Bilateral consistency and converse consistency in axiomatizations and implementation of the nucleolus

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Abstract

We address whether bilateral consistency and converse consistency principles could deepen our understanding of difference between the nucleolus and other rule in the problem of sharing the cost of a public facility among agents who have different needs for it. We propose Right-endpoint Subtraction (RS) bilateral consistency and RS converse consistency, which are counterparts of Left-endpoint Subtraction (LS) bilateral consistency and LS converse consistency introduced by Hu et al. (2012). As we show, axiomatizations of the nucleolus can be obtained by replacing LS bilateral consistency (LS converse consistency) with RS bilateral consistency (RS converse consistency) in Hu et al. (2012)’s axiomatizations of the Constrained Equal Benefits (CEB) rule. Besides, the nucleolus can be implemented by a game obtained by revising Hu et al. (2012)’s game, based on the proposed

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properties, that implements the CEB rule and exploits \textit{LS bilateral consistency} and \textit{LS converse consistency}. \textit{Journal of Economic Literature} Classification Numbers: C71; C72; D63; D70.

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\section{Introduction}

The nucleolus (Schmeidler, 1969) is a central solution concept for Transferable Utility (TU) games. Intuitively, an allocation chosen by the solution minimizes the “complaints” of coalitions in the lexicographic order. It has been investigated from several properties in a variety of models such as TU games, bankruptcy problems, surplus sharing problems \textit{etc}.\textsuperscript{1} However, little attention has been paid to exploring the nucleolus from the two widely applied principles: bilateral consistency and converse consistency.\textsuperscript{2} The aim of our paper is to shed light on the key roles the principles play in deepening our understanding of difference between the nucleolus and other solution.

Consider a problem and an allocation chosen for it. Imagine that all agents, except for two agents, leave with their “components of the allocation”. We can derive a “two-agent reduced problem” by re-evaluating the feasible allocations for the two remaining agents. Bilateral consistency principle says that the restriction of the allocation to this two-agent subgroup should be chosen for the associated reduced problem. Converse consistency principle says that a certain allocation should be chosen for some problem if for each two-agent reduced problem, its restriction to this subgroup is chosen.

Our paper provides positive answers to the following questions. First, in TU games, as shown by Chang and Hu (2007), the Shapley value is the only solution satisfying “the basic properties” and “bilateral self* consistency” (or “converse self* consistency”). Besides, replacing \textit{bilateral self* consistency}

\textsuperscript{1}For examples, Solobev (1975) axiomatizes it for TU games, Aumann and Maschler (1985) for bankruptcy problems, and Moulin (1985) for surplus sharing problems.

\textsuperscript{2}The two principles have been applied to several models such as taxation, bargaining, social choice \textit{etc}. For references, see Lensberg (1987), Peleg (1986), Young (1988), Chun (2002), and Yeh (2006). For a survey on the principles, see Thomson (2010).
(converse self* consistency) with “bilateral complement consistency” (“converse complement consistency”) in the above characterization leads to that the Equal Allocation of Non-Separable Cost (EANSC) value stands out as the most desirable one. Namely, the Shapley value and the EANSC value can be axiomatized by different bilateral consistency and converse consistency properties in TU games. We ask whether such differentiation between the nucleolus and other solution can be obtained in other allocation problems. Second, as suggested by Krishna and Serrano (1996), the properties of a solution can be used as guides in designing a non-cooperative game that implements the solution. We ask whether an implementation of the nucleolus can be obtained by revising a game, based on bilateral consistency and converse consistency of the nucleolus, that implements other solution and exploits different bilateral consistency and converse consistency.

To serve our purpose, we restrict attention to the class of “airport problems”. Several airlines need to build a runway and use it jointly. Different airlines need runways of different lengths. If an airline operates bigger planes than other airlines, it needs a longer runway. If a runway serves a given plane, it can also serve any smaller plane at no extra cost. To accommodate all airlines, the length of the runway must be long enough for the biggest plane any airline operates. How should the cost of the runway be shared among the airlines? A “rule” is a function that assigns each airport problem an allocation of the cost of the runway, called a “contributions vector”.

We introduce “Right-endpoint Subtraction (RS) bilateral consistency” and “RS converse consistency”, which are applications of bilateral consistency and converse consistency principles to the airport problem and natural
counterparts of “Left-endpoint Subtraction (LS) bilateral consistency” and “LS converse consistency” proposed by Hu et al. (2012) for the same model. Hu et al. (2012) show that the “Constrained Equal Benefits (CEB) rule”\(^6\) (Potters and Sudhölter, 1999) is the only LS bilaterally consistent (or LS conversely consistent) rule satisfying “equal treatment of equals” and “last-agent cost additivity”. We show that replacing LS bilateral consistency (LS conversely consistent) with RS bilateral consistency (RS converse consistency) in the above axiomatization leads to that the nucleolus stands out as the most desirable one.

We next introduce a 3-stage extensive form game to implement the nucleolus. Our game is obtained by revising, based on RS bilateral consistency and RS converse consistency, Hu et al. (2012)’s game that implements the CEB rule and exploits LS bilateral consistency and LS converse consistency.

**Stage 1:** Each agent, except for the responder (one of the agents with the largest need), proposes her own contribution to the total cost (the cost of satisfying an agent with the largest need).

**Stage 2:** The responder either accepts their proposals, in which case she contributes the residual cost, or rejects them. In the case of rejection, she takes one agent to the next stage to re-negotiate their contributions. All the others contribute the amounts they proposed.

**Stage 3:** The two agents contribute expected amounts specified as follows. There is a fair coin to select one of the two agents. The chosen one picks one agent, say agent $j$, whose need is no smaller than hers from all the others and forms a coalition with all agents whose needs are no larger than agent $j$’s need. She then takes the contributions already made by the coalition to cover the cost of building the part of the runway she and all agents in this coalition can use. The other then takes the leftover\(^7\) and the remaining contributions already made to cover the difference between the total cost and the cost of the part of the runway already built.

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\(^6\)The rule chooses a contributions vector that equalizes agents’ benefits (the benefit of an agent is the difference between her cost and contribution) subject to no one receiving a subsidy. Potters and Sudhölter (1999) name it as the “modified nucleolus”. Our terminology is due to Thomson (2007).

\(^7\)By leftover, we mean the difference between the sum of the contributions made by the coalition and the cost of the part of the runway already built if positive; otherwise, zero.
We show that for each airport problem, there is a unique Subgame Perfect Equilibrium (SPE) outcome and it is the nucleolus contributions vector.

In the literature on Nash program\textsuperscript{8}, it is typical to introduce a game to implement a solution or a class of solutions. However, there was no attempt to construct a game that does not only implement a solution but also shed light into fundamental differences between the solution and others. It is worthwhile noticing that our game does not only implement the nucleolus but also illustrate the key roles bilateral consistency and converse consistency principles play in differentiation between the nucleolus and the CEB rule.

The paper is organized as follows. Section 2 introduces the model, central rules, and properties. Section 3 bases axiomatizations. Section 4 establishes an implementation. Section 5 is concluding remarks.

\section{Notation and definitions}

\subsection{The model}

Let $U \subseteq \mathbb{N}$ be a universe of agents with at least two elements, where $\mathbb{N}$ is the set of natural numbers. An airport problem, or simply a problem, is a pair $(N, c)$ where $N \subseteq U$ is a finite nonvoid agent set and $c \equiv (c_i)_{i \in N} \in \mathbb{R}_+^N$ is the profile of agents’ costs. Let $\mathcal{A}$ be the class of all problems on $U$.

A contributions vector of $(N, c) \in \mathcal{A}$ is a vector $x \in \mathbb{R}^N$ satisfying the so-called “efficiency” condition: the sum of all contributions should be equal to the total cost. Formally, $\sum_{i \in N} x_i = \max_{i \in N} c_i$. Let $X(N, c)$ be the set of all contributions vectors for $(N, c) \in \mathcal{A}$.\textsuperscript{9} A rule is a function defined on $\mathcal{A}$ that associates with each problem $(N, c) \in \mathcal{A}$ a vector $x \in X(N, c)$. Let $n \equiv |N|$ and for simplicity, assume that $N \equiv \{1, \ldots, n\}$ and $c_1 \leq \cdots \leq c_n$.

