Inefficient Sorting under Output Sharing

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Abstract

I study sorting in a frictional market. Asset owners post their terms, then workers direct their search. When the owners switch from prices to shares, the competition between workers is handicapped. The unique equilibrium features inefficient positive assortative matching. The queue lengths are distorted, even though the Hosios efficiency condition holds for every pair of types. For any distribution of types, all workers pair up with better assets. The best workers suffer while the weakest workers gain; the opposite occurs on the asset side. Competition drives the asset owners to post flatter contracts. It leads to constrained efficiency whenever prices are feasible. Otherwise, handicapped competition results in inefficient sorting.

JEL: C78, D82, D83, D86, G32
Keywords: Assortative matching, Sorting, Directed search, Hosios condition, Linkage principle, Linear contracts

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1 Introduction

This paper studies sorting in a frictional market where two sides are vertically differentiated. The two sides enter into contracts specifying how the payment is contingent on a certain outcome. One side is often better informed about their types. Little is known about how the use of contingent payment affects the equilibrium sorting pattern and the distribution of matching surpluses in the presence of private types.

I address this question in a directed search framework. There are double continuums of assets and workers. A worker's productivity is privately known whereas the quality of an asset is public. Asset owners first post their shares of the future outputs. Workers then decide which type of assets and contract to search for. The meeting is bilateral. The constrained efficient allocation always features positive assortative matching (PAM) despite the search friction.[1] I identify a novel channel of inefficiency and when it arises. I analyze its effects on the sorting pattern and the distribution of surpluses.

Channel of inefficiency Let us first consider the setting that owners post prices at which a worker buys out the asset upfront and assumes the residual claim. Eeckhout and Kircher (2010), henceforth EK, study sorting in goods market using this framework. They show that constrained efficient allocations can be decentralized. In equilibrium, everyone receives her social value, the shadow price in the utilitarian planner's problem. I refer to their benchmark by *price competition*.

The constrained efficient allocations can no longer be decentralized using output shares. Output shares are steeper than prices: a better worker pays more than a weaker worker when conceding a larger share to the owner. The use of shares handicaps the competition between workers for the *same* type of assets, increasing the asset owners' payoff. Holding the allocation *unchanged*, the shift in the divisions of outputs can be attributed to the linkage principle in auction theory (DeMarzo et al., 2005; Milgrom and Weber, 1982). PAM gives rise to an additional spillover. As workers pay more for their supposed partners, they find deviations to better assets even more attractive, further intensifying the competition for better assets. To support a constrained efficient allocation, the set of incentive compatible contracts must provide the asset side with higher payoffs than in the price competition.

As a result, the private benefit for an asset to get matched is above the social benefit.

[1] In the presence of search friction, PAM is said to occur if better workers always search for better assets.

The wedge is largest at the top. Facing search friction, owners of the best assets increase their matching probability by inducing an inefficiently long queue of workers. This leads to the unraveling of the constrained efficient allocation.

The preceding discussion begs three questions. Under PAM, the lower quality assets are left with a pool of weaker workers amid a longer queue of workers for the better assets. Despite higher shares of the outputs, their owners gain less from matching with weaker workers and may induce shorter queues of workers instead. Hence, distortions in the queue lengths and the sorting pattern are intertwined. The first question is what form of distortion *always* arises in equilibrium.

There is another potential source of inefficiency. Unlike a fixed price, an asset owner's payoff now depends on her partner' type. She may attempt to screen out better workers by means of rationing. Chang (2017); Inderst and Müller (2002); Guerrieri et al. (2010); Guerrieri and Shimer (2014) study how distortion in queue lengths helps screening using directed search frameworks. The second question is how the distortion caused by handicapped competition differs from those in the screening literature. In this light, I consider a setting that all types of workers have the *same* preference over the contract term and the matching probability given the asset quality. Consequently, asset owners never use queue length to screen out better workers. The distortion in queue lengths is entirely attributed to the handicapped competition.

Third, the form of contracts determines the distribution of surpluses. If asset owners may choose from different classes of contracts, what contracts do they post in equilibrium? Or when do they post output shares?

Main results I first characterize the unique equilibrium when only output shares are feasible. It features PAM. One type of workers pairs up with one type of assets, and vice versa. Owners of better assets post less generous terms but still induce longer queues. For every pair of matched types, Hosios (1990) efficiency condition holds, the division of the output is given by the elasticity of the matching function.

A key contribution of this paper is the qualitative features of the distortion which are universal for *all* distributions. As one may expect, the queue length for the best assets is always inefficiently high. The novel result is that all (but the highest type of) assets always pair up with weaker workers. There is either excessive entry of workers or insufficient entry of assets. In comparison with the price competition or the complete information case, the best workers suffer while the weakest workers gain; the opposite is true for the asset side.

The owners of the best assets *collectively* gain if the asset side posts output shares. Yet, an owner profits from posting a fixed price to poach better workers. The competition between the owners drives them to all post prices, whenever feasible. In this sense, this paper spells out when constrained efficiency is attained or not. Inefficiency arises from handicapped competition between workers if they are prevented from buying out the assets upfront. The infeasibility could be caused by workers' liquidity constraints, risk aversion, or incentive provision for others.[2]

I extend the previous observation to general contracts. Introducing steeper contracts has no effect on the set of equilibrium allocations and payoffs. The asset side posts only from the flattest class in an equilibrium. Other equilibria feature the same allocation and payoffs. In particular, suppose the asset side chooses from the affine contracts which specifies a wage payment as well as a share of the output. They post only the shares in equilibrium, resulting in the aforementioned distortion.

Applications The market for top executives is a well-documented example. Firms are ranked by their size. Candidates know their productivity better. Gabaix and Landier (2008); Terviö (2008) study how PAM accounts for the empirical distribution of CEOs' pay in the largest U.S. listed companies. Frydman and Jenter (2010) document that from 2000 to 2008, base salary makes up less than 20% of the CEOs' pay in S&P 500 firms. Over half of the remuneration is option and restricted stock grants.

In the preceding discussion, the informed party receives the output and pays the uninformed.[3] In the labor market, the payment flows in the opposite direction. A fixed price can be seen as a high-powered incentive contract whereas an output share is a low-powered incentive. Lemieux et al. (2009) document that the fraction of performance-pay jobs in U.S. increased between late 1970s and 1990s, contributing to the wage inequality. This paper predicts that firms tend to hire better workers when the availability of high-powered incentive contracts improves. While it is well-known that PAM increases inequality (e.g., Rosen, 1981), I show how the combination of incentive contracts and PAM further amplifies the inequality across workers.[4]

[2] We cannot circumvent the mentioned problems by providing the workers a fixed fraction of the residual claim. In this case, the asset owners still place a higher private value on their matching probabilities than the social benefit. The constrained efficient allocation will not be decentralized.

[3] This convention is in line with the existing literature on security-bid auction and assortative matching with private types on one side (e.g., Mailath et al., 2016).

[4] It is not the same as *wage* inequality as the expected payoff also depends on the matching probability.

Rhodes-Kropf and Robinson (2008) document PAM between acquiring and target firms in M&A. This paper also explains how the sorting pattern depends on whether target firms receive cash or equity shares of the acquiring firms.

Related literature This paper adds to the vast literature on assortative matching with search friction: Burdett and Coles (1997); Chade (2006); Damiano et al. (2005); Eeckhout (1999); Jacquet and Tan (2007); Shi (2001); Shimer and Smith (2000); Shimer (2005); Smith (2006). My model is built on EK. The distribution-free results in the literature center on the equilibrium matching pattern. My contribution is the distribution-free features on the *distortion* in the matching pattern. This is not trivial as both the equilibrium and constrained efficient allocations vary with the entire distribution of types.[5]

Chiappori and Reny (2016); Kaya and Vereshchagina (2014); Legros and Newman (2007); Schulhofer-Wohl (2006); Serfes (2005) consider the use of contracts in assortative matching with public types. They study how risk sharing or incentive provision shapes the payoff functions and provide conditions for PAM and negative assortative matching (NAM). My distribution-free results are about the *changes* in the matching pattern and the distribution of surpluses when the form of contract changes.

In the security-bid auction literature (e.g., DeMarzo et al., 2005; Hansen, 1985; Sogo et al., 2016), the seller in a private value auction always benefits from handicapping the competition using steeper securities. In my setting, the wedges in the divisions of the outputs cause adjustments in the queue lengths and the sorting pattern. Consequently, the competition from the best assets makes owners of the lowest quality assets worse off!

This paper contributes to the efficiency conditions in the search and matching models. Albrecht et al. (2010); Julien and Mangin (2017) generalize the analysis in Hosios (1990) to account for other externalities *within* the same market where multiple types are pooled. I study the spillover effects across markets for different types. Looking at each pair of matched types in isolation, the equilibrium queue length maximizes the joint payoffs. However, the queue length and the posted share for one type of assets affect the remaining pool of workers and their information rent, resulting in inefficient sorting.

[5] The above complication does not arise in the textbook competitive screening models and the framework of Guerrieri et al. (2010) where there is free entry of homogeneous principals. In these models, the set of feasible contracts and allocations does not depend on the type distribution. The support of the type distribution alone pins down all incentive compatible conditions. Consequently, the set of separating equilibria is invariant to the type distribution.

Layout Section 2 details the baseline setting and the equilibrium definition. Section 3 characterizes the unique equilibrium under output shares. Section 4 characterizes the constrained efficient allocation. Section 5 studies the form of distortion. Section 6 considers the setting where different classes of contracts are feasible. Section 7 remarks on the setting and the results. All omitted proofs are relegated to Appendix.

2 Model Setting

Production Every male worker privately knows his productivity $p \in [0, 1]$. Assets are ranked by their public quality $q \in [0, 1]$. Each female owner has a unit of asset. All parties are risk neutral. A worker may operate an asset to produce an expected output $Y(p, q)$, where $Y : [0, 1]^2 \to \mathbb{R}_{++}$ is positive, strictly increasing and C^2.

Assumption (Y)**.** *$Y(p, q)$ is strictly log-supermodular (log-SPM) in p and q.*

Assumption (Y) states that a better worker generates a greater *percentage* increase in the output using a better asset. It has two important implications even in a frictionless setting. First, it is stronger than strict supermodularity (SPM). Without search friction, aggregate surplus is maximized under perfect PAM. Second, the use of output shares raises the hurdle for PAM as a better worker now must accept a lower percentage of the output in exchange for a better asset. Hence, Assumption (Y) is necessary for decentralizing PAM using output shares.

Example (O-ring production in Kremer (1993))**.** *Assumption (Y) is satisfied if $Y(p, q) = \underline{y} + pq(\overline{y} - \underline{y})$, where $\overline{y} > \underline{y} > 0$: Production is composed of two tasks. The probabilities of success for the tasks are p and q. Production yields a high output $\overline{y}$ only if both tasks are successful. Otherwise, a base output $\underline{y}$ is produced. The probabilities of success can be generalized to any strictly increasing functions in p and q.*

The types are continuously distributed. $F(p)$ denotes the measure of workers of productivity below p. $G(q)$ is the measure of assets with qualities below q. The total measures of assets and workers may differ. F and G are C^2 and their derivatives are denoted by f and g, respectively. f and g are strictly positive and bounded.

Contracts The output Y is stochastic and contractible. A contract $T : \mathbb{R}_+ \to \mathbb{R}$ specifies the payment $T(y)$ to the asset owner contingent on the realized output y. We consider classes of contracts $T(y;x)$ which can be indexed by some contract term $x \in [\underline{x}, \overline{x}]$. To save on notation, $T(p,q,x) := E(T(Y;x)|p,q)$ denotes the expected payment for a pair of types. The worker expects to keep $Y(p,q) - T(p,q,x)$ after production. I require that $Y(p,q) \geq T(p,q,x) \geq 0$, and both $T(p,q,x)$ and $Y(p,q) - T(p,q,x)$ are C^1 and strictly increasing in p and q.[6]

Definition (DeMarzo, Kremer, and Skrzypacz, 2005)**.** *$T(y;x)$ for $x \in [\underline{x}, \overline{x}]$ is an ordered set of securities if*

1. *$T(p,q,x)$ is continuous and strictly increasing in x; and*
2. *For any (p,q), $Y(p,q) - T(p,q,\overline{x}) \leq \underline{V}$ and $T(p,q,\underline{x}) \leq \underline{U}$.*

x parameterizes the division of the expected output. The first condition means that a higher x always represents a less generous term for the worker. The second condition means that the set of securities can accommodate any mutually acceptable split of the output. It is fulfilled if $T(p,q,\underline{x}) = 0$ and $T(p,q,\overline{x}) \geq Y(p,q)$. Examples are fixed prices $T(y;t) = t$ where $t \in [0, Y(1,1)]$, and output shares $T(y;s) = s$ where $s \in [0,1]$. To follow the terminologies in DeMarzo et al. (2005), I use the terms contract and security interchangeably.

Matching The asset side may post from an ordered set of securities $T(y;x)$. As the asset quality is public, (sub-)markets are indexed by $(q,x) \in [0,1] \times [\underline{x}, \overline{x}]$. An owner of asset quality q may go to one of the markets (q,x) while a worker may visit any market.

They may choose their outside options instead. The values of the workers' and assets' outside options are positive and given by $\underline{V}$ and $\underline{U}$, respectively.

The timing of the events is as follows: Asset owners first make their decisions simultaneously. Observing how asset owners are distributed across the markets, the workers make their decisions. The two sides of each market then pair up randomly.

Define queue length $\lambda \in [0, \infty]$ as the ratio of workers to assets in a (sub-)market. A worker gets matched with probability $\eta(\lambda)$ while the matching probability for an asset owner is $\delta(\lambda)$. Meeting is bilateral, so $\delta(\lambda) \leq \min\{\lambda, 1\}$ and $\lambda\eta(\lambda) = \delta(\lambda)$. The payoffs for those who are left unmatched are normalized to zero.

[6] DeMarzo et al. (2005) provide the primitive conditions on $T(y;x)$ and the output distribution for the mentioned properties in this subsection.

$\delta(\lambda)$ is C^2, strictly increasing and strictly concave while $\eta(\lambda)$ is strictly decreasing. Hence, the elasticities $\frac{d\ln\delta}{d\ln\lambda}$ and $-\frac{d\ln\eta}{d\ln\lambda}$ are always in the unit interval. Following EK, I assume a decreasing elasticity for $\delta(\lambda)$, which holds for common matching functions.

Assumption (M)**.** $\dfrac{d\ln\delta(\lambda)}{d\ln\lambda}$ *is decreasing.*

Searching for partner is costly as the agent forgoes her outside option. I assume

$$\max_{\lambda\geq 0}[\delta(\lambda)Y(1,1)-\lambda\underline{V}-\underline{U}]>0, \tag{1}$$

so that it is efficient to have the best agents participating in matching.

2.1 Equilibrium definition

$K(q,x)$ is the measure of asset owners in the markets $(q',x')\leq(q,x)$. $L(p,q,x)$ is the measure of workers with types $p'\leq p$ in the markets $(q',x')\leq(q,x)$. The marginal distributions are denoted with the corresponding variables as subscripts. For examples, $L_{pq}(p,q)$ is the measure of workers of types $p'\leq p$ in all markets (q',x') where $q'\leq q$; $K_q(q)$ is the measure of assets of types $q'\leq q$ in all markets. We call (L_{pq},K_q) an allocation.

Definition. (K,L) *is feasible if* $K_q\leq G$ *and* $L_p\leq F$*.*

$G(q)-K_q(q)$ and $F(p)-L_p(p)$ are respectively the measures of assets below q and workers below p taking the outside option. $\text{supp}(K)$ is the support of K and $\text{supp}(L)$ denotes that of L. A market is active if it is in $\text{supp}(K)$; otherwise, it is inactive. Workers never get matched if deviating to an inactive market unilaterally. Therefore, L_{qx} is required to be absolutely continuous w.r.t. K.

The equilibrium concept follows the literature on large games e.g., Mas-Colell (1984). An agent's payoff depends on her own decision and others' actions only through (K,L). K and L are in turn derived from the optimal decisions of all agents.

