for this type of bidder the stage two payoff is

$$\tilde{u}_F^{(2)}(x, y|\text{no}, r_1) = \begin{cases} (x - \tilde{b}_F^{(2)}(y|\text{no}, r_1))\tilde{G}(y|\text{no}, r_1) & \text{if } y \in [r_2, \gamma) \\ (x - \tilde{b}_F^{(2)}(y|r_1, r_1))\tilde{G}(y|\text{no}, r_1) & \text{if } y \in [\gamma, \lambda] \end{cases}$$

and

$$\frac{\partial \tilde{u}_F^{(2)}(x, y|\text{no}, r_1)}{\partial y} = \begin{cases} \tilde{G}(y|\text{no}, r_1)\left(-\frac{\partial \tilde{b}_F^{(2)}(y|\text{no},r_1)}{\partial y} + (x - \tilde{b}_F^{(2)}(y|no, r_1))\frac{\tilde{g}(y|\text{no},r_1)}{\tilde{G}(y|\text{no},r_1)}\right) & \text{if } y \in [r_2, \gamma) \\ \tilde{G}(y|\text{no}, r_1)\left(-\frac{\partial \tilde{b}_F^{(2)}(y|r_1,r_1)}{\partial y} + (x - \tilde{b}_F^{(2)}(y|r_1, r_1))\frac{\tilde{g}(y|\text{no},r_1)}{\tilde{G}(y|\text{no},r_1)}\right) & \text{if } y \in (\gamma, \lambda] \end{cases}$$

Then notice that $\tilde{b}_F^{(2)}(\cdot|\text{no}, r_1)$ and $\tilde{b}_F^{(2)}(\cdot|r_1, r_1)$ satisfy the following differential equations in $[r_2, \gamma)$ and in $(\gamma, \lambda]$, respectively:

$$\frac{\partial \tilde{b}_F^{(2)}(y|\text{no}, r_1)}{\partial y} = (y - \tilde{b}_F^{(2)}(y|\text{no}, r_1))\frac{\tilde{g}(y|\text{no}, r_1)}{\tilde{G}(y|\text{no}, r_1)} \qquad \text{for } y \in [r_2, \gamma) \tag{81}$$

$$\frac{\partial \tilde{b}_F^{(2)}(y|r_1, r_1)}{\partial y} = (y - \tilde{b}_F^{(2)}(y|r_1, r_1))\frac{\tilde{g}(y|r_1, r_1)}{\tilde{G}(y|r_1, r_1)} \qquad \text{for } y \in (\gamma, \lambda] \tag{82}$$

and find that

$$\frac{\partial \tilde{u}_F^{(2)}(x,y|\text{no},r_1)}{\partial y} = \begin{cases} (x - y)\tilde{g}(y|\text{no}, r_1) & \text{if } y \in [r_2, \gamma) \\ \tilde{G}(y|\text{no}, r_1)\left(-(y - \tilde{b}_F^{(2)}(y|r_1, r_1))\frac{\tilde{g}(y|r_1,r_1)}{\tilde{G}(y|r_1,r_1)} + (x - \tilde{b}_F^{(2)}(y|r_1, r_1))\frac{\tilde{g}(y|\text{no},r_1)}{\tilde{G}(y|\text{no},r_1)}\right) & \text{if } y \in (\gamma, \lambda] \end{cases}$$

Consider a type $x \in [r_2, \gamma)$. Then $\frac{\partial \tilde{u}_F^{(2)}(x,y|\text{no},r_1)}{\partial y}$ is positive for $y \in [r_2, x)$, negative for $y \in (x, \gamma)$, and negative also for $y \in (\gamma, \lambda]$ because $\frac{\tilde{g}(y|r_1,r_1)}{\tilde{G}(y|r_1,r_1)} > \frac{\tilde{g}(y|\text{no},r_1)}{\tilde{G}(y|\text{no},r_1)}$ for $y \in (\gamma, \lambda]$ implies $-(y-\tilde{b}_F^{(2)}(y|r_1, r_1))\frac{\tilde{g}(y|r_1,r_1)}{\tilde{G}(y|r_1,r_1)} + (x-\tilde{b}_F^{(2)}(y|r_1, r_1))\frac{\tilde{g}(y|\text{no},r_1)}{\tilde{G}(y|\text{no},r_1)} < (x-y)\frac{\tilde{g}(y|\text{no},r_1)}{\tilde{G}(y|\text{no},r_1)} < 0$ given $x < \gamma < y$. Hence the optimal y is equal to x, i.e. the optimal bid is $\tilde{b}_F^{(2)}(x|\text{no}, r_1)$.

Now consider a type $x \in [\gamma, \bar{x}]$. Then $\frac{\partial \tilde{u}_F^{(2)}(x,y|\text{no},r_1)}{\partial y} > 0$ for $y \in [r_2, \gamma)$, hence the optimal y is in $[\gamma, \lambda]$, as specified by (76). Moreover, we have seen above that $\frac{\partial \tilde{u}_F^{(2)}(x,y|\text{no},r_1)}{\partial y} \leq (x - y)\tilde{g}(y|\text{no}, r_1)$ for $y \in (\gamma, \lambda]$, hence for $x = \gamma$ the optimal y is equal to γ.

Step 1.3: The bidding function $\tilde{b}_F^{(2)}(\cdot|r_1, r_1)$. Consider a bidder of type $x \geq r_2$ who has bid r_1 at stage one, and has learned that another bidder has won at stage one with a bid r_1.

[15] In view of $\tilde{G}(\cdot|\text{no}, r_1)$, he expects that no losing bidder has value greater than λ.