Thus, the agents are ordered in terms of their costs. We refer to agent 1 (the agent with the smallest cost and the lowest index) as the first agent and to agent $n$ (the agent with the largest cost and the highest index) as the last

\textsuperscript{8}Nash (1953) initiates this research agenda. It is a study of understanding cooperative solutions through non-cooperative games. For a survey, see Serrano (2005).

\textsuperscript{9}In contrast to Hu et al. (2012), we do not impose the so-called “reasonableness” condition on the set of contributions vectors. The condition says that each agent should not receive a subsidy and should not contribute more than her cost. Formally, for each $i \in N$, $0 \leq x_i \leq c_i$. 

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agent. Our generic notation for rules is $\varphi$. For each $N' \subset N$, we denote $(c_i)_{i \in N'}$ by $c_{N'}$, $(\varphi_i(N,c))_{i \in N'}$ by $\varphi_{N'}(N,c)$, and so on.

2.2 Central rules and properties

We now formally introduce important rules. The first rule is defined only for two-agent problems. It says that the agent with the smaller cost contributes half of her cost, and the other contributes the remainder. Formally,

**Standard rule, $St$:** For each $(\{i, j\}, (c_i, c_j)) \in A$ with $c_i \leq c_j$,

\[
St_i(\{i, j\}, (c_i, c_j)) \equiv \frac{c_i}{2}, \quad St_j(\{i, j\}, (c_i, c_j)) \equiv c_j - St_i(\{i, j\}, (c_i, c_j)).
\]

Our second rule suggests first that each agent in $N$ contributes equally until there are $\lambda^1 \in \mathbb{R}_+$ and a group of agents $\{1, \cdots, l^1\}$ (if there are several such groups, pick the one containing the agent with the largest index)\(^{10}\) who is indifferent between cooperating with $N$ (i.e., each agent in $\{1, \cdots, l^1\}$ contributes $\lambda^1$) and building their own runway (i.e., $c_{l^1}$). Namely, $\lambda^1 l^1 = c_{l^1}$. The rule next suggests that each agent in $\{l^1 + 1, \cdots, n\}$ contributes equally until there are $\lambda^2 \in \mathbb{R}_+$ and a group of agents $\{l^1 + 1, \cdots, l^2\}$ (if there are several such groups, pick the one containing the agent with the largest index) who is indifferent between cooperating with $N$ (i.e., each agent in $\{l^1, \cdots, l^2\}$ then contributes $\lambda^2$ and cooperating with $\{1, \cdots, l^1\}$ (i.e., $c_{l^2} - c_{l^1}$). Namely, $\lambda^2 (l^2 - l^1) = c_{l^2} - c_{l^1}$. Continue this process until the total cost $c_n$ is covered. This algorithm amounts to the following formula.\(^{11}\)

**Constrained Equal Contributions rule, $CEC$:** For each $(N, c) \in A$,

\[
CEC_1(N, c) \equiv \min_{1 \leq k \leq n} \left\{ \frac{c_k}{k} \right\}, \\
CEC_i(N, c) \equiv \min_{i \leq k \leq n} \left\{ \frac{c_k - \sum_{p=1}^{i-1} CEC_p(N,c)}{k-i+1} \right\} \quad \text{where } 2 \leq i \leq n - 1, \\
CEC_n(N, c) \equiv c_n - \sum_{p=1}^{n-1} CEC_p(N, c).
\]

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\(^{10}\)This requirement is just for convenience. If it is not imposed, the same $\lambda$ is obtained at the next step.

\(^{11}\)Aadland and Koplin (1998) name it as the restricted average cost share rule. The terminology is due to Thomson (2007).
Our next rule is a direct application of the nucleolus in TU games, where its name is due to.\textsuperscript{12} For general games, the payoff vector chosen by the nucleolus is difficult to compute since it involves a sequence of linear programs. However, for the airport problem, its contributions vector can be obtained by the following algorithm, which shares a similar spirit with the one for the CEC rule. The algorithm starts by requiring that all agents in $N \backslash \{n\}$ should contribute equally until there are $\beta^1 \in \mathbb{R}_+$ and a group of agents $\{1, \ldots, p^1\}$ (if there are several such groups, pick the one containing the agent with the largest index)\textsuperscript{13} together with the last agent such that $\beta^1 (p^1 + 1) = c_{p^1}$. Each agent in $\{1, \ldots, p^1\}$ then contributes $\beta^1$. The algorithm next requires that all agents in $\{p^1 + 1, \ldots, n - 1\}$ should contribute equally until there are $\beta^2 \in \mathbb{R}_+$ and a group of agents $\{p^1 + 1, \ldots, p^2\}$ (if there are several such groups, pick the one containing the agent with the largest index) together with the last agent such that $\beta^2 (p^2 - p^1 + 1) = c_{p^2} - c_{p^1}$. Each agent in $\{p^1, \ldots, p^2\}$ then contributes $\beta^2$. Continue this process until $c_n$ is covered. The algorithm amounts to Sönmez (1994)’s formula defined next.

**Nucleolus, $Nu$:** For each $(N, c) \in A$,

\[
 Nu_1(N, c) \equiv \min_{1 \leq k \leq n-1} \left\{ \frac{c_k}{k+1} \right\} \\
 Nu_i(N, c) \equiv \min_{i \leq k \leq n-1} \left\{ \frac{c_k - \sum_{p=1}^{i-1} Nu_p(N, c)}{k-i+2} \right\} \quad \text{where} \quad 2 \leq i \leq n - 1 \\
 Nu_n(N, c) \equiv c_n - \sum_{p=1}^{n-1} Nu_p(N, c).
\]

**Remark 1:** Several rules including the nucleolus and the “CEB rule”\textsuperscript{14} coincide with the standard rule. However, the CEC rule is an exception.

\textsuperscript{12} A solution for TU games can be used to provide recommendations for airport problems. One natural way is to transform an airport problem into an associated TU game by defining the worth of a coalition as the largest cost in that coalition. A solution in TU games is then applied to solve the associated TU game. This gives us a payoff vector. We take this payoff vector as the contributions vector for the airport problem. When we adopt this transformation, as Aadland and Koplin (1998) point out, the contributions vector chosen by the CEC rule coincides with that prescribed by a central solution in TU games, “Dutta-Ray solution” (Dutta and Ray, 1989).

\textsuperscript{13} This requirement is again just for convenience.

\textsuperscript{14} For a verbal definition of the rule, see Footnote 6. Formally, for each $(N, c) \in A$ and each $i \in N$, $CEB_i(N, c) \equiv \max \{c_i - \beta, 0\}$, where $\beta \in \mathbb{R}_+$ is chosen such that $\sum_{i \in N} CEB_i(N, c) = c_n$.
There are four agents. For simplicity, assume that $c_1 < c_2 < c_3 < c_4$. Agent 1 uses segment $A$. Agent 2 uses segments $A$ and $B$. Agent 3 uses segments $A$, $B$, and $C$. Agent 4 uses segments $A$, $B$, $C$, and $D$. The cost of segment $A$ is $c_1$. The cost of segment $B$ is $c_2 - c_1$. The cost of segment $C$ is $c_3 - c_2$. The cost of segment $D$ is $c_4 - c_3$. The total cost is $c_4$.

Figure 1: Four-agent problem

In other models of fair allocation, there is only one natural way to formulate a reduced problem. However, in the airport problem, the runway is composed of segments and agents use segments differently. Thus, segments are not homogenous and several formulations of a reduced problem become available. Potters and Sudhölter (1999) propose two natural (left-endpoint and right-endpoint) formulations. The authors use the right-endpoint (left-endpoint) formulation to define the reduced problem underlying “RS consistency” (“LS consistency”) by only considering the possibility of one agent leaving. We extend this idea to the departure of a group of agents by considering one agent leaving at a time. However, it can be verified that the ordering of the departures of the agents plays no role in the formulation of our two-agent reduced problem.