Each market (q,x) is associated with a queue length $\Lambda(q,x;K,L)$ and a composition of workers $F(p|q,x;K,L)$. For active markets, Λ is the Radon-Nikodym derivative, $\frac{dL_{qx}}{dK}$ and $F(p|q,x;K,L)$ is derived using Bayes' law. In an active market (q,x), a worker of type p receives an expected payoff of

$$\eta(\Lambda(q,x;K,L))[Y(p,q)-T(p,q,x)], \tag{2}$$

while an asset owner receives an expected payoff

$$\delta(\Lambda(q,x;K,L))\int T(p,q,x)dF(p|q,x;K,L). \tag{3}$$

Given (K, L), a worker of type p can attain

$$V(p; K, L) = \max\{\underline{V}, \sup_{(q,x)\in \text{supp}(K)} \eta(\Lambda(q, x; K, L))[Y(p, q) - T(p, q, x)]\}.$$

$V(.; K, L)$ in turn determines the asset owners' deviating payoff. For an inactive market, $\Lambda(q, x; K, L)$ and $F(p|q, x; K, L)$ are the belief regarding the queue length and the composition of workers attracted after an owner of asset q deviates to that market. If $V(p; K, L) \geq \eta(0)[Y(p, q) - T(p, q, x)]$ for all types, then no workers will be attracted and $F(p|q, x; K, L)$ is set to be degenerate at $p = 0$. If $V(p; K, L) < \eta(0)[Y(p, q) - T(p, q, x)]$ for some types, $\Lambda(q, x; K, L)$ is pin down by the lowest matching probability some workers are willing to endure. Only the lowest type among these workers will be attracted. Formally,

$$\Lambda(q, x; K, L) = \inf\{\lambda \in [0, \infty] : V(p; K, L) \geq \eta(\lambda)[Y(p, q) - T(p, q, x)], \forall p \in [0, 1]\}.$$

If $\Lambda(q, x; K, L) > 0$, then $F(p|q, x; K, L)$ is degenerate at

$$\inf\{p \in [0, 1] : V(p; K, L) \leq \eta(\Lambda(q, x; K, L))[Y(p, q) - T(p, q, x)]\}.$$

Facing $\Lambda(q, x; K, L)$ and $F(.|q, x; K, L)$, an owner of asset quality q can attain

$$U(q; K, L) = \max\{\underline{U}, \sup_{(q,x)\in[0,1]\times[\underline{x},\overline{x}]} \delta(\Lambda(q, x; K, L)) \int T(p, q, x) dF(p|q, x; K, L)\}.$$

Definition. *An equilibrium is a pair of feasible distributions $(\widetilde{K}, \widetilde{L})$ satisfying:*

- *Asset owners' optimality: $(q, x) \in \text{supp}(\widetilde{K})$ only if x maximizes the asset owner's expected payoff (3). $\widetilde{K}'_q(q) = g(q)$ if $U(q; \widetilde{K}, \widetilde{L}) > \underline{U}$.*
- *Workers' optimality: $(p, q, x) \in \text{supp}(\widetilde{L})$ only if (q, x) maximizes the worker's expected payoff (2). $\widetilde{L}'_p(p) = f(p)$ if $V(p; \widetilde{K}, \widetilde{L}) > \underline{V}$.*

The optimality conditions are interpreted as follows: Everyone takes the queue length and the composition of workers in the markets as given. If participating, they go to the markets where their expected payoff is the highest. $f(p) > L'_p(p)$ indicates some workers of type p take their outside options. It never happens if the workers can get higher expected payoffs in some market. Similarly for the asset side.

The belief restriction for the inactive markets follows Guerrieri et al. (2010). EK adopt the same restriction on the queue length.

Fix an equilibrium $(\widetilde{K}, \widetilde{L})$, $V(p; \widetilde{K}, \widetilde{L})$ and $U(q; \widetilde{K}, \widetilde{L})$ are the equilibrium payoffs. The arguments $\widetilde{K}$ and $\widetilde{L}$ will be omitted if no confusion arises.

We say (K, L) induces voluntary participation of the asset owners if their expected payoff (3) in any active market is no less than $\underline{U}$. (K, L) is incentive compatible (IC) if it satisfies workers' optimality condition in the equilibrium definition. These two notions will be useful for discussing the utilitarian planner's problem.

Assortative Matching

Definition. *A pair of distributions* (K, L) *features positive assortative matching (PAM) if there exists a pair of threshold types* $(\underline{p}, \underline{q}) < (1, 1)$ *and an increasing function* $\kappa : [\underline{p}, 1] \to [\underline{q}, 1]$ *such that* $\kappa(\underline{p}) = \underline{q}$, $\kappa(1) = 1$ *and* $L_{pq}(p, \kappa(p)) = F(p) - F(\underline{p})$.

$\kappa(p)$ is the quality of assets assigned to type p. The definition of PAM also requires that every worker above $\underline{p}$ participates in matching. The requirement is not restrictive. A better worker gains more in the same market. The outside option is only optimal for the lowest types. On the asset side, monotonic participation is efficient. I will show that monotonic participation occurs in equilibrium as well. In both cases, κ is strictly increasing, and its inverse is denoted by $r : [\underline{q}, 1] \to [\underline{p}, 1]$. $r(q)$ is the type of workers assigned to quality q.

3 Equilibrium Characterization

We turn to the setting that only output shares $T(y; s) = s$ is feasible. s and $1 - s$ are the shares for the asset owner and worker, respectively. The workers' expected payoff (2) is separable into $Y(p, q)$ and $\eta(\lambda)(1 - s)$. Only the former depends on the private type.

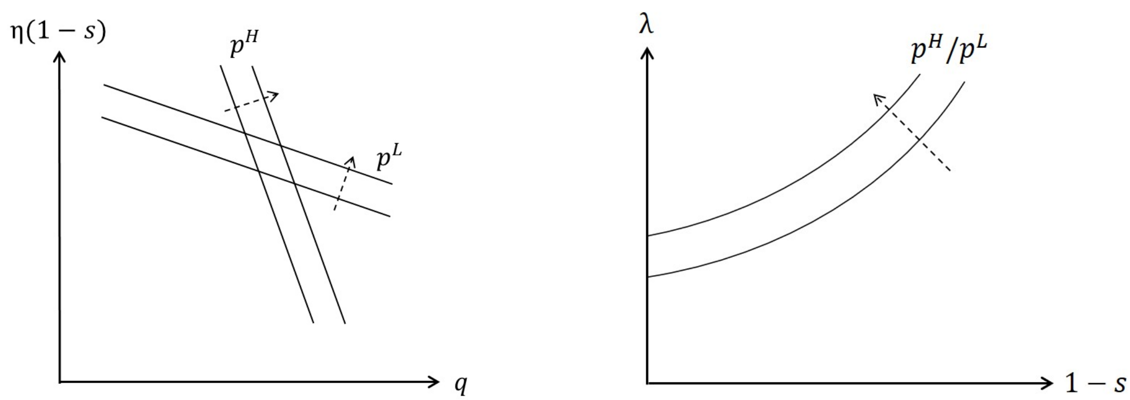

Figure 1: Properties of workers' indifference curves

There are two important implications. First, workers' preferences over q and $\eta(1-s)$ satisfy a strict single crossing property (SCP) under Assumption (Y). The sorting of workers into the active markets turns into a one-dimensional problem. Second, fix the asset quality, workers of all types have the same preference over their matching probability and the contract term.

Matching pattern The SCP implies that if a worker prefers a market with better assets to another market with lower quality assets, then all better workers strictly prefer the one with better assets, and vice versa. This holds irrespective of the queue lengths and the contract terms in the two markets. Therefore, the participants must match assortatively in any equilibrium.

Another implication of the SCP is the monotonic participation on the asset side. Suppose some type of workers goes to an active market. An owner of a better asset can find a less generous term which provides these workers the same payment. Such a poaching offer will not draw weaker workers because of the SCP. Hence, the owners' equilibrium payoff must increase with the quality of their assets.

Let $(\underline{\widetilde{p}}, \underline{\widetilde{q}}, \widetilde{\kappa})$ represent PAM in the equilibrium under consideration. $\widetilde{r}$ denotes the inverse of $\widetilde{\kappa}$. It satisfies the boundary condition

$$\widetilde{r}(\underline{\widetilde{q}}) = \underline{\widetilde{p}}; \widetilde{r}(1) = 1. \tag{4}$$

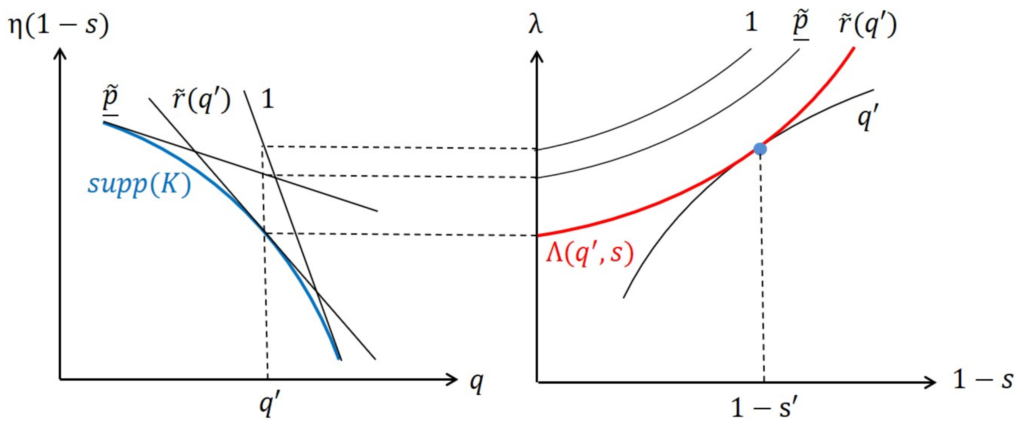

Figure 2: Characterization of active markets and the Hosios condition

Contract term and queue length The left panel plots the indifference curves over q and $\eta(1-s)$ of the participating workers, which yield their equilibrium payoff. The lower envelope of the indifference curves must be $\eta(\Lambda(q,s))(1-s)$ for the active markets. The workers of $\widetilde{r}(q')$ accept a lower matching probability in the active market (q', s') than anybody else. They also endure the lowest matching probability in *all* markets for assets q' because all types have the same preference. Hence, a deviating offer always attracts the same type $\widetilde{r}(q')$ in equilibrium. The competition from these workers will result in a long enough queue driving away all other types.

Thus an owner of asset q' only trades off between the queue length and her output share as in the right panel of Figure 2. The tangent point of the indifference curves of the pair is the optimal contract and the associated queue length for the asset owners. This is exactly the Hosios (1990) efficiency condition.[7] Hence, the owners of assets $q \geq \underline{q}$ all post the share

$$s = 1 - \frac{d \ln \delta(\widetilde{\lambda}(q))}{d \ln \lambda} = 1 - \frac{\delta'(\widetilde{\lambda}(q))}{\eta(\widetilde{\lambda}(q))}$$

where $\widetilde{\lambda}(q)$ is the equilibrium queue length.

Abusing the terminology, I call $(\underline{\widetilde{p}}, \underline{\widetilde{q}}, \widetilde{\lambda}, \widetilde{r})$ the equilibrium allocation. I recover the corresponding $(\widetilde{K}, \widetilde{L})$ in Appendix A.2.

PAM and monotonic participation requires that for any $q \geq \underline{\widetilde{q}}$,

$$\int_q^1 \widetilde{\lambda}(q') dG(q') = F(1) - F(\widetilde{r}(q)),$$

or equivalently

$$\widetilde{r}'(q) = \frac{g(q)}{f(\widetilde{r}(q))} \widetilde{\lambda}(q). \tag{5}$$

Equilibrium payoffs Substituting the expression of s into the expected payoffs (2) and (3), we can restate the Hosios condition in term of the workers' equilibrium payoff

$$V(\widetilde{r}(q)) = \delta'(\widetilde{\lambda}(q)) Y(\widetilde{r}(q), q), \tag{6}$$

or in term of the asset owners' equilibrium payoff

$$U(q) = [\delta(\widetilde{\lambda}(q) - \delta'(\widetilde{\lambda}(q))\widetilde{\lambda}(q)] Y(\widetilde{r}(q), q). \tag{7}$$

(6) and (7) hold for *every* type above $\underline{q}$. I simply call the whole set of conditions as the Hosios condition.

[7] $\widetilde{\lambda}(q)$ maximizes $\delta(\lambda)Y(\widetilde{r}(q), q) - \lambda V(\widetilde{r}(q))$. Hence, the joint payoffs for the pair of types attains its maximum at the equilibrium queue length, subject to free entry of workers at their equilibrium payoff.

By the same token, the optimality condition for workers above $\underset{\sim}{p}$ can be rewritten as

$$V(\widetilde{r}(q)) = \max_{q' \in [\underset{\sim}{q}, 1]} \delta'(\widetilde{\lambda}(q'))Y(\widetilde{r}(q), q').$$

After accounting for the choices of contracts, the sorting of workers is induced by the variation in the queue length. The better the asset, the longer the queue. Under assumption (M), the Hosios condition in turn requires the posted share to increase with the asset quality. Applying the envelope theorem,

$$\frac{dV(\widetilde{r}(q))}{dp} = \delta'(\widetilde{\lambda}(q))\frac{\partial Y(\widetilde{r}(q), q)}{\partial p}. \tag{8}$$

With strict SCP over q and $\eta(1-s)$, (8) is not only necessary but also sufficient. Abusing the terminology, I call it workers' IC condition.

Threshold types The strict SCP also implies that the active market for assets $\underset{\sim}{q}$ is the most profitable deviation for workers below $\underset{\sim}{p}$. By continuity, workers of $\underset{\sim}{p}$ must be indifferent about participation. This yields the boundary condition for the workers side,

$$\underset{\sim}{p}(V(\underset{\sim}{p}) - \underline{V}) = 0. \tag{9}$$

Now consider an owner below $\underset{\sim}{q}$. Under the strict SCP, $\underset{\sim}{p}$ is the highest type her deviating offer can attract. Her deviating payoff must be below $U(\underset{\sim}{q})$. By continuity, the boundary condition for the asset side mirrors that for the workers,

$$\underset{\sim}{q}(U(\underset{\sim}{q}) - \underline{U}) = 0. \tag{10}$$

Proposition 1. *There is a unique equilibrium. This equilibrium features PAM. There exist strictly increasing and C^1 functions $(\widetilde{r}(q), \widetilde{\lambda}(q), \widetilde{v}(p))$ and threshold types $(\underset{\sim}{p}, \underset{\sim}{q})$ satisfying the conditions (4)–(6) and (8)–(10) such that*

1. *Asset owners (workers) participate if and only if $q \geq \underset{\sim}{q}$ ($p \geq \underset{\sim}{p}$);*

2. *Workers of $p \geq \underset{\sim}{p}$ have equilibrium payoffs $\widetilde{v}(p)$;*

3. *$\widetilde{\lambda}(q)$ is the queue length for assets q. Owners of assets $q \geq \underset{\sim}{q}$ post $s = 1 - \frac{d \ln \delta(\widetilde{\lambda}(q))}{d \ln \lambda}$. The output share posted and the queue length strictly increases with the asset quality;*

4. *$F(p|q, s)$ is degenerate at $\widetilde{r}(q)$ if $q \geq \underset{\sim}{q}$ and $\Lambda(q, s) > 0$.*

The differential equations (5), (6) and (8) and the boundary conditions (4), (9) and (10) form a boundary value problem.[8] I show that it has a unique solution in the proof.

4 Constrained Efficient Allocation

Consider a utilitarian planner who has complete information and dictates the market each type goes to. She chooses a pair of feasible distributions to maximize the aggregate surplus

$$\max_{K,L} \int \eta(\frac{dL_{qs}}{dK})Y(p,q)dL + [F(1) - L_p(1)]\underline{V} + [G(1) - K_q(1)]\underline{U}.$$

The presence of search friction is conducive to NAM because the most efficient way to increase the matching probability for high types is assigning them to a market flooded with low types from the other side. Assumption (M) caps the benefit from such an arrangement. It states that either side sees a diminishing gain in the matching probability when there are more participants on the opposite side. Assumption (Y) puts a lower bound on the gain from production complementarity. They jointly ensure that the utilitarian planner's solution always features PAM.