Then his beliefs on the highest value among the other losing bidders at stage one are given by $\tilde{G}(s|r_1,r_1)$ in (45) and we prove that it is optimal for him to bid $\tilde{b}_F^{(2)}(x|r_1,r_1)$ as specified in (77) if he expects each other losing bidder with value in $[r_2,\gamma)$ to bid according to $\tilde{b}_F^{(2)}(\cdot|\text{no},r_1)$ in (76), and each other losing bidder with value in $[\gamma,\lambda]$ to bid according to $\tilde{b}_F^{(2)}(\cdot|r_1,r_1)$.[16]

Arguing as in the proof of Step 1.2, we can write the bidder's payoff at stage two as a function of y as follows:

$$\tilde{u}_F^{(2)}(x,y|r_1,r_1)=\begin{cases}(x-\tilde{b}_F^{(2)}(y|\text{no},r_1))\tilde{G}(y|r_1,r_1) & \text{if } y\in[r_2,\gamma)\\ (x-\tilde{b}_F^{(2)}(y|r_1,r_1))\tilde{G}(y|r_1,r_1) & \text{if } y\in[\gamma,\lambda]\end{cases}$$

and

$$\frac{\partial\tilde{u}_F^{(2)}(x,y|r_1,r_1)}{\partial y}=\begin{cases}\tilde{G}(y|r_1,r_1)\left(-\frac{\partial\tilde{b}_F^{(2)}(y|\text{no},r_1)}{\partial y}+(x-\tilde{b}_F^{(2)}(y|\text{no},r_1))\frac{\tilde{g}(y|r_1,r_1)}{\tilde{G}(y|r_1,r_1)}\right) & \text{if } y\in[r_2,\gamma)\\ \tilde{G}(y|r_1,r_1)\left(-\frac{\partial\tilde{b}_F^{(2)}(y|r_1,r_1)}{\partial y}+(x-\tilde{b}_F^{(2)}(y|r_1,r_1))\frac{\tilde{g}(y|r_1,r_1)}{\tilde{G}(y|r_1,r_1)}\right) & \text{if } y\in(\gamma,\lambda]\end{cases}$$

Then we use (81)-(82) plus $\frac{\tilde{g}(y|\text{no},r_1)}{\tilde{G}(y|\text{no},r_1)}=\frac{\tilde{g}(y|r_1,r_1)}{\tilde{G}(y|r_1,r_1)}$ for $y\in[r_2,\gamma)$ to find

$$\frac{\partial\tilde{u}_F^{(2)}(x,y|r_1,r_1)}{\partial y}=\begin{cases}(x-y)\tilde{g}(y|r_1,r_1) & \text{if } y\in[r_2,\gamma)\\ (x-y)\tilde{g}(y|r_1,r_1) & \text{if } y\in(\gamma,\lambda]\end{cases}$$

This reveals that the optimal y is equal to x for each $x\in[r_2,\lambda]$; and it is equal to λ, for each $x\in(\lambda,\bar{x}]$. Hence, in either case, the optimal bid is $\tilde{b}_F^{(2)}(x|r_1,r_1)$.

Step 1.4: The bidding function $\tilde{b}_F^{(2)}(\cdot|\mathfrak{b},\tilde{b}_F^{(1)}(z))$. If $\mathfrak{b}_w=\tilde{b}_F^{(1)}(z)$ for some $z\in(\lambda,\bar{x}]$, then the beliefs of each losing bidder are given by the c.d.f. $\tilde{G}(\cdot|\mathfrak{b},\tilde{b}_F^{(1)}(z))$ in (80). Then essentially the argument relative to $\hat{b}_F^{(2)}(x|\text{no,no})$ in (22) applies in this case. We find that

$$\frac{\tilde{g}(s|\mathfrak{b},\tilde{b}_F^{(1)}(z))}{\tilde{G}(s|\mathfrak{b},\tilde{b}_F^{(1)}(z))}=\frac{(n-2)f(s)}{F(s)}\quad\text{for } s\in(r_2,z)$$

hence (1) reveals that the equilibrium bidding function for $x\in[r_2,z)$ is $\beta_{n-1,r_2}(x)$, as specified by $\tilde{b}_F^{(2)}(\cdot|\mathfrak{b},\tilde{b}_F^{(1)}(z))$. Finally, given $\mathfrak{b}_w=\tilde{b}_F^{(1)}(z)$, a type $x\in[z,\bar{x}]$ expects each other bidder to have value smaller than z, and $\beta_{n-1,r_2}(z)$ is his payoff maximizing bid, as prescribed by (78).

Step 2: Proof for stage one

Here we consider the point of view of a bidder at stage one, given (75)-(78), and prove that it is profitable for him to bid as specified in $\tilde{b}_F^{(1)}$ in (74), if he expects the other bidders to do so. For

[16] In view of $\tilde{G}(\cdot|r_1,r_1)$, he expects that no losing bidder has value greater than λ.

each x and y in $[\underline{x}, \bar{x}]$, we use $\tilde{u}_F(x, y)$ to denote the total payoff of type x from bidding $\tilde{b}_F^{(1)}(y)$ in stage one, and prove that the following inequalities hold:[17]

$$\text{for each } x \in [r_2, \gamma), \quad \tilde{u}_F(x, \underline{x}) \geq \max\{\tilde{u}_F(x, \gamma), \tilde{u}_F(x, y)\}, \text{ for each } y \in (\lambda, \bar{x}] \tag{83}$$

$$\text{for each } x \in [\gamma, \lambda], \quad \tilde{u}_F(x, \gamma) \geq \max\{\tilde{u}_F(x, \underline{x}), \tilde{u}_F(x, y)\}, \text{ for each } y \in (\lambda, \bar{x}] \tag{84}$$

$$\text{for each } x \in (\lambda, \bar{x}] \text{ and } y \in (\lambda, \bar{x}], \quad \tilde{u}_F(x, x) \geq \max\{\tilde{u}_F(x, \underline{x}), \tilde{u}_F(x, \gamma), \tilde{u}_F(x, y)\} \tag{85}$$

The proof of (83)-(85) is closely linked to the proof of (58)-(60). The first step consists in using (75)-(78) to prove that

$$\tilde{u}_F(x, \underline{x}) = \tilde{u}_S(x, \underline{x}) \qquad \text{for each } x \in [r_2, \gamma) \tag{86}$$

$$\tilde{u}_F(x, \gamma) = \tilde{u}_S(x, \gamma) \qquad \text{for each } x \in [r_2, \bar{x}] \tag{87}$$

$$\tilde{u}_F(x, y) = \tilde{u}_S(x, y) \qquad \text{for each } x \in [r_2, \bar{x}] \text{ and each } y \in (\lambda, \bar{x}] \tag{88}$$

Given (58), it follows that (86)-(88) imply that (83) is satisfied. In order to prove (84)-(85), we use (87)-(88) and then prove directly that $\tilde{u}_F(x, \gamma) \geq \tilde{u}_F(x, \underline{x})$ for $x \in [\gamma, \lambda]$, and $\tilde{u}_F(x, x) \geq \tilde{u}_F(x, \underline{x})$ for $x \in (\lambda, \bar{x}]$.