To illustrate how the formulation is adopted here, consider the four-agent problem in Figure 1 and a contributions vector $x$ chosen by a rule for it. Imagine the departures of agents 1 and 3 with their contributions $x_1$ and $x_3$. To see how the situation is reassessed from the viewpoints of agents 2 and 4,

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15For detailed discussions on these formulations, see Thomson (2007).
16Potters and Sudhölter (1999) refer to RS consistency and LS consistency as $\nu$-consistency and $\psi$-consistency, respectively. For a reference on RS consistency, see Hwang and Yeh (2012).
This is the three-agent reduced problem. Agent 2 uses segment \( \overline{B} \). Agent 3 uses segments \( \overline{B} \) and \( \overline{C} \). Agent 4 uses segments \( \overline{B} \), \( \overline{C} \), and \( \overline{D} \). The cost of segment \( \overline{B} \) is \( c_2 - x_1 \). The cost of segment \( \overline{C} \), is \( c_3 - c_2 \). The cost of segment \( \overline{D} \) is \( c_4 - c_3 \). The total cost is \( c_4 \), which is the sum of \( c_2 - x_1 \) (the cost of segment \( \overline{B} \)), \( c_3 - c_2 \) (the cost of segment \( \overline{C} \)), and \( c_4 - c_3 \) (the cost of segment \( \overline{D} \)).

**Figure 2: Three-agent reduced problem**

suppose first that agent 1 pays her contribution \( x_1 \) and “leave”. Agents 2, 3, and 4 are now left with the amount \( x_1 \) that can help cover part of the total cost \( c_4 \). Instead of thinking of \( x_1 \) as covering an abstract part of the runway, it is natural to think of \( x_1 \) as intended to help cover the segment agent 1 uses (namely, segment \( A \)). Since all agents use segment \( A \), each of the remaining agents benefits from agent 1’s contribution by the amount \( x_1 \). The formulation suggests to revise their costs down by \( x_1 \). Namely, the revised costs are \( c_2 - x_1 \), \( c_3 - x_1 \), and \( c_4 - x_1 \).

Consider the three-agent reduced problem in Figure 2. Suppose now that agent 3 pays her contribution \( x_3 \) and leave. Agents 2 and 4 are now left with the amount \( x_3 \) that can help cover part of the total cost of \( c_4 - x_1 \). Think of \( x_3 \) as intended to help cover the segments agent 3 uses (namely, segments \( \overline{B} \), \( \overline{C} \) and \( \overline{D} \)). A natural question arises: should which segmental cost be covered first? The formulation suggests to first cover the cost of segment \( \overline{C} \) (the segment agents 3 and 4 use but agent 2 does not); unless \( x_3 > c_3 - c_2 \), in which case, \( c_3 - c_2 \) is completely covered and the remainder \(( x_3 - (c_3 - c_2) )\) would be used to help cover the cost of segment \( \overline{B} \) (the segment agents 2, 3, and 4 use). This implies that if \( c_3 - c_2 > x_3 \), since

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\(^{17}\)The left-endpoint formulation also suggests that in this case, the revised costs of the remaining agents are also \( c_2 - x_1 \), \( c_3 - x_1 \), and \( c_4 - x_1 \).
$x_3$ is too little to cover the cost of segment $\bar{C}$, agent 2 benefits nothing from agent 3’s contribution and her cost is no longer revised (namely, her final revised cost is $c_2 - x_1$); otherwise, $c_3 - c_2$ is completely covered and the remainder $(x_3 - (c_3 - c_2))$ would be used to help cover the cost of segment $\bar{B}$. Thus, agent 2’s cost is revised down further by $x_3 - (c_3 - c_2)$ and her final revised cost is $c_3 - (x_1 + x_3)$ (namely, $c_2 - x_1 - (x_3 - (c_3 - c_2))$). Altogether, agent 2’s final revised cost is the minimum of $c_3 - x_1$ and $c_3 - (x_1 + x_3)$. Since contributing to the segments agent 3 uses implies contributing to the segments agent 4 uses, agent 4’s cost is then revised down further by $x_3$. Namely, agent 4’s final revised cost is $c_4 - (x_1 + x_3)$.

Formally, let $(N, c) \in A$ with $|N| \geq 2$, $i \in N \setminus \{n\}$ and $x \in X(N, c)$. The reduced problem of $(N, c)$ with respect to $N' \equiv \{i, n\}$ and $x$, $(N', r_{N'}^x)$, is defined by setting

$$
(r_{N'}^x)_i \equiv \min_{i \leq k, k \neq n} \left\{ c_k - \sum_{m \leq k, m \neq i, n} x_m \right\} \text{ and }
(r_{N'}^x)_n \equiv \min_{n \leq k, k \neq i} \left\{ c_k - \sum_{m \leq k, m \neq i, n} x_m \right\} = c_n - \sum_{m \neq i, n} x_m.
$$

A rule is “RS bilaterally consistent” if a contributions vector $x$ is chosen by the rule for a problem, then for the two-agent reduced problem just defined, the components of $x$ pertaining to the remaining agents should still be chosen by the rule.

**Right-endpoint Subtraction (RS) bilateral consistency:** For each $(N, c) \in A$ with $|N| \geq 2$ and each $i \in N \setminus \{n\}$, if $x = \varphi(N, c)$, then

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The left-endpoint formulation suggests to cover the cost of segment $\bar{B}$; unless $x_3 > c_2 - x_1$, in which case, the cost of segment $\bar{B}$ is completely covered and the remainder $(x_1 + x_3 - c_2)$ would be used to help cover the cost of segment $\bar{C}$ (the segment agents 2, 3, and 4 use). Agent 2’s final revised cost is $c_2 - (x_1 + x_3)$. Since contributing to the segments agent 3 uses implies contributing to the segments agent 4 uses, agent 4’s cost is then revised down further by $x_3$. Namely, agent 4’s final revised cost is $c_4 - (x_1 + x_3)$. Since no agent’s cost can be a negative, for each agent $j \in \{2, 4\}$, agent $j$’s final revised cost is the maximum of $c_j - (x_1 + x_3)$ and zero. For a reference on this formulation, see also Hu et al. (2012).
\[(\{i, n\}, r_{\{i,n\}}^x) \in \mathcal{A} \text{ and } x_{\{i,n\}} = \varphi (\{i, n\}, r_{\{i,n\}}^x) .\]

A rule is “RS conversely consistent” if a contributions vector \(x\) is such that for each two-agent reduced problem involving agent \(n\), the restriction of \(x\) to this subgroup is chosen by the rule, then \(x\) should be chosen by the rule for the initial problem.\(^{19}\)

**RS converse consistency:** For each \((N, c) \in \mathcal{A}\) with \(|N| > 2\) and each \(x \in X(N, c)\), if for each \(N' \subset N\) with \(|N'| = 2\) and \(n \in N'\), \(x_{N'} = \varphi (N', r_{N'}^x)\), then \(x = \varphi (N, c)\).

In the formulation of **RS consistency**, Potter and Sudhölter (1999) includes the possibility of the last agent’s leaving. To make a direct comparison between our results and Hu et al. (2012)’s results, we exclude this possibility in the formulations of **RS bilateral consistency** and **RS converse consistency**. However, this exclusion is reasonable since due to the nature of the airport problem (agents are linearly ordered in terms of their needs for the facility), agent \(l\) would be enthusiastic at reaching an agreement with agent \(l - 1\) and so on. This suggests that the last agent should play the role of coordinator.

## 3 Axiomatizations

As Potter and Sudhölter (1999) point out, the nucleolus and the CEC rule are **RS consistent**. Therefore, they are **RS bilaterally consistent**.

**Lemma 1:** The nucleolus and the CEC rule are **RS bilaterally consistent**.

One may wonder whether the nucleolus and the CEC rule are **RS conversely consistent** as well. The next result provides a positive answer for the nucleolus.

**Lemma 2:** The nucleolus is **RS conversely consistent**.

\(^{19}\)It can be shown that **RS bilateral consistency** and **RS converse consistency** are logically independent.
Proof. Let \((N, c) \in \mathcal{A}\) and \(x \in X(N, c)\). For each \(k \in N \setminus \{n\}\), let \((\{k, n\}, r^x_{(k,n)}) \in \mathcal{A}\) and \(x_{\{k,n\}} = Nu(\{k, n\}, r^x_{(k,n)})\). We show that \(x = Nu(N, c)\). Let \(h_0 \equiv 0\). For each \(l \in \{1, \cdots, p+1\}\) with \(p+1 < n\), let

\[ h_l = \max \left\{ i \in N \setminus \{n\} \middle| \exists i \in \arg\min_{h_{l-1} + 1 \leq k \leq n-1} \left\{ c_k - \sum_{m \leq k, m \neq h_{l-1}+1, n} x_m \right\} \right\}. \]

Clearly, \(1 \leq h_1 < \cdots < h_{p+1} < n\). The proof is by induction on \(\{1, \cdots, p+1\}\).