Theorem (Eeckhout and Kircher, 2010)**.** *The constrained efficient allocation features PAM for all distributions of types.*

Proof. The n-root-supermodularity condition is met under Assumption (Y) and (M). Proposition 4 in EK applies.[9] □

Abusing the terminology, I use $(r_{CE}, \lambda_{CE}, \underline{p}_{CE}, \underline{q}_{CE})$ to represent a constrained efficient allocation.[10] It satisfies (4) and (5). Let $v_{CE}(p)$ and $u_{CE}(q)$ respectively denote the shadow values of workers of p and assets of q in the planner's problem. EK state the first-order conditions in term of $v_{CE}(p)$. For $q \geq \underline{q}_{CE}$, v_{CE} satisfies (6) and

$$\frac{dv_{CE}(r_{CE}(q))}{dp} = \eta(\lambda_{CE}(q))\frac{\partial Y(r_{CE}(q), q)}{\partial p}. \tag{11}$$

[8] $V(p)$ is defined over $[0,1]$. $\widetilde{v}$ is the restriction of V over $[\underline{\widetilde{p}}, 1]$, part of a solution to the boundary value problem.

[9] Unlike EK, I assume the outside options are positive and add the boundary conditions at the bottom. I also assume a weakly, in place of strictly, decreasing elasticity of $\delta(\lambda)$. Their results remain valid because I strengthen the assumption on $Y(p,q)$ to strict log-SPM. Lemma 1 will show that the constrained efficient allocation is unique under Assumption (Y) and (M).

[10] The aggregate surplus is independent of the divisions of outputs, so a continuum of (K,L) solves the utilitarian planner's problem. They all have the same $(r, \lambda, \underline{p}, \underline{q})$ but different sets of active markets.

At the threshold types, $v_{CE}(\underline{p}_{CE})$ and $u_{CE}(\underline{q}_{CE})$ satisfy (9) and (10).

The differential equations (5), (6) and (11) and the boundary conditions (4), (9) and (10) form another boundary value problem with a unique solution. Compared to Proposition 1, the only difference is that (11) replaces (8) in the boundary value problem.

4.1 Price competition

Price competition is the benchmark setting whereby asset owners post prices $T(y;t) = t$.

Theorem (Eeckhout and Kircher, 2010)**.** *A constrained efficient allocation is always decentralized in price competition. The equilibrium payoffs are given by v_{CE} and u_{CE}.*

Proof. Again by Proposition 4 in EK. □

Let $t_{CE}(q)$ denote the posted price for assets q in equilibrium. We can obtain t_{CE} from v_{CE} and r_{CE}. Under Assumption (Y) and (M), the first-order condition (11) is the same as the workers' IC condition in price competition,

$$v_{CE}(r_{CE}(q)) = \max_{q' \in [\underline{q}_{CE}, 1]} \eta(\lambda_{CE}(q'))[Y(r_{CE}(q), q') - t_{CE}(q')], q \geq \underline{q}_{CE}.$$

Bilateral meeting implies that $\eta(\lambda) > \delta'(\lambda)$. Under the *same* allocation, (8) and (11) indicate that the information rent for workers grows at a slower rate when output shares are posted. This is an extension of the linkage principle in the security-bid auction literature. The catch is that the two equilibrium allocations are always different.

5 Form Of Distortion

As indicated by (5), the exact distortion under output shares varies with the distribution of types. I now establish the form of the distortion which arises for all distributions.

Proposition 2. *Compared with the constrained efficient allocation, the equilibrium allocation has the following features:*

1. $\lambda_{CE}(1) < \widetilde{\lambda}(1)$*;*

2. $r_{CE}(q) > \widetilde{r}(q)$ *for* $q \in (\widetilde{\underline{q}}, 1)$*;*

3. $\underline{p}_{CE} \geq \widetilde{\underline{p}}$*, equality holds if and only if* $\underline{p}_{CE} = 0$*;*

4. $\widetilde{\underline{q}} \geq \underline{q}_{CE}$*, equality holds if and only if* $\widetilde{\underline{q}} = 0$*.*

Corollary 1. $v_{CE}(1) > V(1)$*,* $v_{CE}(\underline{p}_{CE}) < V(\underline{p}_{CE})$*; and* $U(1) > u_{CE}(1)$*,* $U(\widetilde{\underline{q}}) < u_{CE}(\widetilde{\underline{q}})$*.*

Longer queue for the best assets It is instructive to start with the equilibrium in price competition. Suppose we replace the equilibrium prices $t_{CE}(q)$ with the output shares $s_{CE}(q)$ which keep the same divisions of the outputs. A fixed share of the output costs more to the better workers but less to the low types. The workers have higher deviating payoffs from the active markets for better assets. In particular, a worker of $r_{CE}(q)$ always profits from searching for assets incrementally above q.

Nevertheless, deviating to a market for even higher quality assets can be more profitable. In particular, if $\lambda_{CE}(q)$ is decreasing at q, the market for better assets features a lower queue length and a lower share s_{CE}.[11] Only the best assets always draw a longer queue of workers. In response, their owners post a greater share to partially offset the increase in the queue length. On the other side, the best workers suffer from the reductions in both the output share and the matching probability.

All workers pair up with better assets Under PAM, a pool of weaker workers is left to the lower quality assets amid a longer queue for the better assets. The asset owners in the intermediate range face two counteracting forces. First, the asset owners offer less generous terms as the local competition between workers is intensified. However, they are also left with weaker workers. In comparison with price competition, these asset owners' gain from a match may go in either direction, so does the distortion in the queue lengths. The relative strengths of the two forces depend on the distribution of types.

Surprisingly, all assets but $q = 1$ must settle with weaker partners. Suppose, to the contrary, that we move down from the top and find that asset $\widehat{q}$ continues pairing with the same type $\widehat{p}$. As workers just above $\widehat{p}$ now match with better assets, the queue for assets $\widehat{q}$ must be shorter, $\lambda_{CE}(\widehat{q}) \geq \widetilde{\lambda}(\widehat{q})$, under PAM. Figure 3 depicts the situation.

[11] Under Assumption (Y) and (M), $t_{CE}(q)$ must be monotonic in q while $s_{CE}(q) = 1 - \frac{d \ln \delta(\lambda_{CE}(q))}{d \ln \lambda}$ increases with $\lambda_{CE}(q)$, and needs not be monotonic.

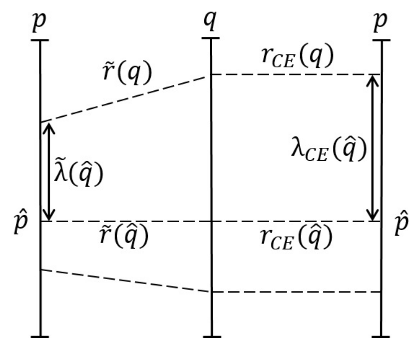

Figure 3: Implication of same matched pair of types on queue length

Now consider a thought experiment of removing all workers and assets above $\widehat{p}$ and $\widehat{q}$. The constrained efficient allocation remains the same. Otherwise, the utilitarian planner would have improved on it in the first place. Replacing $\widetilde{r}(1) = 1$ with $\widetilde{r}(\widehat{q}) = \widehat{p}$, the same set of equilibrium conditions in Proposition 1 applies to the truncated distribution of types. Although workers of $\widehat{p}$ now have no better assets to deviate to, owners of $\widehat{q}$ still offer the same contract in order to deter deviation by workers slightly below $\widehat{p}$. We have argued that the queue for the highest quality assets $\widehat{q}$ in the truncated distribution must be inefficiently long, $\lambda_{CE}(\widehat{q}) < \widetilde{\lambda}(\widehat{q})$. This contradicts my previous claim!

Increased participation by workers Let us return to the discussion that the shares $s_{CE}(q)$ have replaced the prices $t_{CE}(q)$. The workers of $\underline{p}_{CE}$ are strictly better off by deviating to slightly better assets. If $\underline{p}_{CE} > 0$, those just below $\underline{p}_{CE}$ pay less under the share $s_{CE}(\underline{q}_{CE})$ and will participate.

Reduced participation by assets At the same time, the asset owners of $\underline{q}_{CE}$ suffer for two reasons. In the case $\underline{p}_{CE} > 0$, they match with weaker workers as those below $\underline{p}_{CE}$ enter the market. In the case $\underline{p}_{CE} = 0$, they compensate their partners with a higher payoff. Otherwise, the workers of $p = 0$ will deviate to better assets. As a result, some owners drop out and the threshold quality increases.

The queue length at the bottom can be distorted in either direction. First, the cost of changing the queue length increases with the workers' equilibrium payoff. Second, the threshold asset quality is higher while the worker's type is lower. The effect on the expected output is ambiguous. I provide an example in the symmetric setting below.

In Appendix A.1, I approach the form of distortion from the utilitarian planner's perspective instead of the agents' incentives. Given the equilibrium allocation, the planner may reassign a chosen agent to another market or the outside option. I study the planner's reassignment for different types of agents, and connect it to Proposition 2. The advantage of this "marginal" approach is to relate the gain from a reassignment directly to the Hosios condition and the workers' IC condition.

Symmetric setting Suppose two sides are symmetric in every aspect: $F = G$; $\underline{V} = \underline{U}$; $Y(p, q) = Y(q, p)$; $\delta(\frac{1}{\lambda}) = \eta(\lambda) = \frac{1}{\lambda}\delta(\lambda)$. The constrained efficient allocation inherits the symmetry of the setup. It features $r_{CE}(q) = q$, $\lambda_{CE}(q) = 1$ and $\underline{p}_{CE} = \underline{q}_{CE}$.

Since $\widetilde{\lambda}(q)$ is strictly increasing, there is a cut-off quality at which assets of higher quality have inefficiently long queues and the opposite for the lower quality assets. It is possible that the queue lengths in active markets are all inefficiently high.

Remark 1. *In the symmetric setting, $\widetilde{\lambda}(\underline{\widetilde{q}}) \leq 1$. $\widetilde{\lambda}(\underline{\widetilde{q}}) = 1$ if and only if $\widetilde{v}(\underline{\widetilde{p}}) = \underline{V}$.*

6 Choice Of Contract

We now consider the setting that contracts other than output shares are also feasible.

Choice between prices and output shares The main result is that the competition for better workers drives the asset owners to all choose prices over output shares, decentralizing the constrained efficient allocation.

Let's start with the equilibrium in the price competition. The equilibrium payoffs satisfy the extension of the stable matching condition in Koopmans and Beckmann (1957).[12]

Lemma 1. *The constrained efficient allocation is unique. For any p, q and λ,*

$$u_{CE}(q) + \lambda v_{CE}(p) \geq \delta(\lambda) Y(p, q) \tag{12}$$

with equality if and only if $\lambda = \lambda_{CE}(q)$ and $p = r_{CE}(q)$.

We then consider whether any asset owner wants to switch to output shares. Lemma 1 implies that when all others are posting prices, an asset owner can increase her payoff

[12] Without search friction, $\delta(\lambda) = min\{\lambda, 1\}$, the set of inequalities (12) collapses to $u_{CE}(q) + v_{CE}(p) \geq Y(p, q)$ for any pair of types.

only at the expense of the workers. But such an offer draws no workers in a directed search environment. Hence, an owner cannot gain from posting output shares. For any p,

$$u_{CE}(q) \geq \delta(\Lambda(q,s))Y(p,q) - \Lambda(q,s)v_{CE}(p) = \delta(\Lambda(q,s))sY(p,q).$$

The above argument applies to other form of contracts as well.

Proposition 3. *If prices are feasible, there is an equilibrium in which only prices are posted. This equilibrium has the same allocation and active markets as that in the price competition.*

The next step is to rule out other equilibria. Suppose we switch the feasible contracts from prices to output shares. Holding the allocation constant, the workers' information rent grows at a slower rate than their social value. Second, the owners pair up with weaker workers in the new equilibrium allocation. Putting together, an owner always profits from poaching slightly better workers when the asset side posts output shares.[13] She cannot attract better workers using output shares. But she will succeed by posting prices, which are flatter.

Now we turn to the case that some asset owners use output shares while others post prices. When characterizing the equilibrium in Section 3, we rely on that the local deviations to other markets (q,s) are unprofitable. The same characterization, conditions (6)-(8) in particular, must hold *locally* in the active markets where shares are posted. The above argument continues to apply.

Proposition 4. *When both prices and output shares are feasible, there are no equilibria in which a positive measure of asset owners post output shares.*

Steepness of contracts The fundamental reason why the asset owners choose prices over output shares is that the latter cost better workers more. DeMarzo et al. (2005) formalize this notion for general contingent payment.

Definition (DeMarzo, Kremer, and Skrzypacz, 2005)**.** *Given $Y|(p,q)$, security T_1 strictly crosses security T_2 from below if $E(T_1(Y)|p,q) = E(T_2(Y)|p,q)$ implies that*

1. $E(T_1(Y)|p^H,q) > E(T_2(Y)|p^H,q)$ *for* $p^H > p$*; and*

[13]Fix the queue length and the share, an asset owner gains from a better partner. However, a shorter queue length or a more generous term is needed to attract the better workers in the first place. In Appendix A.1, I show that an owner profits from poaching better workers but not weaker workers under the conditions (6)-(8).

2. $E(T_1(Y)|p^L, q) < E(T_2(Y)|p^L, q)$ for $p^L < p$.

An ordered set of securities T_1 is steeper than another ordered set T_2 if $T_1(.;x_1)$ strictly crosses security $T_2(.;x_2)$ from below for any x_1 and x_2. We also say T_2 is flatter than T_1.

In words, steeper contracts always cost more to better workers than weaker workers. The output shares are steeper than prices. Assuming strict monotone likelihood ratio property (SMLRP), DeMarzo et al. (2005) rank other standard financial securities, e.g., call options and bonds, by their steepness as well.[14]

When several ordered sets of securities $T_i(y;x_i)$ are available, we index the markets using (q, x_i). Appendix A.7 extends the definitions of equilibrium objects.

We now formalize the notion that the asset side prefers flatter securities. Suppose steeper securities are made feasible. Since steeper securities are more prone to draw weaker workers, an asset owner never profits from switching to one. However, there can be new equilibria. The owners may post the steeper securities under a specific circumstance.

Lemma 2. *Suppose two ordered sets of securities $T_1(y;x_1)$ and $T_2(y;x_2)$ are feasible. $T_1(y;x_1)$ is steeper than $T_2(y;x_2)$. If $(\widehat{q}, \widehat{x}_1)$ is an active market in an equilibrium, then $F(p|\widehat{q}, \widehat{x}_1)$ is degenerate at $\widehat{p}$ and there exists a unique $\widehat{x}_2$ such that*

1. $T_2(\widehat{p}, \widehat{q}, \widehat{x}_2) = T_1(\widehat{p}, \widehat{q}, \widehat{x}_1)$;

2. $\Lambda(\widehat{q}, \widehat{x}_1) = \Lambda(\widehat{q}, \widehat{x}_2)$;

3. $F(p|\widehat{q}, \widehat{x}_2)$ *is also degenerate at* $\widehat{p}$.

We begin with taking $\widehat{p}$ as the highest type in the market $(\widehat{q}, \widehat{x}_1)$. The first condition states that we focus on the contract $T_2(y;\widehat{x}_2)$ which provides workers of $\widehat{p}$ the same expected payment. The deviation to the market $(\widehat{q}, \widehat{x}_2)$ is unprofitable to the asset owners in the market $(\widehat{q}, \widehat{x}_1)$. It follows that the two markets must have the same queue length and the same composition of the workers, be it the market $(\widehat{q}, \widehat{x}_2)$ is active or not.