Step 2.1: Proof of (86). For a type $x \in [r_2, \gamma)$, the payoff from not bidding at stage one is equal to $\tilde{u}_S(x, \underline{x})$ (see (40)) since

$$\begin{aligned}
\tilde{u}_F(x, \underline{x}) &= \Gamma^{n-1}\left(x - \tilde{b}_F^{(2)}(x|\text{no}, \text{no})\right)\tilde{G}(x|\text{no}, \text{no}) + (\Lambda^{n-1} - \Gamma^{n-1})\left(x - \tilde{b}_F^{(2)}(x|\text{no}, r_1)\right)\tilde{G}(x|\text{no}, r_1) \\
&\quad + \int_\lambda^{\bar{x}} \left(x - \tilde{b}_F^{(2)}(x|\text{no}, \tilde{b}_F^{(1)}(z))\right)\tilde{G}(x|\text{no}, \tilde{b}_F^{(1)}(z))dF^{n-1}(z) \\
&= v_n(x) + (n-1)(1-\Gamma)v_{n-1}(x)
\end{aligned}$$

Step 2.2: Proof of (87). For a type $x \in [r_2, \bar{x}]$, the payoff from bidding r_1 is

$$\begin{aligned}
\tilde{u}_F(x, \gamma) &= \tilde{p}(\gamma)(x - r_1) + \tilde{p}_\ell(x - \tilde{b}_F^{(2)}(x|r_1, r_1))\tilde{G}(x|r_1, r_1) \\
&\quad + \int_\lambda^{\bar{x}} (x - \tilde{b}_F^{(2)}(x|r_1, \tilde{b}_F^{(1)}(z)))\tilde{G}(x|r_1, \tilde{b}_F^{(1)}(z))dF^{n-1}(z)
\end{aligned}$$

in which $\tilde{p}(\gamma)$ is the probability to win at stage one for a bidder bidding r_1, and $\tilde{p}_\ell$ is the probability that another bidder bidding r_1 wins at stage one (see (43) and (44)). Routine

[17] Again, it is understood that bidding more than $b_F^{(1)}(\overline{x})$ is an unprofitable option for all x, since it does not increase the probability of winning while increasing the price to pay for the object.

manipulations reveal that

$$\tilde{u}_F(x,\gamma)=\begin{cases}\tilde{p}(\gamma)(x-r_1)+\frac{n-1}{2}(2-\Lambda-\Gamma)v_{n-1}(x) & \text{if } x\in[r_2,\gamma)\\ \tilde{p}(\gamma)(x-r_1)+\frac{n-1}{2}(\Lambda-\Gamma)v_{n-1}(\gamma)+(n-1)(1-\Lambda)v_{n-1}(x) & \\ +\int_\gamma^x \frac{\Lambda-\Gamma}{n}M(F(s),\Gamma)ds & \text{if } x\in[\gamma,\lambda]\\ \tilde{p}(\gamma)(x-r_1)+\tilde{p}_\ell(x-\beta_{n-1,r_2}(\gamma)\tilde{G}(\gamma|r_1,r_1)-\int_\gamma^\lambda s\tilde{g}(s|r_1,r_1)ds)) & \\ +\int_\lambda^x(x-z)dF^{n-1}(z)+(n-1)\int_\lambda^x v_{n-1}(z)f(z)dz+(n-1)[1-F(x)]v_{n-1}(x) & \text{if } x\in(\lambda,\bar{x}]\end{cases}$$

From (47) it is immediate to see that $\tilde{u}_F(x,\gamma)=\tilde{u}_S(x,\gamma)$, for $x\in[r_2,\lambda]$. For $x\in(\lambda,\bar{x}]$, the equality holds since (i) $\tilde{u}_F(\lambda,\gamma)=\tilde{u}_S(\lambda,\gamma)=\tilde{p}(\gamma)(\lambda-r_1)+\tilde{p}_\ell\int_\gamma^\lambda\tilde{G}(s|r_1,r_1)ds+\frac{n-1}{2}(\Lambda-\Gamma)v_{n-1}(\gamma)+(n-1)(1-\Lambda)v_{n-1}(\lambda)$; (ii) $\tilde{u}_{F1}(x,\gamma)=\tilde{u}_{S1}(x,\gamma)=F^{n-1}(x)+(n-1)F^{n-2}(x)[1-F(x)]$.

Step 2.3: Proof of (88). For a type $x\in[r_2,\bar{x}]$, the payoff $\tilde{u}_F(x,y)$ from bidding $\tilde{b}_F^{(1)}(y)$, for $y\in(\lambda,\bar{x}]$, is

$$\begin{aligned}&(x-\tilde{b}_F^{(1)}(y))F^{n-1}(y)+\int_y^{\bar{x}}\left(x-\tilde{b}_F^{(2)}(x|\tilde{b}_F^{(1)}(y),\tilde{b}_F^{(1)}(z))\right)\tilde{G}(x|\tilde{b}_F^{(1)}(y),\tilde{b}_F^{(1)}(z))dF^{n-1}(z)\\ &=(x-\tilde{b}_F^{(1)}(y))F^{n-1}(y)\\ &+\begin{cases}\int_y^x(x-z)dF^{n-1}(z)+(n-1)\int_y^x v_{n-1}(z)f(z)dz+(n-1)(1-F(x))v_{n-1}(x) & \text{if } y<x\\ (n-1)(1-F(y))v_{n-1}(x) & \text{if } x\le y\end{cases}\end{aligned}$$

From (48) it is immediate to see that $\tilde{u}_F(x,y)=\tilde{u}_S(x,y)$.