Step 1. For each \(i \in \{1, \cdots, h_1\}\), \(x_i = Nu_i(N, c)\). We first show \(x_1 \leq \cdots \leq x_n\). Let \(i, j \in N \setminus \{n\}\) with \(i < j\). Since for each \(k \in N \setminus \{n\}\), \(x_{\{k,n\}} = Nu(\{k, n\}, r^x_{(k,n)})\), then \(x_i = \frac{1}{2} \min_{j \leq k \leq n-1} \left\{ c_k - \sum_{m \leq k, m \neq j, n} x_m \right\}\) and \(x_j = \frac{1}{2} \min_{j \leq k \leq n-1} \left\{ c_k - \sum_{m \leq k, m \neq j, n} x_m \right\}\). Thus, \(x_i = \min_{i \leq k \leq n-1} \left\{ c_k - \sum_{m \leq k, m \neq i, n} x_m \right\}\) and \(x_j = \min_{i \leq k \leq n-1} \left\{ c_k - \sum_{m \leq k, m \neq i, n} x_m \right\}\). Note that \(x_{n-1, n} = Nu(\{n-1, n\}, r^x_{(n-1,n)})\) and \(x \in X(N, c)\). It follows that \(x_{n-1} = \frac{1}{2} \min_{n-1 \leq k \leq n-1} \left\{ c_k - \sum_{m \leq n-1, m \neq n-1, n} x_m \right\} = \frac{1}{2} \left( c_{n-1} - \sum_{m \leq n-1, m \neq n-1, n} x_m \right) = c_{n-1} - \sum_{m=1}^{n-1} x_m \leq c_n - \sum_{m=1}^{n-1} x_m = x_n\)

We then conclude that \(x_1 \leq \cdots \leq x_n\).

We next show that

\[ x_1 = \cdots = x_{h_1} = \frac{c_{h_1}}{h_1 + 1}. \quad (1) \]

Since for each \(i \in \{1, \cdots, h_1\}\), \(x_{\{i,n\}} = Nu(\{i, n\}, r^x_{(i,n)})\), then by the definition of \(h_1\),

\[ x_i + x_1 = 2x_i - x_i + x_1 = \min_{i \leq k \leq n-1} \left\{ c_k - \sum_{m \leq k, m \neq i, n} x_m \right\} - x_i + x_1 = \min_{i \leq k \leq n-1} \left\{ c_k - \sum_{m \leq k, m \neq i, n} x_m \right\} = c_{h_1} - \sum_{m \leq h_1; m \neq 1, n} x_m. \]
which implies that for each $i \in \{1, \cdots, h_1\}$, $x_i = c_{h_1} - \sum_{m \leq h_1; m \neq i} x_m$. Thus, (1) holds.

We now show that

$$\frac{c_{h_1}}{h_1 + 1} = \min_{1 \leq k \leq n-1} \left\{ \frac{c_k}{k + 1} \right\}. \quad (2)$$

Note that for each $l \in N \setminus \{h_1, n\}$, if $l < h_1$, then by the definition of $h_1$, $c_l - \sum_{m \leq l; m \neq 1} x_m \geq c_{h_1} - \sum_{m \leq h_1; m \neq i} x_m$. By (1), we derive $\frac{c_l}{l+1} \geq \frac{c_{h_1}}{h_1+1}$. If $l > h_1$, then $c_l - \sum_{m \leq l; m \neq 1} x_m > c_{h_1} - \sum_{m \leq h_1; m \neq i} x_m$, which implies $c_l - \sum_{m=h_1+1}^l x_m > c_{h_1}$. Since $x_1 \leq \cdots \leq x_n$ and (1), $c_l - (l - h_1 + 1) \left( \frac{c_{h_1}}{h_1+1} \right) > c_{h_1}$. It follows that $\frac{c_l}{l+1} > \frac{c_{h_1}}{h_1+1}$. We then conclude that for each $l \in N \setminus \{h_1, n\}$, $c_{h_1}/(h_1 + 1) \leq c_l/(l + 1)$. Thus, (2) holds.

By (1) and the formula of the nucleolus, for each $i \in \{1, \cdots, h_1\}$, $x_i = \min_{1 \leq k \leq n-1} \left\{ \frac{c_k}{k+1} \right\} = Nu_i(N, c)$.

**Step 2.** For each $i \in \{1, \cdots, h_{p+1}\}$, $x_i = Nu_i(N, c)$. By induction hypothesis, suppose that for each $i \in \{1, \cdots, h_p\}$, $x_i = Nu_i(N, c)$. We show that for each $i \in \{h_p + 1, \cdots, h_{p+1}\}$, $x_i = Nu_i(N, c)$. We first show that

$$x_{h_{p+1}} = \cdots = x_{h_{p+1}} = \frac{c_{h_{p+1}} - \sum_{j=1}^{h_p} x_j}{h_{p+1} - (h_p + 1) + 2} \quad (3)$$

Since for each $i \in \{h_p + 1, \cdots, h_{p+1}\}$, $x_{\{i, n\}} = Nu \left( \{i, n\}, v_{\{i, n\}} \right)$, then by the definition of $h_{p+1},$

$$x_i + x_{h_{p+1}} = 2x_i - x_i + x_{h_{p+1}}$$

$$= \min_{i \leq k \leq n-1} \left\{ c_k - \sum_{m \leq k; m \neq i, n} x_m \right\} - x_i + x_{h_{p+1}}$$

$$= \min_{i \leq k \leq n-1} \left\{ c_k - \sum_{m \leq k; m \neq h_{p+1}, n} x_m \right\}$$

$$= c_{h_{p+1}} - \sum_{m = h_{p+1}; m \neq h_{p+1}, n} x_m,$$
which implies that for each \( i \in \{h_p+1, \cdots, h_{p+1}\} \), \( x_i = c_{h_{p+1}} - \sum_{m \leq h_{p+1}, m \neq n} x_m \).

Thus, (3) holds.

We next show that

\[
\frac{c_{h_{p+1}} - \sum_{j=1}^{h_p} x_j}{h_{p+1} - (h_p + 1) + 2} = \min \left\{ \frac{c_{k} - \sum_{j=1}^{h_p} x_j}{k - (h_p + 1) + 2} \mid k \leq h_{p+1} \right\}.
\]

(4)

Note that for each \( l \in \{h_p+1, \cdots, n-1\} \setminus \{h_{p+1}\} \), if \( l < h_{p+1} \), then by the definition of \( h_{p+1} \), \( c_l - \sum_{m \leq j, m \neq h_{p+1}, n} x_m \geq c_{h_{p+1}} - \sum_{m \leq h_{p+1}, m \neq h_{p+1}, n} x_m \). By (3), we derive

\[
c_l - \sum_{j=1}^{h_p} x_j = \sum_{m=h_{p+1}}^{l} x_m \geq \left( c_{h_{p+1}} - \sum_{j=1}^{h_p} x_j \right) - \sum_{m=h_{p+1}+1}^{l} x_m.
\]

Since \( x_1 \leq \cdots \leq x_n \) and (3), \( c_l - \sum_{j=1}^{h_p} x_j \leq \frac{c_{h_{p+1}} - \sum_{j=1}^{h_p} x_j}{h_{p+1} - (h_p + 1) + 2} \). We then conclude that for each \( l \in \{h_p+1, \cdots, h_{p+1}\} \), \( \frac{c_{h_{p+1}} - \sum_{j=1}^{h_p} x_j}{h_{p+1} - (h_p + 1) + 2} \leq \frac{c_{h_{p+1}} - \sum_{j=1}^{h_p} x_j}{h_{p+1} - (h_p + 1) + 2} \). Thus, (4) holds.

By (3), the induction hypothesis, and the formula of the nucleolus, we obtain that for each \( i \in \{h_p+1, \cdots, h_{p+1}\} \), \( x_i = Nu_i(N, c) \).