First, the market $(\widehat{q}, \widehat{x}_2)$ must not have a shorter queue because the expected payoff for workers of $\widehat{p}$ cannot exceed their equilibrium payoff. Second, the market $(\widehat{q}, \widehat{x}_2)$ attracts no workers below $\widehat{p}$. These workers strictly prefer the market $(\widehat{q}, \widehat{x}_1)$ because its queue is weakly shorter and $T_1(y;\widehat{x}_1)$ strictly crosses $T_2(y;\widehat{x}_2)$ from below. This in turn rules out

[14] The conditional density function $h(y|p,q)$ satisfies SMLRP if $\frac{h(y^H|p^H,q)}{h(y^L|p^H,q)} > \frac{h(y^H|p^L,q)}{h(y^L|p^L,q)}$ for any $y^H > y^L$ and $p^H > p^L$.

the possibilities that i) the market $(\widehat{q}, \widehat{x}_1)$ attracts workers below $\widehat{p}$, ii) the market $(\widehat{q}, \widehat{x}_2)$ attracts workers above $\widehat{p}$ or iii) the queue in the market $(\widehat{q}, \widehat{x}_2)$ is longer. If any of these occurs, an asset owner would have deviated to the market $(\widehat{q}, \widehat{x}_2)$!

The key implication of Lemma 2 is that if all participants in the market $(\widehat{q}, \widehat{x}_1)$ move to the market $(\widehat{q}, \widehat{x}_2)$, the queue length and the composition of the workers in the latter remain the same. Workers have no profitable deviations as before. The equilibrium payoffs remain unchanged, so do the queue length and the composition in other inactive markets. The now inactive market $(\widehat{q}, \widehat{x}_1)$ retains the same queue length and a possibly lower type of workers. The asset side will not deviate to the market $(\widehat{q}, \widehat{x}_1)$. Applying this procedure on every active market (q, x_1), we obtain an equilibrium with the same allocation and only contracts in T_2 are posted.

Proposition 5. *When a steeper ordered set of securities is made feasible, the set of equilibrium payoffs and allocations remains the same. No one posts from the steeper ordered set of securities in an equilibrium.*

Suppose the feasible contracts are ordered sets of securities fully ranked by the steepness, Proposition 5 states that it is without loss to restrict attention to the flattest class of contracts.[15] This generalizes Proposition 3.

The asset side posts output shares if it is flatter than other feasible contracts. Consider the following example: The asset side receives the output. An asset owner may pay the worker a fixed wage w as well as $(1-s)$ fraction of the output. We represent the situation using affine contracts $T_{\text{affine}}(y; s, w) = sy - w$, where $s \in [0, 1]$ and $w \geq 0$.[16] The restriction $w \geq 0$ captures the worker's liquidity constraint and limited liability. Any affine contract with $w > 0$ strictly crosses an output share from below. Thus, the output shares are the flattest ordered set of securities among the affine contracts.

Corollary 2. *Suppose that only affine contracts T_{affine} are feasible. There is an equilibrium in which the asset side posts output shares and satisfies properties in Proposition 1.*

[15]This result is unlike Gorbenko and Malenko (2011), in which steeper securities may be used in equilibrium. The difference is that in their setting, a buyer commits to a seller before he inspects the good and learns his private value.

[16]Suppose the output level takes binary values, 0 or 1. A high output is produced with probability $Y(p, q)$. Furthermore, there are threats of sabotage on both sides, so that $y - T(y)$ and $T(y)$ are weakly increasing in y. In this setting, all feasible contracts take the form of affine contracts.

7 Discussion

The use of contracts and PAM are well documented in empirical studies. An important question is how the interaction of two affects the efficiency and the distribution of surpluses. The literature centers on the matching pattern when the types are public. This paper is a first step towards answering this question in a market with private types.

I identify a new channel of inefficiency caused by handicapped competition. The contribution is threefold. I propose a stylized setting that singles out the channel of inefficiency from a plethora of market forces. The unique equilibrium has two notable features: PAM and the Hosios condition for every pair of types. Second, I provide clean-cut results on the distortion and the distribution of surpluses. The distribution-free results indicate that the underlying forces are always at play. Lastly, I study the form of contracts chosen in equilibrium. Competition drives the asset side to post from the flattest class of contracts. It leads to constrained efficiency whenever prices are feasible. Otherwise, the competition among workers is handicapped, causing the mentioned distortion. Simply put, this paper spells out the ingredients for the inefficiency.

The remainder of this section remarks on the setting and the results.

Recipe for inefficiency It contains three ingredients: handicapped competition, private types, and search friction. Without any one of them, the equilibrium allocation would be (constrained) efficient.

Inefficiency stems from the handicapped competition between workers. Proposition 3 and 4 extend the result in EK to a richer contract space, indicating that constrained efficiency is attained whenever prices are feasible. Indeed, the constrained efficient allocation is supported by an equilibrium if the feasible contracts include an ordered set of securities which cost weaker workers more than better workers.[17] Otherwise, making prices feasible will expand the set of equilibrium allocations, contradicting Proposition 3 and 5.

If workers' types are contractible, a menu of output shares can require the same payment from every type. We can obtain an equilibrium by replacing the equilibrium prices in EK with menus of output shares requiring the same payments. Lemma 1 holds in this equilibrium. An asset owner will not deviate to other menus of output shares.[18]

[17] Assuming the conditional density function satisfies SMLRP, a contract costs weaker workers more than better workers if the worker's payoff $y - T(y)$ increases at a faster rate than the output level.

[18] The meeting is bilateral, so asset owners do not benefit from inducing different queue lengths for different types. Even if an asset owner meets with multiple workers and makes the selection as in Shi

Suppose there is no search friction. The efficient allocation features perfect PAM and a unit queue length for every matched pair. Efficiency is attained if the asset side post prices. Now we replace the posted prices with the output shares which keep the same divisions of the outputs. The workers again deviate to better assets, increasing the queue length for the best assets. Without search friction, the asset owners cannot improve their matching probability by distorting the queue length. They increase their shares until the queue length returns to unity. Their decisions in turn leave the same pool of workers to lower quality assets. Inductively, the equilibrium allocation remains efficient amid higher payoffs for the asset owners.

Efficiency benchmark with private types The utilitarian planner's solution is widely adopted as an efficiency benchmark in the sorting and directed search literature. A key reason is that it usually coincides with the equilibrium allocations (e.g., Chiappori et al., 2010; Gretsky et al., 1992; Moen, 1997; Shapley and Shubik, 1971). In the current framework, this is indeed the case when prices are posted or workers' types are public.

The constrained efficient allocation remains an appropriate benchmark if the planner does not observe the workers' types. She can induce the same allocation by restricting the set of markets available. The planner first excludes all assets below $\underline{q}_{CE}$. She imposes $\widehat{s}(q)$ for each type of assets. For the threshold type,

$$1 - \widehat{s}(\underline{q}_{CE}) = \frac{d \ln \delta(\lambda_{CE}(\underline{q}_{CE}))}{d \ln \lambda},$$

so that the two sides' payoffs are the same as their social values. For $q > \underline{q}_{CE}$, $\widehat{s}(q)$ satisfies

$$\frac{d}{dq}[(1 - \widehat{s}(q))\eta(\lambda_{CE}(q))Y(p, q)]\bigg|_{p=r_{CE}(q)} = 0.$$

Under Assumption (Y), the above condition ensures incentive compatibility on the workers' side. Since workers' information rent grows at a slower rate than under fixed prices, the asset side's payoff is above the outside option. Their participation is voluntary.

However, $\widehat{s}(q)$ does not meet the Hosios condition. An asset owner earns a payoff above the social value. In a decentralized market, she will deviate to post a lower share improving her matching probability.

(2002); Shimer (2005), we can again replace the posted prices in their equilibria with some output shares. The bottom line is that a share is no different from a price if the worker's type is contractible.

Policy intervention A Pigouvian intervention can induce the asset side to post $\widehat{s}(q)$, decentralizing the constrained efficient allocation. Let $\tau(q)$ denote the tax payment levied on assets q when an owner pairs up with a worker. $\tau(q)$ satisfies

$$\delta'(\lambda_{CE}(q))[Y(r_{CE}(q),q)-\tau(q)] = \eta(\lambda_{CE}(q))(1-\widehat{s}(q))Y(r_{CE}(q),q), q \geq \underline{q}_{CE}.$$

The left hand side is an asset owner's marginal benefit from increasing the queue length while the right hand side is the marginal cost.[19] As the purpose is to lower asset owners' gain from a match, $\tau(q)$ can be implemented as a subsidy if an asset owner is left unmatched, or a combination of both to achieve budget balance. However, the Pigouvian intervention is not distribution-free and requires knowledge of the entire distribution of types.

Outside options The outside option is the opportunity cost of participating in matching. All results hold for arbitrarily small values of outside options. Setting $\underline{U} = 0$ leads to additional equilibria with the same allocation and payoffs. Instead of taking the outside option directly, asset owners below $\widetilde{\underline{q}}$ may post contracts which attract no workers. The assumption $\underline{V} > 0$ restricts the mutually acceptable contract terms to the range where the strict SCP holds. Setting $\underline{V} = 0$ introduces trivial equilibria in which asset owners post only $s = 1$.As workers receive zero payoff in any active market, a deviating offer with $s < 1$ will attract an infinitely long queue of workers of the lowest type. This type of equilibria can feature a plethora of sorting patterns. The characterized equilibrium remains an equilibrium if the outside option for either side becomes zero.

Menu of contracts The analysis remains the same if menus of output shares are feasible. This is because the share cannot be made contingent on the worker's private type. Posting a menu of shares is no different from posting only the lowest share in the menu, which the worker always picks. The conclusion remains valid if the menu may include other contracts steeper than output shares.[20] The underlying reason is that an asset owner meets only one worker and an incentive-compatible menu reveals his type only *upon* the meeting.

Contracts and expected output In my setting, the contract term affects the division but not the size of the expected output. Utility can be transferred between two sides

[19]When an asset owner posts a different share, the type of her partner remains the same as in Section 3.

[20]Formally, the steeper contracts satisfy Assumption A1 and A3 in Guerrieri et al. (2010), their Proposition 5 show that allowing menus of contracts does not change the set of equilibrium allocations and payoffs.

perfectly via the share or imperfectly via the queue length. So the latter is not distorted for the purpose of utility transfer. Moreover, switching from prices to shares leaves the aggregate surplus, hence the constrained efficient allocation, unchanged.

The baseline setting with output shares is consistent with principal-agent models built on Diamond (1998) and Carroll (2015). The worker has an efficient production decision. He may undertake other financial transactions that shift returns across different states. The financial transactions are fair bets at best and may require additional effort. The only restriction on the contingent payment is that the worker's payoff must be non-negative, due to his liquidity constraint. In Diamond (1998), the worker can choose any fair bets to exploit non-linearity in the payment scheme. The author shows that an output share is optimal. Alternatively, we assume the owner knows only the efficient production decision, but not know what financial transactions are feasible or their consequences. An output share guarantees that the principal's payoff is proportional to the agent's. Carroll (2015) shows that it is the unique optimal contract under the maximin criterion. In this special case, the worker always chooses the efficient production decision. Consequently, the expected output is independent of the share, and both sides know the expected payoffs beforehand.

In general, the contract affects the size of the surplus in models of incentive provision or risk sharing. Incorporating such considerations into assignment models with public types involves two known complications. Legros and Newman (2007) point out that when adjusting the contract term to transfer utility, the conversion rate typically depends on types. The lack of complementarity between type and transferability can lead to NAM. Besides, Kaya and Vereshchagina (2014) exemplify how the form of the contract may affect the type dependence of the expected output.

The presence of search friction exacerbates these complications. First, utility is transferred imperfectly via a combination of the contract term and the queue length. The latter in turn intertwines with the sorting pattern. Second, the constrained efficient allocation changes with the form of contract, even if it retains PAM. The above channels hinge on the model of the post-matching stage. They also affect screening of workers with private types. As a first step, this paper abstracts away the common reasons behind the use of output shares and focuses on its consequences on sorting.

Other contracts Suppose the asset side has to use contracts other than prices and output shares. The handicapped competition leads to inefficient sorting as long as a less generous contract term costs better workers more. However, the worker's preference over the queue length and the contract differs across types. An asset owner may screen workers by distorting the queue length. The Hosios condition no longer holds in equilibrium. Otherwise, an incremental distortion in the queue length leads to a second-order loss while the improvement in the partner's type yields a first-order gain. The utilitarian planner's reassignment problem in Appendix A.1 sheds light on the obstacle for distribution-free results. The form of distortion is in line with the planner's reassignment. Without the Hosios condition, there is no direct relation between the planner's gain from a reassignment and the optimality conditions for the two sides. Bridging the gap between competitive screening and assortative matching is left to future research. The results here serve as a useful benchmark.

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A Appendix

A.1 Reassignment by utilitarian planner

Let us start with the equilibrium in Section 3 and consider the following thought experiment. Workers and asset owners are distributed across markets according to $(\widetilde{K}, \widetilde{L})$, and have not formed matches. A worker or asset owner is randomly chosen. The utilitarian planner may reassign the chosen agent to another market or her outside option.[21] I study how the reassignment depends on the agent's type, and its connection to Proposition 2.

Only active markets merit consideration. If an additional worker of $\widehat{p}$ is assigned to the active market for assets q, the aggregate surplus changes by

$$\eta(\widetilde{\lambda}(q))Y(\widehat{p}, q) + \widetilde{\lambda}(q)\eta'(\widetilde{\lambda}(q))Y(\widetilde{r}(q), q).$$

The additional worker contributes $\eta(\widetilde{\lambda}(q))Y(\widehat{p}, q)$ himself. At the same time, his presence decreases the number of matched pairs $(\widetilde{r}(q), q)$, reducing the aggregate surplus by $\widetilde{\lambda}(q)\eta'(\widetilde{\lambda}(q))Y(\widetilde{r}(q), q)$.[22] To maintain symmetry, an asset of $\widehat{q}$ can be assigned to a market for a different quality, say q. The aggregate surplus changes by

$$\delta(\widetilde{\lambda}(q))Y(\widetilde{r}(q), \widehat{q}) - \widetilde{\lambda}(q)\delta'(\widetilde{\lambda}(q))Y(\widetilde{r}(q), q).$$

The increase in the surplus generated by the additional asset is $\delta(\widetilde{\lambda}(q))Y(\widetilde{r}(q), \widehat{q})$, while the loss due to the reduction in the matches of the pair $(\widetilde{r}(q), q)$ is $\widetilde{\lambda}(q)\delta'(\widetilde{\lambda}(q))Y(\widetilde{r}(q), q)$.

Taking $p = \widetilde{\kappa}(\widehat{q})$ and $q = \widehat{q}$, we can see that everyone's equilibrium payoff is equal to her marginal contribution, the reduction in the aggregate surplus if she is removed. This is the implication of the Hosios condition. The utilitarian planner reassigns the agent to a market where the increase in the surplus exceeds her equilibrium payoff.

Lemma 3. *Fix any $q \in [\underline{\widetilde{q}}, 1]$. Then for any $p < \widetilde{r}(q)$,*

$$U(q) + \lambda V(p) > \delta(\lambda)Y(p, q). \tag{A.1}$$

Furthermore, there exists some type $\widehat{p} > \widetilde{r}(q)$ such that for any $p \in (\widetilde{r}(q), \widehat{p})$,

$$U(q) < \max_{\lambda \geq 0}[\delta(\lambda)Y(p, q) - \lambda V(p)]. \tag{A.2}$$

[21] The reassignment of a single agent does not affect the expected payoff of any other agent. It does not matter whether the agents anticipate the reassignment beforehand.

[22] Let $M(l, k)$ denote the number of matches in a market with l workers and k assets. Hence, $M(l, k) = k\delta(\frac{l}{k})$. The number of matched pairs decreases by $\lim_{\epsilon \to 0^+} \frac{1}{\epsilon}\left[M(l, \frac{l}{l+\epsilon}k) - M(l, k)\right] = \frac{l}{k}\eta'(\frac{l}{k})$. The remaining expressions are derived in a similar manner.