Step 2.4: Proof of (84). Note that (87) and (88) imply that $\tilde{u}_F(\gamma,\gamma)=\tilde{u}_S(\gamma,\gamma)$, $\tilde{u}_F(\lambda,\gamma)=\tilde{u}_S(\lambda,\gamma)$ and $\lim_{y\downarrow\lambda}\tilde{u}_F(x,y)=\lim_{y\downarrow\lambda}\tilde{u}_S(x,y)$ for $x\le\lambda$. Moreover, from (87)-(88) and (59), it follows that $\tilde{u}_F(x,\gamma)\ge\tilde{u}_F(x,y)$ for each $y\in(\lambda,\bar{x}]$. Below we prove that $\tilde{u}_F(x,\gamma)\ge\tilde{u}_F(x,\underline{x})$ for $x\in[\gamma,\lambda]$. In fact, the payoff from not bidding at stage one for a type $x\in[\gamma,\lambda]$ is

$$\begin{aligned}\tilde{u}_F(x,\underline{x})&=\Gamma^{n-1}\left(x-\tilde{b}_F^{(2)}(x|\text{no},\text{no})\right)+(\Lambda^{n-1}-\Gamma^{n-1})\max_{y\in[\gamma,\lambda]}(x-\tilde{b}_F^{(2)}(y|r_1,r_1))\tilde{G}(y|\text{no},r_1)\\ &+\int_\lambda^{\bar{x}}(x-\tilde{b}_F^{(2)}(x|\text{no},\tilde{b}_F^{(1)}(z)))\tilde{G}(x|\text{no},\tilde{b}_F^{(1)}(z))dF^{n-1}(z)\\ &=\Gamma^{n-1}(x-\beta_{n,r_2}(\gamma))+(\Lambda^{n-1}-\Gamma^{n-1})\max_{y\in[\gamma,\lambda]}(x-\tilde{b}_F^{(2)}(y|r_1,r_1))\tilde{G}(y|\text{no},r_1)\\ &+(n-1)(1-\Lambda)v_{n-1}(x)\end{aligned}$$

Let $\tilde{y}(x)\in\arg\max_{y\in[\gamma,\lambda]}(x-\tilde{b}_F^{(2)}(y|r_1,r_1))\tilde{G}(y|\text{no},r_1)$, and recall from the analysis in Step 1.2 of the bidding function $\tilde{b}_F^{(2)}(\cdot|\text{no},r_1)$ that $\tilde{y}(\gamma)=\gamma$, hence $\tilde{u}_F(\gamma,\underline{x})=v_n(\gamma)+(n-1)(1-\Gamma)v_{n-1}(\gamma)=$

$\tilde{u}_S(\gamma, \underline{x})$. Moreover, in Step 1.2 we have also proved that $\tilde{y}(x) \leq x$, therefore

$$\tilde{u}_{F1}(x, \underline{x}) \leq \Gamma^{n-1} + (\Lambda^{n-1} - \Gamma^{n-1})\tilde{G}(x|\text{no}, r_1) + (n-1)(1-\Lambda)F^{n-2}(x) = \tilde{u}_{S1}(x, \underline{x})$$

hence $\tilde{u}_F(x, \underline{x}) \leq \tilde{u}_S(x, \underline{x}) \leq \tilde{u}_S(x, \gamma) = \tilde{u}_F(x, \gamma)$ for each $x \in [\gamma, \lambda]$ (the middle inequality follows from (59)).

Step 2.5: Proof of (85)**.** From (87)-(88) and (60) it follows that $\tilde{u}_F(x, x) \geq \max\{\tilde{u}_F(x, \gamma), \tilde{u}_F(x, y)\}$ for each $y \in (\lambda, \bar{x}]$. Below we prove that $\tilde{u}_F(x, x) \geq \tilde{u}_F(x, \underline{x})$.

The payoff from not bidding at stage one is

$$\begin{aligned}
\tilde{u}_F(x, \underline{x}) &= \Gamma^{n-1}\left(x - \tilde{b}_F^{(2)}(x|\text{no}, \text{no})\right) + (\Lambda^{n-1} - \Gamma^{n-1}) \max_{y \in [\gamma, \lambda]}(x - \tilde{b}_F^{(2)}(y|r_1, r_1))\tilde{G}(y|\text{no}, r_1) \\
&\quad + \int_\lambda^{\bar{x}} (x - \tilde{b}_F^{(2)}(x|\text{no}, \tilde{b}_F^{(1)}(z)))\tilde{G}(x|\text{no}, \tilde{b}_F^{(1)}(z))dF^{n-1}(z) \\
&= \Gamma^{n-1}(x - \beta_{n,r_2}(\gamma)) + (\Lambda^{n-1} - \Gamma^{n-1}) \max_{y \in [\gamma, \lambda]}(x - \tilde{b}_F^{(2)}(y|r_1, r_1))\tilde{G}(y|\text{no}, r_1) \\
&\quad + \int_\lambda^x (x - z)dF^{n-1}(z) + (n-1)\int_\lambda^x v_{n-1}(z)f(z)dz + (n-1)(1 - F(x))v_{n-1}(x)
\end{aligned}$$

Let $\tilde{y}(x) \in \arg\max_{y \in [\gamma, \lambda]}(x - \tilde{b}_F^{(2)}(y|r_1, r_1))\tilde{G}(y|\text{no}, r_1)$. From $\tilde{G}(\tilde{y}(x)|\text{no}, r_1) \leq 1$ it follows that $\tilde{u}_{F1}(x, \underline{x}) \leq F^{n-1}(x) + (n-1)F^{n-2}(x)[1 - F(x)]$, which is equal to $\frac{d\tilde{u}_F(x,x)}{dx}$. Since $\lim_{x \downarrow \lambda} \tilde{u}_F(x, x) = \tilde{u}_F(\lambda, \lambda^+) = \tilde{u}_F(\lambda, \gamma) \geq \tilde{u}_F(\lambda, \underline{x})$,[18] we conclude that $\tilde{u}_F(x, x) \geq \tilde{u}_F(x, \underline{x})$ for each $x \in (\lambda, \bar{x}]$.

D.2 Proof of Proposition 5(ii)

The proof is largely given by the proof of Proposition 5(i), after setting $\lambda = \bar{x}$. Precisely, regarding stage two, in the main text we have already taken care of (22) and (25). In order study the case in which $\mathfrak{b}_w = r_1$, we can refer to steps 1.1-1.3 in the proof of Proposition 5(i). Regarding stage one, we use $\hat{u}_F(x, y)$ to denote the total payoff of type x from bidding $\hat{b}_F^{(1)}(y)$. We have proved in the main text that $\hat{u}_S(x, \underline{x}) \geq \max\{\hat{u}_S(x, \gamma), x - r_1\}$ for each $x \in [r_2, \gamma)$. We can argue as in Step 2.2 in the proof of Proposition 5(i) to conclude that (i) $\hat{u}_F(x, \gamma) = \hat{u}_S(x, \gamma)$ for each $x \in [\gamma, \bar{x}]$, hence $\hat{u}_F(x, \gamma) \geq x - r_1$; (ii) $\hat{u}_F(x, \gamma) \geq \hat{u}_S(x, \underline{x})$.