Since \( x \in X(N, c) \) and \( Nu(N, c) \in X(N, c) \), we have \( x_n = Nu_n(N, c) \). By Steps 1 and 2, we conclude that \( x = Nu(N, c) \). \( \square \)

As the next example shows, unfortunately, the CEC rule is not RS conversely consistent.

**Example 1:** Let \( N \equiv \{1, 2, 3\} \) and \( c \equiv (1, 4, 9) \). Thus, \( CEC(N, c) = (1, 3, 5) \). Consider the contributions vector \( x \equiv (0.5, 3.5, 5) \). Let \( N' \equiv \{1, 3\} \) and \( N'' \equiv \{2, 3\} \). Note that \( CEC_1(N', r_N) = 0.5 = x_1 \) and \( CEC_2(N'', r_N) = 3.5 = x_2 \). It follows that \( x \) satisfies the hypothesis of RS converse consistency. However, \( x \neq CEC(N, c) \) in violation of RS converse consistency. \( \square \)

In addition, Example 1 points out that RS converse consistency plays an important role in differentiation between the nucleolus and other rule.

We also study the following additional properties. The first one says that if the costs of two agents are equal, they should contribute equal amounts.
Equal treatment of equals: For each \((N, c) \in \mathcal{A}\) and each pair \(\{i, j\} \subseteq N\), if \(c_i = c_j\), then \(\varphi_i(N, c) = \varphi_j(N, c)\).

Suppose that the cost of the last agent increases by some amount. Since this addition to the runway is only used by this agent, the second property requires that she should be responsible for all of this incremental cost.

Last-agent cost additivity: For each pair \(\{(N, c), (N, c')\}\) of elements of \(\mathcal{A}\) and each \(\delta \in \mathbb{R}_+\), if \(c'_n = c_n + \delta\) and for each \(j \in N\setminus\{n\}\), \(c'_j = c_j\), then \(\varphi_n(N, c') = \varphi_n(N, c) + \delta\).

The nucleolus satisfies equal treatment of equals and last-agent cost additivity (Potter and Sudhölter, 1999). It is also RS bilaterally consistent and RS conversely consistent (Lemmas 1 and 2). Hu et al. (2012) show that the CEB rule is the only LS bilaterally consistent (or LS conversely consistent) rule satisfying equal treatment of equals and last-agent cost additivity.\(^{20}\) We ask whether axiomatizations of the nucleolus can be obtained by replacing LS bilateral consistency (LS converse consistency) with RS bilateral consistency (RS converse consistency) in the above axiomatizations of the CEB rule. Our answer is positive. The proof of this assertion makes use of the following lemma, which is an application of the Elevator Lemma (Thomson, 2010).

Elevator Lemma: If a rule \(\varphi\) is RS bilaterally consistent and coincides with a RS conversely consistent rule \(\varphi'\) in the two-agent case, then \(\varphi\) coincides with \(\varphi'\) in general.

Since the proofs of the axiomatizations of the nucleolus are immediate by invoking Remark 1, Proposition 2 in Hu et al. (2012), and the Elevator Lemma, we omit the proofs.\(^{21}\)

\(^{20}\)Hu et al. (2012) also show that the CEB rule is the only LS bilaterally consistent (or LS conversely consistent) rule satisfying “order preservation for benefits” and “cost monotonicity”. Since the nucleolus violates cost monotonicity, we do not consider these characterizations of the CEB rule.

\(^{21}\)Our axiomatizations of the nucleolus are special applications of the lemma. To demonstrate how the lemma is applied to here, we give a formal proof of Theorem 1. Let \((N, c) \in \mathcal{A}\) with \(N \equiv \{1, \ldots, n\}\) and \(c_1 \leq \cdots \leq c_n\). Clearly, the nucleolus satisfies the properties of the theorem. Conversely, let \(\varphi\) be a RS bilaterally consistent rule satisfying the two properties. Let \(x \equiv \varphi(N, c)\) and \(y \equiv Nu(N, c)\). We show that \(x = y\). By RS
Theorem 1: The nucleolus is the only RS bilaterally consistent rule satisfying equal treatment of equals and last-agent cost additivity.

Theorem 2: The nucleolus is the only RS conversely consistent rule satisfying equal treatment of equals and last-agent cost additivity.

Theorems 1 and 2 together with Hu et al. (2012)’s characterizations of the CEB rule point out that the nucleolus and the CEB rule can be axiomatized by different bilateral consistency and converse consistency properties from axiomatic perspective.

4 Implementation

To serve our purpose, we take Hu et al. (2012)’s game as our benchmark and revise it on the basis of RS bilateral consistency and RS converse consistency. Let \((N, c) \in A\) with \(c \equiv (c_i)_{i \in N} \in \mathbb{R}^N_+\). For each \(j \in N\) and each \(k \in N\), let \(S^k_{-j} \equiv \{1, \cdots, k\}\setminus\{j\}\) be the set of agents, excluding agent \(j\), whose indices are no greater than \(k\). The resulting game is a 3-stage extensive form game denoted by \(\Omega(N, c)\) and its game tree is depicted in Figure 2.

Stage 1: Each agent \(k \in N \setminus \{n\}\) independently announces her own contribution \(x_k \in \mathbb{R}\). Let \(x_n \equiv c_n - \sum_{k \neq n} x_k\). We refer to \(x \equiv (x_k)_{k \in N}\) as the proposal.

Stage 2: Agent \(n\) takes one of the following two actions:

1. Accept \(x_n\) (in short, action \(A\)). The game ends with \(x\) as the outcome.
2. Reject \(x_n\) and pick one agent from \(N \setminus \{n\}\), say agent \(i\) (in short, action \((R, i)\)). The game proceeds to the final stage.

Stage 3: Agent \(i\) picks one group from \(\{S^1_{-i}, S^{i+1}_{-i}, \cdots, S^{n-1}_{-i}\}\), say \(S^j_{-i}\) (in short, action \(S^j_{-i}\)). The game ends with \((\tau^i_n(c, x, j), x)\).

The bilateral consistency of \(\varphi\), for each \(i \in N \setminus \{n\}\), \(\varphi(i, n, r^i_{(i, n)}) = x_{(i, n)}\). By Proposition 2 in Hu et al. (2012) and Remark 1, \(x_{(i, n)} = Nu(i, n, r^i_{(i, n)})\). Since \(\varphi\) is RS bilaterally consistent and the nucleolus is RS conversely consistent, by the Elevator Lemma, \(x = y\).
Stage 1:
Each agent $k \in N \setminus \{n\}$ announces a number $x_k \in \mathbb{R}$. Let $x_n = c_n - \sum_{k \neq n} x_k$.

Stage 2:
Agent $n$ decides to take $A$ (accept $x_n$) or $(R, i)$ (reject $x_n$ and choose one agent from $N \setminus \{n\}$, say agent $i$).

Stage 3:
Agent $i$ picks $S^i$ from $\{S^1_{i-1}, S^2_{i-1}, \ldots, S^{n-1}_{i-1}\}$, where for each $k \in \{i, \ldots, n-1\}$, $S^k_{i-1} \equiv \{1, \ldots, k\} \setminus \{i\}$.

Figure 3: The game tree of $\Omega(N, c)$
as the outcome, where each agent $k \in N \setminus \{i, n\}$ contributes $x_k$, and agents $i$ and $n$ contribute

$$\tau^i(c, x, j) \equiv \frac{1}{2} \left\{ c_j - \sum_{m \leq j \atop m \neq i, n} x_m + x_i + x_n - \left( c_n - \sum_{m \leq n \atop m \neq i, n} x_m \right) \right\} \quad \text{and}$$

$$\tau^n(c, x, j) \equiv \frac{1}{2} \left\{ c_n - \sum_{m \leq n \atop m \neq i, n} x_m + x_i + x_n - \left( c_j - \sum_{m \leq j \atop m \neq i, n} x_m \right) \right\} ,$$

respectively.

We elaborate $\Omega(N, c)$ and discuss the differences between our game and Hu et al. (2012)’s game.

1. In Hu et al. (2012)’s game, each agent $k \in N \setminus \{n\}$ proposes her own contribution $x_k \in \mathbb{R}$ such that $0 \leq x_k \leq c_k$ in Stage 1. However, we do not impose any upper or lower bound on agents’ proposals in our game. Moreover, it can be shown that even if the condition (namely, for each agent $k \in N \setminus \{n\}$, $0 \leq x_k \leq c_k$) is imposed, our implementation results hold still.