Proof. Fix some $q \in [\underline{\widetilde{q}}, 1]$. Define

$$\widetilde{U}(p,q) = \max_{\lambda \geq 0}\{\delta(\lambda)Y(p,q) - \lambda V(p)\}.$$

The unique maximizer $\widetilde{\Lambda}(p,q)$ satisfies $\delta'(\widetilde{\Lambda}(p,q))Y(p,q) = V(p)$. For $p \in [\underline{\widetilde{p}}, 1]$, (6) implies that $\widetilde{\Lambda}(p,q) > (<)\widetilde{\lambda}(\widetilde{\kappa}(p))$ if $q > (<)\widetilde{\kappa}(p)$. By envelope theorem and (8),

$$\begin{aligned}
\frac{\partial}{\partial p}\widetilde{U}(p,q) &= \delta(\widetilde{\Lambda}(p,q))\frac{\partial}{\partial p}Y(p,q) - \widetilde{\Lambda}(p,q)\frac{\partial}{\partial p}V(p) \\
&= \delta(\widetilde{\Lambda}(p,q))\frac{\partial}{\partial p}Y(p,q) - \widetilde{\Lambda}(p,q)\delta'(\widetilde{\lambda}(\widetilde{\kappa}(p)))\frac{\partial Y(p,\widetilde{\kappa}(p))}{\partial p} \\
&> \widetilde{\Lambda}(p,q)\left[\eta(\widetilde{\Lambda}(p,q))\frac{\partial}{\partial p}Y(p,q) - \eta(\widetilde{\lambda}(\widetilde{\kappa}(p)))\frac{\partial Y(p,\widetilde{\kappa}(p))}{\partial p}\right] \\
&= \widetilde{\Lambda}(p,q)V(p)\left[\frac{\eta(\widetilde{\Lambda}(p,q))}{\delta'(\widetilde{\Lambda}(p,q))}\frac{\partial \ln Y(p,q)}{\partial p} - \frac{\eta(\widetilde{\lambda}(\widetilde{\kappa}(p)))}{\delta'(\widetilde{\lambda}(\widetilde{\kappa}(p)))}\frac{\partial \ln Y(p,\widetilde{\kappa}(p))}{\partial p}\right]
\end{aligned}$$

Under Assumption (Y) and (M), $\frac{\partial}{\partial p}\widetilde{U}(p,q) > 0$ for $p \leq \widetilde{r}(q)$. $U(q) = \widetilde{U}(\widetilde{r}(q), q)$ under (7). Hence, we obtain the inequalities (A.1) and (A.2). □

Lemma 3 describes how the equilibrium payoffs violate the inequality (12) systematically. Its proof indicates the role of Assumption (Y) and (M) in shaping the distortion.

We first consider the case that a worker of $\widehat{p} < \underline{\widetilde{p}}$ is chosen. He is contributing $\underline{V}$. If he is reassigned to the active market for assets q, the aggregate surplus changes by $\eta(\widetilde{\lambda}(q))Y(\widehat{p},q) + \widetilde{\lambda}(q)\eta'(\widetilde{\lambda}(q))Y(\widetilde{r}(q),q)$. The planner never reassigns him because

$$\begin{aligned}
\underline{V} &> \delta'(\widetilde{\lambda}(q))Y(\widehat{p},q) = \eta(\widetilde{\lambda}(q))Y(\widehat{p},q) + \widetilde{\lambda}(q)\eta'(\widetilde{\lambda}(q))Y(\widehat{p},q) \\
&> \eta(\widetilde{\lambda}(q))Y(\widehat{p},q) + \widetilde{\lambda}(q)\eta'(\widetilde{\lambda}(q))Y(\widetilde{r}(q),q).
\end{aligned}$$

The first inequality comes from the worker's IC condition. The second inequality holds because the reassigned worker displaces better workers $\widetilde{r}(q)$.

Now consider a worker of type $\widehat{p} = \widetilde{r}(\widehat{q}) > \underline{\widetilde{p}}$. The planner would not reassign him to the outside option because $V(\widehat{p}) > \underline{V}$. The worker is not reassigned to better assets either. For $q^H > \widehat{q}$, the change in the aggregate surplus is given by

$$\begin{aligned}
&\eta(\widetilde{\lambda}(q^H))Y(\widehat{p},q^H) + \widetilde{\lambda}(q^H)\eta'(\widetilde{\lambda}(q^H))Y(\widetilde{r}(q^H),q^H) \\
<&\delta'(\widetilde{\lambda}(q^H))Y(\widehat{p},q^H) < \delta'(\widetilde{\lambda}(\widehat{q}))Y(\widehat{p},\widehat{q}) = V(\widehat{p}).
\end{aligned}$$

Again, the first inequality is due to the displacement of better workers, while the second inequality is due to the workers' IC condition. Instead, the planner reassigns him to some lower quality assets q^L. The aggregate surplus changes by

$$\begin{aligned}&\eta(\widetilde{\lambda}(q^L))Y(\widehat{p},q^L)+\widetilde{\lambda}(q^L)\eta'(\widetilde{\lambda}(q^L))Y(\widetilde{r}(q^L),q^L)-\delta'(\widetilde{\lambda}(\widehat{q}))Y(\widehat{p},\widehat{q})\\ =&\eta(\widetilde{\lambda}(q^L))[Y(\widehat{p},q^L)-Y(\widetilde{r}(q^L),q^L)]-[\delta'(\widetilde{\lambda}(\widehat{q}))Y(\widehat{p},\widehat{q})-\delta'(\widetilde{\lambda}(q^L))Y(\widetilde{r}(q^L),q^L)].\end{aligned}$$

As $\eta(\widetilde{\lambda})(\widehat{q}) > \delta'(\widetilde{\lambda}(\widehat{q}))$, the above expression is positive when q^L is sufficiently close to $\widehat{q}$. The root cause is that under output shares, the information rent grows slower with the worker's type. By continuity, the planner cannot gain from reassigning a worker of $\underline{\widetilde{p}}$, be it $\underline{\widetilde{p}} > 0$ or $V(\underline{\widetilde{p}}) > \underline{V}$.

We now carry out the same exercise for the asset side. First, the utilitarian planner reassigns an asset of $\widehat{q} \geq \underline{\widetilde{q}}$ to better workers. Lemma 3 states that there are some types, say $q^H > \widehat{q}$, satisfying

$$U(\widehat{q}) < \delta(\widetilde{\lambda}(q^H))Y(\widetilde{r}(q^H),\widehat{q}) - \widetilde{\lambda}(q^H)V(\widetilde{r}(q^H)).$$

The aggregate surplus increases if reassigning the asset to the active market for assets q^H. For the active market for assets q^L in between $\widehat{q}$ and $\underline{\widetilde{q}}$, let us rearrange the inequality (A.1),

$$U(\widehat{q}) > \delta(\widetilde{\lambda}(q^L))Y(\widetilde{r}(q^L),\widehat{q}) - \widetilde{\lambda}(q^L)\delta'(\widetilde{\lambda}(q^L))Y(\widetilde{r}(q^L),q^L).$$

The aggregate surplus drops if the asset is reassigned to markets with weaker workers. As $U(\widehat{q}) \geq \underline{U}$, the outside option is ruled out. By continuity, the above reassignment also applies to assets of quality slightly below $\underline{\widetilde{q}}$.

Connection to the form of distortion Like Proposition 2, the conclusion on the reassignment applies to all distributions of types. Let us reconcile the two results. The equilibrium features excessive entry of workers and inefficiently low participation on the asset side. This is consistent with the planner's decision to leave a worker of type below $\underline{\widetilde{p}}$ to his outside option, but reassign some assets slightly below $\underline{\widetilde{q}}$ to match with workers.

The planner reassigns a worker above $\underline{p}_{CE}$ to lower quality assets or an asset above $\underline{\widetilde{q}}$ to better workers. This observation is consistent with two features of the distortion. First, the participating workers pair up with better assets than in the constrained efficient allocation. Second, the equilibrium queue length is inefficiently high for the best assets, as the planner reassigns the best workers but not the best assets.

In comparison, it is less obvious why the planner reassigns a worker of $p \in (\underline{\widetilde{p}}, \underline{p}_{CE})$ to lower quality assets instead of his outside option. Insufficient participation on the asset side is the reason. The quality of assets to which the worker is reassigned is still above the efficient threshold $\underline{q}_{CE}$. Therefore, the reassignment increases the aggregate surplus.

Poaching by posting price Lemma 3 also sheds light on Proposition 4. Again we start with the equilibrium in Section 3. Suppose that prices are now feasible to an asset owner. When the owner switches to a fixed price, the attracted workers are compensated with their equilibrium payoff. Which type will she target? The inequality (A.1) implies that the owner never poaches weaker workers. The inequality (A.2) indicates that poaching workers slightly better than the current partner $\widetilde{r}(q)$ is always profitable. The competition for better workers eventually drives the owners to all post prices.

A.2 Proof of Proposition 1

I first characterize the equilibrium. I then show that the boundary value problem admits a unique solution.

Every equilibrium satisfies the properties in Proposition 1 Fix an equilibrium $(\widetilde{K}, \widetilde{L})$. The assumption (1) ensures that the set of active markets is non-empty.

Step 1: The equilibrium allocation must feature PAM.

From the discussion in the main text, the participation on the workers' side must be monotonic as $Y(p,q)$ is strictly increasing in p. The workers and asset owners in the active markets must match assortatively because of the strict SCP.

Step 2: $U(q)$ is strictly increasing for $q \geq \underline{q}$.

Suppose $(q^L, s^L) \in \text{supp}(\widetilde{K})$ and p^L is the highest type in $\text{supp}(F(p|q^L, s^L))$. Fix any $q^H > q^L$. Consider the inactive market (q^H, s^H), where s^H is given by $(1-s^H)Y(p^L, q^H) = (1-s^L)Y(p^L, q^L)$. Note that $s^H > s^L$. $\Lambda(q^H, s^H) \geq \Lambda(q^L, s^L)$ because $V(p^L) \geq \eta(\Lambda(q^H, s^H))(1-s^H)Y(p^L, q^H)$. For all $p < p^L$,

$$V(p) \geq \eta(\Lambda(q^L, s^L))(1-s^L)Y(p, q^L) > \eta(\Lambda(q^H, s^H))(1-s^H)Y(p, q^H).$$

The strict inequality follows from Assumption (Y). Hence, $\text{supp}(F(p|q^H, s^H))$ contains no types below p^L. An asset owner of q^H can obtain

$$\delta(\Lambda(q^H, s^H))s^H \int Y(p, q^H) dF(p|q^H, s^H) > \delta(\Lambda(q^L, s^L))s^L Y(p^L, q^L) \geq U(q^L).$$

Therefore, the participation of the asset side is monotonic and $r(q)$ is well defined.

Step 3: For any $p \in [0,1]$ *and* $(q,s) \in \text{supp}(\widetilde{K})$,

$$V(p) > \eta(\Lambda(q,s))(1-s)Y(p,q) \text{ if } q \neq \kappa(p).$$

Consider two active markets (q^H, s_1) and (q^L, s_0), where $q^H > q^L$. Suppose a worker of $r(q^H)$ is indifferent between these two markets. His decision is optimal only if

$$\begin{aligned}\eta(\Lambda(q^L,s_0))(1-s_0)Y(r(q^H),q^L) =&\eta(\Lambda(q^H,s_1))(1-s_1)Y(r(q^H),q^H)\\ \geq&\eta(\Lambda(q,s'))(1-s')Y(r(q^H),q)\end{aligned}$$

for all active markets (q,s') with $q \in (q^L, q^H)$. Strict SCP implies that all workers below $r(q^H)$ strictly prefer the market (q^L, s_0) to any active market with $q \in (q^L, q^H)$. PAM then implies that all such active markets have zero queue lengths. Hence, $U(q') = \underline{U} \leq U(q^L)$, contradicting my previous claim. A symmetric argument rules out that a worker of $r(q^L)$ is indifferent between (q^L, s_0) and another active market (q^H, s_1).

As the distribution of types is atomless, $\kappa(p)$ and $r(q)$ are strictly increasing and C^1.
Step 3: Characterize the active markets $\text{supp}(\widetilde{K})$.

Lemma 4. *Suppose* $(q,s') \in \text{supp}(\widetilde{K})$ *and for any* $p \in [0,1]$,

$$V(p) \geq \eta(\Lambda(q,s'))(1-s')Y(p,q),$$

and equality holds if and only if $q = \kappa(p)$. *Then, for any* $s \in [0,1)$, $F(p|q,s)$ *is degenerate at* $r(q)$ *if* $\Lambda(q,s) > 0$. *Furthermore, an owner of asset* q *has no profitable deviations if and only if* $\Lambda(q,s')$ *satisfies*

$$\delta'(\Lambda(q,s')) = \eta(\Lambda(q,s'))(1-s'). \tag{A.3}$$

For $s \in [0,1)$ and $p \neq r(q)$,

$$\frac{V(p)}{V(r(q))} > \frac{Y(p,q)}{Y(r(q),q)} = \frac{\eta(\Lambda(q,s))(1-s)Y(p,q)}{\eta(\Lambda(q,s))(1-s)Y(r(q),q)}.$$

Suppose $\Lambda(q,s) > 0$. Then, $V(p) = \eta(\Lambda(q,s))(1-s)Y(p,q)$ if and only if $p = r(q)$, and hence $F(p|q,s)$ is degenerate at $r(q)$. In this case, $\Lambda(q,s)$ is determined by

$$V(r(q)) = \eta(\Lambda(q,s'))(1-s')Y(r(q),q) = \eta(\Lambda(q,s))(1-s)Y(r(q),q).$$

An asset owner has no profitable deviations if and only if $U(q) \geq \delta(\Lambda(q,s))sY(r(q),q)$. This is equivalent to (A.3), which ensures

$$\Lambda(q,s') = \arg\max_{\lambda \in [0,\infty]} \delta(\lambda) - \lambda\eta(\Lambda(q,s'))(1-s').$$

Step 4: There exists a strictly increasing function $\widetilde{\lambda}:[0,1]\to(0,\infty)$ *such that*

$$\text{supp}(\widetilde{K})=\Big\{(q,s):q\in[\underline{q},1],s=1-\frac{d\ln\delta(\widetilde{\lambda}(q))}{d\ln\lambda}\Big\},$$

and $\Lambda(q,s)=\widetilde{\lambda}(q)$ *for* $(q,s)\in\text{supp}(\widetilde{K})$*.* $V(p)$ *satisfies (6) and (8).*

Participation is monotonic on both sides, so there is at least one active market (q,s) for any $q\in[\underline{q},1]$. Substituting (A.3) into (2), workers' optimality condition requires

$$V(r(q'))=\max_{(q,s)\in\text{supp}(\widetilde{K})}\{\delta'(\Lambda(q,s))Y(r(q'),q)\}.$$

The envelope theorem implies that V is C^1. As $r(q)$, $\delta'(\lambda)$ and $Y(p,q)$ are C^1, there must exist a C^1 function $\lambda:[0,1]\to(0,\infty)$ satisfying (8). This also establishes that for any $q\geq\underline{q}$, there is exactly one active market (q,s) where $\Lambda(q,s)=\lambda(q)$ and $s=1-\frac{d\ln\delta(\lambda(q))}{d\ln\lambda}$. Combining with (A.3), we obtain (6).

From (6) and (8), $\widetilde{\lambda}(q)$ is strictly increasing to offset the gain from a better asset.

Step 5: Establish the boundary value problem.

We have discussed (4), (5) and (9) in the main text. We now turn to (10). Suppose $\underline{q}>0$ and $U(\underline{q})>\underline{U}$. Let $(\underline{q},\underline{s})$ be the active market for assets $\underline{q}$. If $\underline{p}=0$, an owner of q' slightly below $\underline{q}$ can secure a payoff $\delta(\lambda(\underline{q}))s'Y(0,q')>\underline{U}$ by posting s' satisfying $\eta(\lambda(\underline{q}))(1-s')Y(0,q')=V(0)$. For the case $\underline{p}>0$, $V(\underline{p})=\eta(\lambda(\underline{q}))(1-\underline{s})Y(\underline{p},\underline{q})=\underline{V}$. By continuity, for q' slightly below $\underline{q}$, there must exist $s'<\underline{s}$ and $p'<\underline{p}$ satisfying both $\delta(\lambda(\underline{q}))s'Y(p',q')>\underline{U}$ and $\eta(\lambda(\underline{q}))(1-s')Y(p',q')=\underline{V}$. By construction, $\Lambda(q',s')\geq\lambda(\underline{q})$, and $\eta(\Lambda(q',s'))(1-s')Y(p,q')<\underline{V}$ for any $p<p'$. Workers below p' will not be attracted. The deviating payoff must be above $\underline{U}$! Therefore, $U(\underline{q})=\underline{U}$ if $\underline{q}>0$.