E Addendum to Step 2.3 in the proof of Proposition 2(i): Proof that there exists a unique r_1 such that $\ell(r_1) = 0$

In the proof of Proposition 2(i) Step 2.3 we have defined $\ell(r_1)$ from (57) as $B(\gamma, \bar{x}) = \frac{n(n-1)}{2}v_{n-1}(\gamma) - M(1, \Gamma)(\bar{x} - r_1) + \int_\gamma^{\bar{x}} M(F(s), \Gamma)ds$, in which γ is determined as the unique solution in $(r_1, \bar{x})$

[18] The second equality follows from (16); the inequality follows from (84).

to $A(\gamma, \lambda) = 0$, given $\lambda = \bar{x}$, where $A(\gamma, \lambda)$ is from (51). Since γ depends on r_1, we use the notation $\gamma(r_1)$, hence

$$\ell(r_1) = \frac{n(n-1)}{2} v_{n-1}(\gamma(r_1)) - M(1, \Gamma)(\bar{x} - r_1) + \int_{\gamma(r_1)}^{\bar{x}} M(F(s), \Gamma) ds$$

in which $\Gamma = F(\gamma(r_1))$. It is enough here to prove that if r_1 is such that $\ell(r_1) = 0$, then $\ell'(r_1) > 0$ because this implies that no more than one r_1 exists such that $\ell(r_1) = 0$. Remark we have denoted such r_1 with $\tilde{r}_1$ in Proposition 2 and elsewhere in the paper. The derivative of ℓ with respect to r_1 is

$$\ell'(r_1) = M(1, \Gamma) + \left[\int_{\gamma(r_1)}^{\bar{x}} M_2(F(s), \Gamma) ds - M_2(1, \Gamma)(\bar{x} - r_1) \right] f(\gamma(r_1))\gamma'(r_1) \tag{89}$$

and if r_1 satisfies $\ell(r_1) = 0$, then

$$\bar{x} - r_1 = \frac{\frac{n(n-1)}{2} v_{n-1}(\gamma(r_1)) + \int_{\gamma(r_1)}^{\bar{x}} M(F(s), \Gamma) ds}{M(1, \Gamma)}$$

Inserting this equality in (89) yields

$$\begin{aligned}\ell'(r_1) = M(1, \Gamma) + f(\gamma(r_1))\gamma'(r_1) \int_{\gamma(r_1)}^{\bar{x}} M_2(F(s), \Gamma) ds \\ - f(\gamma(r_1))\gamma'(r_1) \frac{M_2(1, \Gamma)}{M(1, \Gamma)} \left(\frac{n(n-1)}{2} v_{n-1}(\gamma(r_1)) + \int_{\gamma(r_1)}^{\bar{x}} M(F(s), \Gamma) ds \right)\end{aligned}$$

Moreover, from $A(\gamma, \bar{x}) = 0$ we obtain

$$\gamma'(r_1) = \frac{\hat{p}(\gamma(r_1))}{Z + \left(\frac{n-1}{2} v_{n-1}(\gamma(r_1)) + (\gamma(r_1) - r_1) \frac{M(\Gamma, 1)}{n} \right) f(\gamma(r_1))} > 0$$

with $\hat{p}(\gamma(r_1)) = \frac{1-\Gamma^n}{n(1-\Gamma)}$ and $Z = \hat{p}(\gamma(r_1)) - \Gamma^{n-1} - \frac{n-1}{2}(1 - \Gamma)\Gamma^{n-2} > 0$, hence

$$\begin{aligned}\ell'(r_1) &= \int_{\gamma(r_1)}^{\bar{x}} \left(M_2(F(s), \Gamma) - \frac{M_2(1, \Gamma)}{M(1, \Gamma)} M(F(s), \Gamma) \right) ds f(\gamma(r_1))\gamma'(r_1) \\ &+ \left(M(1, \Gamma) - \frac{M_2(1, \Gamma)}{M(1, \Gamma)} \frac{\frac{n(n-1)}{2} v_{n-1}(\gamma(r_1)) f(\gamma(r_1)) \hat{p}(\gamma(r_1))}{Z + \left(\frac{n-1}{2} v_{n-1}(\gamma(r_1)) + (\gamma(r_1) - r_1) \frac{M(\Gamma, 1)}{n} \right) f(\gamma(r_1))} \right)\end{aligned}$$

The rest of the proof consists in showing that both the terms in $\ell'(r_1)$ are positive.

E.1 Proof that the first term in $\ell'(r_1)$ is positive

Here we prove that $\int_{\gamma(r_1)}^{\bar{x}} \left(M_2(F(s),\Gamma) - \frac{M_2(1,\Gamma)}{M(1,\Gamma)} M(F(s),\Gamma) \right) ds f(\gamma(r_1))\gamma'(r_1) > 0$ by showing that $M_2(F(s),\Gamma) - \frac{M_2(1,\Gamma)}{M(1,\Gamma)} M(F(s),\Gamma) > 0$ for each $s \in (\gamma(r_1), \bar{x}]$. For the sake of brevity, we write b instead of Γ and a instead of $F(s)$ and prove that $M_2(a,b) - \frac{M_2(1,b)}{M(1,b)} M(a,b) \geq 0$ for each a, b such that $0 < b < a < 1$. We find that

$$M_2(a,b) - \frac{M_2(1,b)}{M(1,b)} M(a,b) = \frac{C_b(a)}{b\,(1-b)\,(a-b)^3\,(b^n - bn + n - 1)}$$

in which C_b is a function of a, given a fixed b in $(0,1)$:

$$\begin{aligned} C_b(a) \;=\; & \left(2b^n - n\,(n-1)\,b^2 + 2n\,(n-2)\,b - (n-1)\,(n-2)\right) b^{n+1} \\ & + \left((n^2+n)b^2 - 2b^{n+1} - 2(n^2-1)b + n^2 - n\right) b^n a \\ & + n\left((n-1)b^{n+1} - (n+1)b^n + (n+1)b - (n-1)\right) b^2 a^{n-1} \\ & + \left(2(n^2-1)b^n - 2(n^2-2n)b^{n+1} - (n^2+n)b^2 + (n-1)(n-2)\right) b a^n \\ & + (n-1)\left((n-2)b^n - nb^{n-1} + nb - n + 2\right) b a^{n+1} \end{aligned}$$