2. In contrast to Hu et al. (2012)’s game, we introduce an additional stage. This addition is necessary since due to the definition and the essence of the reduced problem underlying $RS$ bilateral consistency and $RS$ converse consistency, the agent chosen by agent $n$ in Stage 2 is entitled to minimize her contribution by choosing a group of agents and using this group’s contributions to cover the cost of building the part of the runway she and all agents in the group can use.

3. In Hu et al. (2012)’s game, the contributions of agent $n$ and her partner, say agent $i$, are specified on the basis of the reduced problem underlying $LS$ bilateral consistency and $LS$ converse consistency. We specify their contributions on the basis of the reduced problem underlying $RS$ bilateral consistency and $RS$ converse consistency as follows. Note that $S_{i-1}^i = S_{-i}^i$. Imagine that there is a fair coin to select one of the two agents. The chosen one, say agent $l \in \{i, n\}$, picks a group, say $S_{l-1}^l$,
from \( \{S_{i-1}^{l}, \ldots, S_n^{l-1}\} \) and takes \( \sum_{k \in S_{i-1}^{j}} x_k \) to cover \( \max_{k \in S_{i-1}^{j}} \{c_l, c_k\} \) (the cost of building the part of the runway agent \( l \) and all agents in \( S_{i-1}^{j} \) can use). The other then uses \( \max \left\{ \sum_{k \in S_{i-1}^{j}} x_k - \max_{k \in S_{i-1}^{j}} \{c_l, c_k\}, 0 \right\} \) and \( \sum_{k \in N \setminus (S_{i-1}^{j} \cup \{i, n\})} x_k \) (the sum of the remaining contributions already made) to cover the remainder \( c_n - \max_{k \in S_{i-1}^{j}} \{c_l, c_k\} \). In Hu et al. (2012)’s game, agent \( l \) has no choice but to pick the group \( S_{i-1}^{n-1} \) and takes the sum of their contributions \( \sum_{k \in S_{i-1}^{n-1}} x_k \) to cover her cost \( c_l \). The other then uses \( \max \left\{ \sum_{k \in N \setminus \{i, n\}} x_k - c_l, 0 \right\} \) to cover the remainder \( c_n - c_l \).

4. The design of the feasible groups for agent \( i \) (namely, \( \{S_{i-1}^{i}, \ldots, S_n^{i-1}\} \)) is reasonable. To see this, note again that \( S_{i-1}^{i-1} = S_{i-1}^{i} \). We argue that agent \( i \) does not pick a group from \( \{S_{i-1}^{1}, \ldots, S_{i-2}^{i-1}\} \). Suppose, by contradiction, that agent \( i \) picks \( S_{i-1}^{k} \) with \( k \leq i - 2 \). Then she can use \( \sum_{j \in S_{i-1}^{k}} x_j \) to cover \( c_i \) (the cost of building the part of the runway agent \( i \) and all agents in \( S_{i-1}^{k} \) can use). Since a runway that serves agent \( i \) can also serve all agents in \( S_{i-1}^{i-1} \), agent \( i \) can contribute less by picking \( S_{i-1}^{i-1} \) rather than \( S_{i-1}^{k} \) and taking \( \sum_{j \in S_{i-1}^{i-1}} x_j \) instead of \( \sum_{j \in S_{i-1}^{k}} x_j \), to cover \( c_i \). By the same reasoning, agent \( i \) does not pick a group \( S \) such that if \( k \in S \), then there is \( p \notin S \) with \( p < k \).

The following notation is used to construct a SPE of \( \Omega(N, c) \). Let \( x \in \mathbb{R}^N \) and \( i \in N \setminus \{n\} \). When \( x \) is the proposal, agent \( n \) takes agent \( i \) in Stage 2, and agent \( i \) takes action \( S_{i-1}^{k} \) in Stage 3, the contributions of agents \( i \) and \( n \) in Stage 3, denoted respectively by \( \tau^i_x (c, x, k) \) and \( \tau^i_n (c, x, k) \), are defined as follows:

\[
\tau^i_x (c, x, k) \equiv \frac{1}{2} \left( c_k - \sum_{m \leq k, m \neq i} x_m \right)
\]

and

\[
\tau^i_n (c, x, k) \equiv x_i + x_n - \tau^i_x (c, x, k).
\]

Let \( \tau^i (c, x, k) \equiv (\tau^i_x (c, x, k), \tau^i_n (c, x, k)) \) be the contributions vector of agents \( i \) and \( n \) in Stage 3. The expected contributions of agents \( i \) and \( n \) in
Stage 2, denoted respectively by $\tau^i_1(c, x)$ and $\tau^i_n(c, x)$, are defined as follows:

$$\tau^i_1(c, x) \equiv \min_{i \leq k < n} \tau^i_1(c, x, k)$$
and

$$\tau^i_n(c, x) \equiv x_i + x_n - \tau^i_1(c, x).$$

Let $\tau^i(c, x) \equiv (\tau^i_1(c, x), \tau^i_n(c, x))$ be the expected contributions vector of agents $i$ and $n$ in Stage 2.

Note that the nucleolus and the constrained equal contributions rule are \emph{RS bilaterally consistent} and \emph{RS conversely consistent}. As shown next, there is one and only one SPE outcome of $\Omega(N, c)$ and moreover, it is the allocation chosen by the nucleolus. As usual, we solve $\Omega(N, c)$ by backward induction.

**Theorem 3:** Let $(N, c) \in A$ with $c \equiv (c_i)_{i \in N} \in \mathbb{R}^N_{++}$. There is a SPE of $\Omega(N, c)$ with outcome $Nu(N, c)$.

**Proof.** Let $(N, c) \in A$ with $c \equiv (c_i)_{i \in N} \in \mathbb{R}^N_{++}$. The proof is by construction of a strategy profile $f$ that generates $Nu(N, c)$ as the outcome. The profile $f$ is defined as follows.

**Stage 1:** Each agent $k \in N \setminus \{n\}$ announces $Nu_k(N, c)$.

**Stage 2:** Let $x$ be the proposal. Let $\mu \equiv \min_{i \in N \setminus \{n\}} \tau^i_n(c, x)$. If $x_n \leq \mu$, agent $n$ takes action $A$; otherwise, she takes action $(R, i)$ such that $\tau^i_n(c, x) = \mu$.

**Stage 3:** Agent $i$ takes action $S^k_{\neq i}$ such that $\tau^i_1(c, x, k) = \min_{i \leq j < n} \tau^i_1(c, x, j)$.

We first show that following $f$, the outcome is the allocation chosen by the nucleolus, and then conclude the proof by showing that $f$ is a SPE.

**Step 1:** Following $f$, the outcome is $Nu(N, c)$. Following $f_k$, each agent $k \in N \setminus \{n\}$ announces $Nu_k(N, c)$ in Stage 1. Let $\bar{x} \equiv Nu(N, c)$. Then $\bar{x}$ is the proposal. Since the nucleolus is \emph{RS bilaterally consistent} and coincides with the standard rule for two-agent problems, then for each $i \in N \setminus \{n\}$,

$$\tau^i_1(c, \bar{x}) = \frac{1}{2} \min_{i \leq k < n} \left( c_k - \sum_{m \leq k, m \neq i} \bar{x}_m \right) = \frac{1}{2} (r^i_{k,n})_i = \bar{x}_i$$
and

\[ \tau_n^i(c, \bar{x}) = \bar{x}_n, \]

which implies that \( \bar{x}_n = \mu \). Following \( f_n \), agent \( n \) takes action \( A \) and the game ends with outcome \( Nu(N, c) \).