The preceding analysis verifies all properties in Proposition 1.

An allocation is an equilibrium if it satisfies the properties in Proposition 1. Fix a solution $(\underline{\widetilde{p}},\underline{\widetilde{q}},\widetilde{r},\widetilde{\lambda},\widetilde{v})$ to the boundary value problem. We can recover a candidate equilibrium $(\widetilde{K},\widetilde{L})$ satisfying all properties listed in Proposition 1. Define $\widetilde{s}(q)=1-\frac{d\ln\delta(\widetilde{\lambda}(q))}{d\ln\lambda}$. As $\widetilde{\lambda}(q)$ is continuous and strictly increasing, $\widetilde{s}(q)$ is also continuous and increasing in q under Assumption (M). If $q'\geq\underline{\widetilde{q}}$ and $s'\geq\widetilde{s}(\underline{\widetilde{q}})$,

$$\widetilde{K}(q',s')=G(\sup\{q\leq q':\widetilde{s}(q)\leq s'\})-G(\underline{\widetilde{q}}).$$

Otherwise, $\widetilde{K}(q',s')=0$. $\widetilde{\kappa}$ is the inverse of $\widetilde{r}$. If $p\geq\underline{\widetilde{p}}$, $q'\geq\underline{\widetilde{q}}$ and $s'\geq\widetilde{s}(\underline{\widetilde{q}})$,

$$\widetilde{L}(p',q',s')=F(\sup\{p\leq p':\widetilde{\kappa}(p)\leq q',\widetilde{s}(\widetilde{\kappa}(p))\leq s'\})-F(\underline{\widetilde{p}}).$$

Otherwise, $\widetilde{L}(p', q', s') = 0$.

I first verify that workers' optimality condition. Combining (6) and (8),

$$\frac{d \ln \widetilde{v}(p)}{dp} = \frac{\partial \ln Y(p, \widetilde{\kappa}(p))}{\partial p}, p \geq \underline{\widetilde{p}}.$$

$V(p) = \widetilde{v}(p)$ for $p > \underline{\widetilde{p}}$. Consider any $p_0 \geq \underline{\widetilde{p}}$ and $p_1 \neq p_0$,

$$\ln V(p_1) - \ln \delta'(\widetilde{\kappa}(p_0)) Y(p_1, \widetilde{\kappa}(p_0)) = [\ln V(p_1) - \ln \widetilde{v}(p_0)] - [\ln Y(p_1, \widetilde{\kappa}(p_0)) - \ln Y(p_0, \widetilde{\kappa}(p_0)]$$

$$= \int_{p_0}^{p_1} \frac{d \ln V(p)}{dp} - \frac{\partial \ln Y(p, \widetilde{\kappa}(p_0))}{\partial p} dp \geq \int_{p_0}^{\max\{p_1, \underline{\widetilde{p}}\}} \frac{\partial \ln Y(p, \widetilde{\kappa}(p))}{\partial p} - \frac{\partial \ln Y(p, \widetilde{\kappa}(p_0))}{\partial p} dp > 0.$$

The last strict inequality is due to assumption (Y) and $\widetilde{\kappa}$ is strictly increasing. $\eta(\Lambda(q, \widetilde{s}(q)))(1 - \widetilde{s}(q)) = \delta'(\widetilde{\lambda}(q))$ holds for any active market $(q, \widetilde{s}(q))$. A worker of $p = \widetilde{r}(q)$ receives his highest expected payoff in the market $(q, \widetilde{s}(q))$. The outside option is optimal for the workers of $p < \underline{\widetilde{p}}$ because $V(\underline{\widetilde{p}}) = \underline{V}$.

I now turn to the asset side. By Lemma 4, the optimality condition is met for $q \geq \underline{\widetilde{q}}$ as

$$U(q) = [\delta(\widetilde{\lambda}(q)) - \widetilde{\lambda}(q) \delta'(\widetilde{\lambda}(q))] Y(\widetilde{r}(q), q) \geq \underline{U}.$$

Now suppose $\underline{\widetilde{q}} > 0$. It is not optimal for an owner of $q < \underline{\widetilde{q}}$ to participate. Consider an inactive market (q^L, s') where $q^L < \underline{\widetilde{q}}$. The strict SCP implies that workers above $\underline{\widetilde{p}}$ strictly prefer the market $(\underline{\widetilde{q}}, \widetilde{s}(\underline{\widetilde{q}}))$ to the market (q^L, s'). $F(p|q^L, s')$ must be degenerate at some $p^L \leq \underline{\widetilde{p}}$.

The case $\Lambda(q^L, s') = 0$ is trivial. Suppose $\Lambda(q^L, s') > 0$, $\eta(\Lambda(q^L, s'))(1 - s') Y(p^L, q^L) = V(p^L) = \underline{V}$ holds. If deviating to market (q^L, s'), an asset owner will receive

$$\begin{aligned}\delta(\Lambda(q^L, s')) s' Y(p^L, q^L) =& \delta(\Lambda(q^L, s')) Y(p^L, q^L) - \Lambda(q^L, s') V(p^L) \\ \leq& \max_{\lambda} [\delta(\lambda) Y(p^L, q^L) - \lambda \underline{V}] < U(\underline{\widetilde{q}}) = \underline{U}.\end{aligned}$$

The boundary value problem admits a unique solution

First notice that $(\widetilde{r}, \widetilde{\lambda}, \widetilde{v})$ in any solution must be continuously differentiable and strictly increasing. By differentiating (6) w.r.t. q and combining with (8), I obtain

$$\frac{d \ln \delta'(\widetilde{\lambda}(q))}{dq} = -\frac{\partial \ln Y(\widetilde{r}(q), q)}{\partial q} \tag{A.4}$$

There exists a unique pair of $\underline{\lambda}$ and $\overline{\lambda}$ satisfying $[\delta(\underline{\lambda}) - \delta'(\underline{\lambda}) \underline{\lambda}] Y(1, 1) = \underline{U}$ and $\delta'(\overline{\lambda}) Y(1, 1) = \underline{V}$, respectively. Note that $\overline{\lambda} > \underline{\lambda}$.

We first consider the following initial value problem (IPV-λ(1)): $r(q)$ and $\lambda(q)$ satisfy

(5) and (A.4) with initial values $r(1) = 1$ and $\lambda(1) = \lambda^1 \in [\underline{\lambda}, \overline{\lambda}]$.

As (5) and (A.4) are locally Lipschitz, Picard's existence theorem ensures that IPV-$\lambda(1)$ (in the downward direction) admits a unique solution $\{r(q;\lambda^1), \lambda(q;\lambda^1)\}$ over the interval $[\underline{q}(\lambda^1), 1]$, where $\underline{q}(\lambda^1)$ is the first level of q where either of the following cases occurs:

$$0 = r(q;\lambda^1)[\delta'(\lambda(q;\lambda^1))Y(r(q;\lambda^1), q) - \underline{V}], \text{ or} \tag{A.5}$$

$$0 = q[(\delta(\lambda(q;\lambda^1)) - \delta'(\lambda(q;\lambda^1))\lambda(q;\lambda^1))Y(r(q;\lambda^1), q) - \underline{U}]. \tag{A.6}$$

Furthermore, $\underline{q}(\lambda^1)$ and $\underline{p}(\lambda^1) \coloneqq r(\underline{q}(\lambda^1);\lambda^1)$ are continuous in λ^1.

For $p \in [\underline{p}(\lambda^1), 1], \kappa(p;\lambda^1)$ denote the inverse of $r(q;\lambda^1)$. Define

$$v(p;\lambda^1) = \delta'(\lambda(\kappa(p;\lambda^1);\lambda^1))Y(p, \kappa(p;\lambda^1)), p \in [\underline{p}(\lambda^1), 1];$$

$$u(q;\lambda^1) = [\delta(\lambda(q;\lambda^1)) - \delta'(\lambda(q;\lambda^1))\lambda(q;\lambda^1)]Y(r(q;\lambda^1), q), q \in [\underline{q}(\lambda^1), 1].$$

Note that for all $q > \underline{q}(\lambda^1)$, $\lambda(q;\lambda^1)$, $r(q;\lambda^1)$, $r(q;\lambda^1)[v(r(q;\lambda^1);\lambda^1) - \underline{V}]$ and $q[u(q;\lambda^1) - \underline{U}]$ are positive and strictly increasing in q.[23]

Existence of a solution The boundary value problem has a solution if there exists some λ^1 such that the solution to the IPV-$\lambda(1)$ with $\lambda(1) = \lambda^1$ satisfies both (A.5) and (A.6) at $q = \underline{q}(\lambda^1)$. By construction, (A.5) holds at $q = \underline{q}(\overline{\lambda})$ for $\lambda^1 = \overline{\lambda}$ and (A.6) holds at $q = \underline{q}(\underline{\lambda})$ for $\lambda^1 = \underline{\lambda}$. Consider $\widehat{\lambda} = \inf\{\lambda' \geq \underline{\lambda} : [v(\underline{p}(\lambda^1);\lambda^1) - \underline{V}]\underline{p}(\lambda^1) = 0, \forall \lambda^1 \geq \lambda'\}$. By continuity, $[v(\underline{p}(\widehat{\lambda});\widehat{\lambda}) - \underline{V}]\underline{p}(\widehat{\lambda}) = 0$. If $\widehat{\lambda} = \underline{\lambda}$. Then, I have argued that (A.6) also holds at $q = \underline{q}(\widehat{\lambda})$. Suppose $\widehat{\lambda} > \underline{\lambda}$, the construction of $\widehat{\lambda}$ ensures that there is a convergent sequence $\{\lambda^1_n\}$ with limit $\widehat{\lambda}$ such that $\lambda^1_n < \widehat{\lambda}$ and only (A.6) holds at $q = \underline{q}(\lambda^1_n)$ for $\lambda^1 = \lambda^1_n$. By continuity, $\underline{q}(\widehat{\lambda})[u(\underline{q}(\widehat{\lambda});\widehat{\lambda}) - \underline{U}] = 0$ must hold as well. Therefore, the solution to the IPV-$\lambda(1)$ with $\lambda(1) = \widehat{\lambda}$ solves the boundary value problem.

Uniqueness of the solution *Suppose* $\lambda^H > \lambda^L$. *For* $p \in [\max\{\underline{p}(\lambda^H), \underline{p}(\lambda^L)\}, 1)$, $\kappa(p;\lambda^H) > \kappa(p;\lambda^L)$, $v(p;\lambda^H) < v(p;\lambda^L)$ *and* $\lambda(\kappa(p;\lambda^H);\lambda^H) > \lambda(\kappa(p;\lambda^L);\lambda^L)$.

As $r'(1;\lambda^H) > r'(1;\lambda^L)$, $\kappa(p;\lambda^H) > \kappa(p;\lambda^L)$, $\lambda(\kappa(p;\lambda^H);\lambda^H) > \lambda(\kappa(p;\lambda^L);\lambda^L)$ and $v(p;\lambda^H) < v(p;\lambda^L)$ must hold in some neighborhood of $p = 1$.

Consider the case that $\kappa(.;\lambda^H)$ and $\kappa(.;\lambda^L)$ intersect somewhere in $[\max\{\underline{p}(\lambda^H), \underline{p}(\lambda^L)\}, 1)$. $p_\kappa = \max\{p < 1 : \kappa(p;\lambda^H) = \kappa(p;\lambda^L)\}$ is then well defined and, by construction, $\kappa(p;\lambda^H) > \kappa(p;\lambda^L)$ for all $p \in (p_\kappa, 1)$. I now argue that $v(.;\lambda^H)$ and $v(.;\lambda^L)$ must in-

[23] $v(r(q;\lambda^1);\lambda^1)$ is strictly increasing in q because $\frac{\partial \ln \delta'(\lambda(q;\lambda^1))}{\partial q} + \frac{\partial \ln Y(r(q;\lambda^1), q)}{\partial q} = 0$.

tersect somewhere in $[p_\kappa, 1)$. Suppose not, for all $p \in (p_\kappa, 1]$,

$$\delta'(\lambda(\kappa(p;\lambda^H);\lambda^H))Y(p,\kappa(p;\lambda^H)) = v(p;\lambda^H) < v(p;\lambda^L) = \delta'(\lambda(\kappa(p;\lambda^L);\lambda^L))Y(p,\kappa(p;\lambda^L)),$$

and hence $\lambda(\kappa(p;\lambda^H);\lambda^H) > \lambda(\kappa(p;\lambda^L);\lambda^L)$. This contradicts (5) required by PAM,

$$\begin{aligned} 0 > & \int_{p_\kappa}^{1} \frac{1}{\lambda(\kappa(p;\lambda^H);\lambda^H)} - \frac{1}{\lambda(\kappa(p;\lambda^L);\lambda^L)} dF \\ = & [G(1) - G(\kappa(p_\kappa;\lambda^H))] - [G(1) - G(\kappa(p_\kappa;\lambda^L))] = 0! \end{aligned}$$

Consider the case that $v(.;\lambda^H)$ and $v(.;\lambda^L)$ intersect somewhere in $[\max\{\underline{p}(\lambda^H), \underline{p}(\lambda^L)\}, 1)$. Define $p_v = \max\{p < 1 : v(p;\lambda^H) = v(p;\lambda^L)\}$. As $v(p;\lambda^H) < v(p;\lambda^L)$ for $p > p_v$,

$$\frac{\partial \ln Y(p_v, \kappa(p_v;\lambda^H))}{\partial p} = \frac{d \ln v(p_v;\lambda^H)}{dp} \leq \frac{d \ln v(p_v;\lambda^L)}{dp} = \frac{\partial \ln Y(p_v, \kappa(p_v;\lambda^L))}{\partial p}.$$

Assumption (Y) implies that $\kappa(p_v;\lambda^H) \leq \kappa(p_v;\lambda^L)$. So $\kappa(.;\lambda^H)$ and $\kappa(.;\lambda^L)$ intersect somewhere in $[p_v, 1]$.

We conclude that $\kappa(p;\lambda^H) > \kappa(p;\lambda^L)$ and $v(p;\lambda^H) < v(p;\lambda^L)$ throughout the interval $[\max\{\underline{p}(\lambda^H), \underline{p}(\lambda^L)\}, 1)$. Otherwise, p_v and p_κ are well defined, satisfying $p_v > p_\kappa$ and $p_\kappa \geq p_v$.

$\lambda(\kappa(p;\lambda^H);\lambda^H) > \lambda(\kappa(p;\lambda^L);\lambda^L)$ then follows from the definition of $v(p;\lambda^H)$.

There is a unique initial value of $\lambda(1)$ *for which the solution to the IPV-*$\lambda(1)$ *satisfies both (A.5) and (A.6) at* $q = \underline{q}(\lambda^1)$.

Suppose not, the solutions to the IPV-$\lambda(1)$ with $\lambda(1) = \lambda^H$ and $\lambda(1) = \lambda^L$ satisfy both (A.5) and (A.6) at $q = \underline{q}(\lambda^1)$. Assume $\lambda^H > \lambda^L$.

The first case is $\underline{p}(\lambda^L) > \underline{p}(\lambda^H)$. As $\underline{p}(\lambda^L) > 0$, $\underline{V} = v(\underline{p}(\lambda^L);\lambda^L) > v(\underline{p}(\lambda^L);\lambda^H)$. (A.5) cannot be met at $\underline{p}(\lambda^H)$! The second case is $\underline{p}(\lambda^L) \leq \underline{p}(\lambda^H)$. Then $\underline{q}(\lambda^H) = \kappa(\underline{p}(\lambda^H);\lambda^H) > \kappa(\underline{p}(\lambda^H);\lambda^L) \geq \underline{q}(\lambda^L)$ and $\lambda(\underline{q}(\lambda^H);\lambda^H) > \lambda(\kappa(\underline{p}(\lambda^H);\lambda^L);\lambda^L)$. This is impossible because $\underline{q}(\lambda^H) > 0$ implies $\underline{U} = u(\underline{q}(\lambda^H);\lambda^H) > u(\kappa(\underline{p}(\lambda^H);\lambda^L);\lambda^L)$. (A.6) cannot be met at $\underline{q}(\lambda^L)$!