Hence, the sign of $M_2(a,b) - \frac{M_2(1,b)}{M(1,b)} M(a,b)$ is equal to the sign of $C_b(a)$ and now we show that $C_b(a) > 0$ for each $b \in (0,1)$, $a \in (b,1)$. First notice that $C_b(b) = 0$, $C_b(1) = 0$, and then we prove that there exists an $\bar{a}$ in the interval $(b,1)$ such that $C_b'(a) > 0$ for $a \in (b, \bar{a})$, and $C_b'(a) < 0$ for $a \in (\bar{a}, 1)$. This implies that $C_b(a) > 0$ for each $a \in (b,1)$. Precisely,

$$\begin{aligned} C_b'(a) \;=\; & \left((n^2+n)b^2 - 2b^{n+1} - 2(n^2-1)b + n^2 - n\right) b^n \\ & + (n^2-n)\left((n-1)b^{n+1} - (n+1)b^n + (n+1)b - (n-1)\right) b^2 a^{n-2} \\ & + n\left(2(n^2-1)b^n - 2(n^2-2n)b^{n+1} - (n^2+n)b^2 + (n-1)(n-2)\right) b a^{n-1} \\ & + (n^2-1)\left((n-2)b^n - nb^{n-1} + nb - n + 2\right) b a^n \end{aligned}$$

and $C_b'(b) = 0$. Moreover,

$$C_b''(a) = n(n-1)a^{n-3}\hat{C}_b(a)$$

with

$$\begin{aligned} \hat{C}_b(a) \;=\; & (n-2)\left((n-1)b^{n+1} - (n+1)b^n + (n+1)b - (n-1)\right) b^2 \\ & + \left(2(n^2-1)b^n - 2(n^2-2n)b^{n+1} - (n^2+n)b^2 + (n-1)(n-2)\right) b a \\ & + (n+1)\left((n-2)b^n - nb^{n-1} + nb - n + 2\right) b a^2 \end{aligned}$$

Notice that $\hat{C}_b$ is a second degree concave polynomial in a, and $a_1 = b$ is a solution for the equation $\hat{C}_b(a) = 0$ since $\hat{C}_b(b) = 0$. Moreover, there exists another solution a_2 such that

$b < a_2 < 1$, given that $\hat{C}_b''(b) > 0$ and $\hat{C}_b(1) < 0$ (precisely, $a_2 = \frac{\frac{n-1}{n+1} - b + b^n - \frac{n-1}{n+1}b^{n+1}}{1 - \frac{n}{n-2}b + \frac{n}{n-2}b^{n-1} - b^n}$).[19] This implies $C_b''(a) > 0$ for $a \in (b, a_2)$, $C_b''(a) < 0$ for $a \in (a_2, 1)$. Therefore C_b' is strictly increasing for $a \in (b, a_2)$, is strictly decreasing for $a \in (a_2, 1)$. Since $C_b'(b) = 0$, if follows that $C_b'(a) > 0$ if $a \in (b, \bar{a})$, for some $\bar{a}$ between a_2 and 1, and $C_b'(a) < 0$ if $a \in (\bar{a}, 1)$.[20]

E.2 Proof that the second term in $\ell'(r_1)$ is positive

The second term in $\ell'(r_1)$ is equal to

$$\frac{M^2(1,\Gamma)\left(Z + (\gamma(r_1) - r_1)\frac{M(\Gamma,1)}{n}f(\gamma(r_1))\right) + \left(M^2(1,\Gamma) - nM_2(1,\Gamma)\hat{p}(\gamma(r_1))\right)\frac{n-1}{2}v_{n-1}(\gamma(r_1))f(\gamma(r_1))}{M(1,\Gamma)\left(Z + \left(\frac{n-1}{2}v_{n-1}(\gamma(r_1)) + (\gamma(r_1) - r_1)\frac{M(\Gamma,1)}{n}\right)f(\gamma(r_1))\right)}$$

It is immediate that the denominator and $M^2(1,\Gamma)\left(Z + (\gamma(r_1) - r_1)\frac{M(\Gamma,1)}{n}f(\gamma(r_1))\right)$ are both positive. We here prove that $M^2(1,\Gamma) - nM_2(1,\Gamma)\hat{p}(\gamma(r_1)) > 0$, but we use x rather than Γ for the sake of brevity. We obtain that $M^2(1,x) - nM_2(1,x)\hat{p} = \frac{q(x)}{(1-x)^4}$, with

$$q(x) = n^2 - 3n + 3 - n(2n-3)x + n^2x^2 - nx^{n-1} + 2(2n-3)x^n - 3nx^{n+1} + nx^{2n-1} - (n-3)x^{2n}$$

In the following of the proof we show that $q(x) > 0$ for each $x \in (0,1)$.

Step 1: The proof for $n = 3, 4, 5$

Here we write down $q(x)$ for $n = 3, 4, 5$, and it is immediate that $q(x) > 0$ for each $x \in (0,1)$. For $n = 3$, $q(x) = (3 + 3x)(1-x)^4$. For $n = 4$, $q(x) = \left(7 + 8x + 6x^2 - x^4\right)(1-x)^4$. For $n = 5$, $q(x) = \left(13 + 17x + 15x^2 + 10x^3 - 3x^5 - 2x^6\right)(1-x)^4$.

Step 2: The proof for $n \geq 6$

We prove below that $q''(x) > 0$ for each $x \in (0,1)$. Therefore q' is strictly increasing in $[0,1]$, and since $q'(1) = 0$, it follows that $q'(x) < 0$ for each $x \in (0,1)$. Hence, q is strictly decreasing

[19] The inequality $\hat{C}_b''(b) > 0$ is equivalent to $t(b) > 0$, with $t(b) = (n-2)(n-1) - 2(n+1)(n-2)b + n(n+1)b^2 - 2(n+1)b^n + 2(n-2)b^{n+1}$. We find that $t'''(b) < 0$ for each $b \in (0,1)$, hence t'' is strictly decreasing. Since $t''(1) = 0$, it follows that $t''(b) > 0$ for each $b \in (0,1)$, hence t' is strictly increasing. Since $t'(1) = 0$, it follows that $t'(b) < 0$ for each $b \in (0,1)$, hence t is strictly decreasing. Since $t(1) = 0$, it follows that $t(b) > 0$ for each $b \in (0,1)$.