**Step 2: \( f \) is a SPE.** Let \( i \in N \setminus \{n\} \). It is easy to see that following \( f_i \) in Stage 3 and \( f_n \) in Stage 2 are best responses for agents \( i \) and \( n \). We show that following \( f_i \) in Stage 1 is a best response for agent \( i \). Let \( \varepsilon \in \mathbb{R} \) be such that \( 0 < \varepsilon \leq \bar{x}_i \). Suppose that agent \( i \) deviates in Stage 1 by announcing \( x'_i = \bar{x}_i - \varepsilon \). Let \( x' \) be the proposal after agent \( i \)'s deviation. Then \( x'_n = \bar{x}_n + \varepsilon \), and for each \( j \in N \setminus \{i, n\} \), \( x'_j = \bar{x}_j \). Since \( \bar{x}_n = \tau_n^i(c, \bar{x}) \) and for each \( l \in N \setminus \{n\} \), \( \bar{x}_l = \tau_l^i(c, \bar{x}) \), then for each \( j \in N \setminus \{i, n\} \),

\[
\tau_j^l(c, x') = \min_{j \leq k < n} \left( \tau_j^l(c, x', k) \right) \\
= \frac{1}{2} \min_{j \leq k < n} \left( c_k - \sum_{m \leq k, m \neq j} x'_m \right) \\
\leq \frac{1}{2} \min_{j \leq k < n} \left( c_k - \sum_{m \leq k, m \neq j} \bar{x}_m + \varepsilon \right) \\
= \tau_j^l(c, \bar{x}) + \frac{\varepsilon}{2}
\]

and

\[
\tau_n^i(c, x') = x'_j + x'_n - \tau_j^l(c, x') \\
\geq \bar{x}_j + \bar{x}_n + \varepsilon - \left\{ \tau_j^l(c, \bar{x}) + \frac{\varepsilon}{2} \right\} \\
= \bar{x}_n + \frac{1}{2} \varepsilon \\
> \bar{x}_n \\
= \tau_n^i(c, \bar{x}).
\]

Following \( f_n \), agent \( n \) takes action \((R, i)\) in Stage 2. Then agent \( i \)'s contribution is at most \( \bar{x}_i \), she is not better off deviating. Let \( \varepsilon \in \mathbb{R} \) be such that \( 0 < \varepsilon \leq c_i - \bar{x}_i \). Suppose that agent \( i \) deviates in Stage 1 by announcing
\[ x'_i = \bar{x}_i + \varepsilon. \] Then, it can be shown that following \( f_n \), agent \( n \) takes action \( A \). Since agent \( i \)'s contribution is \( x'_i > \bar{x}_i \), she is not better off deviating. \( Q.E.D. \)

It is not difficult to see that there are many SPE of the game. However, our next result shows that there is one and only one SPE outcome and moreover, it is the nucleolus contributions vector.

**Theorem 4:** Let \((N, c) \in A \) with \( c \equiv (c_i)_{i \in N} \in \mathbb{R}^N_+ \). Each SPE outcome of \( \Omega(N, c) \) is \( Nu(N, c) \).

**Proof.** Let \((N, c) \in A \) with \( c \equiv (c_i)_{i \in N} \in \mathbb{R}^N_+ \) and \( g \) be an SPE of \( \Omega(N, c) \). Following \( g_k \), each agent \( k \in N \setminus \{n\} \) announces \( x_k \) in Stage 1. Let \( x_n \equiv \sum_{k \neq n} x_k \) and \( x \equiv (x_i)_{i \in N} \) be the proposal. In Stage 2, following \( g_n \), agent \( n \) takes either action \( A \) or action \( (R, i) \) for some \( i \in N \setminus \{n\} \). We consider two cases.

**Case 1:** Agent \( n \) takes action \( A \). The game ends with outcome \( x \). We show that \( x = Nu(N, c) \). To see this, we first prove that for each \( k \in N \setminus \{n\} \), \( x_{\{k,n\}} = \tau^k(c, x) \). Let \( k \in N \setminus \{n\} \). Suppose, by contradiction, that \( x_{\{k,n\}} \neq \tau^k(c, x) \). Since by subgame perfection, \( x_n \leq \tau^k_n(c, x) \), thus \( x_n < \tau^k_n(c, x) \). Let \( \varepsilon \in \mathbb{R} \) be such that \( 0 < \varepsilon < \tau^k_n(c, x) - x_n \). We show that agent \( k \) is better off announcing \( x_k - \varepsilon \) in Stage 1. Let \( x' \) be the proposal after agent \( k \)'s deviation. Then, \( x'_k = x_k - \varepsilon, \) \( x'_n = x_n + \varepsilon \), and for each \( j \in N \setminus \{k, n\} \), \( x'_j = x_j \). Since \( \tau^k_n(c, x') = \frac{1}{2} \min_{k \leq j < n} \left( c_j - \sum_{m \leq j, m \neq k} x_m \right) \equiv \tau^k_n(c, x) \), thus \( x_n = x_n + \varepsilon < \tau^k_n(c, x) = \tau^k_n(c, x') \). Following \( g_n \), agent \( n \) does not take action \( (R, k) \), which implies that agent \( k \)'s contribution is \( x_k - \varepsilon < x_k \). Since she is better off deviating, the assumption that \( g \) is an SPE is violated. Thus, for each \( k \in N \setminus \{n\} \), \( x_{\{k,n\}} = \tau^k(c, x) \). Since for each \( k \in N \setminus \{n\} \), \( \tau^k(c, x) = Nu\left( \{k, n\}, r^i_{\{k,n\}} \right) \), by RS converse consistency, \( x = Nu(N, c) \).

**Case 2:** Agent \( n \) takes action \( (R, i) \) for some \( i \in N \setminus \{n\} \). Let \( y \equiv (\tau^i(c, x), x_N \setminus \{i, n\}) \) be the outcome. Note that \( \tau^i_1(c, x) = \min_{i \leq k < n} \tau^i_1(c, x, k) \). We show that \( y = Nu(N, c) \). If \(|N| = 2\), since \( N \equiv \{i, n\} \), then \( y_i = \tau^i_1(c, x) = \frac{1}{2} c_i \) and \( y_n = \tau^i_1(c, x) = x_i + x_n - \tau^i_1(c, x) = c_n - \frac{1}{2} c_i \), which
implies that $y = Nu(N, c)$. Let $|N| \geq 3$. Note that for each $k \in N \setminus \{i, n\}$, by subgame perfection, $\tau^i_n(c, x) \leq \tau^k_n(c, x)$. The proof is by the following claims.

**Claim 1:** For each $k \in N \setminus \{i, n\}$, $\tau^i_n(c, x) = \tau^k_n(c, x)$. Suppose, by contradiction, that there is $k \in N \setminus \{i, n\}$ such that $\tau^i_n(c, x) \neq \tau^k_n(c, x)$. By subgame perfection, $\tau^i_n(c, x) < \tau^k_n(c, x)$. Let $\delta$ be a number such that $0 < \delta < \tau^k_n(c, x) - \tau^i_n(c, x)$. We show that agent $k$ is better off announcing $x_k - \delta$. Let $x''$ be the proposal after agent $k$’s deviation. Then, $x''_k = x_k - \delta$, $x''_n = x_n + \delta$, and for each $j \in N \setminus \{k, n\}$, $x''_j = x_j$. It follows that $\tau^k(c, x'') = \tau^k(c, x)$ and for some $p$ with $i \leq p < k$,

$$\tau^i(c, x'') = \frac{1}{2} \min_{1 \leq i \leq n} \left( c_i - \sum_{m \leq i, m \neq i} x''_m \right) \geq \frac{1}{2} \left( c_p - \sum_{m \leq p, m \neq i} x''_m \right).$$

If $k \leq p$, then by the definition of $\tau^i(c, x)$,

$$\tau^i(c, x'') = \frac{1}{2} \left( c_p - \sum_{m \leq p, m \neq i} x''_m \right) + \frac{1}{2} \delta \geq \tau^i(c, x) + \frac{1}{2} \delta. \quad (5)$$

If $k > p$, then by the definition of $\tau^i(c, x)$,

$$\tau^i(c, x'') = \frac{1}{2} \left( c_p - \sum_{m \leq p, m \neq i} x''_m \right) = \frac{1}{2} \left( c_p - \sum_{m \leq p, m \neq i} x''_m \right) \geq \tau^i(c, x). \quad (6)$$

It follows that $\tau^i(c, x'') \geq \tau^i(c, x)$ and

$$\tau^i_n(c, x'') = x''_i + x''_n - \tau^i(c, x'') \leq x_i + x_n + \delta - \tau^i(c, x) = \tau^i_n(c, x) + \delta < \tau^k_n(c, x) = \tau^k(c, x'').$$
Following $g_n$, agent $n$ does not take action $(R, k)$, which implies that agent $k$ contributes $x''_k < x_k$. Since she is better off deviating, the assumption that $g$ is a SPE is violated. Thus, for each $k \in N \setminus \{i, n\}$, $\tau^i_n(c, x) = \tau_k^i(c, x)$.