A.3 Proof of Proposition 2

$(\widetilde{\underline{p}}, \widetilde{\underline{q}}, \widetilde{r}, \widetilde{\lambda}, \widetilde{v}, \widetilde{u})$ satisfies (4)–(6) and (8)–(10) whereas $(r_{CE}, \lambda_{CE}, \underline{p}_{CE}, \underline{q}_{CE}, v_{CE}, u_{CE})$ satisfies (4)–(6) and (9)–(11). The assumption (1) ensures that $\underline{p} < 1$ and $\underline{q} < 1$. Combining

(6) into (8) and (11), we can respectively rewrite the two conditions as

$$\begin{aligned}\frac{d\ln\widetilde{v}(\widetilde{r}(q))}{dp} &= \frac{\partial \ln Y(\widetilde{r}(q),q)}{\partial p},\\ \frac{d\ln v_{CE}(r_{CE}(q))}{dp} &= \frac{\eta(\lambda_{CE}(q))}{\delta'(\lambda_{CE}(q))}\frac{\partial \ln Y(r_{CE}(q),q)}{\partial p}.\end{aligned} \tag{A.7}$$

The subsequent arguments require only $\frac{d\widetilde{v}(\widetilde{r}(q))}{dp} < \eta(\widetilde{\lambda}(q))\frac{\partial Y(\widetilde{r}(q),q)}{\partial p}$ but not other features of the sharing contracts.

Lemma 5. *Suppose* $\widehat{q} = \widetilde{\kappa}(\widehat{p}) = \kappa_{CE}(\widehat{p})$ *for some* $\widehat{p} > \max\{\underline{p}_{CE}, \underline{\widetilde{p}}\}$*, then* $\lambda_{CE}(\widehat{q}) < \widetilde{\lambda}(\widehat{q})$.

Proof. Suppose to the contrary that $\lambda_{CE}(\widehat{q}) \geq \widetilde{\lambda}(\widehat{q})$. (6) implies $v_{CE}(\widehat{p}) \leq \widetilde{v}(\widehat{p})$ and (A.7) implies $\frac{d\ln\widetilde{v}(\widehat{p})}{dp} < \frac{d\ln v_{CE}(\widehat{p})}{dp}$. Hence, $v_{CE}(p) < \widetilde{v}(p)$ and $\kappa_{CE}(p) > \widetilde{\kappa}(p)$ in some interval $(\widehat{p}-\epsilon, \widehat{p})$ where $\epsilon > 0$.[24]

Consider the case where $\widetilde{\kappa}$ and κ_{CE} intersect in $[\max\{\underline{p}_{CE}, \underline{\widetilde{p}}\}, \widehat{p})$. p_κ denotes the highest intersection point of $\widetilde{\kappa}$ and κ_{CE} in $[\max\{\underline{p}_{CE}, \underline{\widetilde{p}}\}, \widehat{p})$, so that $\kappa_{CE}(p) > \widetilde{\kappa}(p)$ for all $p \in (p_\kappa, \widehat{p})$. Then, v_{CE} and $\widetilde{v}$ must intersect somewhere in between p_κ and $\widehat{p}$. Otherwise, for all $p \in (p_\kappa, \widehat{p})$, (6) implies $\delta'(\widetilde{\lambda}(\widetilde{\kappa}(p)))Y(p,\widetilde{\kappa}(p)) > \delta'(\lambda_{CE}(\kappa_{CE}(p)))Y(p_v, \kappa_{CE}(p))$, and hence $\widetilde{\lambda}(\widetilde{\kappa}(p)) < \lambda_{CE}(\kappa_{CE}(p))$. This contradicts PAM and (5),

$$0 < \int_{p_\kappa}^{\widehat{p}} \frac{1}{\widetilde{\lambda}(\widetilde{\kappa}(p))} - \frac{1}{\lambda_{CE}(\kappa_{CE}(p))} dF = [G(\widehat{q}) - G(\widetilde{\kappa}(p_\kappa))] - [G(\widehat{q}) - G(\kappa_{CE}(p_\kappa))] = 0!$$

Consider the case where v_{CE} and $\widetilde{v}$ intersect in $[\max\{\underline{p}_{CE}, \underline{\widetilde{p}}\}, \widehat{p})$. Let p_v be the highest intersection point of v_{CE} and $\widetilde{v}$ in $[\max\{\underline{p}_{CE}, \underline{\widetilde{p}}\}, \widehat{p})$. $v_{CE}(p) < \widetilde{v}(p)$ for all $p \in (p_v, \widehat{p})$. Then $\widetilde{\kappa}$ and κ_{CE} must intersect in between p_v and $\widehat{p}$. Suppose not, $\kappa_{CE}(p) > \widetilde{\kappa}(p)$ for all $p \in (p_\kappa, \widehat{p})$. In the interval $(p_\kappa, \widehat{p})$, $\widetilde{\lambda}(\widetilde{\kappa}(p)) < \lambda_{CE}(\kappa_{CE}(p))$ because of (6) and

$$\begin{aligned}\frac{d\ln v_{CE}(p)}{dp} &= \frac{\eta(\lambda_{CE}(\kappa_{CE}(p)))}{\delta'(\lambda_{CE}(\kappa_{CE}(p)))}\frac{\partial \ln Y(p,\kappa_{CE}(p))}{\partial p}\\ &> \frac{\eta(\widetilde{\lambda}(\widetilde{\kappa}(p)))}{\delta'(\widetilde{\lambda}(\widetilde{\kappa}(p)))}\frac{\partial \ln Y(p,\widetilde{\kappa}(p))}{\partial p} > \frac{\partial \ln \widetilde{v}(p)}{\partial p}.\end{aligned}$$

The first strict inequality is due to Assumption (Y) and (M). Hence, $v_{CE}(p_v) < \widetilde{v}$ contradicting the initial claim that they intersect at p_v!

It follows that $v_{CE}(p) < \widetilde{v}(p)$ and $\kappa_{CE}(p) > \widetilde{\kappa}(p)$ throughout $[\max\{\underline{p}_{CE}, \underline{\widetilde{p}}\}, \widehat{p})$. Otherwise, p_v and p_κ will co-exist, satisfying $\widehat{p} > p_\kappa > p_v$ and $\widehat{p} > p_v > p_\kappa$! (6) then requires $\widetilde{\lambda}(\widetilde{\kappa}(p)) < \lambda_{CE}(\kappa_{CE}(p))$ throughout $[\max\{\underline{p}_{CE}, \underline{\widetilde{p}}\}, \widehat{p})$.

[24]For the case where $\lambda_{CE}(\widehat{q}) = \widetilde{\lambda}(\widehat{q})$, one can show $\frac{\partial \ln \delta'(\lambda_{CE}(\widehat{q}))}{\partial q} > \frac{\partial \ln \delta'(\widetilde{\lambda}(\widehat{q}))}{\partial q}$ by differentiating (6) w.r.t. q and combining it with (A.7).

These conclusions contradict the boundary conditions. If $\underset{\sim}{\widetilde{p}} > \underline{p}_{CE}$, (9) requires $\underline{V} = \widetilde{v}(\widetilde{\underline{p}}) > v_{CE}(\widetilde{\underline{p}})$! If $\widetilde{\underline{p}} \leq \underline{p}_{CE}$, then $\underline{q}_{CE} = \kappa_{CE}(\underline{p}_{CE}) > \widetilde{\kappa}(\underline{p}_{CE}) \geq \widetilde{\underline{q}}$ and $\lambda_{CE}(\underline{q}_{CE}) > \widetilde{\lambda}(\widetilde{\kappa}(\underline{p}_{CE}))$. $\underline{q}_{CE} > 0$ and (10) require $\underline{U} = u_{CE}(\underline{q}_{CE}) > \widetilde{u}(\widetilde{\kappa}(\underline{p}_{CE})) \geq \widetilde{u}(\widetilde{\underline{q}})$! □

Step 1: $\lambda_{CE}(1) < \widetilde{\lambda}(1), v_{CE}(1) > \widetilde{v}(1)$ *and* $u_{CE}(1) < \widetilde{u}(1)$.

Apply Lemma 5 with $\widehat{p} = \widehat{q} = 1$.

Step 2: $\widetilde{\kappa}(p) > \kappa_{CE}(p)$ *for any* $p \in (\max\{\underline{p}_{CE}, \widetilde{\underline{p}}\}, 1)$.

From (5), $\lambda_{CE}(1) < \widetilde{\lambda}(1)$ implies $r'_{CE}(1) < \widetilde{r}'(1)$, and hence $\widetilde{\kappa}(p) > \kappa_{CE}(p)$ for sufficiently large p. Suppose $\widetilde{\kappa}$ and κ_{CE} intersect somewhere in $(\max\{\underline{p}_{CE}, \widetilde{\underline{p}}\}, 1)$. Consider the highest intersection point $\widehat{p}$ and $\widehat{q} = \widetilde{\kappa}(\widehat{p}) = \kappa_{CE}(\widehat{p})$. Lemma 5 states that $\lambda_{CE}(\widehat{q}) < \widetilde{\lambda}(\widehat{q})$. By construction, $r_{CE}(q) > \widetilde{r}(q)$ for $q > \widehat{q}$. It implies $r'_{CE}(\widehat{q}) \geq \widetilde{r}'(\widehat{q})$ and contradicts (5)!

Step 3: $\underline{p}_{CE} \geq \widetilde{\underline{p}}$ *and* $\widetilde{\underline{q}} \geq \underline{q}_{CE}$.

First, suppose that $\widetilde{\underline{p}} > \underline{p}_{CE}$. Then $\widetilde{\underline{q}} = \widetilde{\kappa}(\widetilde{\underline{p}}) \geq \kappa_{CE}(\widetilde{\underline{p}}) > \kappa_{CE}(\underline{p}_{CE}) = \underline{q}_{CE}$. Either (9) or (10) must be violated in such a case.

$$\begin{aligned}\underline{U} &= \widetilde{u}(\widetilde{\underline{q}}) + \widetilde{\lambda}(\widetilde{\underline{q}})[\widetilde{v}(\widetilde{\underline{p}}) - \underline{V}] \\ &= \max_{\lambda \geq 0}[\delta(\lambda)Y(\widetilde{\underline{p}}, \widetilde{\underline{q}}) - \lambda\underline{V}] > \max_{\lambda \geq 0}[\delta(\lambda)Y(\underline{p}_{CE}, \underline{q}_{CE}) - \lambda v_{CE}(\underline{p}_{CE})] \\ &= u_{CE}(\underline{q}_{CE})!\end{aligned} \tag{A.8}$$

The first equality combines (9) and (10) given that $\widetilde{\underline{q}} > 0$ and $\widetilde{\underline{p}} > 0$ while the second and last equalities are both derived from (6) and (7). We have shown $\underline{p}_{CE} \geq \widetilde{\underline{p}}$ and now proceed to $\widetilde{\underline{q}} \geq \underline{q}_{CE}$. Interchanging the roles of $(\underline{p}_{CE}, \underline{q}_{CE})$ and $(\widetilde{\underline{p}}, \widetilde{\underline{q}})$ in the inequality (A.8), the case $\underline{p}_{CE} > \widetilde{\underline{p}}$ and $\underline{q}_{CE} > \widetilde{\underline{q}}$ is ruled out. For any p slightly above $\widetilde{\underline{p}}$, $\widetilde{\kappa}(p) > \kappa_{CE}(p)$, we rule out the case $\underline{p}_{CE} = \widetilde{\underline{p}}$ and $\underline{q}_{CE} > \widetilde{\underline{q}}$ based on the continuity of $\widetilde{\kappa}$. $\underline{p}_{CE} \geq \widetilde{\underline{p}}$ and $\widetilde{\underline{q}} \geq \underline{q}_{CE}$ is the only remaining possibility.

Step 4: $\underline{p}_{CE} = \widetilde{\underline{p}}$ *only if* $\underline{p}_{CE} = \widetilde{\underline{p}} = 0$ *and* $\widetilde{v}(0) > v_{CE}(0)$. $\widetilde{\underline{q}} = \underline{q}_{CE}$ *only if* $\widetilde{\underline{q}} = \underline{q}_{CE} = 0$ *and* $u_{CE}(0) > \widetilde{u}(0)$.

We now consider the case where $\underline{p}_{CE} = \widetilde{\underline{p}}$ and $\widetilde{\underline{q}} = \underline{q}_{CE}$. The first step is to show $\widetilde{\lambda}(\widetilde{\underline{q}}) < \lambda_{CE}(\widetilde{\underline{q}})$. Differentiating the log of (6) and subtracting the expressions with the respective expressions of (A.7), I obtain $\frac{d \ln \delta'(\lambda_{CE}(\widetilde{\underline{q}}))}{dq} > \frac{d \ln \delta'(\widetilde{\lambda}(\widetilde{\underline{q}}))}{dq}$. Suppose, to the contrary, that $\widetilde{\lambda}(\widetilde{\underline{q}}) \geq \lambda_{CE}(\widetilde{\underline{q}})$. For q slightly above $\widetilde{\underline{q}}$, $\widetilde{\lambda}(q) > \lambda_{CE}(q)$ and (5) implies $\widetilde{r}(q) > r_{CE}(q)$. This contradicts the previous claim that $\widetilde{r}(q) < r_{CE}(q)$!

We can further rule out $\underline{p}_{CE} = \widetilde{\underline{p}} > 0$. (6) and (9) are met only if $\widetilde{\lambda}(\widetilde{\underline{q}}) = \lambda_{CE}(\widetilde{\underline{q}})$! A symmetric argument applies to the case $\widetilde{\underline{q}} = \underline{q}_{CE} > 0$. The only remaining possibility is that $\underline{p}_{CE} = \underline{\widetilde{p}} = \underline{\widetilde{q}} = \underline{q}_{CE} = 0$. $\widetilde{v}(0) > v_{CE}(0)$ and $u_{CE}(0) > \widetilde{u}(0)$ because $\widetilde{\lambda}(0) < \lambda_{CE}(0)$.

We now consider the case where $\underline{p}_{CE} = \underline{\widetilde{p}}$ and $\underline{\widetilde{q}} > \underline{q}_{CE}$. From (6), (7) and (10),

$$\max_{\lambda \geq 0} [\delta(\lambda) Y(\underline{p}_{CE}, \underline{q}_{CE}) - \lambda v_{CE}(\underline{p}_{CE})] = u_{CE}(\underline{q}_{CE})$$
$$\geq \underline{U} = \widetilde{u}(\underline{\widetilde{q}}) = \max_{\lambda \geq 0} [\delta(\lambda) Y(\underline{p}_{CE}, \underline{\widetilde{q}}) - \lambda \widetilde{v}(\underline{p}_{CE})].$$

This immediately implies that $\widetilde{v}(\underline{p}_{CE}) > v_{CE}(\underline{p}_{CE}) \geq \underline{V}$, and hence by (9), $\underline{p}_{CE} = \underline{\widetilde{p}} = 0$.

For the case $\underline{p}_{CE} > \underline{\widetilde{p}}$ and $\underline{\widetilde{q}} = \underline{q}_{CE}$, a symmetric argument shows that $\underline{\widetilde{q}} = \underline{q}_{CE} = 0$ and $u_{CE}(0) > \widetilde{u}(0)$.

Note that we never rule out the case $\underline{p}_{CE} > \underline{\widetilde{p}}$ and $\underline{\widetilde{q}} > \underline{q}_{CE}$. Combining step 3 and 4, I show that $\underline{p}_{CE} = \underline{\widetilde{p}}$ if and only if $\underline{p}_{CE} = 0$ and $\underline{\widetilde{q}} = \underline{q}_{CE}$ if only if $\underline{\widetilde{q}} = 0$.