The inequality $\hat{C}_b(1) < 0$ is equivalent to $t(b) > 0$, with $t(b) = 2n - 4 - 2(n+1)b + (n+1)nb^{n-1} - 2(n+1)(n-2)b^n + (n-1)(n-2)b^{n+1}$. We find that $t''(b) > 0$ for each $b \in (0,1)$, hence t' is strictly increasing. Since $t'(1) = 0$, it follows that $t'(b) < 0$ for each $b \in (0,1)$, hence t is strictly decreasing. Since $t(1) = 0$, it follows that $t(b) > 0$ for each $b \in (0,1)$.

[20] It is impossible that $C_b'(a) > 0$ for each $a \in (b,1)$ since $C_b(b) = C_b(1)$.

in $[0,1]$, and since $q(1)=0$, it follows that $q(x)>0$ for each $x\in(0,1)$. In detail, we find

$$\begin{aligned}
q'(x) &= -n(2n-3)+2n^2x-n(n-1)x^{n-2}+2(2n-3)nx^{n-1}-3n(n+1)x^n+n(2n-1)x^{2n-2}\\
&\quad -2n(n-3)x^{2n-1}\\
q''(x) &= 2n^2-n(n-1)(n-2)x^{n-3}+2(2n-3)n(n-1)x^{n-2}-3n^2(n+1)x^{n-1}\\
&\quad +n(2n-1)(2n-2)x^{2n-3}-2n(n-3)(2n-1)x^{2n-2}
\end{aligned}$$

Step 2.1: For each $n\geq 6$, $q''(x)>0$ for each $x\in(0,\frac{1}{2}]$. For each $m\geq 6$ we have that $2^m\geq m^2$, that is $-(\frac{1}{2})^m\geq -\frac{1}{m^2}$. Hence, for each $x\in(0,\frac{1}{2}]$ the following inequalities hold: $-n(n-1)(n-2)x^{n-3}\geq -n(n-1)(n-2)\frac{1}{(n-3)^2}$, $-3n^2(n+1)x^{n-1}\geq -3n^2(n+1)\frac{1}{(n-1)^2}$, $-2n(n-3)(2n-1)x^{2n-2}\geq -2n(n-3)(2n-1)\frac{1}{4(n-1)^2}$. They imply

$$\begin{aligned}
q''(x) &> 2n^2-\frac{n(n-1)(n-2)}{(n-3)^2}-\frac{3n^2(n+1)}{(n-1)^2}-\frac{2n(n-3)(2n-1)}{4(n-1)^2}\\
&= \frac{n}{2\left(n^2-4n+3\right)^2}\left((4n^3-2n^2+27n+125)(n-5)^2+652(n-5)+104\right)
\end{aligned}$$

for each $x\in(0,\frac{1}{2}]$, which is positive for each $n\geq 6$.

Step 2.2: For each $n\geq 6$, $q''(x)>0$ for each $x\in(\frac{1}{2},x_1]$, with $x_1=\frac{2n^2-5n+3+r}{3n^2+3n}$ and $r=\sqrt{n^4-8n^3+46n^2-36n+9}$. We write $q''(x)$ as follows:

$$\begin{aligned}
q''(x) &= 2n(2n-1)\left(n-1-(n-3)x\right)x^{2n-3}\\
&\quad +2n^2+\left(-n(n-1)(n-2)+2n(2n-3)(n-1)x-3n^2(n+1)x^2\right)x^{n-3}
\end{aligned}$$

Then notice that $2n(2n-1)\left(n-1-(n-3)x\right)x^{2n-3}>0$ for each $x\in(\frac{1}{2},1)$, and

$$\begin{aligned}
& 2n^2+\left(-n(n-1)(n-2)+2n(2n-3)(n-1)x-3n^2(n+1)x^2\right)x^{n-3}\\
> \;& 2n^2x^{n-3}+\left(-n(n-1)(n-2)+2n(2n-3)(n-1)x-3n^2(n+1)x^2\right)x^{n-3}\\
= \;& n\left(5n-n^2-2+2(2n-3)(n-1)x-3n(n+1)x^2\right)x^{n-3}
\end{aligned}$$

is non-negative for x between $\frac{1}{2}$ and x_1.

Step 2.3: For each $n\geq 6$, $q''(x)>0$ for each $x\in(x_1,1)$. Here we use the third derivative of q:

$$\begin{aligned}
q'''(x) &= -n(n-1)(n-2)(n-3)x^{n-4}+2(2n-3)n(n-1)(n-2)x^{n-3}-3n^2(n+1)(n-1)x^{n-2}\\
&\quad +n(2n-1)(2n-2)(2n-3)x^{2n-4}-2n(n-3)(2n-1)(2n-2)x^{2n-3}\\
&= n(n-1)x^{n-4}h(x)
\end{aligned}$$

with

$$h(x) = -(n-2)(n-3)+2(2n-3)(n-2)x-3n(n+1)x^2+2(2n-1)(2n-3)x^n-4(n-3)(2n-1)x^{n+1}$$

and

$$\begin{aligned} h'(x) &= 2(2n-3)(n-2) - 6n(n+1)x + 2(2n-1)(2n-3)nx^{n-1} - 4(n-3)(2n-1)(n+1)x^n \\ h''(x) &= -6n(n+1) + 2(2n-1)(2n-3)n(n-1)x^{n-2} - 4(n-3)(2n-1)(n+1)nx^{n-1} \\ h'''(x) &= (n-1)(2n-1)\left(2n(2n-3)(n-2) - 4(n-3)(n+1)nx\right)x^{n-3} \end{aligned}$$

Step 2.3.1: For each $n \geq 6$, we have that $1 - \frac{4n-6}{n^2+n} < x_1 < 1 - \frac{4n-23/3}{n^2+n}$. These inequalities follow from simple manipulations.

Step 2.3.2: There exists $x_2 \in (x_1, 1)$ such that h'' is strictly increasing in (x_1, x_2), h'' is strictly decreasing in $(x_2, 1)$. It is immediate that $h'''(x) > 0$ for $x \in (x_1, x_2)$, and $h'''(x) < 0$ for $x \in (x_2, 1)$, with $x_2 = \frac{(2n-3)(n-2)}{(2n+2)(n-3)}$. Then, $x_2 - \left(1 - \frac{4n-23/3}{n^2+n}\right) = \frac{(15n-41)^2+389}{90n(n+1)(n-3)} > 0$ implies $x_2 > x_1$.