**Claim 2:** $x_{\{i,n\}} = \tau^i(c, x)$. Suppose, by contradiction, that $x_{\{i,n\}} \neq \tau^i(c, x)$. By subgame perfection, $x_n \geq \tau^i_n(c, x)$. Thus, $x_n > \tau^i_n(c, x)$. Since $\tau^i_1(c, x) + \tau^i_n(c, x) = x_i + x_n$, then $x_i < \tau^i_1(c, x)$. Let $\varepsilon$ be a number such that $0 < \varepsilon < \tau^i_1(c, x) - x_i$. We show that agent $i$ is better off announcing $x_i + \varepsilon$. Let $\hat{x}$ be the proposal after agent $i$’s deviation. Then, $\hat{x}_i = x_i + \varepsilon$, $\hat{x}_n = x_n - \varepsilon$, and for each $j \in N \setminus \{i, n\}$, $\hat{x}_j = x_j$. Thus, $\tau^i(c, \hat{x}) = \tau^i(c, x)$. Let $k \in N \setminus \{i, n\}$. By the similar arguments as those for (5) and (6), we have

$$\tau^k_n(c, \hat{x}) = \frac{1}{2} \min_{l \leq i < n} \left( c_l - \sum_{m \leq i, m \neq k} \hat{x}_m \right) \geq \tau^k(c, x) - \frac{1}{2} \varepsilon. \quad (7)$$

By (7) and Claim 1,

$$\tau^k_n(c, \hat{x}) = \hat{x}_k + \hat{x}_n - \tau^k_n(c, \hat{x}) \\
\leq x_k + x_n - \varepsilon - \left( \tau^k(c, x) - \frac{1}{2} \varepsilon \right) \\
= \tau^k_n(c, x) - \frac{1}{2} \varepsilon \\
= \tau^i_n(c, x) - \frac{1}{2} \varepsilon \\
< \tau^i_n(c, x) \\
= \tau^i(c, \hat{x}).$$

Following $g_n$, agent $n$ does not take action $(R, i)$, which implies that agent $i$ contributes $x_i + \varepsilon < \tau^i_1(c, x)$. Since she is better off deviating, the assumption that $g$ is a SPE is violated. Thus, $x_{\{i,n\}} = \tau^i(c, x) = y_{\{i,n\}}$ and by the definition of $y$, $x = y$.

**Claim 3:** For each $j \in N \setminus \{i, n\}$, $x_{\{j,n\}} = \tau^j(c, x)$. Suppose, by contradiction, that there is $j \in N \setminus \{i, n\}$ such that $x_{\{j,n\}} \neq \tau^j(c, x)$. If $x_n > \tau^j_n(c, x)$, agent $n$ is better off taking action $(R, j)$ and contributing $\tau^j_n(c, x) < x_n = y_n$, which violates the assumption that $g$ is a SPE. If $x_n < \tau^j_n(c, x)$, then let $\eta$ be a number such that $0 < \eta < \tau^j_n(c, x) - x_n$. We show that agent $j$ is better off announcing $x_j - \eta$. Let $\tilde{x}$ be the proposal after agent
Then, \( \tilde{x}_j = x_j - \eta, \tilde{x}_n = x_n + \eta \), and for each \( l \in N \setminus \{ j, n \}, \tilde{x}_l = x_l \). Thus, \( \tau^j (c, \tilde{x}) = \tau^j (c, x) \). By the similar arguments as those for (5) and (6), we derive that for each \( k \in N \setminus \{ j, n \}, \tau^k (c, \tilde{x}) \geq \tau^k (c, x) \). By Claims 1 and 2,

\[
\tau^k_n (c, \tilde{x}) = \tilde{x}_k + \tilde{x}_n - \tau^k_k (c, \tilde{x}) \\
\leq x_k + x_n + \eta - \tau^k_k (c, x) \\
= \tau^n_k (c, x) + \eta \\
= \tau^n_i (c, x) + \eta \\
= x_n + \eta \\
< \tau^n_i (c, x).
\]

Following \( g_n \), agent \( n \) does not take action \( (R, j) \), which implies that agent \( j \) contributes \( x_j - \eta \). Since she is better off deviating, the assumption that \( g \) is an SPE is violated. Thus, for each \( l \in N \setminus \{ i, n \}, x_{\{l,n\}} = \tau^l (c, x) \).

By Claims 2 and 3, we conclude that for each \( j \in N \setminus \{ n \}, x_{\{k,n\}} = \tau^k (c, x) \). Since for each \( j \in N \setminus \{ n \}, \tau^j (c, x) = Nu \left( \{ j, n \}, r^*_j \right) \), by RS converse consistency, \( x = Nu (N, c) \). By Claim 2, \( x = y \), We then conclude that \( y = Nu (N, c) \).

Q.E.D.

Theorems 3 and 4 together with Hu et al. (2012)’s implementation of the CEB rule show that different bilateral consistency and converse consistency properties play important roles in implementing the nucleolus and the CEB rule. This is the first paper to point out this surprising phenomenon. It would be interesting to investigate whether such phenomenon occurs between other rules in other allocation problems.

Arin et al. (2009) introduce a \( n \)-stage sequential game in the spirit of Dagan et al. (1997)’s game that implements the nucleolus in the airport problem. In Stage 1, agent \( n \) announces a contributions vector \( x \). If \( x \) is unanimously accepted, it is the outcome; otherwise, bilateral negotiations take place. Responders reply sequentially according to a specified order on the set of responders. Acceptors contribute their components of \( x \) and each rejector \( j \) negotiates her contribution with agent \( n \) by invoking a particular rule to solve the two-agent reduced problem underlying the definitions of RS
bilateral consistency and RS converse consistency. They show that there is a unique Nash equilibrium outcome of their game and it is the nucleolus allocation.

Our game differs from theirs: (1) in our game, there are \( n - 1 \) proposers who proposes her contribution (namely, each agent \( i \in N \setminus \{n\} \) announces a number \( x_i \)) and one responder thinks of whether or not to contribute the residual cost \( (x_n \equiv c_n - \sum_{j \in N \setminus \{n\}} x_j) \); (2) our implementation of the nucleolus neither relies on any specific order on the set of agents nor any particular rule in solving any two-agent reduced problem; and (3) our equilibrium concept is SPE.

5 Concluding remarks

We show that axiomatizations of the nucleolus can be obtained from Hu et al. (2012)’s axiomatizations of the CEB rule by replacing LS bilateral consistency (or LS converse consistency) with RS bilateral consistency (or RS converse consistency). In addition, an implementation of the nucleolus can be established by revising Hu et al. (2012)’s game, based on RS bilateral consistency and RS converse consistency, that implements the CEB rule and exploits LS bilateral consistency and LS converse consistency. To our best knowledge, this is the first paper to observe such surprising phenomenon from non-cooperative perspective. In sum, our paper sheds light into the important roles bilateral consistency and converse consistency principles play in deepening our understanding of the fundamental differences between the nucleolus and the CEB rule from axiomatic as well as strategic perspectives.

In the formulation of the reduced problem underlying RS bilateral consistency and RS converse consistency, we exclude the possibility of the last agent leaving. One may wonder what if we include such possibility and define a stronger version of RS bilateral consistency and a weaker version of RS converse consistency. Clearly, the nucleolus and the CEC rule satisfy both strong RS bilateral consistency and weak RS converse consistency. Moreover, it can be shown that a game obtained by slightly revising our game, based on strong RS bilateral consistency and weak RS converse consistency, implements the nucleolus. However, an implementation of the CEC rule cannot be obtained in a similar way. Since the CEC rule does not coincide with the
standard rule in the two-agent problem, introducing a fair coin in specifying expected contributions of the responder and her partner is no longer available. This is the reason why the game that implements the nucleolus and exploits strong RS bilateral consistency and weak RS converse consistency does not implement the CEC rule. This fact points out that the coincidence between the nucleolus and the standard rule in the two-agent problem also plays an important role in implementing the nucleolus.

It would be interesting to investigate whether such phenomenon occurs between other rules in other allocation problems from axiomatic and strategic perspectives.

References