Step 5: $\widetilde{v}(\underline{p}_{CE}) > v_{CE}(\underline{p}_{CE})$ and $u_{CE}(\underline{\widetilde{q}}) > \widetilde{u}(\underline{\widetilde{q}})$

I have shown in the cases $\underline{p}_{CE} = \underline{\widetilde{p}}$ or $\underline{\widetilde{q}} = \underline{q}_{CE}$. Suppose $\underline{p}_{CE} > \underline{\widetilde{p}}$, (9) immediately implies $\widetilde{v}(\underline{p}_{CE}) > v_{CE}(\underline{p}_{CE}) = \underline{V}$. Similarly, $u_{CE}(\underline{\widetilde{q}}) > \widetilde{u}(\underline{\widetilde{q}}) = \underline{U}$ if $\underline{\widetilde{q}} > \underline{q}_{CE}$.

Corollary 1 follows from step 1 and 5.

A.4 Proof of Remark 1

I first consider the case $\delta'(1)Y(0,0) \geq \underline{V}$ in which $\underline{p}_{CE} = \underline{q}_{CE} = 0$. Proposition 2 states that $\underline{\widetilde{p}} = \underline{p}_{CE} = 0$. Corollary 1 states $\widetilde{v}(0) > v_{CE}(0) \geq \underline{V}$. Suppose $\underline{\widetilde{q}} > \underline{q}_{CE}$, (10) implies

$$[\delta(\widetilde{\lambda}(\underline{\widetilde{q}})) - \widetilde{\lambda}(\underline{\widetilde{q}})\delta'(\widetilde{\lambda}(\underline{\widetilde{q}}))]Y(0, \underline{\widetilde{q}}) = \underline{U} \leq u_{CE}(0) = [\delta(1) - \delta'(1)]Y(0,0)$$

Hence, $\widetilde{\lambda}(\underline{\widetilde{q}}) < 1$. Now suppose $\underline{\widetilde{q}} = \underline{q}_{CE} = 0$. From Corollary 1, $U(0) < u_{CE}(0) = [\delta(1) - \delta'(1)]Y(0,0)$, and hence $\widetilde{\lambda}(\underline{\widetilde{q}}) < 1$.

I now turn to the case $\delta'(1)Y(0,0) < \underline{V}$ in which $\underline{p}_{CE} = \underline{q}_{CE} > 0$. Proposition 2 states that $\underline{\widetilde{q}} > \underline{q}_{CE} > 0$. The boundary conditions at the bottom are given by

$$\delta'(\widetilde{\lambda}(\underline{\widetilde{q}}))Y(\underline{\widetilde{p}}, \underline{\widetilde{q}}) = \widetilde{v}(\underline{\widetilde{p}}) \geq \underline{V} = \underline{U} = [\delta(\widetilde{\lambda}(\underline{\widetilde{q}})) - \widetilde{\lambda}(\underline{\widetilde{q}})\delta'(\widetilde{\lambda}(\underline{\widetilde{q}}))]Y(\underline{\widetilde{p}}, \underline{\widetilde{q}})$$

As the matching function is symmetric, $\delta(\lambda) - \lambda\delta'(\lambda) < (=) \delta'(\lambda)$ if and only if $\lambda < (=) 1$. It follows that $\widetilde{\lambda}(\underline{\widetilde{q}}) \leq 1$. Furthermore, $\widetilde{\lambda}(\underline{\widetilde{q}}) = 1$ if and only if $\widetilde{v}(\underline{\widetilde{p}}) = \underline{V}$.

A.5 Proof of Lemma 1

EK show that the boundary value problem for the constrained efficient allocation admits a solution.[25] Fix a solution $(\underline{p}_{CE}, \underline{q}_{CE}, r_{CE}, \lambda_{CE}, v_{CE}, u_{CE})$, I first show that it satisfies (12), and use the inequality to establish uniqueness of the solution. Define

$$\widehat{U}(p,q) = \max_{\lambda \geq 0}\{\delta(\lambda)Y(p,q) - \lambda v_{CE}(p)\}.$$

The unique maximizer, denoted by $\widehat{\Lambda}(p,q)$, satisfies $\delta'(\widehat{\Lambda}(p,q))Y(p,q) = v_{CE}(p)$. By comparing with (6), we deduce that $\widehat{\Lambda}(p,q) > (<)\lambda_{CE}(\kappa_{CE}(p))$ if $q > (<)\kappa_{CE}(p)$.

I first consider $p \geq \underline{p}_{CE}$. By envelope theorem and (11),

$$\begin{aligned}\frac{\partial}{\partial p}\widehat{U}(p,q) &= \delta(\widehat{\Lambda}(p,q))\frac{\partial}{\partial p}Y(p,q) - \widehat{\Lambda}(p,q)\frac{\partial}{\partial p}v_{CE}(p) \\ &= \delta(\widehat{\Lambda}(p,q))\frac{\partial}{\partial p}Y(p,q) - \widehat{\Lambda}(p,q)\eta(\lambda_{CE}(\kappa_{CE}(p)))\frac{\partial}{\partial p}Y(p,\kappa_{CE}(p)) \\ &= \widehat{\Lambda}(p,q)v_{CE}(p)\left[\frac{\eta(\widehat{\Lambda}(p,q))}{\delta'(\widehat{\Lambda}(p,q))}\frac{\partial \ln Y(p,q)}{\partial p} - \frac{\eta(\lambda_{CE}(\kappa_{CE}(p)))}{\delta'(\lambda_{CE}(\kappa_{CE}(p)))}\frac{\partial \ln Y(p,\kappa_{CE}(p))}{\partial p}\right]\end{aligned}$$

Under Assumption (Y) and (M), $\frac{\partial}{\partial p}\widehat{U}(p,q) > (<)0$ if $q > (<)\kappa_{CE}(p)$.

For $p < \underline{p}_{CE}$, $v_{CE}(p) = \underline{V} = v_{CE}(\underline{p}_{CE})$, so

$$\widehat{U}(p,q) < \max\nolimits_{\lambda\geq 0}\{\delta(\lambda)Y(\underline{p}_{CE},q) - \lambda v_{CE}(\underline{p}_{CE})\} = \widehat{U}(\underline{p}_{CE},q).$$

Putting together, for any $q \geq \underline{q}_{CE}$ and any $p \neq r_{CE}(q)$,

$$u_{CE}(q) = \widehat{U}(r_{CE}(q),q) > \max_{\lambda\geq 0}\{\delta(\lambda)Y(p,q) - \lambda v_{CE}(p)\}$$

For $q < \underline{q}_{CE}$, (10) requires that for any p,

$$u_{CE}(q) = \underline{U} = u_{CE}(\underline{q}_{CE}) \geq \max_{\lambda\geq 0}\{\delta(\lambda)Y(p,\underline{q}_{CE}) - \lambda v_{CE}(p)\} > \max_{\lambda\geq 0}\{\delta(\lambda)Y(p,q) - \lambda v_{CE}(p)\}.$$

We have established the inequality (12).

[25] Although EK assume the values of outside options to be zero, their proof is readily extended to the case with positive outside options.

Fix any (K, L). The aggregate surplus is given by

$$\int_{\text{supp}(L)} \eta(\frac{dL_{qs}}{dK})Y(p,q)dL + [F(1) - L_p(1)]\underline{V} + [G(1) - K_q(1)]\underline{U}$$

$$\leq \int_{\text{supp}(L)} \frac{dK}{dL_{qs}} u_{CE}(q) + v_{CE}(p)dL + [F(1) - L_p(1)]\underline{V} + [G(1) - K_q(1)]\underline{U}$$

$$\leq \int u_{CE}(q)dG(q) + \int v_{CE}(p)dF(p) = \int_{\underline{q}}^{1} \delta(\lambda_{CE}(q))Y(r_{CE}(q), q)dG(q) + F(\underline{p})\underline{V} + G(\underline{q})\underline{U}$$

The first inequality is due to (12) and the second one is due to (9) and (10). Equality holds if and only if (K, L) features PAM with $(\underline{p}_{CE}, \underline{q}_{CE}, \kappa_{CE})$ and $\frac{dL_{qs}}{dK} = \lambda_{CE}(q)$ almost everywhere in $\text{supp}(K)$.

A.6 Proof of Proposition 4

Proposition 5 implies that in any equilibrium, workers' equilibrium payoff is given by v_{CE} and the allocation features PAM $(r_{CE}, \lambda_{CE}, \underline{p}_{CE}, \underline{q}_{CE})$. Suppose, to the contrary, that there are active markets (q, s) for almost every q in some interval $[\underline{q}, \overline{q}]$. We allow the possibility that some owners of these types post prices. As in Section 3, the local deviations to other markets (q, s) must be unprofitable. The conditions (6)-(8) continue to hold. For these active markets, $\Lambda(q, s)$ satisfies

$$v_{CE}(r_{CE}(q)) = \delta'(\Lambda(q,s))Y(r_{CE}(q), q); \text{ and}$$
$$\frac{dv_{CE}(r_{CE}(q))}{dp} = \delta'(\Lambda(q,s))\frac{\partial Y(r_{CE}(q), q)}{dp}.$$

Proposition 5 states that v_{CE} and (r_{CE}, λ_{CE}) also satisfy (6) and (11). The former requires $\Lambda(q, s) = \lambda_{CE}(q)$. The latter implies that workers' IC condition (8) cannot be met!

A.7 Equilibrium definition with ordered sets of securities

I provide the definitions for the setting that two ordered sets of securities of $T_1(y; x_1)$ and $T_2(y; x_2)$ are feasible. Extension to the settings where only one or more than two ordered sets of securities are feasible is straightforward.

There are continuums of markets indexed by $(q, x_i) \in [0, 1] \times [\underline{x}_i, \overline{x}_i]$, $i = 1, 2$. $K^i(q, x_i)$ is the measure of assets in the markets $(q', x_i') \leq (q, x_i)$. $L^i(p, q, x_i)$ is the measure of workers of type $p' \leq p$ in the markets $(q', x_i') \leq (q, x_i)$. As before, I denote a marginal distribution with the corresponding variables as subscripts. L^i_{qs} is required to be absolutely continuous w.r.t. K^i.

Definition. (K^1, K^2, L^1, L^2) *is feasible if* $K^1_q + K^2_q \leq G$ *and* $L^1_p + L^2_p \leq F$.

We define $\Lambda(q, x_i; K^i, L^i)$ and $F(p|q, x_i; K^i, L^i)$ in each market as before. For the market (q, x_i), a worker's expected payoff is

$$\eta(\Lambda(q, x_i; K^i, L^i))[Y(p, q) - T_i(p, q, x_i)], \tag{A.9}$$

while an asset owner receives

$$\delta(\Lambda(q, x_i; K^i, L^i))T_i(p, q, x_i). \tag{A.10}$$

The maximal payoffs $V(p; K^1, K^2, L^1, L^2)$ and $U(q; K^1, K^2, L^1, L^2)$ are defined in the same manner.

Definition. *An equilibrium is a pair of feasible distributions* (K^1, K^2, L^1, L^2) *satisfying:*

- *Asset owners' optimality:*

  (i) $(q, x_i) \in \text{supp}(K^i)$ *only if* x_i *maximizes asset owner's expected payoff (A.10);*

  (ii) $\frac{d}{dq}[K^1_q(q) + K^2_q(q)] = g(q)$ *if* $U(q; K^1, K^2, L^1, L^2) > \underline{U}$.

- *Workers' optimality:*

  (i) $(p, q, x_i) \in \text{supp}(L^i)$ *only if* (q, x_i) *maximizes worker's expected payoff (A.9);*

  (ii) $\frac{d}{dq}[L^1_p(p) + L^2_p(p)] = f(p)$ *if* $V(p; K^1, K^2, L^1, L^2) > \underline{V}$.

A.8 Proof of Proposition 5

Suppose $T_1(y; x_1)$ is steeper than $T_2(y; x_2)$. I am comparing the sets of equilibrium payoffs and allocations when only $T_2(y; x_2)$ is feasible and when both $T_1(y; x_1)$ and $T_2(y; x_2)$ are feasible. The proof consists of two parts.

Fix any equilibrium (K^2, L^2) *in the setting where only* $T_2(y; x_2)$ *is feasible. In the setting where both* $T_1(y; x_1)$ *and* $T_2(y; x_2)$ *are feasible, there is a corresponding equilibrium with the same set of active markets and same allocation.*

To construct a corresponding equilibrium, simply set $K^1(1, \overline{x}_1) = L^1(1, 1, \overline{x}_1) = 0$ and keep K^2 and L^2 unchanged. The set of active markets and the equilibrium payoffs remain the same.

It suffices to show that an asset owner cannot profit from deviating to some inactive market (q, x_1). Fix a market $(\widehat{q}, \widehat{x}_1)$. $F(p|\widehat{q}, \widehat{x}_1)$ is degenerate at some type $\widehat{p}$. We can find $\widehat{x}_2$ satisfying $T_2(\widehat{p}, \widehat{q}, \widehat{x}_2) = T_1(\widehat{p}, \widehat{q}, \widehat{x}_1)$. By construction, $\Lambda(\widehat{q}, \widehat{x}_2) \geq \Lambda(\widehat{q}, \widehat{x}_1)$ and $\text{supp}(F(p|\widehat{q}, \widehat{x}_2))$ contains no types below p. Hence, the deviation is not profitable.

$$U(\widehat{q}) \geq \delta(\Lambda(\widehat{q}, \widehat{x}_2)) \int T_2(p, \widehat{q}, \widehat{x}_2) dF(p|\widehat{q}, \widehat{x}_2) \geq \delta(\Lambda(\widehat{q}, \widehat{x}_1)) T_1(\widehat{p}, \widehat{q}, \widehat{x}_1).$$

Suppose two ordered set of securities $T_1(y; x_1)$ and $T_2(y; x_2)$ are both feasible. If $(\hat{K}^1, \hat{K}^2, \hat{L}^1, \hat{L}^2)$ is an equilibrium in which some contracts in $T_1(y; x_1)$ are posted, there is another equilibrium (K^1, K^2, L^1, L^2) in which only contracts in $T_2(y; x_2)$ are posted and features the same equilibrium payoffs and allocation.

The main text contains the argument. Here I formally construct (K^1, K^2, L^1, L^2) using $(\hat{K}^1, \hat{K}^2, \hat{L}^1, \hat{L}^2)$. For any (q, x_1) in $\text{supp}(\hat{K}^1)$, Lemma 2 states that $F(.|q, x_1)$ is degenerate at some type p and $T_2(p, q, x_2) = T_1(p, q, x_1)$ for some term x_2. We then construct two injective mappings $\rho : \text{supp}(\hat{K}^1) \to [0, 1]$ and $\Phi : \text{supp}(\hat{L}^1) \to [0, 1]^2 \times [\underline{x}_2, \overline{x}_2]$. $\rho(q, x_1)$ is the type of workers in the market (q, x_1). Φ maps (p, q, x_1) into (p, q, x_2) satisfying $T_2(\Phi(p, q, x_1)) = T_1(p, q, x_1)$. Define

$$\check{L}^2(p, q, x_2) = \int_{\text{supp}(\hat{L}^1)} \mathbb{1}(\Phi(p', q', x_1') \leq (p, q, x_2)) d\hat{L}^1(p', q', x_1')$$
$$\check{K}^2(q, x_2) = \int_{\text{supp}(\hat{K}^1)} \mathbb{1}(\Phi(\rho(q', x_1'), q', x_1') \leq (\rho(q', x_1'), q, x_2)) d\hat{K}^1(q', x_1')$$

We simply set $K^1(1, \overline{x}_1) = L^1(1, 1, \overline{x}_1) = 0$. Define

$$K^2(q, x_2) = \check{K}^2(q, x_2) + \hat{K}^2(q, x_2)$$
$$L^2(p, q, x_2) = \check{L}^2(p, q, x_2) + \hat{L}^2(p, q, x_2)$$

By construction, the equilibrium payoffs and the allocation remain the same.