Step 2.3.3: $h''(x_1) > 0$, $h''(1) = -4n(n^2 - 8n + 6)$. From the proof of Step 2.3.2 we know that h'' is strictly increasing in the interval $(0, x_2)$, hence $h''(1 - \frac{4n-6}{n^2+n}) < h''(x_1)$ (from Step 2.3.1) and now we prove that $h''(1 - \frac{4n-6}{n^2+n}) > 0$. Precisely,

$$\begin{aligned} h''(1 - \frac{4n-6}{n^2+n}) = &\frac{2\,(2n-1)\,(n^2+n)^2(7n^2-27n+36)}{(n^2-3n+6)^2} \\ &\times \left[\left(1 - \frac{4n-6}{n^2+n}\right)^n - \frac{3(n^2-3n+6)^2}{(n^2+n)\,(2n-1)\,(7n^2-27n+36)}\right] \end{aligned}$$

and $h''(6) = \frac{14\,940}{343} > 0$, $h''(7) = \frac{307\,852\,483}{4302\,592} > 0$, $h''(8) = \frac{10\,409\,940\,767}{90\,699\,264} > 0$, $h''(9) = \frac{42\,860}{243} > 0$. For $n \geq 10$, we notice that $(1 - \frac{4n-6}{n^2+n})^n$ is a decreasing sequence such that $(1 - \frac{4n-6}{n^2+n})^n > e^{-4} > \frac{9}{500}$ and $\frac{9}{500} - \frac{3(n^2-3n+6)^2}{(n^2+n)(2n-1)(7n^2-27n+36)} = \frac{3(2894n+199n^2+42n^3+27\,669)(n-10)^2+845616(n-10)+101\,460}{500n(2n-1)(n+1)(7n^2-27n+36)} > 0$.

Step 2.3.4: $h'(x_1) < 0$, $h'(1) = 2n^2 + 2n > 0$. From Steps 2.3.2 and 2.3.3 we know that $h''(x) > 0$ for $x \in (x_1, x_2)$, hence h' is strictly increasing in the interval $[x_1, x_2]$ and $h'(x_1) < h'(1 - \frac{4n-23/3}{n^2+n})$ (from Step 2.3.1). Now we prove that $h'(1 - \frac{4n-23/3}{n^2+n}) < 0$:

$$\begin{aligned} h'(1 - \frac{4n-23/3}{n^2+n}) = &-\frac{2(2n^2+n-1)\left(27n^2-100n+138\right)}{3n^2-9n+23} \\ &\times \left(\frac{(3n^2-9n+23)(n^2-2n+17)}{(2n^2+n-1)(27n^2-100n+138)} - \left(1 - \frac{4n-23/3}{n^2+n}\right)^n\right) \end{aligned}$$

is negative as (i) $\frac{(3n^2-9n+23)(n^2-2n+17)}{(2n^2+n-1)(27n^2-100n+138)} > \frac{21}{400}$ for each $n \geq 6$, since $\frac{(3n^2-9n+23)(n^2-2n+17)}{(2n^2+n-1)(27n^2-100n+138)} - \frac{21}{400} = \frac{66(n-\frac{701}{44})^2(n-2)^2+\frac{725\,357}{88}(n-2)^2+23\,794n+59\,318}{400(2n-1)(n+1)(27n^2-100n+138)} > 0$; (ii) $\left(1-\frac{4n-23/3}{n^2+n}\right)^n$ is a decreasing sequence such that $\frac{21}{400} > \left(1-\frac{4\cdot 6-23/3}{6^2+6}\right)^6 \geq \left(1-\frac{4n-23/3}{n^2+n}\right)^n$ for each $n \geq 6$.

Step 2.3.5: $h(x_1) < 0$, $h(1) = 0$. From the proof of Step 2.3.3 we know that $h''(x) > 0$ for $x \in [1-\frac{4n-6}{n^2+n}, x_1]$, hence $h'(x) < h'(x_1) < 0$ for $x \in [1-\frac{4n-6}{n^2+n}, x_1]$ and h is strictly decreasing in $[1-\frac{4n-6}{n^2+n}, x_1]$, thus $h(1-\frac{4n-6}{n^2+n}) > h(x_1)$. Now we prove that $h(1-\frac{4n-6}{n^2+n}) < 0$:

$$h(1-\frac{4n-6}{n^2+n}) = -\frac{2\,(2n-1)\,(11n^2-33n+36)}{n\,(n+1)}\left(\frac{(n+1)\,(2n^2-9n+18)}{(2n-1)\,(11n^2-33n+36)} - \left(1-\frac{4n-6}{n^2+n}\right)^n\right)$$

is negative as (i) $\frac{(n+1)(2n^2-9n+18)}{(2n-1)(11n^2-33n+36)} - \frac{1}{12} = \frac{(n+4)((4n-15)^2+279)}{96(2n-1)(11n^2-33n+36)} > 0$; (ii) $\left(1-\frac{4n-6}{n^2+n}\right)^n$ is a decreasing sequence such that $\frac{1}{12} > \left(1-\frac{4\cdot 6-6}{6^2+6}\right)^6 \geq \left(1-\frac{4n-6}{n^2+n}\right)^n$ for each $n \geq 6$.

Step 2.3.6: There exists x_3 between x_1 and 1 such that $h'(x) < 0$ for $x \in (x_1, x_3)$, $h'(x) > 0$ for $x \in (x_3, 1)$. For $n = 6, 7$ we have $h''(1) > 0$. From steps 2.3.2 and 2.3.3 it follows that $h''(x) > 0$ for each $x \in (x_1, 1)$, hence h' is strictly increasing. Then the conclusion follows from Step 2.3.4.

For $n \geq 8$ we have $h''(1) < 0$. From steps 2.3.2 and 2.3.3 it follows that there exists x_4 between x_2 and 1 such that $h''(x) > 0$ for each $x \in (x_1, x_4)$, $h''(x) < 0$ for each $x \in (x_4, 1)$. Hence h' is strictly increasing in (x_1, x_4), strictly decreasing in $(x_4, 1)$ and the conclusion follows from Step 2.3.4.

Step 2.3.7: $q''(x) > 0$ for each $x \in (x_1, 1)$. From Steps 2.3.5 and 2.3.6 it follows that $h(x) < 0$ for each $x \in (x_1, 1)$, hence $q'''(x) < 0$ for each $x \in (x_1, 1)$. Then $q''(1) = 0$ implies that $q''(x) > 0$ for each $x \in (x_1, 1)$.